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ABSTRACT. In this note we prove the existence of a localization/delocalization transition for Landau Hamiltonians randomly perturbed by an electric potential with unbounded amplitude. In particular, with probability one, no Landau gaps survive as the random potential is turned on; the gaps close, filling up partly with localized states. A minimal rate of transport is exhibited in the region of delocalization. To do so, we exploit the a priori quantization of the Hall conductance and extend recent Wegner estimates to the case of unbounded random variables.

1. INTRODUCTION

In this note we prove the existence of a dynamical localization/delocalization transition for Landau Hamiltonian randomly perturbed by an electric potential with unbounded amplitude, extending results from [GKS1, GKS2]. In [GKS1] the perturbation had to be sufficiently small compared to the strength of the magnetic field: the amplitude of the random potential was such that the Landau gaps survived after adding the perturbation. In [GKS2] the Landau gaps where allowed to close, but the random potentials were bounded. In this article we consider random potentials such that, with probability one, all the Landau gaps close as the random potential is turned on, and are shown to be (partially) filled up with localized states. As in [GKS1, GKS2], a minimal rate of transport is exhibited in the region of delocalization.

These results exploit the a priori quantization of the Hall conductance proved in [GKS2]. Many of the results we will need rely on [GK1, GK4], where the random potential was assumed to be bounded. Such a strong assumption is not necessary, and can be replaced by weaker hypotheses, satisfied by the random Landau Hamiltonian with unbounded random couplings studied in this paper. We will require Wegner estimates for these random operators, which are obtained by extending the analysis of [CHK1, CHK2] to the case of unbounded random variables, a result of independent interest.

We now describe the model and the results. We consider a $\mathbb{Z}^2$-ergodic Landau Hamiltonian

$$H_{B,\lambda,\boldsymbol{\omega}} = H_B + \lambda V_{\boldsymbol{\omega}} \quad \text{on} \quad \mathrm{L}^2(\mathbb{R}^2, \mathrm{d}x), \tag{1.1}$$

where H_B is the (free) Landau Hamiltonian,

$$H_B = (-i\nabla - \mathbf{A})^2 \quad \text{with} \quad \mathbf{A} = \tfrac{B}{2}(x_2, -x_1). \tag{1.2}$$

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(**A** is the vector potential and $B > 0$ is the strength of the magnetic field, we use the symmetric gauge and incorporated the charge of the electron in the vector potential), $\lambda \geq 0$ is the disorder parameter, and V_ω is an unbounded ergodic potential: there is a probability space $(\Omega, \mathbb{P})$ equipped with an ergodic group $\{\tau(a);\ a \in \mathbb{Z}^2\}$ of measure preserving transformations, a potential-valued map V_ω on Ω, measurable in the sense that $\langle \phi, V_{\boldsymbol{\omega}} \phi \rangle$ is a measurable function of ω for all $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^2)$. We assume that

$$V_{\boldsymbol{\omega}}(x) = \sum_{j \in \mathbb{Z}^2} \omega_j u(x - j), \tag{1.3}$$

where the single site potential u is a nonnegative bounded measurable function on $\mathbb{R}^d$ with compact support, uniformly bounded away from zero in a neighborhood of the origin, and the ω_j's are independent, identically distributed random variables, whose common probability distribution μ has a bounded density ρ with $\operatorname{supp} \rho = \mathbb{R}$ and fast decay:

$$\rho(\omega) \leq \rho_0 \exp(-|\omega|^\alpha), \tag{1.4}$$

for some $\rho_0 \in]0, +\infty[$ and $\alpha > 0$. We fix constants for u by

$$C_- \chi_{\Lambda_{\delta_-}(0)} \leq u \leq C_+ \chi_{\Lambda_{\delta_+}(0)} \quad \text{with } C_\pm, \delta_\pm \in]0, \infty[, \tag{1.5}$$

and normalize u so that we have $\| \sum_{j \in \mathbb{Z}^2} u_j \|_\infty \leq 1$. (We write $\Lambda_L(x) := x + \left[-\frac{L}{2}, \frac{L}{2}\right[^d$ for the box of side $L > 0$ centered at $x \in \mathbb{R}^2$, with $\chi_{\Lambda_L(x)}$ being its characteristic function. We also write $\chi_x = \chi_{\Lambda_1(x)}$.)

Under these hypotheses, $H_{B,\lambda,\boldsymbol{\omega}}$ is essentially self-adjoint on $\mathcal{C}_c^\infty(\mathbb{R}^d)$ with probability one, with the bound $\left(\langle x \rangle := \sqrt{1 + |x|^2}\right)$

$$H_{B,\lambda,\boldsymbol{\omega}} \geq -c_\omega (\log \langle x \rangle)^\beta, \text{ for all } x \in \mathbb{R}^d, \tag{1.6}$$

for any given $\beta > \alpha^{-1}$, with c_ω depending also on α, β, d. (See Lemma A.1.)

Moreover, the unbounded random potential V_ω satisfies the probability estimate of Lemma A.1, namely (A.1), the condition that replaces the boundedness of the potential in [GK1, GK4]. Note that (A.1) is similar to the condition given in [U, Eq. (3.2)]. Using the Wegner estimate given in Theorem B.1, we can conclude, similarly to the results in [U] for a continuous Gaussian random potential, that the results of [GK1, GK4], and hence also [GK2, GK5], hold for $H_{B,\lambda,\boldsymbol{\omega}}$. (See also Appendix A.) This condition also suffices for the validity of [GKS2, Theorems 1.1 and 1.2]. Thus we just refer to [GK1, GK2, GK4, GK5, GKS2] where appropriate.

The spectrum $\sigma(H_B)$ of the Landau Hamiltonian H_B consists of a sequence of infinitely degenerate eigenvalues, the Landau levels:

$$B_n = (2n - 1)B, \quad n = 1, 2, \ldots . \tag{1.7}$$

For further reference, we also set

$$\mathcal{B}_1 =]-\infty, 2B[, \quad \text{and} \quad \mathcal{B}_n =]B_n - B, B_n + B[, \quad n = 2, 3, \ldots . \tag{1.8}$$

On the other hand, as soon as $\lambda > 0$, the spectrum fills the Landau gaps and we have [BCH]

$$\sigma(H_{B,\lambda,\boldsymbol{\omega}}) = \mathbb{R}, \quad \mathbb{P} - \text{a.s.} \tag{1.9}$$

The fact that the Landau gaps are immediately filled up as soon as the disorder is turned on implies that the approach used in [GKS1] is non applicable. More properties of the Hall conductance are needed in order to perform the simple reasonning

that provides the existence of a dynamical transition. More precisely, it becomes crucial to know a priori that the Hall conductance is an integer in the region of complete localization (which includes the spectral gaps), a fact that was circumvented in [GKS1] by resorting to an open gap condition. That the Hall conductance for ergodic models is integer valued in the localization region was known for discrete Anderson type models since [BeES, AG]. For ergodic Schrödinger operators in the continuum, it was first established in [AvSS] for energies in gaps and extended to the region of complete localization in [GKS2], where the analysis of [AG] has been carried over to the continuum. This property has to be combined with the continuity of the Hall conductance for arbitrary small λ (in order to let λ go to zero). In [GKS2] it is shown that it is actually enough to prove the same continuity property but for the integrated density of states; see [GKS2, Lemma 3.1]. This is done in this note by revisiting the article [HiKS]; see Theorem B.2. But first, we extend the Wegner estimate given in [CHK2] to unbounded random variables; the estimate is given in terms of the concentration function of a measure which is a modification of the single-site probability measure μ. (See Theorem B.1, which has independent interest.)

We state the main result of this note and its corollary. Following [GK4, GK5, GKS1, GKS2], we set $\Xi^{\mathrm{DL}}_{B,\lambda}$ to be the region of complete localization (gaps included), that is, the set of energies where the multiscale analysis applies (or, if applicable, the fractional moment method of [AENSS]). Its complement is the set of dynamical delocalization $\Xi^{\mathrm{DD}}_{B,\lambda}$. An energy $E \in \Xi^{\mathrm{DD}}_{B,\lambda}$ such that for any $\varepsilon > 0$, $[E-\varepsilon, E+\varepsilon] \cap \Xi^{\mathrm{DL}}_{B,\lambda} \neq \emptyset$, is called a dynamical mobility edge.

Theorem 1.1. *Let $H_{B,\lambda,\boldsymbol{\omega}}$ be a random Landau Hamiltonian as above. For each $n = 1, 2, \ldots$, if λ is small enough (depending on n) there exist dynamical mobility edges $\widetilde{E}_{j,n}(B,\lambda) \in \mathcal{B}_n$, $j = 1, 2$, such that*

$$\max_{j=1,2} \left| \widetilde{E}_{j,n}(B,\lambda) - B_n \right| \leq K_n(B) \lambda \, |\log \lambda|^{\frac{1}{\alpha}} \to 0 \quad \text{as } \lambda \to 0, \tag{1.10}$$

with a finite constant $K_n(B)$. (It is possible that $\widetilde{E}_{1,n}(B,\lambda) = \widetilde{E}_{2,n}(B,\lambda)$, i.e., dynamical delocalization occurs at a single energy.)

By the characterization of the region of complete localization established in [GK4], Theorem 1.1 has a consequence in terms of transport properties of the Hall system. Indeed, to measure "dynamical delocalization" as stated in the theorem, we introduce

$$M_{B,\lambda,\omega}(p, \mathcal{X}, t) = \left\| \langle x \rangle^{\frac{p}{2}} \mathrm{e}^{-itH_{B,\lambda,\omega}} \mathcal{X}(H_{B,\lambda,\omega}) \chi_0 \right\|_2^2, \tag{1.11}$$

the random moment of order $p \geq 0$ at time t for the time evolution in the Hilbert-Schmidt norm, initially spatially localized in the square of side one around the origin (with characteristic function χ_0), and "localized" in energy by the function $\mathcal{X} \in C^\infty_{c,+}(\mathbb{R})$. Its time averaged expectation is given by

$$\mathcal{M}_{B,\lambda}(p, \mathcal{X}, T) = \frac{1}{T} \int_0^\infty \mathbb{E} \left\{ M_{B,\lambda,\omega}(p, \mathcal{X}, t) \right\} \mathrm{e}^{-\frac{t}{T}} \, \mathrm{d}t. \tag{1.12}$$

Corollary 1.2. *The random Landau Hamiltonian $H_{B,\lambda,\omega}$ exhibits dynamical delocalization in each Landau band $\mathcal{B}_n(B,\lambda)$: For each $n = 1, 2, \ldots$ there exists at*

least one energy $E_n(B,\lambda) \in \mathcal{B}_n(B,\lambda)$*, such that for every* $\mathcal{X} \in \mathcal{C}^\infty_{c,+}(\mathbb{R})$ *with* $\mathcal{X} \equiv 1$ *on some open interval* $J \ni E_n(B,\lambda)$ *and* $p > 0$*, we have*

$$\mathcal{M}_{B,\lambda}(p,\mathcal{X},T) \geq \; C_{p,\mathcal{X}}\, T^{\frac{p}{4}-6}\;, \tag{1.13}$$

for all $T \geq 0$ *with* $C_{p,\mathcal{X}} > 0$*.*

As mentioned aboved, to prove Theorem 1.1 we extend the Wegner estimate of [CHK2] to measures μ with unbounded support. More precisely, the finite volume operator $H_\omega^{(\Lambda)}$ satisfies extensions of the Wegner estimates of [CH, CHK1, CHK2]. As in [CHK2], we do not require the probability measure μ to have a density. Precise statements and proofs are given in Appendix B.

2. Hall conductance and dynamical delocalization

We start by introducing some notation. Given $p \in [1,\infty)$, $\mathcal{T}_p$ will denote the Banach space of bounded operators S on $\mathrm{L}^2(\mathbb{R}^2,\mathrm{d}x)$ with $\|S\|_{\mathcal{T}_p} = \|S\|_p \equiv (\mathrm{tr}\,|S|^p)^{\frac{1}{p}} < \infty$. A random operator S_ω is a strongly measurable map from the probability space $(\Omega,\mathbb{P})$ to bounded operators on $\mathrm{L}^2(\mathbb{R}^2,\mathrm{d}x)$. Given $p \in [1,\infty)$, we set

$$|\!|\!|S_\omega|\!|\!|_p \equiv \left\{\mathbb{E}\left\{\|S_\omega\|_p^p\right\}\right\}^{\frac{1}{p}} = \left\|\|S_\omega\|_{\mathcal{T}_p}\right\|_{\mathrm{L}^p(\Omega,\mathbb{P})}, \tag{2.1}$$

and

$$|\!|\!|S_\omega|\!|\!|_\infty \equiv \left\|\|S_\omega\|\right\|_{\mathrm{L}^\infty(\Omega,\mathbb{P})}. \tag{2.2}$$

We define the (B,λ,E) parameter set by

$$\Xi = \{(0,\infty)\times[0,\infty)\times\mathbb{R}\} \setminus \cup_{B\in(0,\infty)}\{(B,0)\times\sigma(H_B)\};$$

that is we exclude the Landau levels at no disorder. We set

$$P_{B,\lambda,E,\omega} = \chi_{]-\infty,E]}(H_{B,\lambda,\omega}).$$

The Hall conductance $\sigma_H(B,\lambda,E)$ is given by (e.g.[BeES, AvSS, AG, BoGKS, GKS1, GKS2])

$$\sigma_H(B,\lambda,E) = -2\pi i\,\mathbb{E}\left\{\mathrm{tr}\left\{\chi_0 P_{B,\lambda,E,\omega}\left[[P_{B,\lambda,E,\omega},X_1],[P_{B,\lambda,E,\omega},X_2]\right]\chi_0\right\}\right\}, \tag{2.3}$$

defined for $(B,\lambda,E) \in \Xi$ such that

$$|\!|\!|\chi_0 P_{B,\lambda,E,\omega}\left[[P_{B,\lambda,E,\omega},X_1],[P_{B,\lambda,E,\omega},X_2]\right]\chi_0|\!|\!|_1 < \infty. \tag{2.4}$$

(X_i denotes the operator given by multiplication by the coordinate x_i, $i = 1,2$, and $|X|$ the operator given by multiplication by $|x|$.) In particular, $\sigma_H(B,\lambda,E)$ is well-defined for all (B,λ,E) such that $E \in \Xi^{\mathrm{DL}}_{B,\lambda}$. Moreover it is proved in [GKS2] that $\sigma_H(B,\lambda,E)$ is integer valued for all (B,λ,E) such that $E \in \Xi^{\mathrm{DL}}_{B,\lambda}$. We need to investigate the continuity properties of $\sigma_H(B,\lambda,E)$, as λ tends to zero. In [GKS2] we prove that for any (B,λ,E) such that $E \in \Xi^{\mathrm{DL}}_{B,\lambda}$, for any $p > 1$, there exists a constant $C(p,B,\lambda,E) < \infty$ for any (B',λ',E') in a neighborhood of (B,λ,E),

$$\begin{aligned}&|\sigma_H(B',\lambda',E') - \sigma_H(B,\lambda,E)| \\ &\quad\leq C(p,B,\lambda,E)\sup_{u\in\mathbb{Z}^2}|\!|\!|\chi_0\left(P_{B',\lambda',E',\omega} - P_{B,\lambda,E,\omega}\right)\chi_u|\!|\!|_1^{\frac{1}{p}}.\end{aligned} \tag{2.5}$$

We shall combine this fact with the following proposition, a consequence from Theorem B.2, which includes an extension of [HiKS] to unbounded random variables.

Proposition 2.1. *Let I be an open interval in a spectral gap of H_B. Then for all $\lambda \geq 0$ the Hall conductance is Hölder continuous in $E \in I$, and for any $E \in I$ the Hall conductance at Fermi energy E is Hölder continuous in the disorder parameter $\lambda \geq 0$.*

Proof. The proposition is a direct consequence of Theorem B.2 and (2.5). □

Proof of Theorem 1.1. We set

$$L_B = K_B \sqrt{\tfrac{4\pi}{B}}, \quad \mathbb{N}_B = L_B \mathbb{N}, \quad \text{and} \quad \mathbb{Z}^2_B = L_B \mathbb{Z}^2. \tag{2.6}$$

Note that $L_B \geq 1$ may not be an integer. We consider squares $\Lambda_L(0)$ with $L \in \mathbb{N}_B$ and identify them with the torii $\mathbb{T}_L := \mathbb{R}^2/(L\mathbb{Z}^2)$ in the usual way. We further let $\widetilde{\Lambda}_L(x) = \mathbb{Z}^2 \cap \Lambda_L(x)$. Given $L \in \mathbb{N}_B$ we define finite volume Landau Hamiltonians $H_{B,0,L}$ on $\mathrm{L}^2(\Lambda_L(0))$ as in [GKS1, Section 5], and set

$$\begin{aligned} H_{B,\lambda,0,L,\omega} &= H_{B,0,L} + \lambda V_{0,L,\omega} \quad \text{on} \quad \mathrm{L}^2(\Lambda_L(0)), \\ V_{0,L,\omega}(x) &= \sum_{i \in \widetilde{\Lambda}_{L-\delta_u}(0)} \omega_i \, u(x-i), \end{aligned} \tag{2.7}$$

It follows from (1.4) that

$$\mu(\{|u| \geq \varepsilon\}) \leq C_\alpha \exp\left(-\tfrac{1}{2}|\varepsilon|^\alpha\right) \quad \text{for all} \quad \varepsilon > 0. \tag{2.8}$$

Let $\bar{L} \in \mathbb{N}_B$ (see (2.6)), and let $H_{B,\lambda,0,\bar{L},\omega}$ and $V_{0,\bar{L},\omega}$ be as in (2.7). A straightforward computation shows that uniformly in $\lambda \in [0,1]$,

$$\begin{aligned} &\mathbb{P}\left\{\sigma(H_{B,\lambda,0,\bar{L},\omega}) \subset \bigcup_{n=1}^{\infty} [B_n - \lambda\varepsilon, B_n + \lambda\varepsilon]\right\} \geq \mathbb{P}\left\{|\omega_i| \leq \varepsilon \text{ if } i \in \widetilde{\Lambda}_{\bar{L}-\delta_u}(0)\right\} \\ &\geq \left(1 - C_\alpha \exp\left(-\tfrac{1}{2}|\varepsilon|^\alpha\right)\right)^{(\bar{L}-\delta_u)^2} \geq 1 - C_2 C_\alpha \exp\left(-\tfrac{1}{2}|\varepsilon|^\alpha\right) \bar{L}^2. \end{aligned} \tag{2.9}$$

We now apply the finite volume criterion for localization given in [GK2, Theorem 2.4], in the same way as in [GK2, Proof of Theorem 3.1], with parameters (we fix $q \in]0,1]$) $\eta_{I,\lambda} = \frac{1}{2}\eta_{B,\lambda,I,q} = \frac{1}{2}\eta_{B,1,I,q}$ and $Q_{B,\lambda,I} \leq \tilde{Q}$, for some $\tilde{Q} < \infty$ independent of $\lambda \in [0,1]$ as it follows from Theorem B.1. (Note that the fact that we work with length scales $L \in \mathbb{N}_B$ instead of $L \in 6\mathbb{N}$ only affects the values of the constants in [GK2, Eqs. (2.16) -(2.18)].)

To conduct the multiscale analysis of [GK1, GK2], we note that in finite volume we have, for any given $\eta < 1$, and uniformly in $\lambda \in [0,1]$,

$$\begin{aligned} &\mathbb{P}\left(|\lambda V_\omega(x)| \leq L^\eta, \text{ for all } x \in \Lambda_L(y)\right) && (2.10) \\ &\geq \mathbb{P}\left(|V_\omega(x)| \leq L^\eta, \text{ for all } x \in \Lambda_L(y)\right) && (2.11) \\ &\geq 1 - C_\alpha \exp(-\tfrac{1}{2}L^{\eta\alpha}) L^2, && (2.12) \end{aligned}$$

which is as close to 1 as wanted, provided L is large enough (independently of λ). Probabilistic bounds on the constant in SLI and EDI follow, with constants bounded by $L^{\eta/2}$. Since we are working in spectral gaps, we use the Combes-Thomas estimate of [BCH, Proposition 3.2] (see also [KlK1, Theorem 3.5]–its proof, based on [BCH, Lemma 3.1], also works for Schrödinger operators with magnetic fields), adapted to finite volume as in [GK2, Section 3].

Now fix $n \in \mathbb{N}$, take $I = \mathcal{I}_n(B)$, and set $\bar{L} = \bar{L}(n,B)$ to be the smallest $L \in \mathbb{N}_B$ satisfying [GK2, Eq. (2.16)]. Let $E \in \mathcal{I}_n(B)$, $|E - B_n| \geq 2\lambda\varepsilon$, where $\varepsilon =$

$\varepsilon(n, B, \lambda)) > 0$ will be chosen later. Then, using (2.9) and the Combes-Thomas estimate, we conclude that condition [GK2, Eq. (2.17)] will be satisfied at energy E if

$$\varepsilon \geq C_3 \, (\log \bar{L})^{\frac{1}{\alpha}}, \tag{2.13}$$

$$C_4 \, (\lambda\varepsilon)^{-1} \, \bar{L}^{\eta} \mathrm{e}^{-C_5\sqrt{\lambda\varepsilon}\bar{L}} < 1, \tag{2.14}$$

for appropriate constants $C_j = C_j(n, B)$, $j = 3, 4, 5$, with $C_5 > 0$. This can be done by choosing (in view of (2.9))

$$\varepsilon = C_3 \, (\log \bar{L})^{\frac{1}{\alpha}}, \tag{2.15}$$

and taking $\bar{L}$ large enough to satisfy (2.14) depending on $\lambda \leq 1$. We conclude from [GK2, Theorem 2.4] that

$$\left\{ E \in \mathcal{I}_n(B); \ |E - B_n| \geq C_5 \lambda \, |\log \lambda|^{\frac{1}{\alpha}} \right\} \subset \Xi^{\mathrm{DL}}_{B,\lambda}. \tag{2.16}$$

for all $\lambda \leq 1$. In particular, for all $n \in \mathbb{N}$ there is $\lambda_n > 0$ such that $B_n - B \in \Xi^{\mathrm{DL}}_{B,\lambda}$ for all $\lambda \in [0, \lambda_n]$.

The existence at small disorder of dynamical mobility edges $\widetilde{E}_{j,n}(B, \lambda)$, $j = 1, 2$, satisfying (1.10) now follows from [GKS2] and (2.16). Indeed, since $B_n - B \in \Xi^{\mathrm{DL}}_{B,\lambda}$ for all $\lambda \in [0, \lambda_n]$, the Hall conductance is constant at energy $B_n - B$ for all $\lambda \in [0, \lambda_n]$. Since for $\lambda = 0$, its value is $n - 1$, we can conclude that there is an energy of delocalization between $B_n - B$ and $B_n + B = B_{n+1} - B$ for all $\lambda \in [0, \min\{\lambda_n, \lambda_{n+1}\}]$. Then (2.16) and the constancy of the Hall conductance on sub-intervals of $\Xi^{\mathrm{DL}}_{B,\lambda}$ imply the estimate (1.10). □

Appendices

In these appendices we extend results known for Anderson-type random Schrödinger operator to unbounded random variables. These appendices are of separate interest and independent of the rest of the paper.

We consider a random Schrödinger operator of the form $H_{\lambda,\omega} = H_0 + \lambda V_\omega$ on $\mathrm{L}^2(\mathbb{R}^d, \mathrm{d}x)$, where the random potential V_ω is as in (1.3) and $\lambda \geq 0$. The unperturbed Hamiltonian H_0 will be either the Landau Hamiltonian H_B on $\mathrm{L}^2(\mathbb{R}^2, \mathrm{d}x)$, as in (1.2), or it will have the general form $H_0 = (-i\nabla - A_0)^2 + V_0$ on $\mathrm{L}^2(\mathbb{R}^d, \mathrm{d}x)$, $d \in \mathbb{N}$, where both A_0 and V_0 are regular enough so that H_0 is essentially self-adjoint on $\mathcal{C}_0^\infty(\mathbb{R}^d)$ and bounded from below by some constant $\Theta \in \mathbb{R}$. As a sufficient condition, it is enough to require that the magnetic potential A_0 and the electric potential V_0 satisfy the Leinfelder-Simader conditions (cf. [BoGKS]):

- $A_0(x) \in \mathrm{L}^4_{\mathrm{loc}}(\mathbb{R}^d; \mathbb{R}^d)$ with $\nabla \cdot A_0(x) \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^d)$.
- $V_0(x) = V_{0,+}(x) - V_{0,-}(x)$ with $V_{0,\pm}(x) \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^d)$, $V_{0,\pm}(x) \geq 0$, and $V_{0,-}(x)$ relatively bounded with respect to Δ with relative bound < 1, i.e., there are $0 \leq \alpha < 1$ and $\beta \geq 0$ such that

$$\|V_{0,-}\psi\| \leq \alpha\|\Delta\psi\| + \beta\|\psi\| \quad \text{for all } \psi \in \mathcal{D}(\Delta).$$

We will say that H_0 is periodic if A_0 and V_0 are $\mathbb{Z}^d$-periodic. It has the property (UCP) if it satisfies the unique continuation principle. (H_0 has the (UCP) if A_0 and V_0 are sufficiently regular; see the discussion in [CHK1].)

Appendix A. Applicability of the multiscale analysis

We provide here estimates that are needed for extending the multiscale analysis, more precisely results of [GK1, GK2, GK4, GK5, GKS1, GKS2], from bounded to unbounded random variables, as mentioned in the introduction. Finite volume operators are as defined in those papers. We fix the disorder $\lambda \geq 0$ and omit it from the notation. Note that the constants are all uniform in λ for $\lambda \leq \lambda_0$.

Lemma A.1. *Given a box Λ, there exists L^*, such that for any $L \geq L^*$ we have, for any $\beta > \alpha^{-1}$,*

$$\mathbb{P}\{\|\chi_{\Lambda_L} V_{\boldsymbol{\omega}}\|_\infty \leq C_+(\log L)^\beta\} \geq 1 - C(\alpha, \delta_+, d)\rho_0 \exp(-C(\alpha, \beta, \delta_+, d)|\log L|^{\alpha\beta}). \tag{A.1}$$

Then for $\mathbb{P}$-a.e. ω we have

$$V_{\boldsymbol{\omega}}(x) \geq -c_\omega(\log\langle x\rangle)^\beta \quad \text{for all } x \in \mathbb{R}^d, \tag{A.2}$$

where $c_\omega > 0$ (depending also on d, α, β). As a consequence $H_{\boldsymbol{\omega}}$ satisfies the lower bound

$$H_\omega \geq -c_\omega(\log\langle x\rangle)^\beta, \text{ for all } x \in \mathbb{R}^d, \tag{A.3}$$

for any given $\beta > \alpha^{-1}$ and is essentially self-adjoint on $\mathcal{C}_c^\infty(\mathbb{R}^d)$ with probability one.

Proof. To get (A.1), we note that

$$\mathbb{P}\{\|\chi_{\Lambda_L} V_{\boldsymbol{\omega}}\|_\infty \leq C_+(\log L)^\beta\} \geq 1 - C(2L)^d \mathbb{P}\{|\omega| \geq (\log L)^\beta\}. \tag{A.4}$$

The bound (A.2) then follows from the Borel-Cantelli Lemma. Now in view of (A.2), $H_{B,\omega}$ satisfies the lower bound (A.3) and thus $H_{\boldsymbol{\omega}}$ is essentially self-adjoint on $\mathcal{C}_c^\infty(\mathbb{R}^d)$ with probability one by the Faris-Levine Theorem [RS, Theorem X.38]. □

Bounds on the constant in SLI and EDI follow from (A.1). GEE follows from heat kernel estimates, as given in [BrLM]. As for SGEE, the bound has been derived by Ueki [U] for Gaussian random variables. For the reader's convenience we provide a short proof in the next theorem. Recall that $H_0 \geq \Theta$. We write $E_{H_\omega}(I) = \chi_I(H_\omega)$.

Theorem A.2. *There exist $m(d) > 0$ such that if $\mathbb{E}(|\omega_0|^{m(d)+\alpha}) < \infty$, with $\alpha \geq 0$, then for any bounded interval I we have*

$$\mathbb{E}\left\{|\omega_0|^\alpha \operatorname{tr} \chi_0 E_{H_\omega}(I)\chi_0\right\} \leq C(H_0, d, I, \alpha), \tag{A.5}$$

for some constant $C(H_0, d, I, \alpha) < \infty$. Moreover, $m(1) = 1$ and $m(d) = 2$ for $d = 2, 3$.

Proof. For simplicity, we assume that the support of u_0 is included in the unit cube centered at the origin. If not, straightforward modifications of the argument (as in [CHK2]) yield the result as well. We write $H = H_\omega = H_0 + V_\omega$, with H_0 bounded from below, say $H_0 \geq 0$. We denote by E the center of the interval I. We set $\tilde{I}$ to be the interval I but enlarged by a distance $\tilde{d} := 2|I|$ from above and below: $I \subset \tilde{I}$ and $\operatorname{dist}(I, \tilde{I}^c) = \tilde{d}$. We have

$$\operatorname{tr} \chi_0 E_H(I) = \operatorname{tr} \chi_0 E_H(I) E_{H_0}(\tilde{I}) + \operatorname{tr} \chi_0 E_H(I) E_{H_0}(\tilde{I}^c) \tag{A.6}$$

$$\leq C(|E| + 3|I|)^d + \operatorname{tr} \chi_0 E_H(I) E_{H_0}(\tilde{I}^c). \tag{A.7}$$

Now, with $R_0(z) = (H_0 - z)^{-1}$,

$$\operatorname{tr} \chi_0 E_H(I) E_{H_0}(\tilde{I}^c) = \operatorname{tr} \chi_0 E_H(I)(H_\omega - E - V_\omega) R_0(E) E_{H_0}(\tilde{I}^c) \tag{A.8}$$

$$\leq \frac{|I|}{\tilde{d}} \operatorname{tr} \chi_0 E_H(I) \chi_0 + |\operatorname{tr} \chi_0 E_H(I) V_\omega R_0(E) E_{H_0}(\tilde{I}^c) \chi_0| \tag{A.9}$$

$$\leq \frac{1}{2} \operatorname{tr} \chi_0 E_H(I) \chi_0 + \sum_{j \neq 0} \|\omega_j u_j R_0(E) E_{H_0}(\tilde{I}^c) \chi_0\|_1 \tag{A.10}$$

$$+ |\omega_0| |\operatorname{tr} \chi_0 E_H(I) u_0 R_0(E) E_{H_0}(\tilde{I}^c) \chi_0|, \tag{A.11}$$

so that, for $p > d$ given, taking advantage of $u_j \chi_0 = 0$ if $j \neq 0$ (use Helffer-Sjöstrand formula plus resolvent identities to get trace class operators),

$$\sum_{j \neq 0} \|\omega_j u_j R_0(E) E_{H_0}(\tilde{I}^c) \chi_0\|_1 \leq \mathbb{E}|\omega_0| \sum_{j \neq 0} C_p (1 + |j|)^{-p}. \tag{A.12}$$

Next, if $d = 1$ then $u_0 R_0(E) E_{H_0}(\tilde{I}^c)$ is trace class, and $\mathbb{E}|\omega_0| < \infty$ is a sufficient condition. If $d = 2, 3$ (in the present application $d = 2$), then Cauchy-Schwartz inequality leads to

$$|\operatorname{tr} \chi_0 E_H(I) \omega_0 u_0 R_0(E) E_{H_0}(\tilde{I}^c)| \tag{A.13}$$

$$\leq \|\chi_0 E_H(I)\|_2 \|\omega_0 R_0(\Theta - 1) \chi_0\|_2 \|(H_0 + \Theta + 1) R_0(E) E_{H_0}(\tilde{I}^c)\|_\infty \tag{A.14}$$

$$\leq \left(1 + \frac{|E| + |\Theta| + 1}{\tilde{d}}\right) \|\chi_0 E_H(I)\|_2 \|\omega_0 R_0(\Theta - 1) \chi_0\|_2 \tag{A.15}$$

$$\leq \frac{1}{4} \operatorname{tr} \chi_0 E_H(I) + \left(1 + \frac{|E| + |\Theta| + 1}{\tilde{d}}\right)^2 \omega_0^2 \operatorname{tr} \chi_0 R_0(\Theta - 1)^2. \tag{A.16}$$

The latter trace is finite in dimension $d = 2, 3$, finishing the proof provided $\mathbb{E}\omega_0^2 < \infty$. In higher dimensions, one repeats the very last step as many times as necessary, as in [CHK2]. □

Appendix B. Optimal Wegner estimate with unbounded random variables

In this appendix we extend the analyses of [CHK2] and [HiKS] to unbounded random variables.

Given a finite box $\Lambda \subset \mathbb{R}^d$, we denote by $H_{\lambda,\omega}^{(\Lambda)}$ an appropriate self-adjoint restriction of $H_{\lambda,\omega}$ to Λ, in which case $H_{\lambda,\omega}^{(\Lambda)}$ has a compact resolvent (see [CHK1, CHK2, GKS1]). There is no other restriction on the boundary condition in Theorem B.1(b),(c) below. When we use the (UCP) for H_0 periodic, as in Theorem B.1(a), we assume periodic boundary condition as in [CHK2]. If $H_0 = H_B$, the Landau Hamiltonian, in Theorem B.1(a) we assume finite volume operators as defined in [GKS1, Section 4] and used in [CHK2, Section 4].

If Δ is a Borelian, $E_{H_{\lambda,\omega}^{(\Lambda)}}(\Delta)$ denotes the associated spectral projection for $H_{\lambda,\omega}^{(\Lambda)}$.

In this appendix we assume $0 \leq \lambda \leq 1$ since we are mostly interested in small values of the coupling constant, but arguments easily extend to $\lambda \leq \lambda_0$ for any given λ_0.

Given an arbitrary Borel measure ν on the real line, we set $Q_\nu(s)$ to be a multiple of its concentration function:

$$Q_\nu(s) := 8 \sup_{a\in\mathbb{R}} \nu([a, a+s]) \tag{B.1}$$

Note that $Q_\nu(s) < \infty$ if ν is a finite measure. The Wegner estimate in [CHK2] is stated in terms of Q_μ; in our extension to unbounded measures Q_μ is replaced by $Q_{\mu^{(q)}}$, for an appropriate $q \geq 1$, where $d\mu^{(q)}(s) := |s|^q d\mu(s)$ for $q > 0$.

Theorem B.1. *Consider $H_{\lambda,\omega}$ with $0 < \lambda \leq 1$. There exists $1 \leq m(d) < \infty$, such that if $\mathbb{E}\{|\omega_0|^{m(d)}\} < \infty$, given $E_0 \in \mathbb{R}$:*

(a) *Assume either $H_0 = H_B$ or H_0 is periodic with the (UCP). Then there exists a constant $K_W(\lambda)$, depending also on d, E_0, $\delta_\pm$ and $C_\pm$, such that for any compact interval $\Delta \subset]-\infty, E_0[$ we have*

$$\mathbb{E}\left\{\operatorname{tr} E^{(\Lambda)}_{H_{\lambda,\omega}}(\Delta)\right\} \leq K_W(\lambda) Q_{\mu^{(m(d))}}(|\Delta|)|\Lambda|. \tag{B.2}$$

(b) *Assume the IDS of H_0 is Hölder continuous with exponent $\delta > 0$ in some open interval $\Delta_0 \subset]-\infty, E_0[$, then there exists a constant K_W depending on d, E_0, $\delta_\pm$,$C_\pm$, such that for any $\lambda \leq 1$, $\Delta \subset \Delta_0$ compact, $|\Delta|$ small enough, and any $0 < \gamma < 1$,*

$$\mathbb{E}\left\{\operatorname{tr} E^{(\Lambda)}_{H_{\lambda,\omega}}(\Delta)\right\} \leq K_W \max\left(|\Delta|^{\delta\gamma}, |\Delta|^{-\gamma m(d)} Q_{\mu^{(m(d))}}(|\Delta|)\right)|\Lambda|. \tag{B.3}$$

In particular, if $Q_{\mu^{(m(d))}}(\varepsilon) \leq C\varepsilon^\zeta$, for some $\zeta \in]0, 1]$, then

$$\mathbb{E}\left\{\operatorname{tr} E^{(\Lambda)}_{H_{\lambda,\omega}}(\Delta)\right\} \leq K_W |\Delta|^{\frac{\zeta\delta}{\delta + m(d)}} |\Lambda|. \tag{B.4}$$

(c) *Assume $E \in \Delta_0 \subset (\mathbb{R} \setminus \sigma(H_0)) \cap]-\infty, E_0[$, Δ_0 compact, then there exists a constant K_W, depending on d, E_0, $\delta_\pm$, $C_\pm$ and Δ_0, such that for any $\lambda \leq 1$ and any $\Delta \subset \Delta_0$ centered at E, $|\Delta|$ small enough,*

$$\mathbb{E}\left\{\operatorname{tr} E^{(\Lambda)}_{H_{\lambda,\omega}}(\Delta)\right\} \leq K_W \lambda Q_{\mu^{(m(d))}}(|\Delta|))|\Lambda|. \tag{B.5}$$

We adapt the proof of [CHK2], using the basic spectral averaging estimate proved in [CHK2]: Let H_0 and W be self-adjoint operators on a Hilbert space $\mathcal{H}$, with $W \geq 0$ bounded. Let $H_s := H_0 + sW$ for $s \in \mathbb{R}$. Then, given $\varphi \in \mathcal{H}$ with $\|\varphi\| = 1$, for all Borel measures ν on $\mathbb{R}$ and all bounded intervals $I \subset \mathbb{R}$ we have ([CH, Corollary 4.2], [CHK2, Eq. (3.16)][1]

$$\int \mathrm{d}\nu(s) \, \langle \varphi, \sqrt{W}\chi_I(H_s)\sqrt{W}\varphi\rangle \leq Q_\nu(|I|). \tag{B.6}$$

The result is stated in [CHK2] for a probability measure ν with compact support, but their proof works for an arbitrary Borel measure ν. In particular, for H_ω as in Theorem B.1, we get, for any $\phi \in \mathrm{L}^2(\mathbb{R}^d)$, $j \in \mathbb{Z}^d$, $\alpha > 0$, and any interval I_ε of length $\varepsilon > 0$,

$$\mathbb{E}\{|\omega_j|^\alpha \langle \phi, \sqrt{u_j} E^{(\Lambda)}_{H_{\lambda,\omega}}(I_\varepsilon)\sqrt{u_j}\phi\rangle\} \leq \tfrac{1}{\lambda} Q_{\mu^{(\alpha)}}(\epsilon)\|\phi\|^2. \tag{B.7}$$

[1] There the estimate (B.6) is stated with W instead of $\sqrt{W}$, with the additional hypothesis that $W \leq 1$. But a careful reading of their proof shows that they actually prove (B.6) as stated here.

As a consequence, for any trace class operator $S \geq 0$,

$$\mathbb{E}\left\{|\omega_j|^\alpha \operatorname{tr}\left\{\sqrt{u_j} E_{H^{(\Lambda)}_{\lambda,\omega}}(I_\varepsilon)\sqrt{u_j} S\right\}\right\} \leq \tfrac{1}{\lambda}(\operatorname{tr} S) Q_{\mu^{(\alpha)}}(\epsilon). \tag{B.8}$$

Proof of Theorem B.1. Recall that $H_{\lambda,\omega} = H_0 + \lambda V_\omega$, $\lambda \in]0,1]$, and to alleviate notations we write $E_\Lambda(\Delta) := E_{H^{(\Lambda)}_{\lambda,\omega}}(\Delta)$ and $E^\Lambda_0(\Delta) := E_{H^{(\Lambda)}_{0,\omega}}(\Delta)$. To simplify the exposition we assume that the support of u is smaller than the unit cube; if not the case, the proof can be modified in a straightforward way, as in [CHK2]. In particular, $u_i u_j = 0$ if $i \neq j$. We also introduce χ to be the characteristic function of a cube containing the support of u, contained in the unit cube, such that $\chi_i \chi_j = 0$ if $i \neq j$, where $\chi_j(x) = \chi(x - j)$. With $\Delta \subset \tilde{\Delta}$, and denoting $d_\Delta = \operatorname{dist}(\Delta, \tilde{\Delta}^c)$, we get

$$\operatorname{tr}(E_\Lambda(\Delta)) = \operatorname{tr}(E_\Lambda(\Delta) E^\Lambda_0(\tilde{\Delta})) + \operatorname{tr}(E_\Lambda(\Delta) E^\Lambda_0(\tilde{\Delta}^c)). \tag{B.9}$$

We first consider the term $\operatorname{tr}(E_\Lambda(\Delta)E^\Lambda_0(\tilde{\Delta}^c))$ and take care of the unboundedness of the random variables. We have,

$$\operatorname{tr}(E_\Lambda(\Delta)E^\Lambda_0(\tilde{\Delta}^c)) \leq C_d(\Delta)\lambda^2 \sum_{i,j\in\Lambda} |\omega_i\omega_j| |\operatorname{tr}(u_j E_\Lambda(\Delta) u_i K_{ij})| \tag{B.10}$$

$$\leq C_d(\Delta)\lambda^2 \sum_{i,j\in\Lambda, i\neq j} |\omega_i\omega_j| |\operatorname{tr}(u_j E_\Lambda(\Delta) u_i K_{ij})| \tag{B.11}$$

$$+ C_d(\Delta)\lambda^2 \sum_{i\in\Lambda} |\omega_i|^2 |\operatorname{tr}(u_i E_\Lambda(\Delta) u_i K_{ii})| \tag{B.12}$$

where

$$K_{ij} = \chi_i (H^\Lambda_0 + M)^{-2} \chi_j, \tag{B.13}$$

and

$$\left\| \left(\frac{H^\Lambda_0 + M}{H^\Lambda_0 - E_m}\right)^2 E^\Lambda_0(\tilde{\Delta}^c) \right\| \leq \left(1 + \frac{2(M + \Delta_+)}{d_\Delta} + \frac{(M+\Delta_+)^2}{d^2_\Delta}\right) = C_d(\Delta) \tag{B.14}$$

for some $M < \infty$ such that $H_0 + M \geq 1$, for example $M = 1$ is enough, and where the $\chi_i, \forall i \in \mathbb{Z}^d$ are compactly supported functions, with support slightly larger than the u_i's one such that $\chi_i u_i = u_i$. Note that K_{ij} is trace class as soon as $i \neq j$ (since we assume $\operatorname{supp} u_j \subset \Lambda_1(j)$), as can be seen by a successive use of the resolvent identity, and by Combes-Thomas its trace class norm satisfies $\|K_{ij}\|_1 \leq C_d \mathrm{e}^{-|i-j|}$, for $i \neq j$. It follows, as in [CGK, Eqs (4.1)-(4.4)], that

$$\sum_{i\neq j} |\omega_i\omega_j| \, |\operatorname{tr}(u_j E_\Lambda(\Delta) u_i K_{ij})| \tag{B.15}$$

$$\leq \sum_{i\neq j} \frac{1}{2}\left(|\omega_i|^2 \operatorname{tr}(u_i E_\Lambda(\Delta) u_i |K_{ij}|) + |\omega_j|^2 \operatorname{tr}(u_j E_\Lambda(\Delta) u_j |K^*_{ij}|)\right) \tag{B.16}$$

$$= \sum_i |\omega_i|^2 \, |\operatorname{tr}(u_i E_\Lambda(\Delta) u_i S_i)| \,, \tag{B.17}$$

where

$$S_j = \frac{1}{2}\sum_{i\neq j}(|K_{ij}| + |K^*_{ji}|) \geq 0, \tag{B.18}$$

with

$$\max_{j\in\Lambda} \operatorname{tr} S_j \leq Q_2 < \infty. \tag{B.19}$$

It remains to consider the diagonal term $i = j$, that is $|\omega_i|^2 \operatorname{tr}(u_i E_\Lambda(\Delta) u_i K_{ii})$. Note that K_{ii} is trace class in dimension $d = 1, 2, 3$ but not higher. To deal with the general case of arbitrary dimension we proceed as in [CHK2] and perform successive Cauchy-Schwartz inequalities, getting, for any integer $m \geq 1$, for some constant $K_{d,m} < \infty$,

$$C_d(\Delta)|\omega_i|^2 \operatorname{tr}(u_i E_\Lambda(\Delta) u_i K_{ii}) \tag{B.20}$$

$$\leq \frac{1}{4} \operatorname{tr}(u_i E_\Lambda(\Delta) u_i) + K_{d,m} (C_d(\Delta)|\omega_i|)^{2^m} \operatorname{tr}(u_i E_\Lambda(\Delta) u_i K_{ii}^{2^{m-1}}). \tag{B.21}$$

We chose m so that $K_{ii}^{2^{m-1}}$ is trace class, that is, we take $m(d) := 2^{m+1} > d$, i.e., $m = [\log d / \log 2]$, where $[x]$ stands for the integer part of x. It follows that, using $\sum_j u_j \leq 1$, uniformly in $\lambda \leq 1$,

$$\operatorname{tr}(E_\Lambda(\Delta) E_0^\Lambda(\tilde{\Delta}^c)) \tag{B.22}$$

$$\leq \frac{1}{4} \sum_i \operatorname{tr}(u_i E_\Lambda(\Delta) u_i) + K_{d,m(d)} \lambda^2 \sum_i (C_d(\Delta)|\omega_i|)^{m(d)} \operatorname{tr}(u_i E_\Lambda(\Delta) u_i \tilde{S}_i) \tag{B.23}$$

$$\leq \frac{1}{4} \operatorname{tr} E_\Lambda(\Delta) + K'_{d,m(d)} \lambda^2 \sum_i \left(\frac{|\omega_i|}{d_\Delta} \right)^{m(d)} \operatorname{tr}(u_i E_\Lambda(\Delta) u_i \tilde{S}_i), \tag{B.24}$$

where

$$\tilde{S}_i = S_i + K_{ii}^{2^{m(d)-1}} \geq 0, \tag{B.25}$$

is a trace class operator. We apply (B.8) to finish the bound:

$$\mathbb{E} \operatorname{tr}(E_\Lambda(\Delta) E_0^\Lambda(\tilde{\Delta}^c)) \leq \frac{1}{4} \mathbb{E} \operatorname{tr} E_\Lambda(\Delta) + C'_d \lambda \frac{Q_{\mu^{(m(d))}}}{d_\Delta^{m(d)}}(|\Delta|)|\Lambda|. \tag{B.26}$$

We now turn to the first term of the right hand side in (B.9), that is $\operatorname{tr}(E_\Lambda(\Delta) E_0^\Lambda(\tilde{\Delta}))$. To get the general Wegner estimate (B.2) the latter is treated as in [CHK2], using either the unique continuation principle for the free Hamiltonian, or, in the Landau case, explicit properties of the Landau Hamiltonian. Note that we then incorporate d_Δ in the constant. To get (B.3), we control $\operatorname{tr}(E_\Lambda(\Delta) E_0^\Lambda(\tilde{\Delta}))$ using the hypothesis on the IDS of H_0, that is $\operatorname{tr} E_{H_0}(\tilde{\Delta}) \leq C|\tilde{\Delta}|^\delta |\Lambda|$. In this case, we need d_Δ to be small enough and it then remains to control the growth of the constant in the second term of the r.h.s. of (B.26). Taking $d_\Delta = \varepsilon^\gamma$, with $0 < \gamma < 1$, and using $Q_{\mu^{(m(d))}}(|\Delta|) \leq C\varepsilon^\zeta$ if μ is ζ-Hölder continuous, we get, with a new constant K_W, and ε small enough so that $\tilde{\Delta} \subset \Delta_0$,

$$\mathbb{E} \operatorname{tr} E_\Lambda(\Delta) \leq K_W \max \left(\varepsilon^{\gamma\delta}, \frac{1}{\varepsilon^{m(d)\gamma}} Q_{\mu^{(m(d))}}(\varepsilon) \right) |\Lambda| \tag{B.27}$$

$$\leq K_W \max \left(\varepsilon^{\gamma\delta}, \varepsilon^{\zeta - m(d)\gamma} \right) |\Lambda| \tag{B.28}$$

$$\leq K_W \varepsilon^{\frac{\zeta\delta}{\delta + m(d)}} |\Lambda| \tag{B.29}$$

where we have chosen γ such that $\gamma\delta = \zeta - m(d)\gamma$.

Finally, in the particular case of (B.5), $\operatorname{tr} E_{H_0}(\tilde{\Delta}) = 0$ as long as $\tilde{\Delta} \subset \Delta_0$. □

The following theorem contains an extension of [HiKS] to unbounded random variables. We set, for $E \in \mathbb{R}$, $P_{\lambda,E,\omega} = \chi_{]-\infty,E]}(H_{\lambda,\omega})$, the Fermi projection.

Theorem B.2. *Consider $H_{\lambda,\omega}$ with $0 < \lambda \leq 1$ Assume that the IDS of H_0 is Hölder continuous in $E \in \Delta_0$ an open interval. Then for some $\nu > 0$ and $C_{\Delta_0} < \infty$, for any $E, E' \in \Delta_0$, $|E - E'|$ small enough, we have uniformly in $0 \leq \lambda \leq 1$,*

$$\max_{u\in\mathbb{Z}^2} \mathbb{E}\left\{\|\chi_0 \left(P_{\lambda,E,\omega} - P_{\lambda,E',\omega}\right)\chi_u\|_1\right\} \leq C_{\Delta_0}|E - E'|^{\nu}, \tag{B.30}$$

and for some $\nu' > 0$, for all $E \in \Delta_0$, for all $\lambda', \lambda'' \in [0, 1]$, $|\lambda'' - \lambda'|$ small enough,

$$\max_{u\in\mathbb{Z}^2} \mathbb{E}\left\{\|\chi_0 \left(P_{\lambda',E,\omega} - P_{\lambda'',E,\omega}\right)\chi_u\|_1\right\} \leq C_{\Delta_0}|\lambda'' - \lambda'|^{\nu'}. \tag{B.31}$$

Proof. Eq. (B.30) follows from Cauchy-Schwarz and the continuity of the Integrated Density of States of $H_{\lambda,\omega}$ given by Theorem B.1 Eq. (B.4). We turn to (B.31). Let $E \in \Delta_0$ and $\lambda', \lambda'' \in [\lambda_1, \lambda_2]$ possibly containing 0. We let $\gamma = |\lambda' - \lambda''|^{\alpha}$, where $\alpha \in (0, 1)$ will be chosen later. Let $f(t)$ be a smooth decaying switch function, equal to 1 for $t \leq 0$ and 0 for $t \geq 1$. We set $g(t) = f\left(\frac{t-(E-\gamma)}{\gamma}\right)$; note $g \in C^{\infty}(\mathbb{R})$, with $0 \leq g(t) \leq 1$, $g(t) = 1$ if $t \leq E - \gamma$, $g(t) = 0$ if $t \geq E$. We write

$$\begin{aligned} P_{\lambda',E,\omega} - P_{\lambda'',E,\omega} = &\left\{P_{\lambda',E,\omega} - g^2(H_{\lambda',\omega})\right\} \\ &+ \left\{g^2(H_{\lambda',\omega}) - g^2(H_{\lambda'',\omega})\right\} + \left\{g^2(H_{\lambda'',\omega}) - P_{\lambda'',E,\omega}\right\}. \end{aligned} \tag{B.32}$$

By construction, for any $\lambda \geq 0$ we have

$$0 \leq P_{\lambda,E,\omega} - g^2(H_{\lambda,\omega}) \leq P_{\lambda,E,\omega} - P_{\lambda,E-\gamma,\omega}\,, \tag{B.33}$$

and thus, for $\lambda^{\#} = \lambda', \lambda''$ and any $u \in \mathbb{Z}^2$, we have

$$\begin{aligned} &\left|\left|\left|\chi_0 \left(P_{\lambda^{\#},E,\omega} - g^2(H_{\lambda^{\#},\omega})\right)\chi_u\right|\right|\right|_1 \\ &\leq \left|\left|\left|\chi_0 \left(P_{\lambda^{\#},E,\omega} - g^2(H_{\lambda^{\#},\omega})\right)^{\frac{1}{2}}\right|\right|\right|_2 \left|\left|\left|\left(P_{\lambda^{\#},E,\omega} - g^2(H_{\lambda^{\#},\omega})\right)^{\frac{1}{2}}\chi_u\right|\right|\right|_2 \\ &= \left|\left|\left|\chi_0 \left(P_{\lambda^{\#},E,\omega} - g^2(H_{\lambda^{\#},\omega})\right)\chi_0\right|\right|\right|_1 \\ &\leq \left|\left|\left|\chi_0 \left(P_{\lambda^{\#},E,\omega} - P_{\lambda^{\#},E-\gamma,\omega}\right)\chi_0\right|\right|\right|_1 \leq C_{\Delta_0}\gamma^{\nu}. \end{aligned} \tag{B.34}$$

To control the middle term in the r.h.s. of (B.32), we proceed as in [HiKS, Eq. (3.8)] and sequel. In the Helffer-Söjstrand formula, one needs to go to the $(4+2d)$th order. The term corresponding to [HiKS, Eq. (3.15)] is controled as follows (we denote by $R_{\lambda,\omega}(z)$ the resolvent of $H_{\lambda,\omega}$):

$$\|R_{\lambda,\omega}(z)V_{\omega}R_{\lambda',\omega}(z)V_{\omega}R_{\lambda,\omega}(z)\chi_0\| \tag{B.35}$$

$$\leq \sum_{j,k\in\mathbb{Z}^d} |\omega_j\omega_k| \|R_{\lambda,\omega}(z)u_j R_{\lambda',\omega}(z)u_k R_{\lambda,\omega}(z)\chi_0\| \tag{B.36}$$

$$\leq \sum_{j,k\in\mathbb{Z}^d} |\omega_j\omega_k| |\Im z|^{-3} \mathrm{e}^{-c|\Im z||j-_k|} \mathrm{e}^{-c|\Im z||k|}. \tag{B.37}$$

It follows, using the Combes-Thomas inequality, that

$$\mathbb{E}\|\chi_u g(H_{\lambda,\omega})\|_1 \|R_{\lambda,\omega}(z)V_{\omega}R_{\lambda',\omega}(z)V_{\omega}R_{\lambda,\omega}(z)\chi_0\| \tag{B.38}$$

$$\leq \sum_{j,k\in\mathbb{Z}^d} \left(\mathbb{E}|\omega_j\omega_k| \|\chi_u g(H_{\lambda,\omega})\|_1\right) |\Im z|^{-3} \mathrm{e}^{-c|\Im z||j-_k|} \mathrm{e}^{-c|\Im z||k|} \tag{B.39}$$

$$\leq C(I,d)|\Im z|^{-3} \sum_{j,k\in\mathbb{Z}^d} \mathrm{e}^{-c|\Im z||j-_k|} \mathrm{e}^{-c|\Im z||k|} \tag{B.40}$$

$$\leq C(I,d)|\Im z|^{-3-2d}, \tag{B.41}$$

by Theorem A.2. The term corresponding to [HiKS, Eqs. (3.16)-(3.18)] is controled in a similar way using Theorem A.2. □

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