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# An extension of the Duistermaat-Singer Theorem to the semi-classical Weyl algebra

Yves Colin de Verdière\*

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## Abstract

Motivated by many recent works (by L. Charles, V. Guillemin, T. Paul, J. Sjöstrand, A. Uribe, San Vũ Ngọc, S. Zelditch and others) on the semi-classical Birkhoff normal forms, we investigate the structure of the group of automorphisms of the graded semi-classical Weyl algebra. The answer is quite similar to the Theorem of Duistermaat and Singer for the usual algebra of pseudo-differential operators where all automorphisms are given by conjugation by an elliptic Fourier Integral Operator (a FIO). Here what replaces general non-linear symplectic diffeomorphisms is just linear complex symplectic maps, because everything is localized at a single point<sup>1</sup>.

## 1 The result

Let  $W = W_0 \oplus W_1 \oplus \dots$  be the semi-classical graded Weyl algebra (see Section 2 for a definition) on  $\mathbb{R}^{2d}$ . Let us define  $X_j := W_j \oplus W_{j+1} \oplus \dots$ . We want to prove the:

**Theorem 1.1** *There exists an exact sequence of groups*

$$0 \rightarrow \mathcal{I} \rightarrow_1 \text{Aut}(W) \rightarrow_2 \text{Sympl}_{\mathbb{C}}(2d) \rightarrow 0$$

where

- $\text{Sympl}_{\mathbb{C}}(2d)$  is the group of linear symplectic transformations of  $\mathbb{C}^{2d} = \mathbb{R}^{2d} \otimes_{\mathbb{C}} \mathbb{C}$

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\*Institut Fourier, Unité mixte de recherche CNRS-UJF 5582, BP 74, 38402-Saint Martin d'Hères Cedex (France); yves.colin-de-verdiere@ujf-grenoble.fr

<sup>1</sup>Thanks to Louis Boutet de Monvel for his comments and suggestions, in particular the proofs of Lemmas 6.1 and 6.2. As he said, the main result is not surprising!

- $\text{Aut}(W)$  is the group of automorphisms  $\Phi$  of the semi-classical graded<sup>2</sup> Weyl algebra preserving  $\hbar$
- $\mathcal{I}$  is the group of “inner” automorphisms  $\Phi_S$  of the form  $\Phi_S = \exp(iadS/\hbar)$ , i.e.  $\Phi_S(w) = \exp(iS/\hbar) \star w \star \exp(-iS/\hbar)$  as a formal power series, with  $S \in X_3$
- The arrow  $\rightarrow_2$  is just given from the action of the automorphism  $\Phi$  on  $W_1 = (\mathbb{R}^{2d})^* \otimes \mathbb{C}$  modulo  $X_2$

The proof follows [3] and also the semi-classical version of it by H. Christianson [2]. This result is implicitly stated in Fedosov’s book [4] in Chapter 5, but it could be nice to have an direct proof in a simpler context. The result is a consequence of Lemmas 4.1, 6.1 and 6.2.

## 2 The Weyl algebra

The elements of the “Weyl algebra” are the formal power series in  $\hbar$  and  $(x, \xi)$

$$W = \bigoplus_{n=0}^{\infty} W_n$$

where  $W_n$  is the space of complex valued homogeneous polynomials in  $z = (x, \xi)$  and  $\hbar$  of total degree  $n$  where the degree of  $\hbar^j z^\alpha$  is  $2j + |\alpha|$ . The Moyal  $\star$ -product

$$a \star b := \sum_{j=0}^{\infty} \frac{1}{j!} \left( \frac{\hbar}{2i} \right)^j a \left( \sum_{p=1}^d \overleftarrow{\partial}_{\xi_p} \overrightarrow{\partial}_{x_p} - \overleftarrow{\partial}_{x_p} \overrightarrow{\partial}_{\xi_p} \right)^j b = ab + \frac{\hbar}{2i} \{a, b\} + \dots$$

(where  $\{a, b\}$  is the Poisson bracket of  $a$  and  $b$ ) gives to  $W$  the structure of a graded algebra: we have  $W_m \star W_n \subset W_{m+n}$  and hence, for the brackets,  $\frac{i}{\hbar} [W_m, W_n]^\star \subset W_{m+n-2}$ .

The previous grading of  $W$  is obtained by looking at the action of  $W$  on the (graded) vector space  $\mathcal{S}$  of *symplectic spinors* (see [5]): if  $F \equiv \sum_{j=0}^{\infty} \hbar^j F_j(X)$  with  $F_j \in \mathcal{S}(\mathbb{R})$ , we define  $f_\hbar(x) = \hbar^{-d/2} F(x/\hbar)$  whose micro-support is the origin.  $W$  acts on  $\mathcal{S}$  in a graded way as differential operators of infinite degree: if  $w \in W$ ,  $w.f = \text{OP}_\hbar(w)(f)$ .

## 3 A remark

We assumed in Theorem 1.1 that  $\hbar$  is fixed by the automorphism. If not, the symplectic group has to be replaced by the homogeneous symplectic group: the group of the linear automorphisms  $M$  of  $(\mathbb{R}^{2d}, \omega)$  which satisfies  $M^* \omega = c\omega$ . We then have to take into account a multiplication of  $\hbar$  by  $c$ . For  $c = -1$ , it is a semi-classical version of the transmission property of Louis Boutet de Monvel.

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<sup>2</sup>“graded” means that  $\Phi(W_n) \subset W_n \oplus W_{n+1} \oplus \dots$

## 4 Surjectivity of the arrow $\rightarrow_2$

**Lemma 4.1** *The arrow  $\rightarrow_2$  is surjective.*

*Proof.*–

Let us give  $\chi \in \text{Symp}_{\mathbb{C}}(2d)$ . The map  $a \rightarrow a \circ \chi$  is an automorphism of the Weyl algebra: the Moyal formula is given only in terms of the Poisson bracket.

□

## 5 The principal symbols

Let  $\Phi$  be an automorphism of  $W$ . Then  $\Phi$  induces a linear automorphism  $\Phi_n$  of  $W_n$ : if  $w = w_n + r$  with  $w_n \in W_n$  and  $r \in X_{n+1}$  and  $\Phi(w) = w'_n + r'$  with  $w'_n \in W_n$  and  $r' \in X_{n+1}$ ,  $\Phi_n(w_n) := w'_n$  is independent of  $r$ . The polynomial  $w_n$  is the *principal symbol* of  $w \in X_n$ . We have  $\Phi_{m+n}(w_m \star w_n) = \Phi_m(w_m) \star \Phi_n(w_n)$ . Hence  $\Phi_n$  is determined by  $\Phi_1$  because the algebra  $W$  is generated by  $W_1$  and  $\hbar$ . The linear map  $\Phi_1$  is an automorphism of the complexified dual of  $\mathbb{R}^{2d}$ . Let us show that it preserves the *Poisson bracket* and hence is the adjoint of a linear symplectic mapping of  $\mathbb{C}^{2d}$ . We have:

$$\Phi([w, w']^{\star}) = [\Phi(w), \Phi(w')]^{\star} .$$

By looking at principal symbols, for  $w, w' \in X_1$ , we get

$$\{\Phi_1(w), \Phi_1(w')\} = \{w, w'\} .$$

## 6 Inner automorphisms

The kernel of  $\rightarrow_2$  is the group of automorphisms  $\Phi$  which satisfy  $\Phi_n = \text{Id}$  for all  $n$ , i.e. for any  $w_n \in W_n$

$$\Phi(w_n) = w_n \text{ mod } X_{n+1} .$$

The following fact is certainly well known:

**Lemma 6.1** *If  $\Phi \in \ker(\rightarrow_2)$ ,  $\Phi = \exp(D)$  where  $D : W_n \rightarrow X_{n+1}$  is a derivation of  $W$ .*

*Proof [following a suggestion of Louis Boutet de Monvel]*– We define  $\Phi^s$  for  $s \in \mathbb{Z}$ . Let  $\Phi_{p,n}^s : W_n \rightarrow W_{n+p}$  be the degree  $(n+p)$  component of  $(\Phi^s)_n : W_n \rightarrow X_n$ . Then  $\Phi_{p,n}^s$  is polynomial w.r. to  $s$ . This allows to extend  $\Phi^s$  to  $s \in \mathbb{R}$  as a 1-parameter group of automorphisms. We put  $D = \frac{d}{ds}(\Phi^s)|_{s=0}$ . We have  $\Phi^s = \exp(sD)$ . We deduce that  $D$  is a derivation. □

We need to show the:

**Lemma 6.2** *Every derivation  $D$  of  $W$  sending  $W_1$  into  $X_2$  is an inner derivation of the form*

$$Dw = \frac{i}{\hbar}[S, w]$$

with  $S \in X_3$ .

*Proof [following a suggestion of Louis Boutet de Monvel]–*

Let  $(\zeta_k)$  the basis of  $W_1$  dual to the canonical basis  $(z_k)$  for the star bracket, i.e. satisfying  $[\zeta_k, z_l] = \frac{\hbar}{i}\delta_{k,l}$ . We have  $[\zeta_k, w] = \frac{\hbar}{i}\frac{\partial w}{\partial z_k}$ . Put  $y_k = D\zeta_k \in X_2$ . As the brackets  $[\zeta_k, \zeta_l]$  are constants, we have:  $[D\zeta_k, \zeta_l] + [\zeta_k, D\zeta_l] = 0$ , or  $\partial y_k/\partial z_l = \partial y_l/\partial z_k$ . There exists an unique  $S$  vanishing at 0 so that:

$$[S, \zeta_k] = -\frac{\partial S}{\partial z_k} = \frac{\hbar}{i}y_k .$$

Hence  $D = (i/\hbar)[S, \cdot]$ . Because  $y_k \in X_2$ ,  $S$  is in  $X_3$ . □

## 7 An homomorphism from the group $G$ of elliptic FIO's whose associated canonical transformation fixes the origin into $\text{Aut}(W)$

Each  $w \in W$  is the Taylor expansion of a Weyl symbol  $a \equiv \sum_{j=0}^{\infty} \hbar^j a_j(x, \xi)$  of a pseudo-differential operator  $\hat{a}$ . Let us give an elliptic Fourier Integral Operator  $U$  associated to a canonical transformation  $\chi$  fixing the origin. The map  $\hat{a} \rightarrow U^{-1}\hat{a}U$  induces a map  $F$  from  $S^0$  into  $S^0$  which is an automorphism of algebra (for the Moyal product).

**Lemma 7.1** *The Taylor expansion of  $F(a)$  only depends on the Taylor expansion of  $a$ .*

This is clear from the explicit computation and the stationary phase expansions.

As a consequence,  $F$  induces an automorphism  $F_0$  of the Weyl algebra graded by powers of  $\hbar$ .

**Lemma 7.2**  *$F_0$  is an automorphism of the algebra  $W$  graded as in Section 2.*

*Proof.–*

We have to check the  $F_0(W_n) \subset X_n$ . Because  $F_0$  preserves the  $\star$ -product, it is enough to check that  $F_0(W_1) \subset X_1$ . It only means that the (usual) principal symbol of  $F(a)$  vanishes at the origin if  $a$  does. It is consequence of Egorov Theorem.

□

Summarizing, we have constructed a group morphism  $\alpha$  from the group  $G$  of elliptic FIO's whose associated canonical transformation fixes the origin in the group  $\text{Aut}(W)$ .

**Definition 7.1** *An automorphism  $\Phi$  of  $W$  is said to be real ( $\Phi \in \text{Aut}_{\mathbb{R}}(W)$ ) if the mapping  $\Phi \bmod(\hbar W)$  is real.*

**Theorem 7.1** *The image of the group  $G$  by the homomorphism  $\alpha$  is  $\text{Aut}_{\mathbb{R}}(W)$ . In particular, any  $\Phi \in \text{Aut}_{\mathbb{R}}(W)$  can be “extended” to a semi-classical Fourier Integral Operator.*

*Proof.*–

The image of  $\alpha$  is in the sub-group  $\text{Aut}_{\mathbb{R}}(w)$  because the canonical transformation  $\chi$  is real.

We have still to prove that the image of  $\alpha$  is  $\text{Aut}_{\mathbb{R}}(W)$ . Using the Theorem 1.1 and the metaplectic representation, it is enough to check that the automorphism  $\exp(iadS/\hbar)$  comes from an FIO. Let  $H$  be a symbol whose Taylor expansion is  $S$  (the principal symbol of  $H$  is a real valued Hamiltonian). The OIF  $U = \exp\left(i\hat{H}/\hbar\right)$  will do the job.

□

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