General curvature estimates for stable H-surfaces in 3-manifolds and applications

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GENERAL CURVATURE ESTIMATES FOR STABLE H-SURFACES IN 3-MANIFOLDS AND APPLICATIONS

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Abstract. We obtain an estimate for the norm of the second fundamental form of stable H-surfaces in Riemannian 3-manifolds with bounded sectional curvature. Our estimate depends on the distance to the boundary of the surface and on the bound on the sectional curvature but not on the manifold itself. We give some applications, in particular we obtain an interior gradient estimate for H-sections in Killing submersions.

1. Introduction

Let (M, g) be a smooth Riemannian 3-manifold. Consider an immersed surface $\Sigma \looparrowright M$ with trivial normal bundle. Call A its second fundamental form and N a global unit normal on it. We denote by H the length of the mean curvature vector of Σ. When Σ has constant mean curvature H, we say that Σ is an H-surface.

Recall that an H-surface Σ is said *strongly stable* if for any $u \in C_0^\infty(\Sigma)$ we have

$$\int_\Sigma |\nabla u|^2 d\Sigma \geq \int_\Sigma (|A|^2 + \mathrm{Ric}(N))u^2 d\Sigma,$$

where $d\Sigma$ and ∇ stand respectively for the area element and the gradient on Σ, and Ric denotes the Ricci curvature of M. Throughout this paper, we will use the term *stable* to mean strongly stable.

Curvature estimates for stable H-surfaces in 3-manifolds are an important tool in the study of H-surfaces, see for instance Meeks-Perez-Ros [12]. Graphs are stable and more generally, surfaces transverse to an ambient Killing field are stable. Curvature estimates for graphs were obtained in 1952 by Heinz [10]. In 1983, Schoen [19] obtained an estimate for the norm of the second fundamental form of stable minimal surfaces in a 3-manifold M depending on the distance to the boundary on the surface and an upper bound on the curvature tensor of M and its covariant derivative. In particular, in $\mathbb{R}^3$, he showed the existence of a constant $C > 0$ such that for any stable and orientable minimal surface Σ one has

$$|A(p)| \leq \frac{C}{d_\Sigma(p, \partial\Sigma)}, \tag{1}$$

for any $p \in \Sigma$, where d_Σ denotes the intrinsic distance on Σ.

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In 1999, Bérard and Hauswirth [1] extended Schoen's work to stable H-surfaces with trivial normal bundle in space forms. In 2005, using different methods based on ideas of Colding and Minicozzi [3], Zhang [23] extended Schoen's result to stable H-surfaces (with trivial normal bundle) in a general 3-manifold, again the estimate depends on H, an upper bound on the sectional curvature and on the covariant derivative of the curvature tensor of M.

In this paper we obtain an estimate for the norm of the second fundamental form of stable H-surfaces in Riemannian 3-manifolds assuming only a bound on the sectional curvature. Our estimate depends on the distance to the boundary of the surface and only on the bound on the sectional curvature of the ambient manifold. More precisely, our main result is the following:

Main Theorem. *Let (M,g) be a complete smooth Riemannian 3-manifold of bounded sectional curvature $|K| \leq \Lambda < +\infty$.*

Then there exists a universal constant C which depends neither on M nor on Λ, satisfying the following:

For any immersed stable H-surface $\Sigma \looparrowright M$ with trivial normal bundle, and for any $p \in \Sigma$ we have

$$|A(p)| \leq \frac{C}{\min\left\{d(p,\partial\Sigma), \frac{\pi}{2\sqrt{\Lambda}}\right\}}.$$

We also obtain a *local* version of our result, see Theorem 2.5. Our method of proof is completely different from the above mentioned works and is based on a blow-up argument. For the readers convenience we sketch the idea of the proof.

Assuming by contradiction the result is not true, we have for each n a stable H_n-surface Σ_n in some 3-manifold M_n with bounded sectional curvature, $|K| \leq \Lambda$, admitting a point $p_n^* \in \Sigma_n$ satisfying $|A_n(p_n^*)| \min\left\{d(p,\partial\Sigma), \frac{\pi}{2\sqrt{\Lambda}}\right\} > n$. In addition, there exists a real number $r_n > 0$ such that the geodesic disk D_n^* on Σ_n centered at p_n^* with radius r_n lies in the domain of some chart of M_n which allows us to treat this disk as a surface in a Euclidean ball of fixed radius in $\mathbb{R}^3$ endowed with the pull-back metric, with p_n^* at the origin. Furthermore in our construction we can make $r_n|A_n(p_n^*)| \to +\infty$. We then blow-up the latter metric multiplying by the factor $|A_n(p_n^*)|$. This sequence of new metrics converges to the Euclidean metric on compact sets of $\mathbb{R}^3$. The sequence (D_n^*) gives rise to a new sequence of surfaces still denoted by (D_n^*) with controlled geometry and second fundamental form at the origin of norm 1. We can then construct a complete surface S, passing through the origin, in the accumulation set of the sequence (D_n^*), whose universal cover $\widetilde{S}$ is a stable H-surface in the Euclidean space $\mathbb{R}^3$. It is well-known that S is then a plane which contradicts the fact that the second fundamental form of S has norm 1 at the origin.

However, this blow-up argument requires some care because of technical difficulties we explain in the proof.

It is worth noticing that an estimate of the type (1) cannot be expected in general, even for minimal surfaces, as such an estimate would imply that a complete stable H-surface is totally geodesic. For example, in $\mathbb{H}^3$ there are complete non-totally geodesic stable H-surfaces, see Silveira [20]. Also in $\mathbb{H}^2 \times \mathbb{R}$, for any $0 \leq H \leq 1/2$ there are non-constant entire vertical H-graphs, and therefore stable, see Nelli and Rosenberg, [13] and [14].

As a consequence of the Main Theorem we have the following:

***Corollary* 1.1.** *Let (M, g) be a complete smooth Riemannian 3-manifold of bounded sectional curvature $|K| \leq \Lambda < +\infty$.*

Then there exists a constant C, which depends neither on M nor on Λ, such that for any immersed complete stable H-surface $\Sigma \looparrowright M$, with trivial normal bundle, we have:

- *if Σ is non compact, then $|A(p)| \leq C\sqrt{\Lambda}$ for any $p \in \Sigma$,*
- *if Σ is compact, then $|A(p)| \min\{\operatorname{diam}\Sigma, \frac{\pi}{\sqrt{\Lambda}}\} \leq C$ for any $p \in \Sigma$, where* $\operatorname{diam}\Sigma$ *denotes the intrinsic diameter of Σ.*

Proof. For non compact surfaces, this follows immediately from the Main Theorem.

Suppose now that Σ is compact. Set $d = \operatorname{diam}\Sigma$ and let $p \in \Sigma$, then the geodesic disk D in Σ of center p and radius $d/2$ satisfies $d(p, \partial D) = d/2$. Applying the Main Theorem to D we get $|A(p)| \min\{\frac{d}{2}, \frac{\pi}{2\sqrt{\Lambda}}\} \leq C$. The result follows. $\square$

This corollary was enunciated by Ros [18] for minimal surfaces (even with non trivial normal bundle) with the additional hypotheses that the derivatives of the curvature tensor are bounded and the injectivity radius is bounded from below.

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2. Proof of the Main Theorem

In the blow-up argument we sketched before, the first idea which comes to mind is to use a chart given by the exponential map, that is, to use geodesic normal coordinates. However in those coordinates bounded geometry guarantees only a C^0-control of the metric. For example consider the metrics $g_n, n \in \mathbb{N}$, on $\mathbb{R}^2$ given in polar coordinates by

$$g_n = dr^2 + G_n^2(r, \theta) d\theta^2,$$

with

$$G_n(r, \theta) = r(1 + ar^2 + e^{-1/r^2} \cos n\theta),$$

where $a > 0$ is big enough. In Cartesian coordinates $x = r\cos\theta, y = r\sin\theta$, we have

$$\begin{aligned} g_n &= \Big(\frac{x^2}{x^2+y^2} + \frac{y^2G_n^2}{(x^2+y^2)^2}\Big)dx^2 + 2xy\Big(\frac{1}{x^2+y^2} - \frac{G_n^2}{(x^2+y^2)^2}\Big)dxdy \\ &\qquad + \Big(\frac{y^2}{x^2+y^2} + \frac{x^2G_n^2}{(x^2+y^2)^2}\Big)dy^2 \\ &= g_{11}^n dx^2 + 2g_{12}^n dxdy + g_{22}^n dy^2. \end{aligned}$$

It is easily seen that g_n is a complete and smooth metric on $\mathbb{R}^2$ and that $g_{ij}^n(0) = \delta_{ij}$ for any $n \in \mathbb{N}$, so that, (x, y) are geodesic normal coordinates at the origin for the metrics g_n. The Gauss curvature of g_n is given by $K_n = -\frac{1}{G_n}\frac{\partial^2 G_n}{\partial r^2}$. Thus, a computation shows that

$$K_n = -\frac{6a + (\frac{4}{r^6} - \frac{2}{r^4})e^{-1/r^2}\cos n\theta}{1 + ar^2 + e^{-1/r^2}\cos n\theta}.$$

Therefore, for $a > 0$ big enough, the Gaussian curvatures K_n are negative and uniformly bounded on $\mathbb{R}^2$ with respect to n. Consequently, the injectivity radius of $(\mathbb{R}^2, g_n)$ is infinite for any $n \in \mathbb{N}$. We can show directly (or as a consequence of Rauch comparaison theorem) that for any $r > 0$ there exists $Q_0 > 0$ depending on r, a and not on n such that on the geodesic balls of $(\mathbb{R}^2, g_n)$ centered at the origin with radius r we have

$$Q_0^{-1}\delta_{ij} \leq g_{ij}^n \leq Q_0\delta_{ij}, \quad \text{as quadratic forms,}$$

this gives a uniform C^0-control of the metrics g_n. Nevertheless we do not have uniform C^1-control. Indeed, consider for example the coefficient g_{11}^n, we have

$$\frac{\partial g_{11}^n}{\partial x} = \frac{\partial}{\partial x}\big(\frac{x^2}{x^2+y^2}\big) + \frac{\partial}{\partial x}\big(\frac{y^2}{(x^2+y^2)^2}\big)G_n^2 + 2G_n\frac{y^2}{(x^2+y^2)^2}\frac{\partial G_n}{\partial x}.$$

Observe that at any fixed point $(x, y) \neq (0, 0)$, there is only one term which is not uniformly bounded with respect to n: the one involving the partial derivative of G_n with respect to x. In coordinates (x, y) we have

$$G_n(x, y) = \sqrt{x^2+y^2} + a(x^2+y^2)^{3/2} + \sqrt{x^2+y^2}e^{-1/(x^2+y^2)}\frac{\mathrm{Re}(x+iy)^n}{(x^2+y^2)^{n/2}}.$$

Also, at any fixed point $(x, y) \neq (0, 0)$, in the partial derivative $\partial G_n/\partial x$, the only term which is not uniformly bounded with respect to n is the one involving the derivative of $\mathrm{Re}(x+iy)^n/(x^2+y^2)^{n/2}$. We have, using the relations $x = r\cos\theta, y = r\sin\theta$,

$$\begin{aligned} \frac{\partial}{\partial x}\frac{\mathrm{Re}(x+iy)^n}{(x^2+y^2)^{n/2}} &= n\frac{\mathrm{Re}(x+iy)^{n-1}}{(x^2+y^2)^{n/2}} - nx\frac{\mathrm{Re}(x+iy)^n}{(x^2+y^2)^{1+n/2}} \\ &= \frac{n}{r}\cos(n-1)\theta - \frac{n}{r}\cos\theta\cos(n-1)\theta \\ &= \frac{n}{r}\sin\theta\sin n\theta. \end{aligned}$$

This shows that at any fixed point $(x, y) \neq (0, 0)$ whose argument θ is not rational with π the partial derivative $(\partial g_{11}^n/\partial x)(x, y)$ is not uniformly bounded with respect to n. Consequently uniform estimates of the Gaussian curvature do not imply a local C^1-control of the metric in geodesic normal coordinates.

For this reason we will instead use *harmonic coordinates*. More precisely, we will need the following result which can be deduced from [9, Theorem 6].

Theorem 2.1. *Let $\alpha \in (0, 1)$ and $\delta > 0$. Let (M, g) be a smooth Riemannian 3-manifold, without boundary, of bounded sectional curvature $|K| \leq \Lambda < +\infty$ and Ω an open subset of M. Set*

$$\Omega(\delta) = \big\{x \in M \mid d_g(x, \Omega) < \delta\big\}$$

where d_g is the distance asociated to g. Suppose there exists $i > 0$ such that for all $x \in \Omega(\delta)$ we have $inj_{(M,g)}(x) \geq i$, where $inj_{(M,g)}(x)$ is the injectivity radius at x.

Then there exist a constant $Q_0 > 1$ and a real number $r_0 > 0$, depending only on i, δ, Λ and α, and not on M, such that for any $x \in \Omega$, there exists a harmonic coordinate chart $(U, \varphi, B(x, r_0))$, U being an open subset of $\mathbb{R}^3$ containing the origin and $B(x, r_0)$ the geodesic ball in M centered at x of radius r_0, with $\varphi(0) = x$, and such that the metric tensor g is $C^{1,\alpha}$-controlled. Namely the components g_{ij}, $i, j = 1, 2, 3$ of g satisfy:

$$Q_0^{-1}\delta_{ij} \leq g_{ij} \leq Q_0\delta_{ij}, \quad \textit{as quadratic forms,} \tag{2}$$

$$\sum_{k=1}^{3} \sup_{y\in U} |\partial_k g_{ij}(y)| + \sum_{k=1}^{3} \sup_{y\neq z} \frac{|\partial_k g_{ij}(y) - \partial_k g_{ij}(z)|}{d_g(y, z)^\alpha} \leq Q_0. \tag{3}$$

We will also need the following result.

Lemma 2.2. *Let (M, g) be a smooth complete Riemannian manifold of bounded sectional curvature, $|K| \leq \Lambda < +\infty$ – thus for each $x \in M$ the exponential map $\exp_x : B_{\vec{0}}(\frac{\pi}{\sqrt{\Lambda}}) \subset T_xM \to B(x, \frac{\pi}{\sqrt{\Lambda}})$ is a local diffeomorphism.*

Then in the closed ball $\overline{B}_{\vec{0}}(\frac{\pi}{4\sqrt{\Lambda}})$, the injectivity radius in $B_{\vec{0}}(\frac{\pi}{\sqrt{\Lambda}})$ endowed with the pull-back metric $\exp_x^* g$ is at least $\frac{\pi}{4\sqrt{\Lambda}}$.

Proof. For any $p \in \overline{B}_{\vec{0}}(\frac{\pi}{4\sqrt{\Lambda}})$ we denote by $i(p)$ the injectivity radius of p and by $C(p)$ its cut-locus in $(B_{\vec{0}}(\frac{\pi}{\sqrt{\Lambda}}), \exp_x^* g)$. We denote by i_0 the infimum of $i(p)$, $p \in \overline{B}_{\vec{0}}(\frac{\pi}{4\sqrt{\Lambda}})$.

Assume by contradiction that $i_0 < \frac{\pi}{4\sqrt{\Lambda}}$. Let $(p_n) \in B_{\vec{0}}(\frac{\pi}{4\sqrt{\Lambda}})$ be a sequence such that $i(p_n) \to i_0$ and $i(p_n) < \frac{\pi}{4\sqrt{\Lambda}}$. For each n we take a point $q_n \in B_{\vec{0}}(\frac{\pi}{2\sqrt{\Lambda}})$ such that $i(p_n) = d(p_n, q_n)(= d(p_n, C(p_n))$, where d is the distance in $(B_{\vec{0}}(\frac{\pi}{\sqrt{\Lambda}}), \exp_x^* g)$, we can assume that $q_n \in B_{\vec{0}}(\frac{3\pi}{8\sqrt{\Lambda}} + \frac{i_0}{2})$. Up to choosing subsequences, we can assume that (p_n)

converges to $p_\infty \in \overline{B}_{\vec{0}}(\frac{\pi}{4\sqrt{\Lambda}})$ and that (q_n) converges to $q_\infty \in \overline{B}_{\vec{0}}(\frac{3\pi}{8\sqrt{\Lambda}}+\frac{i_0}{2}) \subset B_{\vec{0}}(\frac{\pi}{2\sqrt{\Lambda}})$ and so $d(p_\infty, q_\infty) = i_0$.

Note that $i_0 > 0$. Indeed, there is a neighborhood $W \subset B_{\vec{0}}(\frac{\pi}{\sqrt{\Lambda}})$ of p_∞ and there exists $\delta > 0$ such that for any points x, y in W there is a unique minimizing geodesic of length $< \delta$ joining x to y, see do Carmo [4, Remark 3.8, p.72]. It follows easily that there exist a neighborhood $W' \subset W$ of p_∞ and $\delta' > 0$ such that for any $x \in W'$ we have $i(x) \geq \delta'$, and so $i_0 \geq \delta'$.

Since $d(p_n, q_n) < d(p_n, \partial B_{\vec{0}}(\frac{\pi}{\sqrt{\Lambda}}))$, there exists for each n a minimizing geodesic α_n in $B_{\vec{0}}(\frac{\pi}{\sqrt{\Lambda}})$ joining p_n to q_n. Actually, as $q_n \in C(p_n)$ and is not conjugate to p_n (since $d(p_n, q_n) < \frac{\pi}{4\sqrt{\Lambda}}$) there is another minimizing geodesic β_n joining p_n to q_n. As $d(p_n, q_n) = i(p_n)$, by a usual argument, cf. for instance Petersen [16, Lemma 16, p.142], α_n and β_n fit smoothly together at q_n to form a geodesic loop γ_n (with possibly a corner at p_n). As the length of γ_n is bounded from below by $2i_0$, up to taking a subsequence, γ_n converges to a geodesic loop γ_∞ at p_∞ of length $2i_0 = 2d(p_\infty, q_\infty)$ with midpoint q_∞. Therefore $q_\infty \in C(p_\infty)$ and consequently $i(p_\infty) = i_0$.

We now discuss two cases.

Case 1. Assume that $q_\infty \notin \overline{B}_{\vec{0}}(\frac{\pi}{4\sqrt{\Lambda}})$.

Consider the geodesic sphere S centered at $\vec{0}$ of smallest radius enclosing the loop γ_∞. Then γ_∞ is tangent to S at some smooth point z. We now see this is contradictory. For $x \in B_{\vec{0}}(\frac{\pi}{2\sqrt{\Lambda}})$, set $r(x) = d(x, \vec{0})$. Consider the function $f(t) = r(\gamma_\infty(t))$, where t is an arc length parameter of γ_∞ with $\gamma_\infty(0) = z$.

We have $f'(0) = 0$ and $f''(t) = \nabla^2 r(\gamma'_\infty, \gamma'_\infty) + \langle \nabla r, \frac{D}{dt}\gamma'_\infty \rangle = \nabla^2 r(\gamma'_\infty, \gamma'_\infty)$. By comparison theorems we know that $\nabla^2 r$ is positive definite since $r(z) \leq \frac{\pi}{4\sqrt{\Lambda}} + i_0 \leq \frac{\pi}{2\sqrt{\Lambda}}$, Petersen [16, Chapter 6, Section 5, Theorem 27]. Therefore f has a local strict minimum at 0 contradicting the fact that S encloses the loop γ_∞.

Case 2. Assume that $q_\infty \in \overline{B}_{\vec{0}}(\frac{\pi}{4\sqrt{\Lambda}})$.

By the symmetry of the cut-locus, $p_\infty \in C(q_\infty)$. Observe that we have by construction $i_0 \leq i(q_\infty) \leq d(p_\infty, q_\infty) = i_0$. Therefore $i(q_\infty) = d(p_\infty, q_\infty)$ and so p_∞ is a closest point to q_∞ in $C(q_\infty)$. As before we deduce that γ_∞ is also smooth at p_∞. Taking the smallest geodesic sphere centered at $\vec{0}$ enclosing the closed geodesic γ_∞, we arrive to a contradiciton as in case 1. $\square$

We now start the proof of our Main Theorem.

We first show that the conclusion of the theorem is true with a constant $C(\Lambda)$ depending on Λ. Later we will show that this constant may be chosen independant of Λ.

Assume by contradiction that for any $n \in \mathbb{N}^*$ there are: a complete smooth Riemannian 3-manifold (M_n, g_n) with bounded sectional curvature, $|K| \leq \Lambda$, a stable

H_n-surface $\Sigma_n \looparrowright M_n$ and a point $p_n \in \Sigma_n$ satisfying

$$|A_n(p_n)| \min(d_{\Sigma_n}(p_n, \partial\Sigma_n), \frac{\pi}{2\sqrt{\Lambda}}) \geq n, \tag{4}$$

A_n being the second fundamental form of Σ_n.

We denote by $D_n \subset \Sigma_n$ the open geodesic disk of radius $\min(d_{\Sigma_n}(p_n, \partial\Sigma_n), \frac{\pi}{2\sqrt{\Lambda}})$ centered at p_n.

For any $n \in \mathbb{N}^*$ let us denote by $\vec{0}_n$ the origin in $T_{p_n}M_n$. Call $\widetilde{D}_n$ the connected component, containing $\vec{0}_n$, of $\exp_{p_n}^{-1}(D_n) \cap B_{\vec{0}_n}(\frac{\pi}{8\sqrt{\Lambda}}) \subset T_{p_n}M_n$. We endow the ball $B_{\vec{0}_n}(\frac{\pi}{\sqrt{\Lambda}})$ with the pull-back metric $\exp_{p_n}^* g_n$. By abuse of notation, we still denote by g_n this metric and note that its sectional curvature satisfies $|K| \leq \Lambda$.

Observe that $\widetilde{D}_n$ is a stable H_n-surface with trivial normal bundle in $B_{\vec{0}_n}(\frac{\pi}{\sqrt{\Lambda}})$. Indeed, by a result of Fischer-Colbrie and Schoen [6], there exists a positive Jacobi function u on D_n, therefore $u \circ \exp_{p_n}$ is a positive Jacobi function on $\widetilde{D}_n$, this implies stability of $\widetilde{D}_n$ (cf. [6]). By abuse of notation we still denote by A_n the second fundamental form of $\widetilde{D}_n$ in $(B_{\vec{0}_n}(\frac{\pi}{\sqrt{\Lambda}}), g_n)$.

Note that

$$|A_n(\vec{0}_n)| \min\{d_{\widetilde{D}_n}(\vec{0}_n, \partial\widetilde{D}_n), \frac{\pi}{2\sqrt{\Lambda}}\} \geq n. \tag{5}$$

Indeed, we have $|A_n(\vec{0}_n)| = |A_n(p_n)|$ and one can check that for any $x \in \partial\widetilde{D}_n$ we have $d_{\widetilde{D}_n}(\vec{0}_n, x) \geq d_{D_n}(p_n, \partial D_n)$ (consider the two cases $x \in \partial B_{\vec{0}_n}(\frac{\pi}{4\sqrt{\Lambda}})$ and $x \notin \partial B_{\vec{0}_n}(\frac{\pi}{4\sqrt{\Lambda}})$). Then we conclude by (4).

Let $\alpha \in (0,1)$ be a fixed number. Consider the constants $Q_0 > 1$ and r_0 given in Theorem 2.1 applied to $(M,g) = (B_{\vec{0}_n}(\frac{\pi}{\sqrt{\Lambda}}), g_n)$, $\Omega = B_{\vec{0}_n}(\frac{\pi}{8\sqrt{\Lambda}})$, $\delta = \frac{\pi}{8\sqrt{\Lambda}}$ and $i = \frac{\pi}{4\sqrt{\Lambda}}$ (cf. Lemma 2.2). We can assume that $r_0 \leq \frac{\pi}{4\sqrt{\Lambda}}$. Let p_n^* be a point in the closure of $\widetilde{D}_n$ where the function $f : \widetilde{D}_n \to \mathbb{R}$ defined by $f(x) = |A_n(x)| \min\{d_{\widetilde{D}_n}(x, \partial\widetilde{D}_n), r_0\}$ reaches its maximum. Note that p_n^* is an interior point of $\widetilde{D}_n$ since $f \equiv 0$ on $\partial\widetilde{D}_n$.

One can check that $f(\vec{0}_n) \geq r_0 \frac{\sqrt{\Lambda}}{\pi} n$. As $f(\vec{0}_n) \leq f(p_n^*) \leq |A_n(p_n^*)| r_0$ we get

$$|A_n(p_n^*)| \geq \frac{\sqrt{\Lambda}}{\pi} n. \tag{6}$$

Put $\rho_n = \min\{d_{\widetilde{D}_n}(p_n^*, \partial\widetilde{D}_n), r_0\}$. Consider the geodesic disk $D_n^* \subset \widetilde{D}_n$ of radius $\rho_n/2$ centered at p_n^*.

Consider for any $n \in \mathbb{N}^*$ the harmonic chart of M_n, $(U_n, \varphi_n, B(p_n^*, r_0))$, given by Theorem 2.1, where $U_n \subset \mathbb{R}^3$ is an open neighborhood of the origin and $\varphi_n(0) = p_n^*$. By the property (2) the Euclidean ball of radius r_0/Q_0 centered at the origin, $B_e(0, r_0/Q_0)$, is contained in U_n for any n.

Call $\lambda_n = |A_n(p_n^*)|$, by (6), $\lambda_n \to +\infty$. Since $\lambda_n = |A_n(p_n^*)| \geq H_n$, the sequence H_n/λ_n is bounded and so, up to taking a subsequence, we can assume that

$$\frac{H_n}{\lambda_n} \to H^* < \infty. \tag{7}$$

We can also assume that (λ_n) is nondecreasing. Let F_n be the homothety of $\mathbb{R}^3$ of ratio $1/\lambda_n$. Put $V_n = F_n^{-1}(U_n)$ and observe that $B_e(0, \lambda_n r_0/Q_0) \subset V_n \subset B_e(0, Q_0\lambda_n r_0)$ for any n. Therefore $\cup_n V_n = \mathbb{R}^3$ and, up to taking a subsequence, we can assume that the sequence (V_n) is monotone for the inclusion.

We call (x_1, x_2, x_3) the cartesian coordinates of $x \in U_n$ and (y_1, y_2, y_3) those of $y = F_n^{-1}(x) \in V_n$. Now we use a blow-up argument, we endow V_n with the metric $h_n = \lambda_n^2 F_n^*(\varphi_n^* g_n)$. In the coordinates (x_1, x_2, x_3) the metric $\varphi_n^* g_n$ reads as

$$(\varphi_n^* g_n)(x) = \sum_{i,j} g_{ij}^n(x) dx_i dx_j.$$

For any $C^{1,\alpha}$-function w on an open set $\Omega \subset \mathbb{R}^3$ we set:

$$\|w\|_{C^{1,\alpha}(\Omega)} = \|w\|_\infty + \sum_{k=1}^{3} \sup_{y\in\Omega} \Big|\partial_k w(y)\Big| + \sum_{k=1}^{3} \sup_{y\neq z} \frac{\big|\partial_k w(y) - \partial_k w(z)\big|}{|y-z|^\alpha}.$$

Observe that, because of property (2) of Theorem 2.1, up to passing to a subsequence, we can assume that the sequence of inner products $(g_{ij}^n(0))$ converges to some inner product on $\mathbb{R}^3$. Up to a linear change of coordinates we can assume that this limit is the Euclidean inner product (δ_{ij}).

From properties (2) and (3) of Theorem 2.1, there exists a constant $Q > 1$ such that :

$$\|g_{ij}^n\|_{C^{1,\alpha}(U_n)} \leq Q. \tag{8}$$

For each n and each $y \in V_n$ we have

$$h_n(y) = \sum_{i,j} g_{ij}^n(\frac{y}{\lambda_n}) dy_i dy_j.$$

Thus the components h_{ij}^n of h_n in the coordinates $y = (y_1, y_2, y_3)$ are given by $h_{ij}^n(y) = g_{ij}^n(\frac{y}{\lambda_n})$ for any $y \in V_n$. Observe that $\frac{\partial h_{ij}^n}{\partial y_k}(y) = \frac{1}{\lambda_n}\frac{\partial g_{ij}^n}{\partial x_k}(\frac{y}{\lambda_n})$, so we get on V_n:

$$\Big|\frac{\partial h_{ij}^n}{\partial y_k}\Big| \leq \frac{Q}{\lambda_n}, \quad k = 1, 2, 3. \tag{9}$$

Therefore we have for any $y \in V_n$

$$|h_{ij}^n(y) - \delta_{ij}| \leq \sqrt{3}\frac{Q}{\lambda_n}|y| + |h_{ij}^n(0) - \delta_{ij}|. \tag{10}$$

Thus on any Euclidean ball $B_R \subset V_n$ of radius $R > 0$ centered at the origin, we get

$$\|h_{ij}^n - \delta_{ij}\|_{C^{1,\alpha}(B_R)} \leq \sqrt{3}\frac{Q}{\lambda_n}R + 3\frac{Q}{\lambda_n} + \frac{Q^{1+\alpha}}{\lambda_n^{1+\alpha}} + |h_{ij}^n(0) - \delta_{ij}|. \tag{11}$$

It follows that the sequence (V_n, h_n) converges uniformly on compact subsets of $\mathbb{R}^3$ for the $C^{1,\alpha}$-Euclidean topology to $(\mathbb{R}^3, g_{euc})$, where g_{euc} stands for the Euclidean metric.

Composing with the diffeomorphism $(\varphi_n \circ F_n)^{-1}$ we view D_n^* as immersed in V_n. Note that the image $(\varphi_n \circ F_n)^{-1}(p_n^*) = 0 \in \mathbb{R}^3$ for any n and that D_n^* is a geodesic disk of radius $\lambda_n\rho_n/2$ centered at p_n^* for the metric induced by h_n.

Note that $\lambda_n\rho_n = f(p_n^*) \geq f(\vec{0}_n) \geq r_0\frac{\sqrt{\Lambda}}{\pi}n$, thus

$$\lambda_n\rho_n \to +\infty. \tag{12}$$

For $x \in D_n^*$ we have $d_{\widetilde{D}_n}(p_n^*, \partial\widetilde{D}_n) \leq 2d_{\widetilde{D}_n}(x, \partial\widetilde{D}_n)$. Since $f(x) \leq f(p_n^*)$ we deduce that the second fundamental form A_n^* of D_n^* in (V_n, h_n) satisfies

$$|A_n^*(x)| = \frac{|A_n(x)|}{\lambda_n} \leq \frac{\min\{d_{\widetilde{D}_n}(p_n^*, \partial\widetilde{D}_n), r_0\}}{\min\{d_{\widetilde{D}_n}(x, \partial\widetilde{D}_n), r_0\}} \leq 2. \tag{13}$$

It is important to observe that

$$|A_n^*(p_n^*)| = 1 \tag{14}$$

for any $n \in \mathbb{N}^*$.

Let us call II_n the second fundamental form of D_n^* in $(\mathbb{R}^3, g_{euc})$.

Given $m \in \mathbb{N}^*$ denote by $\Delta_{n,m}$ the connected component of $D_n^* \cap B_m$ passing through 0 (that is, containing p_n^*). As the sequence of metrics h_n converges on compact sets of $\mathbb{R}^3$ to g_{euc} for the $C^{1,\alpha}$-Euclidean topology, we deduce from Proposition 4.1 in the appendix that there exists $n_m \in \mathbb{N}^*$ such that for any $n \geq n_m$ and any $x \in \Delta_{n,m}$, we have

$$\Big||\mathrm{II}_n(x)| - |A_n^*(x)|\Big| \leq \frac{1}{m}.$$

Furthermore we have $d_{\Delta_{n,m}}(p_n^*, \partial\Delta_{n,m}) \geq m$, if n_m is big enough, since $\lambda_n\rho_n \to +\infty$. Observe that if $m' > m$ then $\Delta_{n,m} \subset \Delta_{n,m'}$. Moreover we can assume that the sequence (n_m) is increasing. We set $\Delta_m = \Delta_{n_m,m}$. Thus, we have constructed in this way a sequence of connected surfaces $\Delta_n \looparrowright \mathbb{R}^3$ passing through 0 (that is, containing p_n^*) with the following property: for any $k \in \mathbb{N}^*$ there exists n_k such that for any $n \geq n_k$ and any $x \in \Delta_n \cap B_k$ we have

$$\Big||\mathrm{II}_n(x)| - |A_n^*(x)|\Big| \leq \frac{1}{k} \quad \text{and} \quad d_{\Delta_n}(p_n^*, \partial\Delta_n) \geq k. \tag{15}$$

In particular we get for $n \geq n_k$

$$\Big| |\mathrm{II}_n(p_n^*)| - 1 \Big| \leq \frac{1}{k}. \tag{16}$$

Furthermore, for any k, any $n \geq n_k$ and any $x \in \Delta_n \cap B_k$ we get from (13) and (15):

$$|\mathrm{II}_n(x)| \leq 5. \tag{17}$$

We will use the following well known result which can be deduced from Colding-Minicozzi [2] or Perez-Ros [15].

Proposition 2.3. *Let $\Sigma \looparrowright \mathbb{R}^3$ be an immersed surface whose second fundamental form A satisfies $|A| < \frac{1}{4\delta}$ for some constant $\delta > 0$. Then for any $x \in \Sigma$ with $d_\Sigma(x, \partial\Sigma) > 4\delta$ there is a neighborhood of x in Σ which is a graph of a function u over the Euclidean disk of radius $\sqrt{2}\delta$ centered at x in the tangent plane of Σ at x. Moreover u satisfies*

$$|u| < 2\delta, \quad |\nabla u| < 1 \quad \textit{and} \quad |\nabla^2 u| < \frac{1}{\delta}. \tag{18}$$

We deduce from the estimate (17) and Proposition 2.3 that for each n big enough, a part of the surface Δ_n is the graph of a function u_n over a Euclidean disk of radius $\sqrt{2}\delta$, with $\delta = 1/20$, centered at the origin in the tangent plane of Δ_n at the origin. Furthermore the functions u_n satisfy the uniform estimates (18). Up to passing to a subsequence, still denoted Δ_n, and up to a rotation in $\mathbb{R}^3$, we can assume that the tangent planes $T_0\Delta_n$ converge to the horizontal plane P through the origin. Consequently, for n big enough, a part of Δ_n is the graph of a function still denoted u_n, over the Euclidean disk D_δ of radius δ centered at the origin in P. By continuity, note that the new functions u_n satisfy the following uniform estimates for n big enough:

$$|u_n| < 3\delta, \quad |\nabla u_n| < 2 \quad \text{and} \quad |\nabla^2 u_n| < \frac{1}{\delta} \qquad (\text{with } \delta = \frac{1}{20}). \tag{19}$$

Thus we have obtained uniform C^2-estimates for the functions u_n on D_δ. To go further we need $C^{2,\alpha}$-estimates, $0 < \alpha < 1$. This is the content of the next lemma.

Lemma 2.4. *For any $\delta' \in (0, \delta)$ there exits a constant C which does not depend on n such that for n big enough we have*

$$\|u_n\|_{C^{2,\alpha}(D_{\delta'})} < C, \tag{20}$$

where $D_{\delta'}$ denotes the disk in P of radius δ' centered at the origin.

Proof. Since the mean curvature of Δ_n for the metric h_n is H_n/λ_n, we infer, see for instance Colding-Minicozzi [2, p. 99–100], that the function u_n is solution of an elliptic PDE of the form:

(21)

$$L(u) := a^{ij}(u, \nabla u, h_n)u_{ij} + b^i(u, \nabla u, h_n, \partial_m h_n)u_i = 2\frac{H_n}{\lambda_n}W(\nabla u, h_n) + c(u, \nabla u, h_n, \partial_m h_n),$$

where by h_n we mean the components $(h_n)_{\alpha\beta}$ of the metric h_n and by $\partial_m h_n$ we mean the partial derivatives $\frac{\partial (h_n)_{\alpha\beta}}{\partial y_m}$, $1 \leq \alpha, \beta, m \leq 3$. Moreover the functions a^{ij}, b^i, W and c depend in a smooth way on their arguments.

Recall that the sequence (u_n) has uniform C^2-estimates on D_δ and so has uniform $C^{1,\alpha}$-estimates. Moreover the sequence of metrics (h_n) converges to the Euclidean metric g_{euc} for the $C^{1,\alpha}$-topology on compact sets.

Recall also that H_n/λ_n is bounded, see (7). Thus the functions $a^{ij}(u, \nabla u, h_n)$, $b^i(u, \nabla u, h_n, \partial_m h_n)$ and $2(H_n/\lambda_n)W(\nabla u, h_n) + c(u, \nabla u, h_n, \partial_m h_n)$ have uniform $C^{0,\alpha}$-estimates on D_δ, that is, $C^{0,\alpha}$-estimates independant on n.

By Schauder estimates, see Gilbarg-Trudinger [7, Chapter 6] or Petersen [16, Chapter 10], for any $\delta' \in (0, \delta)$, there exists a constant $C_0 > 0$ which does not depend on n such that:

$$\|u_n\|_{C^{2,\alpha}(D_{\delta'})} < C_0\Big(\|L(u_n)\|_{C^\alpha(D_\delta)} + \|u_n\|_{C^\alpha(D_\delta)}\Big). \tag{22}$$

Consequently, there exists a constant $C > 0$ which does not depend on n such that:

$$\|u_n\|_{C^{2,\alpha}(D_{\delta'})} < C, \tag{23}$$

which concludes the proof of the lemma. □

Fix some $\delta' \in (0, \delta)$. As the sequence (u_n) is $C^{2,\alpha}$-bounded on the disk $D_{\delta'}$, by Arzela-Ascoli's theorem there is some subsequence, still denoted (u_n), which converges to a C^2-function u in the C^2-topology. Since the mean curvature, H_n/λ_n, of the graph of u_n for the metric h_n tends to H^*, see (7), thanks to the Proposition 4.1 we know that the mean curvature of the graph of u_n for the Euclidean metric also tends to H^*. We deduce that the graph of u, denoted by S, is an H^*-surface for the Euclidean metric.

Note that this graph contains the origin and that its second fundamental form A verifies: $|A(0)| = 1$ and $|A| < 5$ on $D_{\delta'}$ (by (13), (14) and Proposition 4.1).

Let $x_0 \in D_{\delta'}$ at Euclidean distance $\delta'/2$ from 0 and fix once for all some $\delta'' \in (\delta', \delta)$. Note that for n big enough, a part of the surface Δ_n is the graph of a function v_n over the Euclidean disk $D_{\delta''}(x_0)$ centered at $(x_0, u(x_0)) \in S$ in the tangent plane $T_{(x_0,u(x_0))}S$. Using the same arguments as before, we can extract a subsequence of (v_n) whose graphs converge to a H^*-graph over the disk $D_{\delta'}(x_0)$. Observe that this new graph is not contained in S because $|\nabla u| \leq 1$. We have in this way obtained a new H^*-surface extending S. We will still denote this extended H^*-surface by S.

As $d_{\Delta_n}(0, \partial\Delta_n) \to +\infty$, using a standard diagonal process, we obtain a complete H^*-surface S immersed in $\mathbb{R}^3$, passing through the origin and whose second fundamental form A satisfies:

$$|A(0)| = 1 \quad \text{and} \quad |A| \leq 5. \tag{24}$$

Consider the universal covering $\widetilde{S}$ of S, observe that $\widetilde{S}$ is naturally immersed in $\mathbb{R}^3$ and its second fundamental form is bounded. Consequently, there exists an $\varepsilon > 0$

such that the map $\Pi : W := \widetilde{S} \times (-\varepsilon, +\varepsilon) \to \mathbb{R}^3$ given by $\Pi(p,t) = p + t\widetilde{N}(p)$ is an immersion, where $\widetilde{N}$ is the Gauss map of $\widetilde{S}$. Therefore we can endow W with a flat metric which makes Π a local isometry. Note that $\widetilde{S}$ is a complete H^*-surface in W.

Claim: $\widetilde{S}$ is stable in W.

For any geodesic disk in S of radius δ', for n big enough, a piece of Δ_n is a graph over this geodesic disk of S. By construction of S, using this fact, for any compact and simply connected domain U in S, a piece of Δ_n will be a graph over U for n big enough. Using a continuation argument, for any compact and simply connected domain $\widetilde{U}$ in $\widetilde{S}$, we can lift a part G_n of Δ_n, n big enough, to get a surface $\widetilde{G}_n$ in W which will be a graph over $\widetilde{U}$. Moreover the graphs $\widetilde{G}_n$ converge to $\widetilde{U}$ in the C^2-topology.

Observe that each graph $\widetilde{G}_n$ is a stable H-surface (with $H = H_n/\lambda_n$) in (a part of) W endowed with the metric $\Pi^*(h_n)$. Indeed, since G_n is two-sided and stable for the metric h_n, by a result of Fischer-Colbrie and Schoen [6], there exists a positive Jacobi function v_n on G_n. The function $\widetilde{v}_n = v_n \circ \Pi$ is thus a positive Jacobi function on $\widetilde{G}_n$ for the metric $\Pi^*(h_n)$. Again by [6] this implies that $\widetilde{G}_n$ is stable.

For n big enough, $\widetilde{G}_n$ is the graph over $\widetilde{U}$ of a function which tends to 0 in the C^2-norm as n goes to $+\infty$. In this way we may see the stability operator of $\widetilde{G}_n$: $J_n = \Delta_n + |\widetilde{B}_n^*|^2 + \mathrm{Ric}_n(\nu)$ as an operator on $\widetilde{U}$. Here $\widetilde{B}_n^*$ stands for the second fundamental form of $\widetilde{G}_n$, Δ_n denotes the Laplacian on $\widetilde{G}_n$ and $\mathrm{Ric}_n(\nu)$ denotes the Ricci curvature in the direction of the unit normal field ν to $\widetilde{G}_n$, all of them with respect to the metric $\Pi^*(h_n)$.

As the metrics $\Pi^*(h_n)$ converge to the flat metric $\Pi^*(g_{euc})$ in the $C^{1,\alpha}$-topology and $|\mathrm{Ric}_n(\nu)| \leq 2\Lambda/\lambda_n^2$, it follows that the domain $\widetilde{U}$ is stable for the flat metric. This implies that $\widetilde{S}$ is stable in W for the flat metric as claimed.

As the immersion Π is a local isometry we infer that the complete immersion $\widetilde{S} \to S \subset \mathbb{R}^3$ is stable. Thanks to results of do Carmo-Peng [5], Fischer-Colbrie and Schoen [6], Pogorelov [17], Lopez-Ros [11] and Silveira [20], we know that S is a plane. This contradicts the fact that $|A(0)| = 1$, see (24).

It remains to check that $C(\Lambda)$ can be chosen independant of Λ.

First we can assume that $C(\Lambda)$ is the infimum among the constants satisfying the conclusion of the theorem. Let Σ be any stable H-surface in (M,g). Observe that Σ is a stable $\frac{H}{\tau}$-surface in $(M, \tau^2 g)$ for any $\tau > 0$ and that the quantity $|A(p)|\min\{d(p,\partial\Sigma), \frac{\pi}{2\sqrt{\Lambda}}\}$ is scale invariant. It follows that $C(\Lambda) = C(\Lambda/\tau^2)$ for any $\tau > 0$ and so $C(\Lambda)$ does not depend on Λ, which concludes the proof of the Main Theorem. $\square$

The same proof gives the following *local* result.

Theorem 2.5. *Let (M, g) be a smooth Riemannian 3-manifold (not necessarily complete), with bounded sectional curvature $|K| \leq \Lambda < +\infty$. Let Ω be an open subset of M such that there exists $\delta > 0$ for which $\Omega(\delta)$ is relatively compact in M (cf. Theorem 2.1 for the notation).*

Then there exists a constant $C = C(\delta^2\Lambda) > 0$ depending only on the product $\delta^2\Lambda$ and neither on M nor on Ω, satisfying the following:

For any immersed stable H-surface $\Sigma \looparrowright \Omega$, with trivial normal bundle, and for any $p \in \Sigma$ we have

$$|A(p)| < \frac{C}{\min\left\{d(p, \partial\Sigma), \frac{\pi}{2\sqrt{\Lambda}}, \delta\right\}}.$$

3. Applications

In this section we consider Riemannian 3-manifolds (M^3, g) which fiber over a Riemannian surface (M^2, h). We assume that the fibration $\Pi : (M^3, g) \to (M^2, h)$ is a Riemannian submersion with the following properties.

(1) Each fiber is a complete geodesic of infinite length.
(2) The fibers of the fibration are the integral curves of a unit Killing vector field ξ on M^3.

A fibration satisfying (1) and (2) will be called a *Killing submersion*. It can be shown that such a fibration is (topologically) trivial. Indeed, there always exists a global section $s : M^2 \to M^3$ (see Steenrod [22, Theorem 12.2]). Considering the flow φ_t of ξ, a trivialization of the fibration is given by the diffeomorphism: $(p, t) \in M^2 \times \mathbb{R} \to \varphi_t(s(p)) \in M^3$.

Notice that there are many such 3-manifolds, including $\mathbb{R}^3$, $\widetilde{\mathrm{PSL}}(2, \mathbb{R})$ (which fibers over the hyperbolic plane $\mathbb{H}^2$), the Heisenberg space Nil_3 (which fibers over the Euclidean plane $\mathbb{R}^2$) and the metric product spaces $M^2 \times \mathbb{R}$ for any Riemannian surface (M^2, h).

Definition 3.1. *Let $\Pi : (M^3, g) \to (M^2, h)$ be a Killing submersion.*

(1) *Let $\Omega \subset M^2$ be a domain. An* H-section *over Ω is an H-surface in M^3 which is the image of a section $s : \overline{\Omega} \to M^3$, with s of class C^2 on Ω and C^0 on $\overline{\Omega}$.*
(2) *Let $\gamma \subset M^2$ be a smooth curve with geodesic curvature $2H$. Observe that the surface $\Pi^{-1}(\gamma) \subset M^3$ has mean curvature H. We call such a surface a* vertical H-cylinder*.*

We may also consider H-sections without boundary.

Remark 3.2. *Let $s : \overline{\Omega} \to \Sigma \subset \Omega$ be a H-section where $\Omega \subset M^2$ is a relatively compact domain. We make the following observations.*

(1) *For any interior point $p \in \Omega$ the H-surface Σ is transversal at $s(p)$ to the fiber. Indeed, assume by contradiction that Σ is tangent to the fiber at the*

interior point $s(p)$. Let $S \subset M^3$ be the vertical H-cylinder tangent to Σ at $s(p)$ with the same mean curvature vector. Then, in a neighborhood of $s(p)$, the intersection $\Sigma \cap S$ is composed of $n \geq 2$ smooth curves passing through $s(p)$. But the union of those curves cannot be a graph, contradicting the assumption that Σ is a graph over $\overline{\Omega}$.

(2) *Let $s_1 : \overline{\Omega} \to \Sigma_1 \subset \Omega$ be another H-section over $\overline{\Omega}$ such that the mean curvature vector field $\vec{H}_1$ points in the same direction as the mean curvature vector field $\vec{H}$ of Σ, that is, the scalar products $g(\vec{H}, \xi)$ and $g(\vec{H}_1, \xi)$ have the same sign. Assume that there exists a constant $C > 0$ such that $|s(p) - s_1(p)| < C$ at any $p \in \partial\Omega$ where $|s(p) - s_1(p)|$ denotes the distance on the fiber over p between the points $s(p)$ and $s_1(p)$. As ξ is a Killing field, the translated copies of Σ along the fibers are also H-surfaces whose mean curvature vector has the same orientation as that of Σ. Then applying the maximum principle to Σ_1 and a translation of Σ by ξ, we deduce that we have also $|s(p) - s_1(p)| < C$ at any $p \in \overline{\Omega}$.*

This gives for any such H-section Σ_1 height estimates, relative to Σ, depending on the vertical distance between the boundaries of Σ and Σ_1.

Our first application is as follows.

Theorem 3.3. *Let $\Pi : (M^3, g) \to (M^2, h)$ be a Killing submersion and let $s : \Omega \to \Sigma \subset M^3$ be an H-section over a domain $\Omega \subset M^2$. Let U_0 be a neighborhood of an arc $\gamma \subset \partial\Omega$ and $s_0 : U_0 \to M^3$ a section.*

Assume that for any sequence (p_n) of Ω which converges to a point $p \in \gamma$, the height of $s(p_n)$ goes to $+\infty$, that is $s(p_n) - s_0(p_n) \to +\infty$.

Then, γ is a smooth curve with geodesic curvature $2H$. If $H > 0$ then γ is convex with respect to Ω if, and only if, the mean curvature vector $\vec{H}$ of Σ points up, that is, if $g(\vec{H}, \xi) > 0$ along Σ. Moreover, Σ converges to the vertical H-cylinder $\Pi^{-1}(\gamma)$ with respect to the C^k-topology for any $k \in \mathbb{N}$; this convergence will be made precise in the proof.

Proof. Let $p \in \gamma$ and (p_n) a sequence in Ω converging to p. Let $\overline{B(p, \rho)}$ be a compact geodesic disc of M^2 centered at p, with the radius ρ small so that $s(\Omega \cap B(p, \rho))$ is far from $\partial\Sigma$.

Let $\widetilde{\Sigma}_n$ be the H-section obtained by translating Σ by the integral curves of ξ so that $s(p_n)$ goes to $s_0(p_n) = \widetilde{p}_n$. After passing to a subsequence, we can assume the tangent planes $T_{\widetilde{p}_n}\widetilde{\Sigma}_n$ converge to a 2-plane $P_{\widetilde{p}}$ of the tangent space $T_{\widetilde{p}}M^3$; here $\widetilde{p} = s_0(p)$.

We deduce from the Main Theorem, Proposition 4.3 (H-sections are stable) and a continuity argument that there exist two real numbers $\delta, \delta_0 > 0$ such that for n big enough, a part of $\widetilde{\Sigma}_n$ is the Euclidean graph (in the sense of Remark 4.2) of some function $\widetilde{u}_n$ over the disk of $P_{\widetilde{p}}$ centered at $\widetilde{p}$ with Euclidean radius δ. Furthermore this part of $\widetilde{\Sigma}_n$ contains the geodesic disk, denoted by $\widetilde{D}_n$, centered at $\widetilde{p}_n$ with radius δ_0.

Since the mean curvature of each $\widetilde{D}_n$ is constant and equal to H, the functions $\widetilde{u}_n$ satisfy an elliptic PDE. Therefore, using the Schauder theory, we can find a subsequence of the previous subsequence converging for the C^k-topology, for any $k \in \mathbb{N}$, to some H-surface $D_{\widetilde{p}}$ passing through $\widetilde{p}$ with tangent plane $P_{\widetilde{p}}$, $T_{\widetilde{p}}D_{\widetilde{p}} = P_{\widetilde{p}}$. Since $D_{\widetilde{p}}$ is the limit of vertical graph and $p = \Pi(\widetilde{p})$ is a boundary point of Ω, the tangent plane at $\widetilde{p}$ must be vertical. Furthermore $D_{\widetilde{p}}$ contains the geodesic disk centered at $\widetilde{p}$ with radius $\delta_0/2$. Denoting by N the unit normal field along $D_{\widetilde{p}}$ given by the limit of the unit normal fields along the disks $\widetilde{D}_n$, observe that the function $g(N, \xi)$ is a Jacobi function on $D_{\widetilde{p}}$. As each $\widetilde{D}_n$ is a vertical graph, this function has a sign, but it vanishes at an interior point of $D_{\widetilde{p}}$: $\widetilde{p}$. Therefore, the maximum principle, see Spivak [21, Chapter 10, Addendum 2, Corollary 19] shows that $g(N, \xi)$ is the null function. This means that the normal field N is horizontal. Clearly, this implies that the limit surface $D_{\widetilde{p}}$ is a part of the vertical cylinder over some curve $\widetilde{\alpha} \subset M^2$. Since $D_{\widetilde{p}}$ has mean curvature H and the fibers are geodesic lines of M^3, the curve $\widetilde{\alpha}$ must be smooth and must have geodesic curvature $2H$ in M^2.

Since each $\widetilde{D}_n$ is, up to a translation along the fibers, part of the graph Σ over Ω, we deduce that:

(1) $\widetilde{\alpha} \subset \overline{\Omega} \subset M^2$.
(2) Each converging subsequence of $\widetilde{D}_n$ must converge to the same H-surface $D_{\widetilde{p}}$.
(3) The limit surface $D_{\widetilde{p}}$ doest not depend on the sequence (p_n) of Ω converging to $p \in \gamma$.

Now we show that we have $\widetilde{\alpha} \subset \gamma \subset \partial\Omega$, that is, the arc γ is smooth and has geodesic curvature $2H$.

Assume by contradiction that there is an interior point $q \in \widetilde{\alpha} \cap \Omega$. Take any point $\widetilde{q} \in D_{\widetilde{p}}$ in the fiber over q, $\Pi(\widetilde{q}) = q$. By construction, $\widetilde{q}$ is the limit of some sequence $(\widetilde{q}_n)$ with $\widetilde{q}_n \in \widetilde{D}_n$. Since the sequence $(\widetilde{D}_n)$ converges to (a part of) the vertical cylinder $\Pi^{-1}(\widetilde{\alpha})$ with the C^k-topology for any $k \in \mathbb{N}$, the surface Σ must be vertical at $s(q)$, contradicting the fact that Σ is transversal to the fibers, see Remark 3.2-(1).

The assertion about the convexity of the arc γ is obvious. □

Remark 3.4. *In the case where M^3 is the metric product space $\mathbb{H}^2 \times \mathbb{R}$ or $\mathbb{S}^2 \times \mathbb{R}$, the result above was shown in an analytic way by Hauswirth, Rosenberg and Spruck* [8].

Remark 3.5. *In a Riemannian product $(M^3, g) = (M^2, h) \times \mathbb{R}$, consider a domain $\Omega \subset M^2$ and a smooth surface $\Sigma \subset M^3$ which is the vertical graph of a function u on Ω. Let N be a unit normal field on Σ and let $\xi = \frac{\partial}{\partial t}$ be the unit vertical field. Then* *we have*

$$|g(N,\xi)| = \frac{1}{\sqrt{1+|\nabla_h u|^2}}.$$

Therefore, bounding $|\nabla_h u|$ from above is equivalent to bounding $|g(N,\xi)|$ from below away from 0.

As a second application we obtain *interior gradient estimates*, see Remark 3.5, for H-sections.

Theorem 3.6. *Let $\Pi : (M^3, g) \to (M^2, h)$ be a Killing submersion. Let $\Omega \subset M^2$ be a relatively compact domain and $s_0 : \overline{\Omega} \to \Sigma_0$ a C^0-section over $\overline{\Omega}$.*

Then, for any $C_1, C_2 > 0$, there exists a constant $\alpha = \alpha(C_1, C_2, \Omega)$ such that for any $p \in \Omega$ with $d(p, \partial\Omega) > C_2$ and for any H-section $s : \overline{\Omega} \to \Sigma \subset M^3$ over $\overline{\Omega}$ satisfying $|s - s_0| < C_1$ on Ω, we have

$$\big|g\big(N,\xi\big)(s(p))\big| > \alpha, \tag{25}$$

where N is a unit normal field along Σ.

Proof. Assume by contradiction that there exist positive constants C_1 and C_2 such that for any $n \in \mathbb{N}^*$ there exists a H_n-section $s_n : \overline{\Omega} \to \Sigma_n \subset M^3$ over $\overline{\Omega}$ with $|s_n - s_0| < C_1$, and there exists a point $p_n \in \Omega$ such that $d(p_n, \partial\Omega) > C_2$ and verifying

$$\big|g\big(N_n,\xi\big)(s_n(p_n))\big| \leq \frac{1}{n}, \tag{26}$$

where N_n is the unit normal field along Σ_n oriented by $\vec{H}_n$.

Since $\overline{\Omega}$ is compact, there exists a subsequence of (p_n) converging to a point $p \in \Omega$, with $d(p, \partial\Omega) \geq C_2$. For each $n \in \mathbb{N}^*$ we denote by $\widetilde{\Sigma}_n$ the vertically translated copy of Σ_n passing through $\widetilde{p}_n := s_0(p_n)$. Observe that the sequence $(\widetilde{p}_n)$ converges to $\widetilde{p} := s_0(p)$.

Since each H_n-surface $\widetilde{\Sigma}_n$ is a vertical graph, $\widetilde{\Sigma}_n$ is stable. Thus we can apply our Main Theorem and Proposition 4.3. Therefore, there exist positive constants δ and δ_0 such that for each $n \in \mathbb{N}^*$ a part $\widetilde{S}_n$ of $\widetilde{\Sigma}_n$ is a Euclidean graph over the disk of $T_{\widetilde{p}_n}\widetilde{\Sigma}_n$ centered at $\widetilde{p}_n$ with Euclidean radius δ_0, furthermore $\widetilde{S}_n$ contains the geodesic disk $\widetilde{D}_n$ of $\widetilde{\Sigma}_n$ centered at $\widetilde{p}_n$ with radius δ.

As usual, up to choosing a subsequence, we can assume that the sequence of tangent planes $(T_{\widetilde{p}_n}\widetilde{\Sigma}_n)$ converges to a 2-plane $P_{\widetilde{p}} \subset T_{\widetilde{p}}M^3$ at $\widetilde{p}$.

Taking into account (26), we deduce that $P_{\widetilde{p}}$ is vertical. Thus, there exists a positive number $\delta_0' < \delta_0$ such that for n big enough, a part of $\widetilde{S}_n$ is a Euclidean graph over the disk of $P_{\widetilde{p}}$ centered at $\widetilde{p}$ with Euclidean radius δ_0' and this part contains the geodesic disk of $\widetilde{\Sigma}_n$ centered at $\widetilde{p}_n$ with radius $\delta/2$.

Observe that the sequence (H_n) is bounded. Indeed, consider the geodesic sphere $S(\widetilde{p})$ in M^3 centered at $\widetilde{p}$ with radius $C_2/2$ and $H_0 > 0$ an upper bound of the absolute mean curvature of $S(\widetilde{p})$.

Using the comparison principle applied to Σ_n and a suitable translated copy of $S(\widetilde{p})$ along the fibers, we get $|H_n| \leq H_0$. Up to taking a subsequence, we can assume that $H_n \to H$.

As in the proof of the Main Theorem, using the Schauder theory, we can prove that a subsequence of $(\widetilde{S}_n)$ converges in the C^k-topology, for any $k \in \mathbb{N}$, to a H-surface $\widetilde{S}$ passing through $\widetilde{p}$ with vertical tangent plane $P_{\widetilde{p}} = T_{\widetilde{p}}\widetilde{S}$. As in the proof of Theorem 3.3, the maximum principle, see Spivak [21, Chapter 10, Addendum 2, Corollary 19], shows that $g(N, \xi)$ is the null function on $\widetilde{S}$, where N is the limit unit normal field on $\widetilde{S}$. This means that $\widetilde{S}$ is part of a vertical H-cylinder over some curve $\widetilde{\gamma} \subset \Omega$, that is, $\widetilde{S} \subset \Pi^{-1}(\widetilde{\gamma})$. Since $\widetilde{S}$ is a H-surface, $\widetilde{\gamma}$ is a smooth curve with geodesic curvature $2H$. Finally, $\widetilde{S}$ contains the geodesic disk of $\Pi^{-1}(\widetilde{\gamma})$ centered at $\widetilde{p}$ with radius $\delta/2$.

Let us call $\widetilde{q} \in \Pi^{-1}(p) \subset M^3$ the point in the same fiber than $\widetilde{p}$ with vertical distance $\delta/4$, lying over $\widetilde{p}$. Observe that, by construction, $\widetilde{q}$ is the limit of some sequence $(\widetilde{q}_n)$, with $\widetilde{q}_n \in \widetilde{D}_n \subset \widetilde{\Sigma}_n$. Observe that $\Pi(\widetilde{q}) \in \Omega$ and $d(\Pi(\widetilde{q}), \partial\Omega) \geq C_2$ since $\Pi(\widetilde{q}) = p$. Therefore, the same arguments used above show that the geodesic disk of the vertical H-cylinder $\Pi^{-1}(\widetilde{\gamma})$ centered at $\widetilde{q}$ with radius $\delta/2$ is the limit of a sequence (Σ_n'), $\Sigma_n' \subset \widetilde{\Sigma}_n$, extending in this way the part $\widetilde{S}$ of $\Pi^{-1}(\widetilde{\gamma})$.

Repeating this argument, we can show that a connected part of the vertical H-cylinder $\Pi^{-1}(\widetilde{\gamma})$, which is as high as we want, is contained in the limit set of the sequence $(\widetilde{\Sigma}_n)$. This gives a contradiction since the vertical distance between Σ_0 and Σ_n is uniformly bounded by C_1 by hypothesis. $\square$

4. Appendix

Proposition 4.1. *Let $U \subset \mathbb{R}^3$ be an open set and let $S \subset U$ be an immersed C^2-surface without boundary. Let g be a metric on U, denote by g_{euc} the Euclidean metric. Let respectively A and $\overline{A}$ be the second fundamental forms of S for the metrics g_{euc} and g. Assume that $|\overline{A}| < C$ for some constant $C > 0$.*

Then, for any $n \in \mathbb{N}^*$, there exists a constant $C_n > 0$ which does not depend on S such that if $\|g_{ij} - g_{euc,ij}\|_{C^1(U)} < C_n$, $1 \leq i, j \leq 3$, then for any $p \in S$ and any nonzero tangent vector $v \in T_pS$ we have:

$$|\overline{\lambda}_p(v) - \lambda_p(v)| < \frac{1}{n} \quad \text{and} \quad |H - \overline{H}|(p) < \frac{1}{n}, \tag{27}$$

where $\lambda_p(v)$, resp. $\overline{\lambda}_p(v)$, denotes the normal curvature of S at p in the tangent direction v with respect to the metric g_{euc}, resp. g, and H, resp. $\overline{H}$, denotes the mean curvature of S at p for the metric g_{euc}, resp. g, (the curvatures of both surfaces being computed with respect to normals inducing the same transversal orientation).

Consequently, for any $n \in \mathbb{N}^*$*, there exists a constant* $C'_n > 0$ *which does not depend on* S *such that if* $\|g_{ij} - g_{euc,ij}\|_{C^1(U)} < C'_n$, $1 \leq i, j \leq 3$*, then we have:*

$$\Big| |A| - |\overline{A}| \Big| < \frac{1}{n}. \tag{28}$$

Proof. Let us denote the normal curvatures by $\lambda(v)$ and $\overline{\lambda}(v)$.

Observe that if we have $\|g_{ij} - g_{euc,ij}\|_{C^1(U)} < C$ for some constant $C > 0$ then, for any Euclidean change of coordinates we certainly have $\|g_{ij} - g_{euc,ij}\|_{C^1(U)} < 9C$ in the new coordinates.

We can choose Euclidean coordinates (x_1, x_2, x_3) of $\mathbb{R}^3$ such that the origin coincides with p, the tangent plane T_pS coincides with the plane $\{x_3 = 0\}$ and v is tangent to the x_1-axis. Thus, a neighborhood S_p of p in S is the graph of a function $x_3 = u(x_1, x_2)$ defined in a neighborhood V of the origin in the plane $\{x_3 = 0\}$. Therefore a parametrization of S_p is given by $(x_1, x_2) \mapsto F(x_1, x_2) = (x_1, x_2, u(x_1, x_2))$, $(x_1, x_2) \in V \subset \{x_3 = 0\}$.

Let us set $F_i = \frac{\partial F}{\partial x_i}$, $u_i = \frac{\partial u}{\partial x_i}$, $i = 1, 2$. We have

$$u_i(0) = 0, \quad i = 1, 2. \tag{29}$$

We denote by $\overline{N}$ the Gauss map of S for the metric g and by N the Gauss map of S for the metric g_{euc}, both oriented so that at 0 we have $N(0) = (0, 0, 1)$ and $g_{euc}(N, \overline{N})(0) > 0$.

We have by definition:

$$\overline{\lambda}(v) = \frac{g(\overline{N}, \overline{\nabla}_{F_1} F_1)}{g(F_1, F_1)}(0) \quad \text{and} \quad \lambda(v) = \frac{u_{11}}{1 + u_1^2}(0),$$

where $\overline{\nabla}$ stands for the Levi-Civita connection with respect to the metric g.

Using (29) we get $F_1(0) = \partial_{x_1}$. Furthermore a straightforward computation shows that:

$$\overline{N}(0) = \frac{1}{\sqrt{g^{33}}}\big(g^{13}\partial_{x_1} + g^{23}\partial_{x_2} + g^{33}\partial_{x_3}\big)(0) \quad \text{and} \quad \big(\overline{\nabla}_{F_1} F_1\big)(0) = \big(\overline{\nabla}_{\partial_{x_1}} \partial_{x_1}\big)(0) + u_{11}(0)\partial_{x_3},$$

Therefore we deduce that

$$\overline{\lambda}(v) = \frac{1}{g_{11}\sqrt{g^{33}}}\Big(g(\overline{\nabla}_{\partial_{x_1}} \partial_{x_1}, g^{13}\partial_{x_1} + g^{23}\partial_{x_2} + g^{33}\partial_{x_3})(0) + u_{11}(0)\Big), \tag{30}$$

$$\lambda(v) = u_{11}(0). \tag{31}$$

Observe that if the metrics g and g_{euc} are close enough on U in the C^1-topology then the expression $g(\overline{\nabla}_{\partial_{x_1}} \partial_{x_1}, g^{13}\partial_{x_1} + g^{23}\partial_{x_2} + g^{33}\partial_{x_3})(0)$ is as close to 0 as we want and the expression $\frac{1}{g_{11}\sqrt{g^{33}}}$ is as close to 1 as we want. Observe that these expressions are independant of u, that is, independant of the surface S. By hypothesis, we have $|\overline{\lambda}(v)| < C$. Therefore there exists a constant $D_n > 0$ which does not depend on

S such that $\|g_{ij} - g_{euc,ij}\|_{C^1(U)} < D_n$, $1 \le i,j \le 3$, implies that $|u_{11}(0)| < 2C$. Consequently we have:

$$|\overline{\lambda}(v) - \lambda(v)| < 2C\Big|1 - \frac{1}{g_{11}\sqrt{g^{33}}}\Big| + \frac{1}{g_{11}\sqrt{g^{33}}}\Big|g(\overline{\nabla}_{\partial_{x_1}}\partial_{x_1}, g^{13}\partial_{x_1} + g^{23}\partial_{x_2} + g^{33}\partial_{x_3})(0)\Big|.$$

For the same reasons as before, there exists a constant $C_n > 0$, with $C_n < D_n$, which does not depend on S such that $\|g_{ij} - g_{euc,ij}\|_{C^1(U)} < C_n$, $1 \le i,j \le 3$, implies that

$$|\overline{\lambda}(v) - \lambda(v)| < \frac{1}{n}.$$

Let us denote by $\overline{\lambda}_{\max}, \overline{\lambda}_{\min}, \lambda_{\max}$ and $\lambda_{\min}$ the principal curvatures of S at P for the metrics g and g_{euc} respectively. It follows that:

$$|\overline{\lambda}_{\max} - \lambda_{\max}| < \frac{1}{n} \qquad \text{and} \qquad |\overline{\lambda}_{\min} - \lambda_{\min}| < \frac{1}{n}.$$

Therefore:

$$|H - \overline{H}| < \frac{1}{n}.$$

Let $k > 0$ be some integer and let C_k be the constant given in the first part of the proposition. Thus, if $\|g_{ij} - g_{euc,ij}\|_{C^1(U)} < C_k$, $1 \le i,j \le 3$, then we have:

$$|\overline{\lambda}_{\max} - \lambda_{\max}| < \frac{1}{k} \quad \text{and} \quad |\overline{\lambda}_{\min} - \lambda_{\min}| < \frac{1}{k}$$

on S. Therefore we get

$$\begin{aligned}\Big||A|^2 - |\overline{A}|^2\Big| &= \Big|\big(\lambda_{\max}^2 + \lambda_{\min}^2\big) - \big(\overline{\lambda}_{\max}^2 + \overline{\lambda}_{\min}^2\big)\Big| \\ &\le \frac{1}{k}\big(2|\overline{\lambda}_{\max}| + \frac{1}{k} + 2|\overline{\lambda}_{\min}| + \frac{1}{k}\big) \\ &< \frac{1}{k}\big(4C + \frac{2}{k}\big).\end{aligned}$$

To conclude the proof just observe that $\Big||A| - |\overline{A}|\Big|^2 \le \Big||A|^2 - |\overline{A}|^2\Big|$. □

Remark 4.2. *Let (M, g) be a smooth Riemannian 3-manifold and let $S \looparrowright M$ be an immersed surface. Let $p \in S$ be an interior point of S. Let $(U, \varphi, B(p, r_0))$ be a harmonic coordinate chart at p, where $B(p, r_0)$ is the geodesic ball in M centered at p with radius r_0 such that $B(p, r_0) \cap \partial S = \emptyset$. Let us denote the harmonic coordinates on U by (x_1, x_2, x_3) and let us consider the Euclidean metric $dx_1^2 + dx_2^2 + dx_3^2$ on U. Viewing the connected component of $B(p, r_0) \cap S$ through p in U observe that a neighborhood of p in S is a Euclidean graph over a Euclidean disk in the tangent plane T_pS centered at p.*

The arguments of the proof of the Proposition 4.1 can be used to show the following result which is very useful for the applications.

Proposition 4.3. *Let (M, g) be a smooth Riemannian 3-manifold (not necessarily complete), with bounded sectional curvature $|K| \leq \Lambda < +\infty$ and let $r > 0$. Let $\Omega \subset M$ be an open subset of M such that the injectivity radius in M at any $x \in \Omega$ is $\geq r$.*

Then for any $C_1 > 0$ and $C_2 > 0$ there exist constants $\delta, \delta_0 > 0$ depending only on C_1, C_2, Λ and r and neither on M nor on Ω satisfying the following:

For any immersed surface $S \looparrowright \Omega$ whose second fundamental form $\overline{A}$ satisfies $|\overline{A}| < C_1$ and for any $p \in S$ such that $d_S(p, \partial S) > C_2$ then a part S_0 of S is contained in the image of a harmonic chart and is a Euclidean graph (see Remark 4.2) over the disk of T_pS centered at p with Euclidean radius δ. Furthermore, the subset S_0 contains the geodesic disk of S centered at p with radius δ_0.

Proof. Let us fix some constants $\alpha \in (0, 1)$ and $Q_0 > 1$. The Theorem 6 in Hebey-Herzlich [9] shows that there exists some $r_0 = r_0(Q_0, \alpha, r, \Lambda) > 0$ such that for any $x \in \Omega$ there exists a harmonic coordinate chart $(U, \varphi, B(x, r_0))$ satisfying the assumption of the Theorem 2.1 where $B(x, r_0)$ is the geodesic ball in M centered at x with radius r_0.

Let C_1, C_2, S and $p \in S$ be as stated. Consider a harmonic coordinate chart $(U, \varphi, B(p, r_0))$ at p, and call $x = (x_1, x_2, x_3)$ the coordinates on $U \subset \mathbb{R}^3$, we have

$$\sum_{k=1}^{3} \sup_{x \in U} |\partial_k g_{ij}(x)| \leq Q_0 \quad \text{and} \quad Q_0^{-1}\delta_{ij} \leq g_{ij} \leq Q_0\delta_{ij} \quad \text{as quadratic forms.}$$

We want to show that there exists a constant $C_3 = C_3(C_1)$ which does not depend on $p \in S$ or on S such that $|A(p)| < C_3$ where A denotes the second fundamental form of $S \cap B(p, r_0)$ for the Euclidean metric $dx_1^2 + dx_2^2 + dx_3^2$. Let $v \in T_pS$ be a nonzero tangent vector. Choose Euclidean coordinates $y = (y_1, y_2, y_3)$ in U such that the tangent plane T_pS coincides with the plane $\{y_3 = 0\}$ and v is tangent to the y_1-axis. In those new coordinates we have

$$\sum_{k=1}^{3} \sup_{y \in U} |\partial_k g_{ij}(y)| \leq 9Q_0 \quad \text{and} \quad Q_0^{-1}\delta_{ij} \leq g_{ij} \leq Q_0\delta_{ij} \quad \text{as quadratic forms.} \tag{32}$$

where ∂_k stands for the partial derivative with respect to y_k.

Let us denote by $\lambda_p(v)$, resp. $\overline{\lambda}_p(v)$, the normal curvature of S at p in the tangent direction v with respect to the Euclidean metric g_{euc}, resp. g. We get from the formulae (30) and (31) established in the proof of the Proposition 4.1 that

$$\lambda_p(v) = (g_{11}\sqrt{g^{33}})(0)\overline{\lambda}(v) - g(\overline{\nabla}_{\partial_{y_1}}\partial_{y_1}, g^{13}\partial_{y_1} + g^{23}\partial_{y_2} + g^{33}\partial_{y_3})(0).$$

Taking into account the formulae (32) we get that the expressions $(g_{11}\sqrt{g^{33}})(0)$ and $|g(\overline{\nabla}_{\partial_{y_1}}\partial_{y_1}, g^{13}\partial_{y_1} + g^{23}\partial_{y_2} + g^{33}\partial_{y_3})(0)|$ are bounded by a constant Q_1 which only depends on Q_0. Therefore the expression $|\lambda_p(v)|$ is bounded by a constant C_4 which depends only on C_1 and Q_0 and which does not depend on $p \in S$ or on S. Consequently we have $|A| < 2C_4$ on S. Now the proof follows from the Proposition 2.3

setting $\delta = 1/8C_4$. The second part of the estimates (32) shows that we can choose $\delta_0 = \delta/Q_0$.

$\square$

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