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A Software package for the Calibration of Robots

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ABSTRACT

This paper presents a software package for the simulation and the realization of the calibration of the geometric parameters of robots. This package which is called GECARO, GEometric CAlibration of RObots, contains new methods which can carry out the calibration of the geometric parameters and the gain transmission ratios of serial robots without the need of external sensors. It contains also classical methods which are based on measuring the position and the orientation of the terminal link using external sensors.

I. INTRODUCTION

The absolute accuracy of a robot depends to a large extent on the calibration of its geometric parameters. Geometric calibration of robots is the process by which the parameters defining the base parameters, link parameters and tool parameters are precisely identified. Classically, the geometric calibration is carried out by solving a system of linear or non-linear equations which is a function of the end-effector pose measurements and of the joint positions [1,...,4]. Recently, some papers have proposed autonomous calibration methods which do not need an external sensor ;

-in [5,6,7,8] the observation matrix and the error vector have been constructed using the fact that different configurations of the robot can give the same pose of the terminal effector .

- in [9] the observation matrix is based on calculating the vector product of a set of vectors in the same plane, which must be equal to zero. Second order terms appearing in the vector products are neglected, thus the obtained observation matrix contains more approximation than the classical one, and the problem of the identifiable parameters is not considered in this paper.

In this paper we present a software package which can simulate and realize the calibration of serial robots. The following methods are available:

- Position measure: which is based on using an external sensor giving the position of the terminal point. The corresponding motors positions are also needed.

- Frame measure : which is based on using an external sensor giving the position and the orientation of the terminal link. The corresponding motors positions are also needed.

- Position link : which is an autonomous method needing no external sensor [8]. It can be applied for robots which can achieve the same position of the terminal point by multiple configurations. The data used in the calibration correspond to the motor variables of some sets of robot configurations, where the configurations of each set correspond to the same Cartesian position of the terminal point of the tool. No external sensor is needed.

- Frame link : which is an autonomous method needing no external sensor [8]. This method can be used for robots which can achieve the same pose (position and orientation) of the terminal link by multiple configurations, thus having at least 6 d.o.f. The data used in the calibration correspond to the motor variables of some sets of robot configurations, where the configurations of each set correspond to the same Cartesian pose.

- Plane link: This is an autonomous method. It uses in the calibration a set of points in a plane. Two methods are available [10] : in the first, the plane coefficients are supposed known, while in the second, the plane coefficients are unknown, they will be identified as well as the geometric parameters.

In all the methods, the geometric parameter are identified using a linearized model which is solved iteratively using least squares criterion and by updating the identified parameters and the observation matrix after each iteration.

The software package GECARO is running on PC computers and developed under MATLAB, using numerical methods. No symbolic model is needed.

The paper is organized as follows : The parameters defining the robot will be presented in section 2. Sections 3 , 4 and 5 describe the identification methods. Section 6 presents the organization of GECARO. Section 7 contains the conclusion.

2.DESCRIPTION OF THE ROBOT

Two types of parameters are used:

The geometric parameters defining the different frames and the geometric of the robot and general parameters

concerning the joints limits, offset, gain rations, and the coupling between the motor and joints variables.

2.1 Description of the geometric parameters

We consider serial robots consisting of n joints and n+1 links. Link 0 is the base and link n is the terminal link, frame j is defined fixed on link j. We denote :
frame -1 : the fixed reference frame,
frame n+1: the tool frame.

The end-effector pose can be calculated with respect to the reference frame by the direct geometric model :

$$ {}^{-1}\mathbf{T}_{n+1} = {}^{-1}\mathbf{T}_0 \, {}^{0}\mathbf{T}_n(\mathbf{q}) \, {}^{n}\mathbf{T}_{n+1} \qquad (1) $$

where ${}^{i}\mathbf{T}_j$ is the 4x4 transformation matrix defining frame j with respect to frame i.
The definition of the link frames will be carried out by Khalil and Kleinfinger notation [11]. Frame j is defined such that $\mathbf{z}_j$ is along the axis of joint j, and $\mathbf{x}_j$ is perpendicular to $\mathbf{z}_j$ and $\mathbf{z}_{j+1}$. Frame j is defined with respect to frame j-1 by the matrix ${}^{j-1}\mathbf{T}_j$, function of the four parameters (α_j, d_j, θ_j, r_j) .

It have been seen that, the calculation of ${}^{-1}\mathbf{T}_{n+1}$ can be obtained as a function of the geometric parameters of n+2 frames represented by four parameters for each frame (α_j, d_j, θ_j, r_j) , except the first frame which is represented by only two parameters different than zero [8,12]. The joint variable q_j is equal to θ_j if joint j is rotational and r_j if joint j is prismatic.

In the calibration process, we have to identify the deviation of the real parameters from the nominal values, thus to identify : $\Delta\alpha_j$, Δd_j, $\Delta\theta_j$, Δr_j (for j = 0, n+1).
If the axis of joint j is parallel to the axis of joint j-1, an additional parameter $\Delta\beta_j$ must be considered, the nominal value of β_j is equal to zero [13].
Using the same procedure, one can identify also the error in the transmission gain of the joints ΔK_j (for j = 1, n_j) [4].
In GECARO the geometric parameters of the base frame, the robot links, and the tool frame will be represented by a unique table where each row, for j=1,…,nf, represents a frame as given in table 1. Beside the parameters β_j, α_j, d_j, θ_j, r_j, we see the following parameters :

. σ_j : which defines the type of the frame,

σ_j = 0 for j rotational, σ_j = 1 for j prismatic, and σ_j = 2 if frame j is fixed.

. K_j : the transmission gain ratio for joint j (not frame j), for j=1,...,nj, with nj is the number of joints.

The number of frames is denoted nf. The difference (nf-nj) gives the number of rows representing fixed frames.

2.2 Definition of general parameters

The following parameters are also needed in the calibration of the geometric parameters :

a- **Offset**: a vector of nf components, containing the nominal values of the Offset of joint variables, for a fixed frame the Offset is equal to zero.

b- **Qmax** and **Qmin:** the vectors of joint limits, for a fixed frame the corresponding elements are equal to zero.

c- **Com**: the (njxnj) coupling matrix of motor variables,

d- **Coa**: the (nfxnf) coupling matrix of joint variables,

e- **K**: the vector of gain transmission,

The joint variables are calculated as a function of motor variables using the following relation :

$$ \mathbf{q} = \mathbf{Coa}*\mathbf{G(K)}*\mathbf{Com}*\mathbf{qm} + \mathbf{Offset} \qquad (2) $$

where :

q is the (nfx1) joint variables vector, for a fixed frame the joint variable is equal to zero,

qm is the (njx1) vector of motor variables ,

G(K) is a (nfxnj) matrix, equal to : $\begin{bmatrix} \mathbf{diag\ (K)} \\ \mathbf{0}_{(nf-nj)xnj} \end{bmatrix}$

where $\mathbf{0}_{ixj}$ is the zero matrix of dimension ixj .

f- The (nfx6) priority matrix. This matrix defines the order of priority of the geometric parameters. Columns 1,...,6 correspond to the parameters α, d,θ, r, β, K respectively. The elements of row j give the priority of the parameters of frame j, for instance the element (3,4) defines the priority of the parameter r_3. During the calculation of the identifiable parameters or the execution of the identification algorithm, the parameters will be arranged as function of the corresponding priority number, the first parameters will be those having the highest priority number. If the elements of the priority matrix are all equal, for instance equal to 1, then the parameters will be arranged in the following order :

$\alpha(1)$, ...,$\alpha(nf)$, d(1),..., d(nf), $\theta(1)$,…, $\theta(nf)$, r(1),…, r(nf), $\beta(1)$,…, $\beta(nf)$, K(1),…K(nj). The importance of this matrix comes from the fact that we construct the base of identifiable parameters as those corresponding to the first independent columns.

j	σ_j	α_j	d_j	θ_j	r_j	β_j	Kj
1	σ_1	α_1	d_1	θ_1	r_1	β_1	K1
…	…	…	…	…	…	…	…
nj	σ_{nj}	α_{nj}	d_{nj}	θ_{nj}	r_{nj}	β_{nj}	Knj
nf	σ_{nf}	α_{nf}	d_{nf}	θ_{nf}	r_{nf}	β_{nf}	0

Table 1. Definition of the robot parameters

3. CLASSICAL CALIBRATION METHODS

The end-effector pose can be calculated with respect to the reference frame by the direct geometric model (DGM) as given in equation (1).

Using a first order Taylor development of the parameters errors, a linear differential model defining the deviation of the end effector pose due to the differential error in the geometric parameters can be obtained as [8,12] :

$$\Delta \mathbf{y} = \mathbf{J}(\mathbf{q})\,\Delta \mathbf{X} \qquad (3)$$

with

$$\Delta \mathbf{X} = \mathbf{X}_r - \mathbf{X}$$

$\Delta \mathbf{y}$ represents the (6x1) vector of the position and orientation error.
J is a (6xNp) matrix,
Np = number of the parameters to be calibrated.
$\Delta \mathbf{X}$ defines the (Npx1) vector of the errors of the geometric parameters.
$\mathbf{X}_r$ is the vector of the real unknown values of the geometric parameters,
X is the vector of the nominal values of the geometric parameters,

The expressions of the columns of **J** are calculated as given in appendix A [8,12].

To identify $\Delta \mathbf{X}$, equation (3) will be applied for a sufficient number of configurations $\mathbf{q}^1,\ldots,\mathbf{q}^m$, the corresponding poses will be measured and the $\Delta \mathbf{y}^i$ will be calculated. The resulting linear system of equations will be represented by :

$$\Delta \mathbf{Y} = \mathbf{W}\ \Delta \mathbf{X} \qquad (4)$$

Equation (4) will be solved iteratively to get the least squares errors solution. After each iteration the geometric parameters will be updated.

If the orientation of the terminal link is not measurable, only the equations corresponding to the position error will be taken into account (the first three equations of Eq.(3)).

It can be seen that if some columns of **J** are dependent in all configurations. Relation (3) can be reduced to [14]:

$$\Delta \mathbf{y} = \mathbf{J_b}(\mathbf{q})\,\Delta \mathbf{X_b} \qquad (5)$$

where $\mathbf{J_b}(\mathbf{q})$ contains the b independent columns of **J**, the corresponding parameters are known as identifiable parameters or base parameters they will be denoted by the vector $\Delta \mathbf{X_b}$.

The determination of the identifiable parameters $\Delta \mathbf{X_b}$ must be done before the identification process. They can be obtained numerically using the QR decomposition of a matrix **W** similar to that defined in equation (4) but obtained using random configurations [14].

4. POSITION LINK AND FRAME LINK CALIBRATION METHODS

The main problem in the classical calibration methods, is the need to have an accurate, fast and not expensive equipment to measure the real end-effector pose. Many sensors have been proposed in the literature but neither fulfill these three conditions.
The position (frame) link method can be applied for robots which can achieve the same position (frame) by more than one configuration [8]. Thus, if $\mathbf{q}^a$ and $\mathbf{q}^b$ represent two different configurations giving the same pose of the tool then:

$$^{-1}\mathbf{T}_{n+1}(\mathbf{q}^a, \mathbf{X}_r) = {}^{-1}\mathbf{T}_{n+1}(\mathbf{q}^b, \mathbf{X}_r) \qquad (6)$$

If the values of the geometric parameters in the model **X** are different from the real values $\mathbf{X}_r$, then using a first order development and by considering only the position equations, to facilitate the presentation, we obtain :

$$[\mathbf{J}_t(\mathbf{q}^b)-\mathbf{J}_t(\mathbf{q}^a)]\Delta \mathbf{X} = [{}^{-1}\mathbf{P}_{n+1}(\mathbf{q}^a,\mathbf{X})-{}^{-1}\mathbf{P}_{n+1}(\mathbf{q}^b,\mathbf{X})] \qquad (7)$$

where $\mathbf{J_t}$ is the position Jacobian, represented by the first three rows of **J.**

Eq.(7) relates the deviation in the geometric parameters to the position error between $\mathbf{q}^a$ and $\mathbf{q}^b$.
To estimate the vector $\Delta \mathbf{X}$ we have to repeat this process for sufficient number of couple of configurations.
If multiple configurations can be obtained for a given position and orientation of the tool, the system (7) will be replaced by :

$[\mathbf{J}(\mathbf{q}^b)-\mathbf{J}(\mathbf{q}^a)]\Delta\mathbf{X} =$

$$\begin{bmatrix} [{}^{-1}\mathbf{P}_{n+1}(\mathbf{q}^a,\mathbf{X}) - {}^{-1}\mathbf{P}_{n+1}(\mathbf{q}^b,\mathbf{X})] \\ \delta_{a,b} \end{bmatrix} \quad (8)$$

where $\delta_{a,b}$ is the differential rotation between the orientation of the tool in the configurations $\mathbf{q}^a$ and $\mathbf{q}^b$.

The calculation of the identifiable parameters can be carried out as given in Appendix B, by studying the QR decomposition of a matrix **W** calculated from (7) or (8) using sufficient random couple of configurations $\mathbf{q}^a(j)$ and $\mathbf{q}^b(j)$.

5. THE PLANAR CALIBRATION [10]

The methods, presented in this section, will be carried out using a set of configurations of the robot, where the terminal point of the robot is in the same plane.

The general equation of a plane is supposed as :

$$a\,x + b\,y + c\,z + 1 = 0 \quad (9)$$

where a, b, c represent the plane coefficients,

As the terminal point of the robot is in the plane, then :

$$a\,Px(\mathbf{q}) + b\,Py(\mathbf{q}) + c\,Pz(\mathbf{q}) + 1 = 0 \quad (10)$$

where Px, Py, Pz represent the coordinates of the position of the terminal point in the world frame.

Two methods are developed.

5.1 The first method

Assuming the coefficients of the plane are known. Using a first order development for **X** in equation (10), we obtain :

$$[a\,\mathbf{J}x(\mathbf{q}) + b\,\mathbf{J}y(\mathbf{q}) + c\,\mathbf{J}z(\mathbf{q})]\,\Delta\mathbf{X} =$$

$$-\,a\,Px(\mathbf{q}) - b\,Py(\mathbf{q}) - c\,Pz(\mathbf{q}) - 1 \quad (11)$$

where :

Jx is the first row of the Jacobian matrix defined in (3),
Jy is the second row of the Jacobian matrix ,
Jz is the third row of the Jacobian matrix,

Equation (11) gives a linear equation in $\Delta\mathbf{X}$, for each configuration**.**

Sufficient number of configurations must be used to obtain a system of equations similar to (4) which will be solved iteratively to identify $\Delta\mathbf{X}$, the geometric parameters will be updated after each iteration.

The calculation of the identifiable parameters can be carried out numerically, by studying the QR decomposition of a matrix **W** calculated from Eq.(11) using sufficient random points in a given plane.

5.2 The second method

Assuming that the coefficients of the plane are unknown, from Eq. (10) the following relation can be obtained :

$$[Px(\mathbf{q})\quad Py(\mathbf{q})\quad Pz(\mathbf{q})\quad a\mathbf{J}x(\mathbf{q})+b\mathbf{J}y(\mathbf{q})+c\mathbf{J}z(\mathbf{q})]\begin{bmatrix} \Delta a \\ \Delta b \\ \Delta c \\ \Delta\mathbf{X} \end{bmatrix}$$

$$= -aPx(\mathbf{q})-bPy(\mathbf{q})-cPz(\mathbf{q})-1 \quad (12)$$

using sufficient number of configurations we can get a linear system as in the standard form (4).

The coefficients of the plane are initialized by calculating the equation of the nearest plane to the terminal points of the given configurations. The Cartesian coordinates of the terminal point are calculated using the direct geometric model of the robot. The solution of the following equation gives the initial values of a, b, c :

$$\begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} Px^1 & Py^1 & Pz^1 \\ \dots & \dots & \dots \\ Px^m & Py^m & Pz^m \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} \quad (13)$$

with Px^j , Py^j , Pz^j are the coordinates of the terminal point calculated by the DGM for the configuration $\mathbf{q}(j)$.

The calculation of the identifiable parameters can be carried out numerically, by studying the QR decomposition of a matrix **W** calculated from (12) using sufficient random points in a given plane.

6. DESCRIPTION OF GECARO

The following menus are available :

Robot, Position measure, Frame measure, Point link, Frame link, Plane link, Help

Robot menu is used to define or select the robot to be treated. The following functions are available as sub-menus :

New, Open, Save, Save as, Quit, Help

The other functions correspond to a given calibration method. Each of them has the following three principal sub-menus

a- Identifiable parameters : Calculation of the identifiable geometric parameters (base parameters).

The identifiable parameters can be obtained from the classical parameters by eliminating those which have no effect on the observation matrix or those whose corresponding elements in the priority matrix are equal to zero, and by regrouping some others (whose corresponding columns in the observation matrix are not independent). These parameters are calculated using QR

decomposition of an observation matrix which can be calculated using either random values, or by using a real experimental data file. This function can also be used to test if some experimental data are sufficiently exciting (giving a good condition number for the observation matrix, and giving the same identifiable parameters as the random values).

It is to be noted that the identifiable parameters are not unique. It is more practical to identify, if possible, the parameters which can be updated easily in the control system without changing the symbolic direct and inverse geometric models of the robot. Therefore we use the following rules to define default priorities as input to this function (the user can define arbitrarily his input priority matrix) :

-the highest is equal to 6 correspond to joint offsets and motor transmission gains.

- the distances R, and D which are not equal to zero take priority 5,

- the angles which are not equal to ±k*pi/2, with k integer, will take the priority 4,

- the parameters of the base location or the tool frames, will get the priority 2,

- the rest of parameters will get the priority 1,

The user can put equal to zero the elements of the priority matrix corresponding to the parameters which he does not like to identify, either because their values are known accurately, or because he cannot make use of their identified values.

The output of this function is the regrouping relations of the identifiable parameters and a new priority matrix which is similar to the input one, but the non identifiable parameters priorities are put equal to zero.

b- Points generation : this function generates a file which can be used to test and simulate the functions of the corresponding method, the generated file contains two matrices :

Qm : of dimension (nptxnj). Each row contains the motor variables of a random robot configuration.

X : contains the Cartesian pose of dimension nptX3 if the position only is needed or of dimension 3nptx4 if the position and orientation are needed.

The number of points denoted (npt) is given by the user, this number must be chosen such that sufficient number of equations can be generated.

c- Identification : This function gives the identified values and the precision of the obtained solution. Two input files are needed :

- the file containing the priority matrix,

- the file containing **Qm** and **X** in classical methods or motor variables matrix **Qm** for autonomous methods, sufficient number of configurations must be given. In the case of position or frame link methods the number of configurations of each set giving the same pose will be given in a vector (whose dimension is equal to the number of sets).

7. CONCLUSION

This paper presents the software package GECARO, GEometric CAlibration of RObots. It contains new methods which can carry out the calibration of the geometric parameters and the gain transmission ratios of serial robots without the need of external sensors. It contains also classical methods which are based on measuring the position and the orientation of the terminal link using external sensors. The parameters defining the robot and the main menus of this package are described in the paper. GECARO is running on PC computers and developed using MATLAB, any general open loop robot can be treated directly.

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APPENDIX A: CALCULATION OF J

Assuming the matrix defining frame j with respect to the fixed frame as:

$$^{-1}\mathbf{T}_j = \begin{bmatrix} \mathbf{s}_j & \mathbf{n}_j & \mathbf{a}_j & \mathbf{P}_j \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \text{(A-1)}$$

The calculation of the columns of the matrix **J** can be done as follows [8,11]:

$$\mathbf{j}\alpha_j = \begin{bmatrix} \mathbf{s}_{j-1} x \mathbf{L}_{j-1,n+1} \\ \mathbf{s}_{j-1} \end{bmatrix}, \quad \mathbf{j}d_j = \begin{bmatrix} \mathbf{s}_{j-1} \\ \mathbf{0}_{(3x1)} \end{bmatrix},$$

$$\mathbf{j}\theta_j = \begin{bmatrix} \mathbf{a}_j x \mathbf{L}_{j,n+1} \\ \mathbf{a}_j \end{bmatrix}, \quad \mathbf{j}r_j = \begin{bmatrix} \mathbf{a}_j \\ \mathbf{0}_{(3x1)} \end{bmatrix},$$

$$\mathbf{j}\beta_j = \begin{bmatrix} \mathbf{n}_{j-1} x \mathbf{L}_{j-1,n+1} \\ \mathbf{n}_{j-1} \end{bmatrix},$$

$$\mathbf{j}k_j = \mathbf{j}q_j \frac{\partial \mathbf{q}}{\partial K_j}$$

Where :

x denotes the vector product,

$\mathbf{L}_{i,n+1}$ is the (3x1) position vector between the origin of frame i and the origin of frame n+1 equal to $P_{n+1} - P_i$,

$\mathbf{0}_{(3x1)}$ is the (3x1) zero vector.

$\mathbf{j}q_j$ is equal to $\mathbf{j}\theta_j$ if j rotational or $\mathbf{j}r_j$ if j prismatic.

All the vectors of equations (A-2…A-5) are referred to the measuring fixed frame.

APPENDIX B: NUMERICAL CALCULATION OF THE IDENTIFIABLE PARAMETERS

Let us rewrite the system (4) as:

$$\Delta\mathbf{Y} = [\ \mathbf{W_1} \quad \mathbf{W_2}\ ] \begin{bmatrix} \Delta\mathbf{X_1} \\ \Delta\mathbf{X_2} \end{bmatrix} \qquad \text{(B-1)}$$

where : $\mathbf{W_1}$ represents b independent columns of **W**,
$\mathbf{W_2}$ represents (c-b) dependent columns of **W**.

where c is the number of columns, and b is the number of independent columns of **W** .

We can write:

$$\mathbf{W_2} = \mathbf{W_1}\,\mathbf{B} \qquad \text{(B-2)}$$

with **B** is (bx(c-b)) matrix with constant elements.

Using (B-2) , equation (4) can be written as:

$$\Delta\mathbf{Y} = \mathbf{W_1}\ \Delta\mathbf{X_b} \qquad \text{(B-3)}$$

with $\Delta\mathbf{X_b} = \Delta\mathbf{X_1} + \mathbf{B}\,\Delta\mathbf{X_2}$ (B-4)

In the identification process, equation (B-3) will be used instead of (4). The solution will give directly $\Delta\mathbf{X_b}$ which is called the base parameters vector. The matrix **B** is not needed in the identification process.

Numerically, the study of the base parameters is equivalent to study the space spanned by the columns of (rxc) matrix **W** with r>> c, calculated from **J** at m random configurations $\mathbf{q}^1,\ldots,\mathbf{q}^m$ satisfying the constraint of the problem (terminal point in a plane, ...)

Using QR decomposition the (rxc) matrix **W** can be decomposed as [15,16]:

$$\mathbf{Q}^T\,\mathbf{W} = \begin{bmatrix} \mathbf{R} \\ \mathbf{0}_{(r-c)xc} \end{bmatrix} \qquad \text{(B-5)}$$

Q is a (rxr) orthogonal matrix,
R is a (cxc) upper triangular matrix.
$\mathbf{0}_{ixj}$ is the (ixj) matrix of zeros.

If the element $|R_{ii}| \le \tau$, the corresponding parameter in $\Delta\mathbf{X}_i$ is not identifiable. τ is the numerical zero which can be taken as [15]:

$$\tau = c.\varepsilon.\ \max.\ |R_{ii}| \qquad \text{(B-6)}$$

Where ε is the machine precision.

<u>Base Parameters as Function of the Standard Parameters</u>

Let us permute the columns of **W** such that the first b columns are independent.

$$[\mathbf{W}\ \mathbf{P}] = [\ \mathbf{W_1} \quad \mathbf{W_2}\ ] \qquad \text{(B-7)}$$

where:

P is a permutation matrix,
$\mathbf{W_1}$ represent the b independent columns of **W.**
$\mathbf{W_2}$ represent the (c-b) dependent columns of **W.**

A QR decomposition of [**W P**] gives :

$$\left[\, \mathbf{W_1} \;\; \mathbf{W_2} \,\right] = \left[\, \mathbf{Q_1} \;\; \mathbf{Q_2} \,\right] \begin{bmatrix} \mathbf{R_1} & \mathbf{R_2} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$$

Where $\mathbf{R_1}$ is a bxb regular matrix. Then it comes:

$$\mathbf{W_2} = \mathbf{W_1}\, \mathbf{R_1}^{-1}\, \mathbf{R_2} \qquad \text{(B-8)}$$

thus : $\mathbf{B} = \mathbf{R_1}^{-1}\, \mathbf{R_2}$ (B-9)