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# Distinction and Asai $L$ -functions for generic representations of general linear groups over $p$ -adic fields

Nadir Matringe

## Abstract

Let  $K/F$  be a quadratic extension of  $p$ -adic fields, and  $n$  a positive integer. A smooth irreducible representation of the group  $GL(n, K)$  is said to be distinguished, if it admits on its space a nonzero  $GL(n, F)$ -invariant linear form. In the present work, we classify distinguished representations of the group  $GL(n, K)$  in terms of inducing quasi-square integrable representations. This has as a consequence the truth of the expected equality between the Rankin-Selberg type Asai  $L$ -function of a generic representation, and the Asai  $L$ -function of its Langlands parameter.

## Introduction

Given  $K/F$  a quadratic extension of  $p$ -adic fields, we denote by  $\sigma$  the non trivial element of the Galois group of  $K$  over  $F$ . We denote by  $\eta_{K/F}$  the character of order 2 of  $F^*$ , trivial on the set of norms of  $K^*$ .

A smooth representation of  $GL(n, K)$  is said to be distinguished if it admits on its space a nonzero linear form, which is invariant under  $GL(n, F)$ . The pair  $(GL(n, K), GL(n, F))$  is known to be a generalized Gelfand pair, which means that for an irreducible representation  $(\pi, V_\pi)$  of  $GL(n, K)$ , the space of  $GL(n, F)$ -invariants linear form on  $V_\pi$  is of dimension at most one. The unitary distinguished representations are the natural space which support the Plancherel measure of the symmetric space  $GL(n, F)\backslash GL(n, K)$ . Hence their understanding is related to harmonic analysis on  $GL(n, F)\backslash GL(n, K)$ .

We classify here distinguished generic representations of  $GL(n, K)$ , in terms of inducing discrete series representations. More precisely we prove the following result.

**Theorem 5.2.** *Let  $\pi$  be a generic representation of the group  $GL(n, K)$ , obtained by normalised parabolic induction of quasi-square-integrable representations  $\Delta_1, \dots, \Delta_t$ . It is distinguished if and only if there exists a reordering of the  $\Delta_i$ 's, and an integer  $r$  between 1 and  $t/2$ , such that we have  $\Delta_{i+1}^\sigma = \Delta_i^\vee$  for  $i = 1, 3, \dots, 2r - 1$ , and  $\Delta_i$  is distinguished for  $i > 2r$ .*

Distinguished representations of  $GL(n, K)$  are also related by a conjecture of Flicker and Rallis, to the base change theory of representations of a unitary group (see for example [A-R]). The main result of this paper could perhaps reduce the proof of this conjecture for generic distinguished representations of  $GL(n, K)$ , to the case of discrete series representations.

Generic distinguished representations are related to the analytic behaviour of meromorphic functions, called Asai  $L$ -functions associated with generic representations. The basic theory of the Asai  $L$ -function of a generic representation  $\pi$  of  $GL(n, K)$ , denoted by  $L_F^K(\pi)$  and defined as the gcd of functions obtained as meromorphic extension of Rankin-Selberg integrals, such as its functional equation, has first been developed by Flicker in [F1] and [F3].

Then in [K], Kable proves that if the Asai  $L$ -function  $L_F^K(\pi)$  of a discrete series representation  $\pi$  admits a pole at zero, then the representation  $\pi$  is distinguished. This, with the equality of

the product of the Asai  $L$ -functions of  $\pi$  and  $\eta\pi$  (for a character  $\eta$  of  $K^*$  extending  $\eta_{K/F}$ ), and of the classical  $L$  function of the pair  $(\pi^\sigma, \pi^\vee)$ , obtained by a global-local method, allows him to prove the so called Jacquet conjecture for discrete series representations. This result states that a discrete series representation  $\pi$  of  $GL(n, K)$  which is Galois autodual (i.e.  $\pi^\vee = \pi^\sigma$ ), is either distinguished or  $\eta_{K/F}$ -distinguished.

This result is precised in [A-K-T], where it is shown that the preceding (either/or) is exclusive, by first proving that the Asai  $L$ -function of a tempered distinguished representation has a pole at zero.

Unfortunately, the Jacquet conjecture doesn't hold anymore for non discrete series representation, as it would be in contradiction with Theorem 5.2. This fact is first noticed in [M1], where the theorem 5.2 is proved for principal series representation. Actually this theorem can be seen as a generalization of Jacquet's conjecture for discrete series representations, as it says that a generic representation of  $GL(n, K)$  is Galois autodual if and only if it is parabolically induced from three representations, one that is distinguished but not  $\eta_{K/F}$ -distinguished, one that is distinguished and  $\eta_{K/F}$ -distinguished, and one that is distinguished but not  $\eta_{K/F}$ -distinguished. Among these, the distinguished are those with purely  $\eta_{K/F}$ -distinguished part equal to zero. The last step before Theorem 5.2, consisting of showing that the representations described in the theorem are indeed distinguished is the main result of [M4].

Concerning Rankin-Selberg type Asai  $L$ -functions, a definitive statement relating their poles and distinction is obtained in [M2], where it is proved that a representation  $\pi$  is distinguished, if and only if its Asai  $L$ -function admits a so called (in the terminology of [C-P]) exceptional pole at zero.

There are two other ways to associate an Asai  $L$ -function to a representation  $\pi$  of the group  $GL(n, K)$ .

The first is by considering the  $n$ -dimensional representation  $\rho$  of the Weil-Deligne  $W'_K$  of  $K$ , associated to  $\pi$  by the local Langlands correspondence. One then defines by multiplicative induction a representation of the Weil-Deligne group  $W'_F$  (which contains  $W'_K$  as a subgroup of index 2), of dimension  $n^2$ , denoted by  $M_{W'_K}^{W'_F}(\rho)$ . The Asai  $L$ -function corresponding to  $\pi$ , and denoted by  $L_F^{K,W}(\pi, s)$  is by definition the classical  $L$ -function of the representation  $M_{W'_K}^{W'_F}(\rho)$ , which we denote by  $L_F^K(\rho)$ .

The second, called the Langlands-Shahidi method, is introduced in [Sh]. We denote by  $L_F^{K,U}(\pi)$  the meromorphic function obtained by this process, the study of its poles is this time related to the fact of knowing when a representation  $\pi$  is obtained by base change lift from a unitary group (see [Go]). It is conjectured that these three functions are actually the same (cf. [He], [K], [A-R]). Henniart proves in [He] that the functions  $L_F^{K,U}$  and  $L_F^{K,W}$  are equal. Anandavardhanan and Rajan prove in [A-R] that the functions  $L_F^K$  and  $L_F^{K,U}$  coincide on representations of the discrete series of  $GL(n, K)$ .

In [M3], which can be used as a survey for local Rankin-Selberg type Asai  $L$ -functions, Theorem 5.2 is stated as a conjecture. It is then showed using a method of Cogdell and Piatetski-Shapiro and the known equality of  $L_F^{K,W}$  and  $L_F^K$  for discrete series representations, that the theorem implies the equality of  $L_F^{K,W}$  and  $L_F^K$  for generic representations. Hence we have the following result.

**Theorem 5.3.** *Let  $\pi$  be a generic representation of the group  $GL(n, K)$ , and let  $\rho$  be the representation of dimension  $n$  of the Weil-Deligne group  $W'_K$  of  $K$ , corresponding to  $\pi$  through Langlands correspondence. The following equality of  $L$ -functions is satisfied:*

$$L_F^K(\pi, s) = L_F^K(\rho, s).$$

Now the main tool for the proof of Theorem 5.2 is just Mackey theory. A generic representation  $\pi$  of the group  $GL(n, K)$ , is obtained by normalized parabolic induction of a discrete series representation  $\Delta = \Delta_1 \otimes \cdots \otimes \Delta_t$  of a standard Levi subgroup of a parabolic subgroup  $P(K)$  of  $GL(n, K)$ . Calling  $R$  a set of representatives of the double classes  $P(K) \backslash GL(n, K) / GL(n, F)$ , the representation  $\pi$  has a factor series, with factors being induced representations of  $\Delta$  to  $uGL(n, F)u^{-1}$  for some  $u$  in  $R$ , from the subgroup  $P(K) \cap uGL(n, F)u^{-1}$ . If the representation  $\pi$  is  $GL(n, F)$ -distinguished, then it is at least the case for one of the factors. But using Frobenius reciprocity law, such a factor is distinguished, if and only if the representation  $\Delta$  is itself ( $\chi$ )-distinguished by  $P(K) \cap uGL(n, F)u^{-1}$ . Studying the structure of this subgroup, implies that the  $\Delta_i$ 's must be of the requested form.

The first part of Section 1 is about definitions and notations of the basic objects we use. The second part concerns itself with results of Bernstein and Zelevinsky about classification of discrete series representations in terms of segments, and the computation of their Jacquet modules.

In the Section 2, we give a natural set of representatives  $R$  of the double classes  $P(K) \backslash GL(n, K) / GL(n, F)$  for a standard parabolic subgroup  $P(K)$  of  $GL(n, K)$  for  $u$  in  $R$ .

The third section is devoted to prove a relation between the modulus characters of the groups  $P(K)$  and  $P(K) \cap uGL(n, F)u^{-1}$ .

We eventually prove Theorem 5.2 in the last section.

## 1 Preliminaries

### 1.1 Notations and definitions

We fix until the end of this paper a local field  $F$  of characteristic zero. We fix a quadratic extension  $K$  of  $F$ .

If  $G$  is a group acting on two vector spaces  $V$  and  $V'$ , then  $Hom_G(V, V')$  designates the space of  $G$ -equivariant maps from  $V$  to  $V'$ .

If  $E$  is a finite extension of  $F$ , we denote by  $v_E$  the discrete valuation of  $E$ , which verifies that  $v_E(\pi_E)$  is 1 if  $\pi_E$  is a prime element of  $E$ . We denote by  $q_E$  the cardinality of the residual field of  $E$ . We denote by  $|\cdot|_E$  the absolute value of  $E$  defined by  $|x|_E = q_E^{-v_E(x)}$ , for  $x$  in  $E$ . We denote by  $R_E$  the valuation ring of  $E$ , and by  $P_E$  the maximal ideal of  $R_E$ . Finally we denote by  $W_E$  the Weil group of  $E$  (cf. [T]), and by  $W'_E$  the Weil-Deligne group of  $E$ . The group  $W'_E$  is the semidirect product group  $W_E \rtimes SL(2, \mathbb{C})$ , with  $W_E$  acting by its quotient group  $q_E^{\mathbb{Z}}$  on  $SL(2, \mathbb{C})$ , such that if we take a Frobenius element  $\phi_E$  in  $W_E$ , the action of  $\phi_E$  on  $SL(2, \mathbb{C})$  is given by conjugation by the matrix  $\begin{pmatrix} q_E & 0 \\ 0 & 1 \end{pmatrix}$ .

Let  $G$  be an affine algebraic group defined on the field  $F$ . If  $E$  is an extension of  $F$ , we denote by  $G(E)$  the group of the points of  $G$  over  $E$ . Such a group is locally compact and totally disconnected, we will call it an  $l$ -group.

Let  $n$  be a positive integer, we denote by  $M_n = M_n(\bar{F})$  the additive group of  $n \times n$  matrices with entries in  $\bar{F}$ , and we denote by  $G_n$  the general linear group  $GL(n, \bar{F})$  of invertible matrices of  $M_n(\bar{F})$ . If  $M$  belongs to  $M_n$ , we denote its determinant by  $det(M)$ .

We call partition of a positive integer  $n$ , a family  $\bar{n} = (n_1, \dots, n_t)$  of positive integers (for a certain  $t$  in  $\mathbb{N} - \{0\}$ ), such that the sum  $n_1 + \cdots + n_t$  is equal to  $n$ . To such a partition, we

associate an algebraic subgroup of  $G_n$  denoted by  $P_{\bar{n}}$ , given by matrices of the form

$$\begin{pmatrix} g_1 & \star & \star & \star & \star \\ & g_2 & \star & \star & \star \\ & & \ddots & \star & \star \\ & & & g_{t-1} & \star \\ & & & & g_t \end{pmatrix},$$

with  $g_i$  in  $G_{n_i}$  for  $i$  between 1 and  $t$ . We call it the standard parabolic subgroup associated with the partition  $\bar{n}$ . We call parabolic subgroup any conjugate of a standard parabolic subgroup. We denote by  $N_{\bar{n}}$  its unipotent radical subgroup, given by the matrices

$$\begin{pmatrix} I_{n_1} & \star & \star \\ & \ddots & \star \\ & & I_{n_t} \end{pmatrix}$$

and by  $M_{\bar{n}}$  its Levi subgroup given by the matrices

$$\begin{pmatrix} g_1 & & \\ & \ddots & \\ & & g_t \end{pmatrix}$$

with the  $g_i$ 's in  $G_{n_i}$ . The group  $P_{\bar{n}}$  identifies with the semidirect product  $N_{\bar{n}} \rtimes M_{\bar{n}}$ .

Let  $G$  be an  $l$ -group (i.e. locally compact totally disconnected), we denote by  $d_G g$  or simply  $d g$  if the context is clear, a left Haar measure  $G$ . For  $x$  in  $G$ , we denote by  $\delta_G(x)$  the positive number defined by the relation  $d g(gx) = \delta_G^{-1}(x)d g(g)$ . The modulus character  $\Delta_G$  defines a morphism from  $G$  into  $\mathbb{R}_{>0}$ . We denote by  $\delta_G$  (which we also call modulus character) the morphism from  $G$  into  $\mathbb{R}_{>0}$  defined by  $x \mapsto \Delta_G(x^{-1})$ .

Let  $G$  be an  $l$ -group, and  $H$  a subgroup of  $G$ , a representation  $(\pi, V)$  of  $G$  is said to be smooth if for any vector  $v$  of the vector space  $V$ , there is a subgroup  $U_v$  of  $G$  stabilizing  $v$  through  $\pi$ . We denote by  $V^H$  subspace of fixed points of  $V$  under  $H$ . The category of smooth representations of  $G$  is denoted by  $\text{Alg}(G)$ . If  $(\pi, V)$  is a smooth representation of  $G$ , we denote by  $\pi^\vee$  its dual representation in the smooth dual space  $\tilde{V}$  of  $V$ .

We will only consider smooth representations of  $l$ -groups.

**Definition 1.1.** *Let  $G$  be an  $l$ -group,  $H$  a closed subgroup of  $G$ , and  $(\pi, V)$  a representation of  $G$ . If  $\chi$  is a character of  $H$ , we say that the representation  $\pi$  is  $\chi$ -distinguished under  $H$ , if it admits on its space a nonzero linear form  $L$ , verifying  $L(\pi(h)v) = \chi(h)L(v)$  for all  $v$  in  $V$  and  $h$  in  $H$ . If  $\chi = 1$ , we say  $H$ -distinguished instead of 1-distinguished. We omit "H-" if the context is clear.*

Let  $X$  be a locally closed space of an  $l$ -group  $G$ , and  $H$  closed subgroup of  $G$ , with  $H.X \subset X$ . If  $\rho$  is a complex representation of  $H$  in  $V_\rho$ , we denote by  $C_c^\infty(H \backslash X, \rho, V_\rho)$  space of smooth functions on  $G$ , with value in  $V_\rho$ , with support compact modulo  $H$  which verify  $f(hx) = \rho(h)f(x)$  for  $h \in H$ , and  $x \in X$ .

If  $\rho$  is a complex representation of  $H$  in  $V_\rho$ , we denote by  $C^\infty(H \backslash X, \rho, V_\rho)$  the space of functions  $f$  from  $X$  to  $V_\rho$ , fixed under the action by right translation of some compact open subgroup  $U_f$  of  $G$ , and which verify  $f(hx) = \rho(h)f(x)$  for  $h \in H$ , and  $x \in X$ . We denote by  $C_c^\infty(H \backslash X, \rho, V_\rho)$  subspace of functions with support compact modulo  $H$  of  $C^\infty(H \backslash X, \rho, V_\rho)$ .

We denote by  $\text{Ind}_H^G(\rho)$  the representation by right translation of  $G$  in  $C^\infty(H \backslash G, \rho, V_\rho)$  and by  $\text{ind}_H^G(\rho)$  the representation by right translation of  $G$  in  $C_c^\infty(H \backslash G, \rho, V_\rho)$ . Let  $n$  be a positive

integer, and  $\bar{n} = (n_1, \dots, n_t)$  be a partition of  $n$ , and suppose that we have a representation  $(\rho_i, V_i)$  of  $G_{n_i}(K)$  for each  $i$  between 1 and  $t$ . We denote by  $Ind'_H{}^G(\rho)$  the normalized induced representation  $Ind'_H{}^G((\Delta_G/\Delta_H)^{1/2}\rho)$  and by  $ind'_H{}^G(\rho)$  the normalized induced representation  $ind'_H{}^G((\Delta_G/\Delta_H)^{1/2}\rho)$ .

Let  $\rho$  be the extension to  $P_{\bar{n}}$  of the natural representation  $\rho_1 \otimes \dots \otimes \rho_t$  of  $M_{\bar{n}} \simeq G_{n_1}(K) \times \dots \times G_{n_t}(K)$ , by taking it trivial on  $N_{\bar{n}}$ . We denote by  $\rho_1 \times \dots \times \rho_t$  the representation  $ind'_{P_{\bar{n}}(K)}{}^{G_n(K)}(\rho) = Ind'_{P_{\bar{n}}(K)}{}^{G_n(K)}(\rho)$ .

## 1.2 Segments and quasi-square integrable representations

From now on we assimilate representations to their isomorphism classes.

In this subsection, we recall results of Bernstein and Zelevinsky about quasi-square integrable representations, more precisely their classification in terms of segments associated to supercuspidal representations, and how to compute their Jacquet modules.

If  $\pi$  is an irreducible representation of  $G_n(K)$ , one denotes by  $c_\pi$  its central character. We recall that an irreducible representation of  $G_n(K)$  is called supercuspidal if all its Jacquet modules associated to proper standard Levi subgroups are zero, which is equivalent to the fact that it has a coefficient with support compact modulo the center  $Z_n(K)$  of the group  $G_n(K)$ .

An irreducible representation  $\pi$  of the group  $G_n(K)$  is called quasi-square-integrable, if there exists a positive character  $\chi$  of the multiplicative group  $K^*$ , such that one of the coefficients  $g \mapsto c(g)$  of  $\pi$  verifies that  $c(g)\chi(\det(g))$  is a square-integrable function for a Haar measure of  $G_n(K)/Z_n(K)$ . One says that the representation  $\pi$  is square-integrable (or belongs to the discrete series of  $G_n(K)$ ) if one can choose  $\chi$  to be trivial.

If  $\rho$  is a supercuspidal representation of  $G_r(K)$  for a positive integer  $r$ , one denotes by  $\rho|_K$  the representation obtained by twist with the character  $|\det(\ )|_K$ .

In general, if  $E$  is an extension of  $F$ , and  $\chi$  is a character of  $E^*$ , we will denote by  $\chi(g)$  the complex number  $\chi(\det(g))$ .

We call segment a list  $\Delta$  of supercuspidal representations of the form

$$\Delta = [\rho|_K^{l-1}, \rho|_K^{l-2}, \dots, \rho]$$

for a positive integer  $l$ . We call length of the segment the integer  $rl$ . We have the following theorem (Theorem 9.3 of [Z]) who classifies quasi-square integrable representations in terms of segments.

**Theorem 1.1.** *Let  $\rho$  be a supercuspidal representation of  $G_r(K)$  for a positive integer  $r$ . The representation  $\rho \times \rho|_F \times \dots \times \rho|_F^{l-1}$  of  $G_{rl}(K)$  is reducible, with a unique irreducible quotient that we denote by  $[\rho|_K^{l-1}, \rho|_K^{l-2}, \dots, \rho]$ . A representation  $\Delta$  of the group  $G_n(K)$  is quasi-square-integrable if and only if there is  $r \in \{1, \dots, n\}$  and  $l \in \{1, \dots, n\}$  with  $lr = n$ , and  $\rho$  a supercuspidal representation of  $G_r(K)$  such that the representation  $\Delta$  is equal to  $[\rho|_K^{l-1}, \rho|_K^{l-2}, \dots, \rho]$ , the representation  $\rho$  is unique.*

A representation of this type is square-integrable if and only if it is unitarizable, or equivalently if and only if  $\rho|_F^{(l-1)/2}$  is unitarizable (i.e. its central character is unitary). We say that two segments are linked if none of them is a subsegment of the other, but their union is still a segment.

Now we allow ourself to call segment a quasi-square-integrable representation, and to denote such a representation by its corresponding segment.

We will also use the following useful notation: if  $\Delta_1$  and  $\Delta_2$  are two disjoint segments, which are linked, and such that  $\Delta_1$  precedes  $\Delta_2$  (i.e. the segment  $\Delta_1$  is of the form  $[\rho_1|_K^{l_1-1}, \rho_1|_K^{l_1-2}, \dots, \rho_1]$ , the segment  $\Delta_2$  is of the form  $[\rho_2|_K^{l_2-1}, \rho_2|_K^{l_2-2}, \dots, \rho_2]$ , with  $\rho_1 = \rho_2|_K^{l_2}$ ), we denote by  $[\Delta_1, \Delta_2]$  the segment  $[\rho_1|_K^{l_1-1}, \dots, \rho_2]$ .

Let  $P$  be a standard parabolic subgroup of  $G_n(K)$ ,  $M$  its standard Levi subgroup, and let  $P'$  be a standard parabolic subgroup of  $M$ , with standard Levi subgroup  $M'$ , and unipotent radical  $N'$ . We recall that the normalized Jacquet module of a representation  $(\rho, V)$  of  $M$ , associated to  $M'$ , which we denote by  $r_{M',M}(\rho)$ , is the representation of  $M'$  on the space  $V/V(N')$  (where  $V(N')$  is the subspace of  $V$  generated by vectors of the form  $v - \pi(n')v$  for  $v$  in  $V$  and  $n'$  in  $N'$ ), defined by  $r_{M',M}(\rho)(m')(v + V(N')) = \delta_{M'}^{-1/2} \rho(m')v + V(N')$ .

The following proposition (Proposition 9.5 of [Z]), explains how to compute normalized Jacquet modules of segments.

**Proposition 1.1.** *Let  $\rho$  a supercuspidal representation of  $G_r(K)$  for a positive integer  $r$ . Let  $\Delta$  be the segment  $[\rho|_K^{l-1}, \rho|_K^{l-2}, \dots, \rho]$ , for a positive integer  $l$ . Let  $M$  be a standard Levi subgroup of  $G_{lr}(K)$  associated with a partition  $(n_1, \dots, n_t)$  of  $lr$ .*

*The representation  $r_{M,G}(\Delta)$  is zero, unless  $(n_1, \dots, n_t)$  admits  $\underbrace{(r, \dots, r)}_{l \text{ times}}$  as a sub partition, in which case  $\Delta$  is of the form  $[\Delta_1, \dots, \Delta_t]$ , with  $\Delta_i$  of length  $n_i$ , and  $r_{M,G}(\Delta)$  is equal to the tensor product  $\Delta_1 \otimes \dots \otimes \Delta_t$ .*

## 2 Double classes $P(K) \backslash G_n(K) / G_n(F)$

Let  $\bar{n}$  be a partition  $(n_1, \dots, n_t)$  of a positive integer  $n$ , we denote by  $P$  the standard parabolic subgroup  $P_{\bar{n}}(K)$  of  $G = G_n(K)$ . We denote by  $H$  the group  $G_n(F)$ .

We study in first place the double classes of  $H \backslash G / P$ . We identify the quotient space  $G/P$  with a flag manifold given by sequences (called  $\bar{n}$ -flags)  $0 \subset V_1 \subset \dots \subset V_{t-1} \subset V = K^n$ , where  $V_j$  is a vector subspace of  $V$ , of dimension  $n_1 + \dots + n_j$ . Studying the double classes of  $H \backslash G / P$ , is then equivalent to understand the  $H$ -orbits of the flag manifold  $G/P$ . This is done in the following theorem.

**Theorem 2.1.** *The  $H$ -orbits of the flag manifold  $G/P$ , are characterized by the integers  $\dim(V_i \cap V_j^\sigma)$ , for  $1 \leq i \leq j \leq t-1$ , which means that two  $\bar{n}$ -flags  $D = 0 \subset V_1 \subset \dots \subset V_{t-1} \subset V$  and  $D' = 0 \subset V'_1 \subset \dots \subset V'_{t-1} \subset V$  are in the same orbit under  $H$ , if and only if  $\dim(V_i \cap V_j^\sigma) = \dim(V'_i \cap V'^\sigma_j)$  for  $1 \leq i \leq j \leq t-1$ .*

*Proof.* We first state the following classical lemma.

**Lemma 2.1.** *Let  $V = K^n$ , and  $V_F = F^n \subset V$ , the  $F$ -subspace of vectors of  $V$  fixed by  $\sigma$ . A vector subspace  $V'$  of  $V$  verifies that  $V' = V'^\sigma$ , if and only if one can choose a basis of  $V'$  in  $V_F$ , in which case one says that  $V'$  is defined over  $F$ . Any subspace defined over  $F$ , has a supplementary subspace defined over  $F$ .*

Now we prove a second lemma about the filtration of  $V$  in terms of  $V_i \cap V_j^\sigma$  for  $V_i$  and  $V_j$  in the set of subspaces defining a  $\bar{n}$ -flag.

Let  $D$  be a  $\bar{n}$ -flag, given by the sequence  $D = 0 \subset V_1 \subset \dots \subset V_{t-1} \subset V$ . We set  $V_0 = 0$  and  $V_t = V$ .

For  $1 \leq i \leq j \leq t$ , we denote by  $S_{i,j}$  a supplementary space of  $V_i \cap V_{j-1}^\sigma + V_{i-1} \cap V_j^\sigma$  in  $V_i \cap V_j^\sigma$ . If  $i = j$ , we add the condition that the supplementary space  $S_{i,i}$  we choose is defined over  $F$ , which is possible according to Lemma 2.1.

Eventually, for  $1 \leq i \leq j \leq t$ , we denote by  $S_{j,i}$ , the space  $S_{i,j}^\sigma$ , which is a supplementary space of  $V_j \cap V_{i-1}^\sigma + V_{j-1} \cap V_i^\sigma$  in  $V_j \cap V_i^\sigma$ .

**Lemma 2.2.** *With these notations, if  $(i, j)$  belongs to  $\{1, \dots, t\}$ , the space  $V_{i-1} + V_i \cap V_j^\sigma$  is equal to the sum*

$$(S_{1,1} \oplus \dots \oplus S_{1,t}) \oplus \dots \oplus (S_{i-1,1} \oplus \dots \oplus S_{i-1,t}) \oplus (S_{i,1} \oplus \dots \oplus S_{i,j}).$$

In particular, the space  $V_i$  is equal to the direct sum

$$(S_{1,1} \oplus \dots \oplus S_{1,t}) \oplus \dots \oplus (S_{i,1} \oplus \dots \oplus S_{i,t}).$$

*Proof.* Let  $x$  belong  $V_{i-1} + V_i \cap V_j^\sigma$ , then one can write  $x = x_{i-1} + y_{i,j}$  for  $v_{i-1}$  in  $V_{i-1}$  and  $y_{i,j}$  in  $V_i \cap V_j^\sigma$ . But then  $y_{i,j} = y_{i-1,j} + y_{i,j-1} + s_{i,j}$  with  $y_{i-1,j}$  in  $V_{i-1} \cap V_j^\sigma$ ,  $y_{i,j-1}$  in  $V_{i-1} \cap V_{j-1}^\sigma$  and  $s_{i,j}$  in  $S_{i,j}$ . Hence we have  $x = x'_{i-1} + y_{i,j-1} + s_{i,j}$ , with  $x'_{i-1} = x_{i-1} + y_{i-1,j}$  belonging to  $V_{i-1}$ . So  $x$  belongs to  $(V_{i-1} + V_i \cap V_{j-1}^\sigma) + S_{i,j}$ . But by definition of  $S_{i,j}$ , the preceding sum is actually direct, i.e.  $x$  belongs  $(V_{i-1} + V_i \cap V_{j-1}^\sigma) \oplus S_{i,j}$ . We thus proved that  $V_{i-1} + V_i \cap V_j^\sigma = (V_{i-1} + V_i \cap V_{j-1}^\sigma) \oplus S_{i,j}$ , and the proof ends by induction.  $\square$

Getting back to the proof of Theorem 2.1, it is obvious that if two  $\bar{n}$ -flags  $D$  and  $D'$  are in the same  $H$ -orbit, then one must have  $\dim(V_i \cap V_j^\sigma) = \dim(V'_i \cap V_j'^\sigma)$  for  $1 \leq i \leq j \leq t$ . Conversely, suppose that two  $\bar{n}$ -flags  $D$  and  $D'$ , satisfy the condition  $\dim(V_i \cap V_j^\sigma) = \dim(V'_i \cap V_j'^\sigma)$  for  $1 \leq i \leq j \leq t$ .

The assumption  $\dim(V_i \cap V_j^\sigma) = \dim(V'_i \cap V_j'^\sigma)$  for  $1 \leq i \leq j \leq t$ , implies that for any couple  $(i, j) \in \{1, \dots, t\}^2$ ,  $S_{i,j}$  and  $S'_{i,j}$  have the same dimension. For  $1 \leq i < j \leq t$ , we choose a  $K$ -linear isomorphism  $h_{i,j}$  between  $S_{i,j}$  and  $S'_{i,j}$ . This defines an isomorphism  $h_{j,i}$  between  $S_{j,i}$  and  $S'_{j,i}$ , verifying  $h_{j,i}(v) = (h_{i,j}(v^\sigma))^\sigma$  for all  $v$  in  $S_{j,i}$ .

Eventually, for each  $l$  between 1 and  $t$ , as  $S_{l,l}$  and  $S'_{l,l}$  are defined over  $F$ , we choose an isomorphism  $h_{l,l}$  between  $S_{l,l}$  and  $S'_{l,l}$ , verifying that  $h_{l,l}(v^\sigma) = h_{l,l}(v)^\sigma$  for all  $v \in V_{l,l}$ .

As the space  $V$  is equal to the sum  $\bigoplus_{(k,l) \in \{1, \dots, t\}^2} S_{k,l}$ , and  $V'$  is equal to  $\bigoplus_{(k,l) \in \{1, \dots, t\}^2} S'_{k,l}$ , the  $K$ -linear isomorphism  $h = \bigoplus_{(k,l) \in \{1, \dots, t\}^2} h_{l,k}$  defines an element of  $H$ , sending  $D$  to  $D'$ , so that  $D$  and  $D'$  are in the same  $H$ -orbit.  $\square$

The proof of the previous theorem has as a consequence the following corollary.

**Corollary 2.1.** *The quotient  $H \backslash G / P$  is a finite set, and its cardinality is equal to the number of sequences of positive or null integers  $(n_{i,j})_{1 \leq i \leq j \leq t}$ , such that if we let  $n_{j,i}$  be equal to  $n_{j,i}$ , then for  $i$  between 1 and  $t$ , one has  $n_i = \sum_{j=1}^t n_{i,j}$ .*

**Definition 2.1.** *We call  $I(\bar{n})$  the set of sequences described in the preceding corollary.*

Now to such a sequence, we are going to associate an element of  $G$ , which will be a representative of the corresponding double coset of  $H \backslash G / P$ . This will thus achieve the description of the set  $H \backslash G / P$ .

First we recall that we denote by  $V$  the space  $K^n$ , and that  $P$  corresponds to a partition  $\bar{n} = (n_1, \dots, n_t)$  of  $n$ . We denote by  $(e_1, \dots, e_n)$  the canonical basis of  $V$ , and by  $D^0$  the canonical  $\bar{n}$ -flag defined over  $F$ , given by  $0 \subset V_1^0 \subset V_2^0 \subset V_{t-1}^0 \subset V$ , with  $V_i^0 = \text{Vect}(e_1, \dots, e_{(n_1 + \dots + n_i)})$ , corresponding to the sequence  $n_{i,j} = 0$  if  $i < j$  and  $n_{i,i} = n_i$ .

**Proposition 2.1.** *(Representatives for  $H \backslash G / P$ ) Let  $(n_{i,j})_{1 \leq i \leq j \leq t}$  be an element of  $I(\bar{n})$ . We denote by  $V_{i,j}^0$  the space*

$$\text{Vect}(e_{(n_1 + \dots + n_{i-1} + n_{i,1} + \dots + n_{i,j-1} + 1)}, \dots, e_{(n_1 + \dots + n_{i-1} + n_{i,1} + \dots + n_{i,j-1} + n_{i,j})}),$$

and we denote by  $B_{i,j}^0$  its canonical basis

$$\{e_{(n_1 + \dots + n_{i-1} + n_{i,1} + \dots + n_{i,j-1} + 1)}, \dots, e_{(n_1 + \dots + n_{i-1} + n_{i,1} + \dots + n_{i,j-1} + n_{i,j})}\}.$$

Hence one has

$$V = (V_{1,1}^0 \oplus \dots \oplus V_{1,t}^0) \oplus (V_{2,1}^0 \oplus \dots \oplus V_{2,t}^0) \oplus \dots \oplus (V_{t,1}^0 \oplus \dots \oplus V_{t,t}^0).$$



We denote by  $u'$  the element of  $G$  sending  $V_{i,i}^0$  onto itself, and  $V_{i,j}^0 \oplus V_{j,i}^0$  onto itself for  $i < j$ , whose restriction to  $V_{i,i}^0$  has matrix  $I_{n_{i,i}}$  in the basis  $B_{i,i}^0$ , and whose restriction to  $V_{\{i,j\}}^0 = V_{i,j}^0 \oplus V_{j,i}^0$  has matrix

$$\begin{pmatrix} \frac{1}{2}I_{n_{i,j}} & \frac{1}{2}I_{n_{i,j}} \\ -\frac{1}{2\delta}I_{n_{i,j}} & \frac{1}{2\delta}I_{n_{i,j}} \end{pmatrix}$$

in the basis  $B_{\{i,j\}}^0 = B_{i,j}^0 \cup B_{j,i}^0$ .

The element  $u'$  is a representative of the double coset of  $H \backslash G / P$  associated with  $(n_{i,j})_{1 \leq i \leq j \leq t}$  in  $I(\bar{n})$ .

*Proof.* If  $B_1 = (v_i)$  and  $B_2 = (w_i)$  are two families of vectors of same finite cardinality in  $V$ , we denote by  $\lambda B_1 + \mu B_2$  the family  $(\lambda v_i + \mu w_i)$  for  $\lambda$  and  $\mu$  in  $K$ . With these notations, the element  $u'$  sends  $V_{i,j}^0$  onto  $S_{i,j} = \text{Vect}(B_{i,j}^0 - \frac{1}{2\delta}B_{j,i}^0)$ , and  $V_{j,i}^0$  onto  $S_{j,i} = S_{i,j}^\sigma$ . Denoting  $V_{i,i}^0$  by  $S_{i,i}$  and one verifies from our choices that for  $1 \leq i \leq t$ , one has  $V_i = u(V_i^0) = (S_{1,1} \oplus \dots \oplus S_{1,t}) \oplus \dots \oplus (S_{i,1} \oplus \dots \oplus S_{i,t})$ , and that  $S_{i,j}$  is a supplementary space of  $V_i \cap V_{j-1}^\sigma + V_{i-1} \cap V_j^\sigma$  in  $V_i \cap V_j^\sigma$ . Hence the  $\bar{n}$ -flag  $D$  corresponds to the sequence  $(n_{i,j})_{1 \leq i \leq j \leq t}$  of  $I(\bar{n})$ .  $\square$

A reformulation of what precedes is the following.

**Proposition 2.2.** (*Representatives for  $P \backslash G / H$* ) A set of representatives of  $P \backslash G / H$  is given by the elements  $u = u'^{-1}$ , where the  $u'$  are as in Proposition 2.1. The representative of the class associated with the sequence  $(n_{i,j})_{1 \leq i \leq j \leq t}$  in  $I(\bar{n})$ , restricts to  $V_{i,i}^0$  with matrix  $I_{n_{i,i}}$  in the basis  $B_{i,i}^0$ , and to  $V_{\{i,j\}}^0$  with matrix

$$\begin{pmatrix} I_{n_{i,j}} & -\delta I_{n_{i,j}} \\ I_{n_{i,j}} & \delta I_{n_{i,j}} \end{pmatrix}$$

in the basis  $B_{\{i,j\}}^0$ .

**Definition 2.2.** We denote by  $R(P \backslash G / H)$  the set of representatives described in Proposition 2.2.

### 3 Structure of the group $P(K) \cap uG_n(F)u^{-1}$

Let  $u$  be an element of  $R(P \backslash G / P)$ , corresponding to a sequence  $(n_{i,j})_{1 \leq i \leq j \leq t}$  in  $I(\bar{n})$ . We want to analyze the structure of the group  $P \cap uHu^{-1}$ , which is the same set as  $P \cap uM_n(F)u^{-1}$ . First we give a bloc decomposition of the group  $uM_n(F)u^{-1}$ . We denote by  $M_P$  the standard Levi subgroup of  $P$ .

A matrix  $M = M_{B^0}(a)$  in  $M_n(K)$  can be viewed as a “disjoint union” of blocks  $\tilde{M}_{\{k,l\},\{i,j\}} = M_{B_{\{i,j\}}^0, B_{\{k,l\}}^0}(a_{\{k,l\},\{i,j\}})$ , for  $1 \leq k \leq l \leq t$  and  $1 \leq i \leq j \leq t$ , where the morphism  $a_{\{k,l\},\{i,j\}}$  is the composition of the projection on  $V_{\{k,l\}}^0$  (where  $V_{\{t,t\}}^0$  designates  $V_{t,t}^0$ ), with respect to the other spaces  $V_{\{k',l'\}}^0$  (for  $1 \leq k' \leq l' \leq t$ , and  $(k', l') \neq (k, l)$ ), and of the restriction of  $a$  to  $V_{\{i,j\}}^0$ .

More visually, the four following cases occur.

- If  $k < l$  and  $i < j$ , then one has  $M =$ 

$$n_{k,l} \left\{ \begin{array}{cccccc} & & \overbrace{\quad}^{n_{i,j}} & & \overbrace{\quad}^{n_{j,i}} & \\ * & * & * & * & * & * \\ * & * & [M_1] & * & * & [M_2] \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & [M_3] & * & * & [M_4] \\ * & * & * & * & * & * \\ * & * & * & * & * & * \end{array} \right\}, \text{ and}$$

$\tilde{M}_{\{k,l\},\{i,j\}}$  is the block  $\begin{bmatrix} M_1 & M_2 \\ M_3 & M_4 \end{bmatrix}$ , which we identify with the matrix  $M_{\{k,l\},\{i,j\}} =$

$$\begin{matrix}
& & \overbrace{\hspace{1.5cm}}^{n_{i,j}} & & \overbrace{\hspace{1.5cm}}^{n_{j,i}} & & \\
& \begin{matrix} n_{k,l} \{ \\ \\ \\ \\ \\ \end{matrix} & \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & [M_1] & 0 & 0 & [M_2] \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} & & & & \\
& \begin{matrix} n_{l,k} \{ \\ \\ \\ \\ \\ \end{matrix} & \begin{bmatrix} 0 & 0 & [M_3] & 0 & 0 & [M_4] \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} & & & & 
\end{matrix}$$

• If  $k = l$  and  $i < j$ , one has  $M = n_{k,k} \left\{ \begin{matrix} * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ * & * & [M_1] & * & * & [M_2] & * \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \end{matrix} \right\}$ , and  $\tilde{M}_{\{k,k\},\{i,j\}}$

is the block  $[M_1 \quad M_2]$ , which we identify with the matrix  $M_{\{k,k\},\{i,j\}} =$

$$n_{k,k} \left\{ \begin{matrix} & & \overbrace{\hspace{1.5cm}}^{n_{i,j}} & & \overbrace{\hspace{1.5cm}}^{n_{j,i}} & & \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & [M_1] & 0 & 0 & [M_2] \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} & & & & & & \end{matrix} \right\},$$

• If  $k < l$  and  $i = j$ , one has  $M = n_{k,l} \left\{ \begin{matrix} * & * & * & * & * & * & * \\ * & * & * & * & [M_1] & * & * \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ * & * & * & * & [M_2] & * & * \\ * & * & * & * & * & * & * \end{matrix} \right\}$ , and  $\tilde{M}_{\{k,l\},\{i,i\}}$  is the

block  $\begin{bmatrix} M_1 \\ M_2 \end{bmatrix}$ , which we identify with the matrix  $M_{\{k,l\},\{i,i\}} =$

$$\begin{matrix}
& & & & \overbrace{\hspace{1.5cm}}^{n_{i,i}} & & \\
& \begin{matrix} n_{k,l} \{ \\ \\ \\ \\ \\ \end{matrix} & \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & [M_1] & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} & & & & \\
& \begin{matrix} n_{l,k} \{ \\ \\ \\ \\ \\ \end{matrix} & \begin{bmatrix} 0 & 0 & 0 & 0 & [M_2] & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} & & & & 
\end{matrix}$$

• If  $k = l$  and  $i = j$ , one has  $M = n_{k,k} \left\{ \begin{array}{ccccccc} & & & & \overbrace{\phantom{M}}^{n_{i,i}} & & \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ * & * & * & * & [M] & * & * \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \end{array} \right\}$ , and  $\tilde{M}_{\{k,k\},\{i,i\}}$  is the

$$\text{block } [M], \text{ which we identify with the matrix } M_{\{k,k\},\{i,i\}} = n_{k,k} \left\{ \begin{array}{ccccccc} & & & & \overbrace{\phantom{M}}^{n_{i,i}} & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & [M] & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right\}.$$

With these notations, a matrix  $M$  is the “disjoint union” of the blocks  $\tilde{M}_{\{k,l\},\{i,j\}}$ , or put in another way, the direct sum of the matrices  $M_{\{k,l\},\{i,j\}}$ .

Denoting by  $\tilde{u}_{\{i,j\}}$  the matrix  $\begin{pmatrix} I_{n_{i,j}} & -\delta I_{n_{i,j}} \\ I_{n_{i,j}} & \delta I_{n_{i,j}} \end{pmatrix}$  if  $i < j$ , and  $I_{n_{i,i}}$  otherwise, the matrix  $uMu^{-1}$  becomes the “disjoint union” of the blocks  $\tilde{u}_{\{k,l\}} \tilde{M}_{\{k,l\},\{i,j\}} \tilde{u}_{\{i,j\}}^{-1}$ .

If we denote by  $u_{\{i,j\}}$  the matrix 
$$\left[ \begin{array}{ccccccc} I_{n_{1,1}} & & & & & & \\ & \ddots & & & & & \\ & & I_{n_{i,j}} & & -\delta I_{n_{i,j}} & & \\ & & & \ddots & & & \\ & & I_{n_{i,j}} & & \delta I_{n_{i,j}} & & \\ & & & & & \ddots & \\ & & & & & & I_{n_{t,t}} \end{array} \right]$$
 of  $M_n(K)$  if

$i < j$ , and the identity matrix of  $M_n(K)$  otherwise, so that the matrix  $u$  is the commutative product of the matrices  $u_{\{i,j\}}$  for  $1 \leq i < j \leq t$ , another way to say this is that  $uMu^{-1}$  is the direct sum of the matrices  $u_{\{k,l\}} M_{\{k,l\},\{i,j\}} u_{\{i,j\}}^{-1}$ .

**Proposition 3.1.** *A matrix  $M$  belongs to  $M_n(F)$ , if and only if the blocks  $\tilde{u}_{\{k,l\}} \tilde{M}_{\{k,l\},\{i,j\}} \tilde{u}_{\{i,j\}}^{-1}$  are of the form  $\begin{bmatrix} M_1 & M_2^\sigma \\ M_2 & M_1^\sigma \end{bmatrix}$  with  $M_1$  and  $M_2$  in  $M_{n_{k,l},n_{i,j}}(K)$  for  $i < j$  and  $k < l$ , the blocks  $\tilde{u}_{\{k,k\}} \tilde{M}_{\{k,k\},\{i,j\}} \tilde{u}_{\{i,j\}}^{-1}$  are of the form  $\begin{bmatrix} M & M^\sigma \end{bmatrix}$  with  $M$  in  $M_{n_{k,k},n_{i,j}}(K)$  for  $i < j$ , the blocks  $\tilde{u}_{\{k,l\}} \tilde{M}_{\{k,l\},\{i,i\}} \tilde{u}_{\{i,i\}}^{-1}$  are of the form  $\begin{bmatrix} M' \\ M'^\sigma \end{bmatrix}$  with  $M'$  in  $M_{n_{k,l},n_{i,i}}(K)$  for  $k < l$ , and the blocks  $\tilde{u}_{\{k,k\}} \tilde{M}_{\{k,k\},\{i,i\}} \tilde{u}_{\{i,i\}}^{-1}$  have coefficients in  $F$ .*

This completes the description of  $uM_n(F)u^{-1}$ .

Now let's see when a matrix of  $uM_n(F)u^{-1}$  is in  $P$ . To do this, let  $M$  be in  $uM_n(F)u^{-1}$ , it is more convenient to look at the blocks  $\tilde{M}_{\{k,l\},\{i,j\}}$ , which are those corresponding to the matrices  $M_{\{k,l\},\{i,j\}} = M_{\{k,l\},\{k,l\}} + M_{\{k,l\},\{i,j\}} + M_{\{k,l\},\{i,j\}} + M_{\{k,l\},\{i,j\}}$  when  $(k,l) \neq (i,j)$ . The matrix  $M$  is thus the “union” (but not disjoint anymore) of those blocks.

The configurations are the following:

**A)**  $k < l \leq i < j$ : the block  $\tilde{M}_{\{k,l,i,j\}}$  has the form

$$\begin{matrix} & \overbrace{\hspace{1.5cm}}^{n_{k,l}} & \overbrace{\hspace{1.5cm}}^{n_{l,k}} & \overbrace{\hspace{1.5cm}}^{n_{j,i}} & \overbrace{\hspace{1.5cm}}^{n_{i,j}} \\ \begin{matrix} n_{k,l} \{ \\ n_{l,k} \{ \\ n_{i,j} \{ \\ n_{j,i} \{ \end{matrix} & \left[ \begin{array}{cc|cc} M_1 & M_2^\sigma & M_3 & M_4^\sigma \\ M_2 & M_1^\sigma & M_4 & M_3^\sigma \\ M_5 & M_6^\sigma & M_7 & M_8^\sigma \\ M_6 & M_5^\sigma & M_8 & M_7^\sigma \end{array} \right] \end{matrix}, \text{ with}$$

the  $M_i$ 's with coefficients in  $K$ .

In this case, remembering that two elementary blocks (i.e of the form  $M_i$  or  $M_i^\sigma$ ) can occur in the same Levi block of  $P$ , if and only if  $l = i$ , in which case the elementary blocks in the same block of  $M_P$  are  $M_1^\sigma, M_4, M_6^\sigma$  and  $M_7$ .

From this we deduce that in any case, if  $M$  is in  $P$ , the block  $\tilde{M}_{\{k,l,i,j\}}$  is of the form

$$\begin{matrix} & \overbrace{\hspace{1.5cm}}^{n_{k,l}} & \overbrace{\hspace{1.5cm}}^{n_{l,k}} & \overbrace{\hspace{1.5cm}}^{n_{i,j}} & \overbrace{\hspace{1.5cm}}^{n_{j,i}} \\ \begin{matrix} n_{k,l} \{ \\ n_{l,k} \{ \\ n_{i,j} \{ \\ n_{j,i} \{ \end{matrix} & \left[ \begin{array}{cc|cc} A_{k,l} & 0 & M & M'^\sigma \\ 0 & A_{l,k} & M' & M^\sigma \\ 0 & 0 & A_{i,j} & 0 \\ 0 & 0 & 0 & A_{i,j} \end{array} \right] \end{matrix}$$

with  $A_{l,k} = A_{k,l}^\sigma$  and  $A_{j,i} = A_{i,j}^\sigma$  being invertible.

**B)**  $k \leq i < l \leq j$ : the block  $\tilde{M}_{\{k,l,i,j\}}$  has the form

$$\begin{matrix} & \overbrace{\hspace{1.5cm}}^{n_{k,l}} & \overbrace{\hspace{1.5cm}}^{n_{i,j}} & \overbrace{\hspace{1.5cm}}^{n_{l,k}} & \overbrace{\hspace{1.5cm}}^{n_{j,i}} \\ \begin{matrix} n_{k,l} \{ \\ n_{i,j} \{ \\ n_{l,k} \{ \\ n_{j,i} \{ \end{matrix} & \left[ \begin{array}{cc|cc} M_1 & M_3 & M_2^\sigma & M_4^\sigma \\ M_5 & M_7 & M_6^\sigma & M_8^\sigma \\ M_2 & M_4 & M_1^\sigma & M_3^\sigma \\ M_6 & M_8 & M_5^\sigma & M_7^\sigma \end{array} \right] \end{matrix}.$$

In this case, two elementary blocks can occur in the same Levi block of  $P$ , if and only if  $k = i$  or  $l = j$ . In the first case, the elementary blocks in the same block of  $M_P$  are  $M_1, M_3, M_5$  and  $M_7$ . In the second case the elementary blocks in the same block of  $M_P$  are  $M_1^\sigma, M_3^\sigma, M_5^\sigma$ , and  $M_7^\sigma$ .

Hence in any case, if  $M$  is in  $P$ , then  $\tilde{M}_{\{k,l,i,j\}}$  is of the form

$$\begin{matrix} & \overbrace{\hspace{1.5cm}}^{n_{k,l}} & \overbrace{\hspace{1.5cm}}^{n_{i,j}} & \overbrace{\hspace{1.5cm}}^{n_{l,k}} & \overbrace{\hspace{1.5cm}}^{n_{i,j}} \\ \begin{matrix} n_{k,l} \{ \\ n_{i,j} \{ \\ n_{l,k} \{ \\ n_{j,i} \{ \end{matrix} & \left[ \begin{array}{cc|cc} A_{k,l} & M & 0 & 0 \\ 0 & A_{i,j} & 0 & 0 \\ 0 & 0 & A_{l,k} & M^\sigma \\ 0 & 0 & 0 & A_{j,i} \end{array} \right] \end{matrix}$$

with  $A_{l,k} = A_{k,l}^\sigma$  and  $A_{j,i} = A_{i,j}^\sigma$  being invertible.

**C)**  $k < i < j < l$ : the block  $\tilde{M}_{\{k,l,i,j\}}$  has the form

$$\begin{matrix} & \overbrace{\hspace{1.5cm}}^{n_{k,l}} & \overbrace{\hspace{1.5cm}}^{n_{i,j}} & \overbrace{\hspace{1.5cm}}^{n_{j,i}} & \overbrace{\hspace{1.5cm}}^{n_{k,l}} \\ \begin{matrix} n_{k,l} \{ \\ n_{i,j} \{ \\ n_{j,i} \{ \\ n_{k,l} \{ \end{matrix} & \left[ \begin{array}{cc|cc} M_1 & N_1 & N_2 & M_2^\sigma \\ N_1' & M_1' & M_2^\sigma & N_1'^\sigma \\ N_2' & M_2' & M_1'^\sigma & N_2'^\sigma \\ M_2 & N_1^\sigma & N_2 & M_1^\sigma \end{array} \right] \end{matrix}.$$

In this case, two elementary blocks can never occur in the same Levi block of  $P$

If  $M$  is in  $P$ , the block  $\tilde{M}_{\{k,l,i,j\}}$  is of the form

$$\begin{matrix} & \overbrace{\hspace{1.5cm}}^{n_{k,l}} & \overbrace{\hspace{1.5cm}}^{n_{i,j}} & \overbrace{\hspace{1.5cm}}^{n_{j,i}} & \overbrace{\hspace{1.5cm}}^{n_{l,k}} \\ \begin{matrix} n_{k,l} \{ \\ n_{i,j} \{ \\ n_{j,i} \{ \\ n_{l,k} \{ \end{matrix} & \left[ \begin{array}{cc|cc} A_{k,l} & 0 & 0 & 0 \\ 0 & A_{i,j} & 0 & 0 \\ 0 & 0 & A_{j,i} & 0 \\ 0 & 0 & 0 & A_{l,k} \end{array} \right] \end{matrix} \text{ with}$$

$A_{l,k} = A_{k,l}^\sigma$  and  $A_{j,i} = A_{i,j}^\sigma$  being invertible.

**D)**  $k = l \leq i < j$ : the block  $\tilde{M}_{\{k,k,i,j\}}$  has the form  $n_{k,k} \left\{ \begin{array}{c} \overbrace{H}^{n_{k,k}} \\ N \\ N^\sigma \end{array} \right\} \left[ \begin{array}{ccc} \overbrace{M}^{n_{i,j}} & \overbrace{M^\sigma}^{n_{j,i}} \\ A & B^\sigma \\ B & A^\sigma \end{array} \right]$  with  $H$  having its coefficients in  $F$ .

In this case, two elementary blocks can occur in the same Levi block of  $P$ , if and only if  $k = i$ , in which case the elementary blocks in the same block of  $M_P$  are  $H$ ,  $M$ ,  $N$  and  $A$ .

In any case, if  $M$  is in  $P$ , then  $\tilde{M}_{\{k,k,i,j\}}$  is of the form  $n_{k,k} \left\{ \begin{array}{c} \overbrace{A_{k,k}}^{n_{k,k}} \\ 0 \\ 0 \end{array} \right\} \left[ \begin{array}{ccc} \overbrace{M}^{n_{i,j}} & \overbrace{M^\sigma}^{n_{j,i}} \\ A_{i,j} & 0 \\ 0 & A_{i,j}^\sigma \end{array} \right]$  with  $A_{j,i} = A_{i,j}^\sigma$  invertible, and  $A_{k,k}$  invertible with coefficients in  $F$ .

**E)**  $i < k = l < j$ : the block  $\tilde{M}_{\{k,k,i,j\}}$  has the form  $n_{i,j} \left\{ \begin{array}{c} \overbrace{A}^{n_{i,j}} \\ N \\ B \end{array} \right\} \left[ \begin{array}{ccc} \overbrace{M}^{n_{k,k}} & \overbrace{B^\sigma}^{n_{j,i}} \\ H & N^\sigma \\ M^\sigma & A^\sigma \end{array} \right]$  with  $H$  having its coefficients in  $F$ .

In this case, two elementary blocks cannot occur in the same Levi block of  $P$ . If  $M$  is in  $P$ ,

then  $\tilde{M}_{\{k,k,i,j\}}$  is of the form  $n_{i,j} \left\{ \begin{array}{c} \overbrace{A_{i,j}}^{n_{i,j}} \\ 0 \\ 0 \end{array} \right\} \left[ \begin{array}{ccc} \overbrace{0}^{n_{k,k}} & \overbrace{0}^{n_{j,i}} \\ A_{k,k} & 0 \\ 0 & A_{j,i} \end{array} \right]$  with  $A_{k,k} = A_{k,k}^\sigma$  and  $A_{j,i} = A_{i,j}^\sigma$  invertible.

**F)**  $k < l \leq i = j$ : the block  $\tilde{M}_{\{k,l,i,i\}}$  has the form  $n_{k,l} \left\{ \begin{array}{c} \overbrace{A}^{n_{k,l}} \\ B \\ N \end{array} \right\} \left[ \begin{array}{ccc} \overbrace{B^\sigma}^{n_{l,k}} & \overbrace{M}^{n_{i,i}} \\ A^\sigma & M^\sigma \\ N^\sigma & H \end{array} \right]$  with  $H$  having its coefficients in  $F$ .

In this case, two elementary blocks can occur in the same Levi block of  $P$ , if and only if  $l = i$ , in which case the elementary blocks in the same block of  $M_P$  are  $A^\sigma$ ,  $M^\sigma$ ,  $N^\sigma$  and  $H$ .

In any case, if  $M$  is in  $P$ , then  $\tilde{M}_{\{k,l,i,i\}}$  is of the form  $n_{k,l} \left\{ \begin{array}{c} \overbrace{A_{l,k}}^{n_{k,l}} \\ 0 \\ 0 \end{array} \right\} \left[ \begin{array}{ccc} \overbrace{0}^{n_{l,k}} & \overbrace{M}^{n_{i,i}} \\ A_{l,k}^\sigma & M^\sigma \\ 0 & A_{i,i} \end{array} \right]$  with  $A_{k,l} = A_{l,k}^\sigma$  invertible, and  $A_{i,i}$  invertible with coefficients in  $F$ .

**G)**  $k = l < i = j$ : the block  $\tilde{M}_{\{k,k,i,i\}}$  has the form  $n_{k,k} \left\{ \begin{array}{c} \overbrace{H_1}^{n_{k,k}} \\ H_3 \end{array} \right\} \left[ \begin{array}{cc} \overbrace{H_2}^{n_{i,i}} \\ H_4 \end{array} \right]$  with the  $H_i$ 's having its coefficients in  $F$ , and if  $M$  is in  $P$ , as two elementary blocks can not occur in the same Levi block of  $P$ , the matrix  $\tilde{M}_{\{k,k,i,i\}}$  has the form  $n_{k,k} \left\{ \begin{array}{c} \overbrace{A_{k,k}}^{n_{k,k}} \\ 0 \end{array} \right\} \left[ \begin{array}{cc} \overbrace{H}^{n_{i,i}} \\ A_{i,i} \end{array} \right]$  with all coeffi-









As the action of such matrices on the space of vectors of the form  $\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$  with  $V_i$  in  $M_{t,1}(F)$ , is given by  $|\cdot|_F$ , the module of the action of  $M = \begin{bmatrix} A & 0 \\ 0 & A^\sigma \end{bmatrix}$  on the space of vectors of the form  $\begin{bmatrix} V \\ V^\sigma \end{bmatrix}$  with  $V$  in  $M_{t,1}(K)$ , is equal to  $|M|_F = |A|_K$ , as  $|\cdot|_F \circ N_{K/F}$  equals  $|\cdot|_K$ .

We then recall that from Proposition 3.2, the group  $P \cap uHu^{-1}$ , is the semi direct product of the group  $M_s(F)$ , and the group  $N_u \cap uHu^{-1} = \left\{ I_n \oplus_{k \leq l, i \leq j} N_{\{k,l\},\{i,j\}} \right\}$ . Hence  $\delta_{P \cap uHu^{-1}}(A)$  which is the modulus of the automorphism  $\text{int}_A$  of the group  $N_u \cap uHu^{-1}$ , is the product of the modulus  $\delta_{\{k,l\},\{i,j\}}(A)$  of the automorphism  $\text{int}_A$  of the vector spaces  $N_{\{k,l\},\{i,j\}}$ . But  $|A_{i_0, j_0}|_K$  contributes with non zero multiplicity only in the modulus  $\delta_{\{k,l\},\{i_0, j_0\}}(A)$  or  $\delta_{\{i_0, j_0\},\{k,l\}}(A)$  with  $k \leq l$ . We are going to count these multiplicities.

- A) If  $k < l \leq i_0 < j_0$ : according to configuration  $A$  of the Section 3,  $|A_{i_0, j_0}|_K$  occurs with multiplicity  $-2n_{k,l}$  in  $\delta_{\{k,l\},\{i_0, j_0\}}(A)$ , and  $N_{\{i_0, j_0\},\{k,l\}}$  is null from Proposition 3.2.
- A') If  $i_0 < j_0 \leq k < l$ : according to configuration  $A$  of the Section 3,  $|A_{i_0, j_0}|_K$  occurs with multiplicity  $2n_{k,l}$  in  $\delta_{\{i_0, j_0\},\{k,l\}}(A)$ , and  $N_{\{k,l\},\{i_0, j_0\}}$  is null from Proposition 3.2.
- B) If  $k \leq i_0 < l \leq j_0$ : according to configuration  $B$  of the Section 3,  $|A_{i_0, j_0}|_K$  occurs with multiplicity  $-n_{k,l}$  in  $\delta_{\{k,l\},\{i_0, j_0\}}(A)$ , and  $N_{\{i_0, j_0\},\{k,l\}}$  is null from Proposition 3.2.
- B') If  $i_0 \leq k < j_0 \leq l$ : according to configuration  $B$  of the Section 3,  $|A_{i_0, j_0}|_K$  occurs with multiplicity  $n_{k,l}$  in  $\delta_{\{i_0, j_0\},\{k,l\}}(A)$ , and  $N_{\{k,l\},\{i_0, j_0\}}$  is null from Proposition 3.2.
- C) If  $k < i_0 < j_0 < l$ : both spaces  $N_{\{k,l\},\{i_0, j_0\}}$  and  $N_{\{i_0, j_0\},\{k,l\}}$  are null from Proposition 3.2.
- C') If  $i_0 < k < l < j_0$ : both spaces  $N_{\{k,l\},\{i_0, j_0\}}$  and  $N_{\{i_0, j_0\},\{k,l\}}$  are null from Proposition 3.2.
- D) If  $k = l \leq i_0 < j_0$ : according to configuration  $D$  of the Section 3,  $|A_{i_0, j_0}|_K$  occurs with multiplicity  $-n_{k,l}$  in  $\delta_{\{k,l\},\{i_0, j_0\}}(A)$ , and  $N_{\{i_0, j_0\},\{k,l\}}$  is null from Proposition 3.2.
- D') If  $i_0 < j_0 \leq k = l$ : according to configuration  $D$  of the Section 3,  $|A_{i_0, j_0}|_K$  occurs with multiplicity  $n_{k,l}$  in  $\delta_{\{i_0, j_0\},\{k,l\}}(A)$ , and  $N_{\{k,l\},\{i_0, j_0\}}$  is null from Proposition 3.2.
- E) If  $i_0 < k = l < j_0$ : both spaces  $N_{\{k,l\},\{i_0, j_0\}}$  and  $N_{\{i_0, j_0\},\{k,l\}}$  are null from Proposition 3.2.

**Eventually, by comparison, the positive real  $|A_{i_0, j_0}|_K$  occurs with same multiplicity in  $\delta_P \delta_{P'_s}(\mathbf{A})$  and in  $\delta_{P \cap uHu^{-1}}^2(\mathbf{A})$ , when  $i_0 < j_0$ .**

**Now we treat the case  $i_0 = j_0$  :**

The occurrence of  $|A_{i_0, i_0}|_K$  in  $\delta_P(A) \delta_{P'_s}(A)$  is equal to the sum  $S = S_1 + S_2 + S_3 + S_4$ , with  $S_1 = -n_1 - \dots - n_{i_0-1}$ ,  $S_2 = -n_{i_0,1} - \dots - n_{i_0, i_0-1}$ ,  $S_3 = n_{i_0, i_0+1} + \dots + n_{i_0, t}$ , and  $S_4 = n_{i_0+1} + \dots + n_t$ . For  $k \leq l$ , we count the number of occurrence of  $n_{l,k}$  in the sum  $S$ . We suppose  $(k, l) \neq (i_0, i_0)$  because  $n_{i_0, i_0}$  doesn't occur in  $S$ .

- D) If  $i_0 \leq k < l$ : then  $n_{k,l}$  occurs with multiplicity 2 in  $S_4$  if  $i_0 < k$ , and multiplicity 1 in  $S_3$  and in  $S_4$  if  $i_0 = k$ . In both cases it occurs with multiplicity  $-2$  in  $S$ .
- E) If  $k < i_0 < l$ : then  $n_{k,l}$  occurs with multiplicity  $-1$  in  $S_1$ , and multiplicity 1 in  $S_4$ , hence with multiplicity 0 in  $S$ .
- F) If  $k < l \leq i_0$ : then  $n_{k,l}$  occurs with multiplicity  $-2$  in  $S_1$  if  $l < i_0$ , and multiplicity  $-1$  in  $S_1$  and in  $S_2$  if  $l = i_0$ . In both cases it occurs with multiplicity  $-2$  in  $S$ .

**G)** If  $k = l < i_0$ : then  $n_{k,k}$  occurs with multiplicity  $-1$  in  $S_1$ , and in  $S$ .

**G')** If  $i_0 < k = l$ : then  $n_{k,k}$  occurs with multiplicity  $1$  in  $S_4$ , and in  $S$ .

Now, as previously, we are going to compute the multiplicity of  $|A_{i_0,i_0}|_K$  in the modulus  $\delta_{\{k,l\},\{i_0,i_0\}}(A)$  or  $\delta_{\{i_0,i_0\},\{k,l\}}(A)$  with  $k \leq l$ . We are going to count these multiplicities.

**D)** If  $i_0 \leq k < l$ : according to configuration  $D$  of the Section 3,  $|A_{i_0,i_0}|_K$  occurs with multiplicity  $n_{k,l}$  in  $\delta_{\{i_0,i_0\},\{k,l\}}(A)$ , and  $N_{\{k,l\},\{i_0,i_0\}}$  is null from Proposition 3.2.

**E)** If  $k < i_0 < l$ : both spaces  $N_{\{k,l\},\{i_0,i_0\}}$  and  $N_{\{i_0,i_0\},\{k,l\}}$  are null from Proposition 3.2.

**F)** If  $k < l \leq i_0$ : according to configuration  $F$  of the Section 3,  $|A_{i_0,i_0}|_K$  occurs with multiplicity  $-n_{k,l}$  in  $\delta_{\{k,l\},\{i_0,i_0\}}(A)$ , and  $N_{\{i_0,i_0\},\{k,l\}}$  is null from Proposition 3.2.

**G)** If  $k = l < i_0$ : according to configuration  $G$  of the Section 3,  $|A_{i_0,i_0}|_F$  occurs with multiplicity  $-n_{k,k}$  in  $\delta_{\{k,l\},\{i_0,i_0\}}(A)$ , and  $N_{\{i_0,i_0\},\{k,l\}}$  is null from Proposition 3.2.

**G')** If  $i_0 < k = l$ : according to configuration  $D$  of the Section 3,  $|A_{i_0,i_0}|_F$  occurs with multiplicity  $n_{k,k}$  in  $\delta_{\{i_0,i_0\},\{k,k\}}(A)$ , and  $N_{\{k,k\},\{i_0,i_0\}}$  is null from Proposition 3.2.

**Reminding that**  $|\frac{\cdot}{F}|^2$  equals  $|\cdot|_K$  on  $F^*$ , **by comparison again, the positive real  $|A_{i_0,i_0}|_K$  occurs with same multiplicity in  $\delta_{\mathbf{P}}\delta_{\mathbf{P}'_s}(\mathbf{A})$  and in  $\delta_{\mathbf{P} \cap \mathbf{uHu}^{-1}}^2(\mathbf{A})$ .**

This concludes the proof. □

## 5 Distinguished generic representations and Asai $L$ -functions

We recall that an irreducible representation  $\pi$  of  $G_n(K)$  is called generic if there is a non trivial character  $\psi$  of  $(K, +)$ , such that the space of linear forms  $\lambda$  on  $V$ , which verify  $\lambda(\pi(n)v) = \psi(n)v$  for  $n$  in  $N_n(K)$  and  $v$  in  $V$ , is of dimension 1.

If  $\pi$  is generic, the previous invariance property holds for any non trivial character  $\psi$  of  $K$ . Hence a generic representation is isomorphic, up to unique (modulo scalars) isomorphism to a submodule of  $\text{Ind}_{N_n(K)}^{G_n(K)}(\psi)$ . We denote  $W(\pi, \psi)$  this model of  $\pi$  on which  $G_n(K)$  acts by right translation, and call it the Whittaker model of  $\pi$ .

In [F4], the Asai  $L$ -function  $L_F^K(\pi)$  of a generic representation  $\pi$  is defined “à la Rankin-Selberg” as the gcd of a family of integrals of functions in  $W(\pi, \psi)$  depending on a complex parameter  $s$ , for  $\psi$  trivial on  $F$ . We refer to Sections 3 and 4 of [M3] for a survey of the main properties the Rankin-Selberg type Asai  $L$ -function of a generic representation.

The following theorem due to Zelevinsky (Th. 9.7 of [Z]), classifies the generic representations of the group  $G_n(K)$  in terms of quasi-square-integrable ones:

**Theorem 5.1.** *Let  $\bar{n} = (n_1, \dots, n_t)$  be a partition of  $n$ , and let  $\Delta_i$  be a quasi-square-integrable of  $G_{n_i}(K)$  for  $i$  between 1 and  $t$ , the representation  $\pi = \Delta_1 \times \dots \times \Delta_t$  of the group  $G_n(K)$  is irreducible if and only if no  $\Delta_i$ 's are linked, in which case  $\pi$  is generic. If  $(m_1, \dots, m_{t'})$  is another partition of  $n$ , and if the  $\Delta'_j$ 's are unlinked segments of length  $m_j$  for  $j$  between 1 and  $t'$ , then the representation  $\pi$  equals  $\Delta'_1 \times \dots \times \Delta'_{t'}$  if and only if  $t = t'$ , and  $\Delta_i = \Delta'_{s(i)}$  for a permutation  $s$  of  $\{1, \dots, t\}$ . Eventually, every generic representation of  $G_n(K)$  is obtained this way.*

Now from Proposition 12 of [F2], an irreducible distinguished representation  $\pi$  of the group  $G_n(K)$  is Galois-autodual, which means that the smooth dual  $\pi^\vee$  is isomorphic to  $\pi^\sigma$ . A consequence of this fact and of Theorem 5.1 is the following. If  $\pi = \Delta_1 \times \dots \times \Delta_t$  is a generic representation as in the statement of Theorem 5.1 and if it is distinguished, then there exists a reordering of the  $\Delta_i$ 's, and an integer  $r$  between 1 and  $t/2$ , such that  $\Delta_{i+1}^\sigma = \Delta_i^\vee$  for  $i = 1, 3, \dots, 2r - 1$ , and

$\Delta_i^\sigma = \Delta_i^\vee$  for  $i > 2r$ . According to Theorem 6 of [K], this means that there exists a reordering of the  $\Delta_i$ 's, and an integer  $r$  between 1 and  $t/2$ , such that  $\Delta_{i+1}^\sigma = \Delta_i^\vee$  for  $i = 1, 3, \dots, 2r-1$ , and such that  $\Delta_i$  is distinguished or  $\eta_{K/F}$ -distinguished for  $i > 2r$ . We are going to prove the following result. We recall that from Corollary 1.6 of [A-K-T], a discrete series representation cannot be distinguished, and  $\eta_{K/F}$ -distinguished at the same time.

**Theorem 5.2.** *Let  $\pi = \Delta_1 \times \dots \times \Delta_t$  a generic representation of the group  $G_n(K)$  as in Theorem 5.1, it is distinguished if and only if there is a reordering of the  $\Delta_i$ 's, and an integer  $r$  between 1 and  $t/2$ , such that  $\Delta_{i+1}^\sigma = \Delta_i^\vee$  for  $i = 1, 3, \dots, 2r-1$ , and  $\Delta_i$  is distinguished for  $i > 2r$ .*

It is a consequence of Proposition 26 of [F3], and of the main result of [M4] that representations of the form  $\Delta_1 \times \dots \times \Delta_t$  with  $\Delta_{i+1}^\sigma = \Delta_i^\vee$  for  $i = 1, 3, \dots, 2r-1$ , and  $\Delta_i$  distinguished for  $i > 2r$ , are distinguished.

Before proving the converse fact (i.e. Theorem 5.2), we recall that from the main result of [M3], this result is known to imply the equality of the Rankin-Selberg type Asai  $L$ -function  $L_F^K(\pi)$  for a generic representation  $\pi$  of  $G_n(K)$ , and of the Asai  $L$ -function  $L_F^K(\rho)$  of the Langlands parameter  $\rho$  of  $\pi$  (see definition 2.4 of [M3]). Hence the following result is also true.

**Theorem 5.3.** *Let  $\pi$  be a generic representation of the group  $G_n(K)$ , and let  $\rho$  be the representation of dimension  $n$  of the Weil-Deligne group  $W'_K$  of  $K$ , corresponding to  $\pi$  through Langlands correspondence. Then we have the following equality of  $L$ -functions:*

$$L_F^K(\pi, s) = L_F^K(\rho, s).$$

From the discussion before and after theorem 5.2, its proof is reduced to show the following fact.

**Theorem 5.4.** *Let  $\pi = \Delta_1 \times \dots \times \Delta_t$  be Galois autodual generic representation of the group  $G_n(K)$ , if it is distinguished, then there is a reordering of the  $\Delta_i$ 's, and an integer  $r$  between 1 and  $t/2$ , such that  $\Delta_{i+1}^\sigma = \Delta_i^\vee$  for  $i = 1, 3, \dots, 2r-1$ , and  $\Delta_i$  is distinguished for  $i > 2r$ .*

*Proof.* Let  $\bar{n} = (n_1, \dots, n_t)$  be the partition of  $n$  corresponding to  $\pi$ . We suppose that the  $\Delta_i$ 's are ordered by length. Moreover as  $\pi$  is Galois autodual, we suppose that inside a subsequence of same length representations, the first to occur are the non Galois autodual, and that at the first occurrence of such a  $\Delta_{i_0}$ , its successors are alternatively isomorphic to  $\Delta_{i_0}^{\vee\sigma}$  and  $\Delta_{i_0}$ , until no representation among the  $\Delta_i$ 's is isomorphic to  $\Delta_{i_0}$  (hence such a sub subsequence begins with a  $\Delta_i$  isomorphic to  $\Delta_{i_0}$ , and ends with a  $\Delta_i$  isomorphic to  $\Delta_{i_0}^{\vee\sigma}$ ).

Now from Lemma 4 of [F4], the  $H$ -module  $\pi$  has a factor series with factors of the representations  $ind_{u^{-1}P \cap uH}^H((\delta_P^{1/2} \Delta)^u)$  (with  $(\delta_P^{1/2} \Delta)^u(x) = \delta_P^{1/2} \Delta(uxu^{-1})$ ) when  $u$  describes  $R(P \backslash G/H)$ , where  $\Delta$  is the representation  $\Delta_1 \otimes \dots \otimes \Delta_t$  of  $P$ . Hence if  $\pi$  is distinguished, one of these representations admits a nonzero  $H$ -invariant linear form on its space. This implies that there is  $u$  in  $R(P \backslash G/H)$  such that the representation  $ind_{P \cap uHu^{-1}}^{uHu^{-1}}(\delta_P^{1/2} \Delta)$  admits a nonzero  $uHu^{-1}$ -invariant linear form on its space. Then Frobenius reciprocity law says that  $Hom_{uHu^{-1}}(ind_{P \cap uHu^{-1}}^{uHu^{-1}}(\delta_P^{1/2} \Delta), 1)$  is isomorphic as a vector space, to  $Hom_{P \cap uHu^{-1}}(\delta_P^{1/2} \Delta, \delta_{P \cap uHu^{-1}}) = Hom_{P \cap uHu^{-1}}(\delta_P^{1/2} / \delta_{P \cap uHu^{-1}} \rho, 1)$ . Hence there is on the space  $V_\Delta$  of  $\Delta$  a linear nonzero form  $L$ , such that for every  $p$  in  $P \cap uHu^{-1}$ , such that for every  $v$  in  $V_\Delta$ , one has  $L(\chi(p)\Delta(p)v) = L(v)$ , where  $\chi(p) = \frac{\delta_P^{1/2}}{\delta_{P \cap uHu^{-1}}}(p)$ . As both  $\delta_P^{1/2}$  and  $\delta_{P \cap uHu^{-1}}$  are trivial on  $N_s \cap uHu^{-1}$ , so is  $\chi$ . Now, if  $s$  is the element of  $I(\bar{n})$  corresponding to  $u$ , let  $n'$  belong to  $N'_s$ , from Corollary 3.1, we can write  $n'$  as a product  $n_s n_0$ , with  $n_s$  in  $N_s \cap uHu^{-1}$ , and  $n_0$  in  $N$ . As  $N$  is included in  $Ker(\Delta)$ , one has  $L(\Delta(n')(v)) = L(\Delta(n_s n_0)(v)) = L(\Delta(n_s)(v)) = L(\chi(n_s)\Delta(n_s)v) = L(v)$ . Hence  $L$  is actually a nonzero linear form on the Jacquet module of  $V_\Delta$  associated with  $N'_s$ . But we also know that  $L(\chi(m_s)\Delta(m_s)v) = L(v)$  for  $m$  in  $M_s(F)$ , which reads according to Lemma 4.1:  $L(\delta_{P'_s}^{-1/2}(m_s)\Delta(m_s)v) = L(v)$ .

This says that the linear form  $L$  is  $M_u(F)$ -distinguished on the normalized Jacquet module  $r_{M_s, M}(\Delta)$  (as  $M_s$  is also the standard Levi subgroup associated with  $N'_s$ ).

The following lemma will conclude the proof of Theorem 5.4.

**Lemma 5.1.** *Let  $\Delta_1, \dots, \Delta_t$  be unlinked segments of respectively  $G_{n_1}(K), \dots, G_{n_t}(K)$ , such that the set  $\{\Delta_1, \dots, \Delta_t\}$ , is stable under the involution  $\Delta \mapsto \Delta^{\vee\sigma}$ , call  $n$  the integer  $n_1 + \dots + n_t$ , and  $\bar{n}$  the sequence  $(n_1, \dots, n_t)$ . Suppose moreover that the  $\Delta_i$ 's are ordered by length, and that inside a subsequence of same length representations, the first to occur are the non Galois autodual, and that at the first occurrence of such a segment  $\Delta_{i_0}$ , its successors are alternatively isomorphic to  $\Delta_{i_0}^{\vee\sigma}$  and  $\Delta_{i_0}$ , until no segment among the  $\Delta_i$ 's is isomorphic to  $\Delta_{i_0}$ . Then if there is  $s = (n_{i,j})_{1 \leq i \leq j \leq t}$  in  $I(\bar{n})$ , such that  $r_{M_s, M}(\otimes_i \Delta_i)$  is  $M_s(F)$ -distinguished, there exists a reordering of the  $\Delta_i$ 's, and an integer  $r$  between 1 and  $t/2$ , such that  $\Delta_{i+1}^\sigma = \Delta_i^\vee$  for  $i = 1, 3, \dots, 2r-1$ , and  $\Delta_i$  is distinguished for  $i > 2r$ .*

*Proof of the Lemma.* We do this by induction on  $t$ . The result for  $t = 1$  follows from the fact theorem Theorem 6 of [K].

Now suppose the result to be true for any  $t' < t$ .

If there is  $s$  in  $I(\bar{n})$ , such that  $r_{M_s, M}(\Delta)$  is  $M_s(F)$ -distinguished, in particular  $r_{M_s, M}(\Delta)$  is nonzero, which from Proposition 1.1, implies that one can write each  $\Delta_i$  under the form  $[\Delta_{i,1}, \dots, \Delta_{i,t}]$ , with  $\Delta_{i,j}$  a subsegment of  $\Delta_i$  of length  $n_{i,j}$ . With these notations,  $r_{M_s, M}(\Delta)$  is equal to the tensor product  $\Delta_{1,1} \otimes \dots \otimes \Delta_{t,t}$ . Hence the fact that  $r_{M_s, M}(\Delta)$  is distinguished by the group  $M_s(F)$  is equivalent to the fact that  $\Delta_{i,i}$  is  $G_{n_{i,i}}(F)$ -distinguished if  $n_{i,i} \neq 0$ , and  $\Delta_{j,i} = \Delta_i, j^{\vee\sigma}$  if  $i < j$  and  $n_{i,j} \neq 0$ .

Let  $i_0$  be the smallest  $i$ , such that  $\Delta_{1,i}$  (or equivalently  $n_{1,i}$ ) is nonzero.

1.  **$i_0 = 1$ :** the representation  $\Delta_{1,1}$  is distinguished, hence Galois autodual. If  $\Delta_1 = [\Delta_{1,1}, \dots, \Delta_{1,t}]$  was not equal to  $\Delta_{1,1}$ , then one would have  $\Delta_1^{\vee\sigma} = [\Delta_{1,t}^{\vee\sigma}, \dots, \Delta_{1,1}] \neq \Delta_1$ . But the segment  $\Delta_1^{\vee\sigma}$  would also occur, and  $\Delta_1$  and  $\Delta_1^{\vee\sigma}$  would be linked, which is absurd. Hence  $\Delta_1 = \Delta_{1,1}$  is distinguished, and  $n_{1,i} = 0$  if  $i > 1$ . We conclude by applying our induction hypothesis to the family  $\Delta_2, \dots, \Delta_t$ , the integer  $n - n_1$  with partition  $(n_2, \dots, n_t)$ , and sub partition  $s' = (n_{i,j} \mid i \geq 2, j \geq 2)$ .
2.  **$i_0 > 1$ :** one has  $\Delta_{i_0,1} = \Delta_{1,i_0}^{\vee\sigma}$ . As the representation  $\Delta_{i_0}$  is either Galois autodual, or coupled with  $\Delta_{i_0}^{\vee\sigma}$ , the representation  $\Delta_{i_0}^{\vee\sigma} = [\Delta_{i_0,t}^{\vee\sigma}, \dots, \Delta_{1,i_0}]$  occurs. But because the representation  $\Delta_1$  has the smallest length among the  $\Delta_i$ 's, the segments  $\Delta_1$  and  $\Delta_{i_0}^{\vee\sigma}$  would be linked unless  $\Delta_1 = \Delta_{1,i_0}$ , which thus must be the case. In particular one has  $n_{1,i} = 0$  for  $i \neq i_0$ .

Two cases occur.

- a)  **$\Delta_1 = \Delta_{1,i_0}$  is Galois autodual:** if  $\Delta_{i_0}$  wasn't equal to  $\Delta_{i_0,1}$ , then the two occurring segments  $\Delta_{i_0} = [\Delta_{i_0,1}, \dots, \Delta_{i_0,t}]$  and  $\Delta_{i_0}^{\vee\sigma} = [\Delta_{i_0,t}^{\vee\sigma}, \dots, \Delta_{i_0,1}]$  would be linked, and that is not the case. Hence we have  $\Delta_{i_0} = \Delta_{i_0,1} = \Delta_1$ , and  $n_{i_0,j} = 0$  for  $j \neq 1$ . We conclude by applying our induction hypothesis to the family  $\Delta_2, \dots, \Delta_{i_0-1}, \Delta_{i_0+1}, \dots, \Delta_t$ , the integer  $n - n_1 - n_{i_0}$  with partition  $(n_2, \dots, n_{i_0-1}, n_{i_0+1}, \dots, n_t)$ , and sub partition  $s' = (n_{i,j} \mid \{i, j\} \neq \{i_0, 1\})$ .

- b)  **$\Delta_1 = \Delta_{1,i_0}$  is not Galois autodual:** in this case  $\Delta_2$  is  $\Delta_1^{\vee\sigma}$  because of our ordering. Let  $j_0$  be the smallest  $j$ , such that  $\Delta_{2,j}$  (or equivalently  $n_{2,j}$ ) is nonzero. If  $j_0 = 2$ , as in the case  $i_0 = 1$ , one has  $\Delta_2 = \Delta_{2,2}$ , and we conclude by applying our induction hypothesis to the family  $\Delta_1, \Delta_3, \dots, \Delta_t$ , the integer  $n - n_2$  with partition  $(n_1, n_3, \dots, n_t)$ , and sub partition  $s' = (n_{i,j} \mid i \neq 2, j \neq 2)$ .

If  $j_0 \neq 2$ , then  $\Delta_2$  must be equal to  $\Delta_{2,j_0}$ , otherwise  $\Delta_2$  and  $\Delta_{j_0}^{\vee\sigma}$  would be linked. This implies that  $n_{2,j} = 0$  for  $j \neq j_0$ . Thus we have  $\Delta_2 = \Delta_{2,j_0} = \Delta_{1,i_0}^{\vee\sigma} = \Delta_1^{\vee\sigma}$ . But

the two occurring segments  $\Delta_{i_0}^{\vee\sigma} = [\Delta_{i_0,t}, \dots, \Delta_1]$  and  $\Delta_{j_0} = [\Delta_{j_0,2} = \Delta_1, \dots, \Delta_{j_0,t}]$  will be linked unless  $\Delta_{i_0} = \Delta_1^{\vee\sigma}$ , in which case we conclude by applying our induction hypothesis to the family  $\Delta_2, \dots, \Delta_{i_0-1}, \Delta_{i_0+1}, \dots, \Delta_t$ , the integer  $n - n_1 - n_{i_0}$  with partition  $(n_2, \dots, n_{i_0-1}, n_{i_0+1}, \dots, n_t)$ , and sub partition  $s' = (n_{i,j} \mid \{i, j\} \neq \{i_0, 1\})$ , or  $\Delta_{j_0} = \Delta_1$ , in which case we conclude by applying our induction hypothesis to the family  $(\Delta_j \mid j \neq 2 \text{ and } j_0)$ , the integer  $n - n_2 - n_{j_0}$  with partition  $(n_j \mid j \neq 2 \text{ and } j_0)$ , and sub partition  $s' = (n_{i,j} \mid \{i, j\} \neq \{2, j_0\})$ .

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## References

- [A-K-T] U. K. Anandavardhanan, A.C. Kable and R.Tandon, *Distinguished representations and poles of twisted tensor L-functions*, Proc. Amer. Math. Soc., **132** (2004), No. 10, 2875-2883.
- [A-R] U.K. Anandavardhanan and C.S. Rajan, *Distinguished representations, base change, and reducibility for unitary groups*, Int. Math. Res. Not., **14** (2005), No. 14, 841-854.
- [B-Z] I.N. Bernstein and A.V.Zelevinsky, *Representations of the group  $GL(n, F)$  where  $F$  is a non-archimedean local field*, Russian Math. Surveys **31:3**, (1976), 1-68.
- [C-P] J. W. Cogdell, I.I. Piatetski-Shapiro, *Derivatives and L-functions for  $GL(n)$* , to appear in The Heritage of B. Moizhezon, IMCP.
- [F1] Y. Flicker, *Twisted tensors and Euler products*, Bull. Soc. Math. France, **116** no.3, (1988), 295-313.
- [F2] Y. Flicker, *On distinguished representations*, J. Reine Angew. Math., **418** (1991), 139-172.
- [F3] Y. Flicker, *Distinguished representations and a Fourier summation formula*, Bull. Soc. Math. France, **120** (1992), 413-465.
- [F4] Y. Flicker, Appendix of *On zeroes of the twisted tensor L-function*, Math. Ann., **297**, (1993), 199-219.
- [Go] D. Goldberg, *Some results on reducibility for unitary groups and local Asai L-functions*, J. Reine Angew. Math., **448**, (1994), 65-95.
- [He] G. Henniart, *Correspondance de Langlands et fonctions L des carres exterieur et symetrique*, preprint, 2003, Institut des Hautes Etudes Scientifiques.
- [K] A. C. Kable, *Asai L-functions and Jacquet's conjecture*, Amer. J. Math., **126**, (2004), 789-820.
- [M1] N. Matringe, *Distinguished principal series representations for  $GL_n$  over a p-adic field*, Pacific J.Math., Vol. 239, No. 1, Jan 2009.
- [M2] N. Matringe, *Distinguished representations and exceptional poles of the Asai-L-function*, preprint, <http://hal.archives-ouvertes.fr/hal-00299528/fr/>, arXiv:0807.2748
- [M3] N. Matringe, *Conjectures about distinction and Asai L-functions of generic representations of general linear groups over local fields*, to appear in IMRN
- [M4] N. Matringe, *Distinction of some induced representations*, preprint, <http://arxiv.org/abs/0811.3733>.

- [Sh] F. Shahidi , *A proof of Langlands conjecture on Plancherel measures; Complementary series for  $p$ -adic groups*, Ann. of Math., **132** (1990), 273-330.
- [T] J. Tate, *Number theoretic background*, in Automorphic Forms, Representations and  $L$ -functions, Proc. Symp. Pure. Math., **33**, part 2 (1979), 3-26.
- [Z] A.V. Zelevinsky, *induced representations of reductive  $p$ -adic groups II*, Ann.Sc.E.N.S., 1980.