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SUB HILBERT SPACES IN THE UNIT BALL OF \mathbb{C}^n

FREDERIC SYMESAK

ABSTRACT. Let $\Phi(z) = (\varphi_1(z), \dots, \varphi_l(z)) : \mathbb{B}^l \rightarrow \mathbb{B}^n$ be holomorphic. We denote by $B_\alpha(z, w)$ the weighted Bergman kernel. We give a condition so that the kernel $(1 - \Phi(z)\overline{\Phi(w)}) B_\alpha(z, w)$ is a reproducing kernel and we study the related Hilbert space.

1. INTRODUCTION

The aim of this note is to give a characterization of some reproducing kernels of spaces of holomorphic functions on the unit ball of \mathbb{C}^n . These kernels are obtained with the weighted Bergman kernel $B_\alpha(z, w)$ and bounded holomorphic functions $\varphi_1(z), \dots, \varphi_l(z)$.

We denote by \mathbb{B} the unit ball of \mathbb{C}^n and by $r(z) = \sum_{i=1}^n |z_i|^2 - 1$ a defining function. Let $\alpha > -1$ and let $dV_\alpha(z) = (-r(z))^\alpha dV(z)$. We denote by \mathbb{A}_α^2 the weighted Bergman space of \mathbb{B} with respect the measure $dV_\alpha(z)$ [3]. It is a Hilbert space and its reproducing kernel is given by $B_\alpha(z, w) = \frac{c_\alpha}{(1 - z\bar{w})^{n+1+\alpha}}$. When $\alpha = 0$, we denote by $B(z, w)$ the usual Bergman kernel and by \mathbb{A}^2 the Bergman space. we have

Theorem 1.1. *Let $\alpha > -1$ and $\Phi(z) = (\varphi_1(z), \dots, \varphi_l(z))$ be holomorphic. The kernel $K_\alpha(z, w) = (1 - \Phi(z)\overline{\Phi(w)}) B_\alpha(z, w)$ is a reproducing kernel if and only if $\sup_{\mathbb{B}} |\Phi(w)| \leq 1$.*

The reproducing properties of kernels of type $(1 - \varphi(z)\overline{\varphi(w)})B(z, w)$ was studied by F. Beatrous and J. Burbea [1], for spaces of holomorphic functions on domains of \mathbb{C}^n and by S. Saitoh for abstract kernels of Hilbert spaces of functions [5]. In the case of the unit disc of \mathbb{C} , when φ is finite Blaschke product, the characterization of the related Hilbert space was obtained by K. Zhu [8] and [9] for Bergman space and by S. Sultanic for weighted Bergman spaces [7]. The same questions was introduced by L. De Branges and J. Rovnyak in the context of Hardy spaces [6]. The following theorem gives the Hilbert space associated to the kernel $K_\alpha(z, w)$. We have

Theorem 1.2. *Let $\alpha \geq 1$ and let $\Phi(z) = (\varphi_1(z), \dots, \varphi_l(z))$ be holomorphic. The Kernel $K_\alpha(z, w)$ is a reproducing kernel of $\mathbb{A}_{\alpha-1}^2$ if and only if there exists $C > 0$ such that $1 - |\Phi(z)|^2 \leq C(-r(z))$, z in \mathbb{B} .*

For $-1 \leq \alpha < 0$, we denote by \mathcal{B}_α^2 the diagonal Besov space. A holomorphic function f belongs to \mathcal{B}_α^2 if and only if $(I + N)f$ is in $\mathbb{A}_{\alpha+1}^2$, where $N = \sum_{k=1}^n z_k \partial_{z_k}$ is the normal complex field [3]. We have

Theorem 1.3. *Let $\Phi(z) = (\varphi_1(z), \dots, \varphi_l(z))$ be holomorphic. If there exists $C > 0$ such that $\sup_{\mathbb{B}} |N\varphi_i(w)| \leq C$ $i, 1 \leq i \leq l$, then*

- (1) *If $0 < \alpha < 1$, $K_\alpha(z, w)$ is a reproducing kernel of $\mathbb{A}_{\alpha-1}^2$.*
- (2) *$K_0(z, w)$ is a reproducing kernel of the Hardy space \mathbb{H}^2 .*
- (3) *If $-1 < \alpha < 0$, $K_\alpha(z, w)$ is a reproducing kernel of \mathbb{B}_α^2 .*

2. SUB-BERGMAN SPACES

Recall that the Bergman projection B_α is the orthogonal projection from $L^2(dV_\alpha)$ onto \mathbb{A}_α^2 . Given a bounded function φ the Toëplitz operator of symbol φ is defined by $T_\varphi f = B_\alpha(\varphi f)$. We consider the self-adjoint operator H given by

Definition 2.1. *Let $\Phi(z) = (\varphi_1(z), \dots, \varphi_l(z))$ be holomorphic. We set*

$$Hf = \sum_{i=1}^l \varphi_i T_{\overline{\varphi}_i} f \quad f \in \mathbb{A}_\alpha^2.$$

We first estimate the norm of H . We have

Proposition 2.2. *Let H as above. Then $\|H\| = \sup_{\mathbb{B}} |\Phi(w)|^2$.*

Proof : Let f in \mathbb{A}_α^2 . Then

$$\|Hf\|_{\mathbb{A}_\alpha^2} \leq \sum_{i=1}^l \|\varphi_i B_\alpha(\overline{\varphi}_i f)\|_{\mathbb{A}_\alpha^2} \leq \sup_{\mathbb{B}} |\Phi(w)| \sum_{i=1}^l \|B_\alpha(\overline{\varphi}_i f)\|_{\mathbb{A}_\alpha^2}.$$

Since B_α is a projection, $\|B_\alpha(\overline{\varphi}_i f)\|_{\mathbb{A}_\alpha^2} \leq \|\varphi_i f\|_{\mathbb{A}_\alpha^2}$, then

$$\sum_{i=1}^l \|B_\alpha(\overline{\varphi}_i f)\|_{\mathbb{A}_\alpha^2} \leq \sup_{\mathbb{B}} |\Phi(w)| \|f\|_{\mathbb{A}_\alpha^2}.$$

Let $f_w(z) = \frac{B_\alpha(z, w)}{\sqrt{B_\alpha(w, w)}}$, u in \mathbb{B} . Then $Hf_w(z) = \sum_{i=1}^l \varphi_i(z) \overline{\varphi}_i(w) f_w(z)$ and $\langle Hf_w, f_w \rangle_\alpha = |\Phi(w)|^2$, where $\langle \cdot, \cdot \rangle_\alpha$ is the inner product of \mathbb{A}_α^2 . The relation $\langle Hf_w, f_w \rangle_\alpha \leq \|H\| \|f\|_{\mathbb{A}_\alpha^2}^2$ and the fact that $\|f_w\|_{\mathbb{A}_\alpha^2} = 1$ finish the proof. \square

Proof of the theorem 1.1: Assume that $\sup_{\mathbb{B}} |\Phi(w)|^2 \leq 1$. Then H is a contraction and the operator $(I - H)^{1/2}$ is well defined. It is a self-adjoint operator and it is well known that $(I - H)^{1/2} \mathbb{A}_\alpha^2$ is a sub-Hilbert space whose reproducing kernel is given by $(I - H)B_\alpha(z, w) = K_\alpha(z, w)$ see [6],[8] for details.

Assume now that $K_\alpha(z, w)$ is a reproducing kernel, then for any finite sequences (w_j) $K_\alpha(w_j, w_k)$ is a definite positive matrix. Let w be a point in the ball, then $K_\alpha(w, w) = (1 - |\Phi(w)|^2) B_\alpha(w, w) \geq 0$, then $|\Phi(w)|^2 \leq 1$. \square

Proof of the theorem 1.2: The proof involves the Douglas criterion [2] we mention here the statement.

Douglas criterion : *Let $\mathcal{H}_1, \mathcal{H}_2$ and \mathcal{H} be Hilbert spaces. Let A from \mathcal{H}_1 into \mathcal{H} and B from \mathcal{H}_2 into \mathcal{H} be bounded operators.*

There exists $\lambda > 0$ such that $AA^ \ll \lambda BB^*$ if and only if $Im(A) \subset Im(B)$.*

In the definition, the relation $AA^* \ll \lambda BB^*$ means that $\lambda BB^* - AA^*$ is a positive operator.

We consider the coordinate functions $\Phi_0(z) = (z_1, \dots, z_n)$ and we denote by H_0 its associated operator. The kernel of the Hilbert space $(I - H_0)^{1/2} \mathbb{A}_\alpha^2$ is given by $(1 - z\bar{w}) B_\alpha(z, w) = c_\alpha B_{\alpha-1}(z, w)$. Then $(I - H_0)^{1/2} \mathbb{A}_\alpha^2 = \mathbb{A}_{\alpha-1}^2$ by the uniqueness of the reproducing kernel.

If we suppose that $K_\alpha(z, w)$ is a reproducing kernel of $\mathbb{A}_{\alpha-1}^2$, from the Douglas criterion, there exists $\lambda > 0$ such that $I - H \ll \lambda(I - H_0)$. Let f in \mathbb{A}_α^2 . Then $\langle (I - H)f, f \rangle_\alpha \leq \lambda \langle (I - H_0)f, f \rangle_\alpha$. Let w in \mathbb{B} . The function $f(z) = B_\alpha(z, w)$ gives $1 - |\Phi(w)|^2 \leq \lambda(-r(w))$.

The sufficient conditions follows from the result :

Proposition 2.3. *Let $\Phi(z) = (\varphi_1(z), \dots, \varphi_l(z))$ be holomorphic. If $1 - |\Phi(z)|^2 \leq C(-r(z))$, there exists $\lambda > 0$ such that*

$$\frac{1}{\lambda}(I - H_0) \ll I - H \ll \lambda(I - H_0).$$

Proof of the proposition 2.3 : Let $\alpha \geq 1$. The right part of the "inequality" follows from two technical lemmas :

Lemma 2.4. *There exists $C > 0$ such that for f in \mathbb{A}_α^2 ,*

$$\langle (I - H)f, f \rangle_\alpha \leq C \|f\|_{\mathbb{A}_{\alpha+1}^2}^2.$$

Proof : Let f in \mathbb{A}_α^2 . Then

$$(1) \quad \langle (I - H)f, f \rangle_\alpha = \int_{\mathbb{B}} \int_{\mathbb{B}} K_\alpha(z, w) f(w) \overline{f(z)} (-r(w))^\alpha (-r(z))^\alpha dV(w) dV(z).$$

Let $K(z, w) = K_\alpha(z, w) (-r(w))^{1/2+\alpha/2} (-r(z))^{1/2+\alpha/2}$ and notice by K the integral operator given by

$$(2) \quad Kg(z) = \int_{\mathbb{B}} K(z, w) g(w) dV(w), \quad g \in L^2(\mathbb{B})$$

Assume for the moment that K is bounded in $L^2(\mathbb{B})$, the Chauchy-Schwarz inequality gives

$$\langle (I - H)f, f \rangle_\alpha \leq \|K(f(-r(\cdot)))^{1/2+\alpha/2}\|_{L^2(\mathbb{B})} \|f\|_{\mathbb{A}_{\alpha+1}^2} \leq C \|f\|_{\mathbb{A}_{\alpha+1}^2}^2.$$

It remains to study the boundedness of K . This follows from the Schur criterion [10]. We have only to prove that there exists $C > 0$ such that $K(z, w) = \frac{C}{|1 - z\bar{w}|^{n+1}}$. Recall that, [4] prop 8.1.4.,

$$(3) \quad \frac{|1 - \Phi(z)\overline{\Phi(w)}|^2}{(1 - |\Phi(z)|^2)(1 - |\Phi(w)|^2)} \leq \frac{|1 - z\bar{w}|^2}{(-r(z))(-r(w))}.$$

Therefore there exists $C > 0$ such that $|1 - \Phi(z)\overline{\Phi(w)}| \leq C|1 - z\bar{w}|$ then $|K(z, w)|$ has the desired pointwise estimate.

We finish the proof of the proposition with the following result :

Lemma 2.5. *Let $\alpha > -1$. There exists $C > 0$ such that*

$$\frac{1}{C} \|f\|_{\mathbb{A}_{\alpha+1}^2}^2 \leq \langle (I - H_0)f, f \rangle_\alpha \leq C \|f\|_{\mathbb{A}_{\alpha+1}^2}^2, \quad f \in \mathbb{A}_\alpha^2.$$

Proof : The proof is direct. Let $f(z) = \sum_m a_m z^m$. Recall that

$$\|z^m\|_{\mathbb{A}_\alpha^2}^2 = c_\alpha \frac{\Gamma(m_1 + 1) \cdots \Gamma(m_n + 1)}{\Gamma(|m| + n + 1 + \alpha)},$$

where $|m| = m_1 + \cdots + m_n$ [4]. Let us remark that $(1 - z\bar{w})B_\alpha(z, w) = cB_{\alpha-1}(z, w)$. Then

$$(I - H_0)f(z) = c \int_{\mathbb{B}} B_{\alpha-1}(z, w) f(w) (-r(w))^\alpha dV(w) = c \sum_m \frac{a_m z^m}{|m| + n + \alpha}.$$

and

$$\langle (I - H_0)f, f \rangle_\alpha = c \sum_m |a_m|^2 \frac{\Gamma(m_1 + 1) \cdots \Gamma(m_n + 1)}{(|m| + n + \alpha)\Gamma(|m| + n + 1 + \alpha)}.$$

Notice that

$$\|f\|_{\mathbb{A}_{\alpha+1}^2}^2 = c \sum_m |a_m|^2 \|z^m\|_{\mathbb{A}_{\alpha+1}^2}^2 = c \sum_m |a_m|^2 \frac{\Gamma(m_1 + 1) \cdots \Gamma(m_n + 1)}{\Gamma(|m| + n + 2 + \alpha)}.$$

The equivalence follows from the relation $\Gamma(z + 1) = z\Gamma(z)$.

For the left part of the "inequality" (of the proposition 2.3), we consider the auxiliary operator \tilde{H} given by $\tilde{H}f = B_\alpha(|\Phi|^2 f)$, $\alpha > -1$. Let us remark that $\|\tilde{H}\| \leq \sup_{\mathbb{B}} |\Phi(w)|^2$ and $\langle Hf, f \rangle_\alpha = \sum_{i=1}^l \langle \varphi_i f, B_\alpha(\varphi_i f) \rangle_\alpha \leq \sum_{i=1}^l \langle \varphi_i f, \varphi_i f \rangle_\alpha = \langle \tilde{H}f, f \rangle_\alpha$, then

$$(4) \quad I - \tilde{H} \ll I - H$$

We have the following result :

Lemma 2.6. *Let \tilde{H} as above. There exists $\lambda > 0$ such that $I - H_0 \ll \lambda(I - \tilde{H})$.*

Proof : Let us remark that $1 - |\Phi(0)|^2 \leq 2|1 - \Phi(z)\overline{\Phi(0)}|$, the relation (3) gives

$$\frac{1}{1 - |\Phi(z)|^2} \leq \frac{4}{1 - |\Phi(0)|^2} \frac{|1 - \Phi(z)\overline{\Phi(0)}|^2}{(1 - |\Phi(z)|^2)(1 - |\Phi(0)|^2)} \leq \frac{C}{(-r(z))}.$$

Then $-r(z) \leq C(1 - |\Phi(z)|^2)$ and $\|f\|_{\mathbb{A}_{\alpha+1}^2}^2 \leq C\langle (I - \tilde{H})f, f \rangle_\alpha$. It remains to apply the lemma 2.5 to achieve the proof. \square

Proof of the theorem 1.3 : The proof is similar for $0 \leq \alpha \leq 1$ but require a minor modification to use the Schur criterion. We recall that for u an v holomorphic functions [4],

$$(5) \quad \begin{aligned} \int_{\mathbb{B}} u(w)\overline{v(w)}(-r(w))^\alpha dV(w) &= \int_{\mathbb{B}} (N + (n + 1 + \alpha)I)u(w)\overline{v(w)}(-r(w))^{\alpha+1} dV(w) \\ &= \int_{\mathbb{B}} u(w)(\bar{N} + (n + 1 + \alpha)I)\overline{v(w)}(-r(w))^{\alpha+1} dV(w), \end{aligned}$$

In this case the kernel of $(I - H_0)^{1/2}\mathbb{A}_\alpha^2$ is given by $(1 - z\bar{w})B_\alpha(z, w) = cB_{\alpha-1}(z, w)$ thus $(I - H_0)^{1/2}\mathbb{A}_\alpha^2$ is the weighted Bergman space $\mathbb{A}_{\alpha-1}^2$ when $0 < \alpha < 1$, the Hardy space \mathbb{H}^2 for $\alpha = 0$ and the Besov space \mathcal{B}_α for $-1 < \alpha < 0$.

For f in \mathbb{A}_α^2 ,

$$(I - H)f(z) = \int_{\mathbb{B}} (\bar{N} + (n + 1 + \alpha)I)K_\alpha(z, w)f(w)(-r(w))^{\alpha+1}dV(w)$$

and the relation (1) becomes

$$\langle (I - H)f, f \rangle_\alpha = \int_{\mathbb{B}} (N + (n + 1 + \alpha)I)(I - H)f(z)\overline{f(z)}(-r(z))^{\alpha+1}dV(z)$$

Let

$$K(z, w) = (-r(z))^{\alpha/2+1/2}(-r(w))^{\alpha/2+1/2} \times \\ (N + (n + 1 + \alpha)I)(\bar{N} + (n + 1 + \alpha)I)K_\alpha(z, w)$$

and K be the integral operator given by (2). Then

$$\langle (I - H)f, f \rangle_\alpha = \int_{\mathbb{B}} K(f(\cdot)(-r(\cdot))^{1/2})(z)\overline{f(z)}(-r(z))^{1/2}dV(z).$$

$$K(z, w) = \sum_{s,t=0}^1 (I + (n + 1)N)^s (I + (n + 1)\bar{N})^t (1 - \Phi(z)\overline{\Phi(w)}) \\ \times (I + (n + 1)N)^{1-s} (I + (n + 1)\bar{N})^{1-t} B_\alpha(z, w).$$

Then there exists $C > 0$ such that

$$|K(z, w)| \leq C(-r(z))^{\alpha/2+1/2}(-r(w))^{\alpha/2+1/2} \left(\frac{|1 - \Phi(z)\overline{\Phi(w)}|}{|1 - z\bar{w}|^{n+1+\alpha+2}} \right. \\ \left. + \frac{\sum_i |N\varphi_i(w)| + \sum_i |N\varphi_i(z)|}{|1 - z\bar{w}|^{n+1+\alpha+1}} + \frac{(\sum_i |N\varphi_i(z)|)(\sum_i |N\varphi_i(w)|)}{|1 - z\bar{w}|^{n+1+\alpha}} \right).$$

Recall that $|1 - \Phi(z)\overline{\Phi(w)}| \leq C|1 - z\bar{w}|$ and $\sup_{\mathbb{B}} |N\varphi_i(w)| \leq C$, then

$$|K(z, w)| \leq \frac{C(-r(z))^{1/2}(-r(w))^{1/2}}{|1 - z\bar{w}|^{n+2}}.$$

Let us remark that $(-r(z))$ and $(-r(w))$ are bounded by a constant times $|1 - z\bar{w}|$. Then $|K(z, w)| \leq \frac{C}{|1 - z\bar{w}|^{n+1}}$. The operator K is bounded in $L^2(\mathbb{B})$ and the relation (1) gives $\langle (I - H)f, f \rangle_\alpha \leq C\|f\|_{\mathbb{A}_{\alpha+1}^2}^2$. The lemma 2.5 gives $I - H \leq \lambda(I - H_0)$. \square

The theorem 1.2 and 1.3 characterize the sub-Hilbert space $(I - H)^{1/2}(\mathbb{A}_\alpha)$, this space is the analogue for the unit ball of \mathbb{C}^n of the complementary space. The space $(I - \tilde{H})^{1/2}(\mathbb{A}_\alpha)$ is also a complementary space. It is a consequence of the relation (4) and the lemma 2.6 that there exists $C > 0$ such that

$$\frac{1}{C}(I - H_0) \ll I - \tilde{H} \ll I - H$$

Under the condition on Φ given in the theorem 1.2 and theorem 1.3, we have

$$(I - \tilde{H})^{1/2}(\mathbb{A}_\alpha) = (I - H)^{1/2}(\mathbb{A}_\alpha).$$

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