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# EXISTENCE OF NON-ALGEBRAIC SINGULARITIES OF DIFFERENTIAL EQUATION

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ABSTRACT. An algebraizable singularity is a germ of a singular holomorphic foliation which can be defined in some local chart by a differential equation with algebraic coefficients. We show that there exist at least countably many saddle-node singularities of the complex plane that are not algebraizable.

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We consider differential equations in the complex plane

$$(0.1) \quad A(x, y) dy = B(x, y) dx$$

near an isolated singularity, which can be conveniently located at  $(0, 0)$  by translation. The coefficients  $A$  and  $B$  are germs of a holomorphic function with a common zero at  $(0, 0)$  and no common factor. We denote by  $\lambda_1$  and  $\lambda_2$  the eigenvalues of the linear part of the equation at  $(0, 0)$ . We will always assume that at least one of those is non-zero, say  $\lambda_2 \neq 0$ , and set  $\lambda := \frac{\lambda_1}{\lambda_2}$ . We recall the following classical result :

**Theorem. (Poincaré and Dulac [4])** *If  $\lambda \notin \mathbb{R}_{\leq 0}$  then there exist two polynomials  $P, Q$  such that the previous differential equation is orbitally equivalent through a local analytic change of coordinates to*

$$P(x, y) dy = Q(x, y) dx .$$

*If moreover  $\lambda \notin \mathbb{N} \cup 1/\mathbb{N}_{\neq 0}$  then we can choose  $P(x, y) = x$  and  $Q(x, y) = \lambda y$  (i.e. the equation is linearizable).*

We recall that two germs of differential equation are *orbitally equivalent* when there exists a germ of biholomorphism conjugating their solutions. It thus turns out that a generic equation is orbitally equivalent to a linear, or at least algebraic, equation. Up to now an open question regarded whether *every* differential equation is algebraic in some local chart. Such an equation will be called *algebraizable*. Geometrically, it is equivalent to ask if any germ of a singularity of foliation in the complex plane can be realized as some singularity of a foliation of  $\mathbb{C}\mathbb{P}^2$ . We aim to prove that it is not so in the case of a saddle-node ( $\lambda = 0$ ), as was expected in [5] for non-linearizable resonant singularities. Notice that these equations are nonetheless *formally* algebraizable.

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**Theorem 1.** *There exist at least countably many non-equivalent saddle-node equations*

$$(0.2) \quad x^2 dy = (y + h.o.t) dx$$

*which are not algebraizable.*

Our proof is based on Martinet-Ramis' theorem about orbital classification of such equations, stating that the equivalence classes of all equations (0.2) under the action of local changes of coordinates is in one-to-one correspondence with the space of germs  $\mathbb{C}\{h\}$ . We will give a more precise statement in Section 3. Our argument boils down to the following : since the space of orbitally equivalent saddle-node equations is in one-to-one correspondence with a functional space of germs and since this space is "big" then the trace of all algebraic equations should reasonably be "meagre", for instance in the sense of Baire. Many problems arise immediately, one of those being that  $\mathbb{C}\{h\}$  cannot be endowed with a topology which would make it a Baire space while at the same time preserving the "nice" structure of the set of algebraic equations. Another problem lies in the fact that  $\mathbb{C}\{h\}$  might not be objectively "big" as it can be the range of a continuous map  $\mathbb{R} \rightarrow \mathbb{C}\{h\}$  and thus set-theoretically equivalent to the field of scalars. Hence both set theory and topology are not sufficient to guarantee that the heuristics works, and we must consider "analytic Baire properties". What makes things work is the fact that Martinet-Ramis' invariant of classification is analytic with respect to the equation, as was already known. The main part of our proof regarding this Baire analyticity property deals with showing that Dulac's prenormalization procedure is analytic too.

What is actually expected is that the *typical* saddle-node equation is non-algebraizable, *i.e.* the set of non-algebraizable equations is a  $G_\delta$ -dense subset of all saddle-node equations, not only that the image of those non-algebraizable equations is a  $G_\delta$ -dense subset of the space of invariants (which is what we prove here). To do so one must consider a finer topology on spaces of germs than the ones used presently and study analyticity and openness of maps from and into these spaces. This requires a lot of additional technical work and is currently being carried out. The authors nonetheless believe this stronger result to be true.

## 1. A TOPOLOGY ON $\mathbb{C}\{z_1, z_2, \dots, z_n\}$

In the sequel we use bold-typed letters to indicate multi-variables  $\mathbf{z} := (z_1, \dots, z_n) \in \mathbb{C}^n$  or multi-indices  $\mathbf{J} := (j_1, \dots, j_n) \in \mathbb{N}^n$ . We use the standard notations  $\mathbf{J}! := \prod_\ell (j_\ell!)$ ,  $|\mathbf{J}| := \sum_\ell j_\ell$  and  $\mathbf{z}^{\mathbf{J}} := \prod_\ell z_\ell^{j_\ell}$ .

### 1.1. Norm on $\mathbb{C}\{\mathbf{z}\}$ .

Let us endow the topological space  $\mathbb{C}\{\mathbf{z}\}$  with the norm

$$\|f\| := \sum_{\mathbf{J}} \frac{|a_{\mathbf{J}}|}{\mathbf{J}!}$$

where  $f(\mathbf{z}) = \sum_{\mathbf{J}} a_{\mathbf{J}} \mathbf{z}^{\mathbf{J}}$ . Since the series  $f$  is convergent  $\|f\|$  is well defined and is a norm on the space  $\mathbb{C}\{\mathbf{z}\}$ . Notice that the space  $(\mathbb{C}\{\mathbf{z}\}, \|\cdot\|)$  is not complete since the sequence  $\left(\sum_{|\mathbf{J}| \leq n} \sqrt{\mathbf{J}!} \mathbf{z}^{\mathbf{J}}\right)_{n \in \mathbb{N}}$  has the Cauchy property but is not convergent in the space of convergent series. It is not even a Baire space. The evaluation

$f \mapsto f(0)$  is continuous as well as the evaluation at 0 of any derivative. Hence, the family of projectors  $J^N$  which associates to  $f$  is  $N$ -jet is continuous.

## 1.2. Analytical functions from $\mathbb{C}^p$ to $\mathbb{C}\{\mathbf{z}\}$ .

**Definition 2.** Let  $\Omega$  be a domain of  $\mathbb{C}^p$  for  $p \in \mathbb{N}_{\neq 0}$ .

- (1) A map  $F : \Omega \rightarrow \mathbb{C}\{\mathbf{z}\}$  is said to be *strongly analytic* if the map  $(\mathbf{x}, \mathbf{z}) \mapsto F(\mathbf{x})(\mathbf{z})$  is analytic with respect to the  $n + p$  complex variables  $x_1, \dots, x_p$  and  $z_1, \dots, z_n$  on a neighbourhood of  $\Omega \times \{0\}$ .
- (2) The map  $F$  is said to be *analytic* if for any point  $\mathbf{x}$  in  $\Omega$ , there exists a linear map  $L : \mathbb{C}^p \rightarrow \mathbb{C}\{\mathbf{z}\}$  such that

$$F(\mathbf{x} + \mathbf{h}) = F(\mathbf{x}) + L(\mathbf{h}) + o(\mathbf{h}) .$$

- (3) A map  $G : \mathbb{C}\{\mathbf{w}\} \mapsto \mathbb{C}\{\mathbf{z}\}$  is said to be strongly analytic if the image of any analytic family of  $\mathbb{C}\{\mathbf{w}\}$  with a lower bounded radius of convergence is an analytic family of  $\mathbb{C}\{\mathbf{z}\}$  with a lower bounded radius of convergence.

**Proposition 3.** *If  $F$  is strongly analytic then it is analytic.*

Notice that there exist analytic maps which are not strongly analytic: the obstruction comes simply from the non-existence of local uniform lower bound for the radius of convergence of series on any open ball of  $\mathbb{C}\{\mathbf{z}\}$  for  $\|\cdot\|$ . The following example, due to J. Duval, illustrates that fact.

**Example 4.** Consider the family of compact sets for  $\varepsilon > 0$

$$K_\varepsilon := \mathbb{D} \setminus \{0 < \text{Im}(z) < \varepsilon\}$$

which is the union of two simply connected, compact and connected sets  $K_\varepsilon^+$  and  $K_\varepsilon^-$  such that, say,  $K_\varepsilon^\pm$  intersects  $\pm i\mathbb{R}_{>0}$ . According to Runge's approximation theorem there exists a sequence of polynomials  $(P_n^\varepsilon)_{n \in \mathbb{N}}$  which is a uniform approximation of the function defined by  $x \in K_\varepsilon^+ \mapsto \frac{1}{x}$  and  $x \in K_\varepsilon^- \mapsto 1$ . There exists a slowly converging sequence  $\varepsilon_n > 0$  such that  $\sup_{x \in \mathbb{D}} |P_n^{\varepsilon_n}(x)| \leq \sqrt{n}$ . We now form the sequence  $P_n := P_n^{\varepsilon_n}$  and consider the map:

$$F : x \in \mathbb{C} \mapsto \sum_{j \in \mathbb{N}} P_j(x)^j z^j .$$

The reader can easily prove that  $F(x) \in \mathbb{C}\{z\}$  for all  $x \in \mathbb{C}$  and that its radius of convergence is  $|x|$  if  $\text{Im}(x) > 0$  and equals 1 otherwise. As a consequence  $F$  cannot be strongly analytic, as  $(x, z) \mapsto F(x, z)$  is analytic on no neighbourhood of  $(0, 0)$ , whereas  $x \mapsto F(x)$  is analytic, for

$$\left\| F(x+h) - F(x) - h \sum_{j \in \mathbb{N}} j P_j'(x) P_j(x)^{j-1} z^j \right\| \leq C |h|^2 \sum_{j \in \mathbb{N}} \frac{\sqrt{j^j}}{j!}$$

if we require that  $x$  belong to a smaller disc  $r\mathbb{D}$ ,  $0 < r < 1$ , thanks to Cauchy's formula as will be detailed further down.

*Proof.* In the proof we assume that  $n = p = 1$  : the general case can be treated in much the same way. Since analyticity is a local property, we can also perform the proof in a neighbourhood of  $0 \in \mathbb{C}$ . Let us write  $F(x)(z) = \sum_{j \geq 0} f_j(x) z^j$ . Since  $F(x)(z)$  is analytic as a map of two variables, the series  $F(x)$  are convergent on a

common open disc centered at  $x = 0$  with radius  $2\rho$ . The Cauchy formula ensures that for any  $j$

$$f_j(x) = \frac{(-1)^j}{2i\pi} \int_{\gamma} \frac{F(x)(\xi)}{\xi^{j+1}} d\xi$$

for any loop  $\gamma$  in the disc of convergence. Substituting  $\gamma := \{|\xi| = \rho\}$  yields

$$\rho^{j+1} |f_j(x)| \leq \|F(x)\|_{\infty, D(0, \rho)},$$

Since  $F(x)(z)$  is bounded on  $D(0, \beta) \times D(0, \rho)$  for some  $\beta$ , there exists a positive number  $C$  such that for any  $j$

$$|f_j(x)| \leq \frac{C}{\rho^j}.$$

Hence on a disc  $D(0, \beta')$  with  $\beta' < \beta$  we have a control of the second derivative of the components of  $f_j(x)$

$$|f_j^{(2)}(x)| \leq \frac{C'}{\rho^j}.$$

As a consequence, we have on a yet smaller disc :

$$|f_j(x+h) - f_j(x) - hf_j^{(1)}(x)| \leq C'' \frac{1}{\rho^j} |h|^2.$$

Defining  $D_x F(h)$  as  $h \sum_{j \geq 0} f_j^{(1)}(x)$ , which is a convergent series, yields

$$\|F(x+h) - F(x) - D_x F(h)\| \leq C'' e^{\frac{1}{\rho}} |h|^2,$$

which ensures the analyticity of  $F$ . □

## 2. ANALYTICAL BAIRE PROPERTY OF $\mathbb{C}\{\mathbf{z}\}$

We haven't been able to find a suitable "nice" and reasonably interesting topology on  $\mathbb{C}\{z\}$  in order to obtain a Baire space, and surely it is not possible to do so if we agree on what "interesting topology" might be... We can prove that  $(\mathbb{C}\{z\}, \|\cdot\|)$  is not Baire. But we can also prove that this space cannot be covered by countably many analytic subspaces, which is the purpose of this paragraph.

### Definition 5.

- (1) An *analytic subspace* of  $\mathbb{C}\{\mathbf{z}\}$  is the range of an analytic map  $F : \Omega \subset \mathbb{C}^p \rightarrow \mathbb{C}\{\mathbf{z}\}$ .
- (2) We say that  $\mathbb{C}\{\mathbf{z}\}$  is an *analytic Baire space* if it cannot be the union of a countable analytic subspaces.

Our main result is the following

**Theorem 6.**  $\mathbb{C}\{\mathbf{z}\}$  is an analytic Baire space.

### 2.1. Annoying facts about $\mathbb{C}\{z\}$ .

We begin with proving the following

**Lemma 7.**  $\mathbb{C}\{z\}$  is in one-to-one correspondence with  $\mathbb{C}$ .

This result is a consequence of the existence of a "Peano-curve" in  $\mathbb{C}\{z\}$  for some relatively natural topology.

*Proof.* The space  $\mathbb{C}\{z\}$  is naturally a subset of  $\mathbb{C}^{\mathbb{N}}$ , which can be endowed with the product topology. The induced topology on  $\mathbb{C}\{z\}$  makes this space a connected and locally connected topological space. Moreover for any  $(p, r) \in \mathbb{N} \times \mathbb{Q}$  the subset of  $\mathbb{C}\{z\}$  defined by

$$A_{p,r} := \left\{ f(h) = \sum_{j \geq 0} a_j h^j : |a_j| \leq pr^j \right\}$$

is compact. The union  $\bigcup_{\mathbb{N} \times \mathbb{Q}} A_{p,r}$  covers the whole  $\mathbb{C}\{z\}$ , which means the latter is  $\sigma$ -compact for the topology under consideration. A theorem of Hahn, Mazurkiewicz, Menger, Moore and Sirpieński [6] states precisely that the continuous images of  $[0, 1]$  are the compact, connected and locally connected spaces. Therefore  $\mathbb{C}\{z\}$  is a continuous image of  $\mathbb{R}$ , and obviously of  $\mathbb{C}$ , for the above not-too-pathological product topology. A weaker consequence is that from a purely set-theoretical point of view  $\mathbb{C}$  and  $\mathbb{C}\{z\}$  are in one-to-one correspondence.  $\square$

Now we show that

**Lemma 8.**  $(\mathbb{C}\{z\}, \|\cdot\|)$  is not a Baire space.

*Proof.* We consider the following example due to R. Schäfke. Consider the subspaces

$$M_N := \left\{ \sum a_j z^j : |a_j| \leq N^j \right\}, \quad N \in \mathbb{N}.$$

Obviously  $\mathbb{C}\{z\} = \bigcup_N M_N$ . Moreover  $M_N = \bigcap_j \{|a_j| \leq N^j\}$  is closed as the association  $f \mapsto f^{(j)}(0)$  is continuous, and its interior is empty as the example 4 shows that no neighbourhood of  $f \in \mathbb{C}\{z\}$  may admit a uniform lower bound for the radius of convergence.  $\square$

As an inductive space  $\mathbb{C}\{z\}$  can also be endowed with the inductive topology : this space becomes complete but not Baire. In particular this topology cannot be induced by a metric.

## 2.2. Preliminaries.

In order to prove Theorem 6 we will need to eventually locate the proof within a Baire space to get a contradiction. Let  $\mathcal{A}$  be the subspace of  $\mathbb{C}\{z\}$  defined by

$$\mathcal{A} := \left\{ f(z) = \sum_{j \geq 0} a_j z^j : |a_j| \text{ is bounded} \right\}$$

together with the norm  $\|\cdot\|_{\infty}$  :

$$\|f\|_{\infty} := \sup_j |a_j|.$$

$(\mathcal{A}, \|\cdot\|_{\infty})$  is a complete metric space and is thus a Baire space because it is isometric to a subspace of  $\mathbb{C}^{\mathbb{N}}$  formed by all bounded sequences equipped with the sup-norm.

**Lemma 9.** Let  $S$  be a closed set in  $\mathbb{C}\{z\}$  for the norm  $\|\cdot\|$ . Then  $S \cap \mathcal{A}$  is closed in  $\mathcal{A}$  for the norm  $\|\cdot\|_{\infty}$ .

*Proof.* Let  $(f_n)$  be a sequence in  $S \cap \mathcal{A}$  which tends to  $f$  when  $n$  tends to infinity for the norm  $\|\cdot\|_\infty$ . Then  $f$  belongs to  $\mathcal{A}$  since it is closed. Moreover, as

$$\|f_n - f\| \leq e \|f_n - f\|_\infty,$$

the sequence is convergent in  $\mathbb{C}\{z\}$  for the norm  $\|\cdot\|$ . Since  $S$  is closed  $f$  must belong to  $S$  too.  $\square$

**Lemma 10.** *A family  $f_1, \dots, f_n \in \mathbb{C}\{z\}$  is free over  $\mathbb{C}$  if, and only if, there exists  $p \in \mathbb{N}$  such that their  $p$ -jets are free over  $\mathbb{C}$ .*

*Proof.* Suppose that for any  $p \in \mathbb{N}$  there exists a non-trivial relation

$$\Lambda_p := (\lambda_{1,p}, \dots, \lambda_{n,p}) \neq 0$$

for the family  $\varphi_p := (J^p(f_1), \dots, J^p(f_n))$ , that is

$$J^p \left( \sum_{j=1}^n \lambda_{j,p} f_j \right) = 0.$$

Up to rescaling  $\Lambda_p$  one can suppose that it belongs to the unit sphere of  $\mathbb{C}^n$  and so consider some adherence value  $(\lambda_{1,\infty}, \dots, \lambda_{n,\infty})$ . Because if  $J^{k+1}(f) = 0$  then  $J^k(f) = 0$ , by taking the limit  $p \rightarrow \infty$  while fixing an arbitrary  $k$  we obtain that  $\Lambda_\infty$  is a non-trivial relation for  $\varphi_k$  by continuity of  $f \mapsto J^k(f)$ , and thus is a non-trivial relation for  $(f_1, \dots, f_n)$ .  $\square$

According to this lemma, if  $F$  is of maximal rank at  $x$ , i.e. its rank is equal to the dimension of the source space, there exists  $N \in \mathbb{N}$  such that the function  $J^N F$  is of maximal rank. Since the space of polynomials of maximal degree  $N$  is of finite dimension, the function  $J^N R$  is locally one-to-one around  $x$ . So is the application  $F$ . Hence the

**Corollary 11.** *Let  $F : \Omega \subset \mathbb{C}^n \rightarrow \mathbb{C}\{\mathbf{z}\}$ .*

- (1) If  $D_x F$  is of rank  $n$  then  $F$  is locally one-to-one near  $x$ .
- (2) If  $D_x F$  is of maximal rank  $p < n$  then there exists a smooth hypersurface  $S$  of dimension  $p$  at  $x$  such that  $F|_S$  is of rank  $p$  and has the same image as  $F$ .

*Proof.* The second part of the corollary is proved using the same result in finite dimension : indeed, if the range of  $F$  were some finite dimensional vector space, one could choose for  $S$  the hypersurface  $\{x_{i_1} = \dots = x_{i_{n-p}} = 0\}$  where  $D_{(x_{j_1}, \dots, x_{j_p})} F$  is of rank  $p$  with  $\{1, \dots, n\} = \{i_1, \dots, i_{n-p}\} \cup \{j_1, \dots, j_p\}$ . Now if the range of  $F$  were  $\mathbb{C}\{\mathbf{z}\}$ , one applies this argument to  $J^N F$  for all  $N$  big enough.  $\square$

The key point to Theorem 3 is the following proposition :

**Proposition 12.** *Let  $F : \Omega \rightarrow \mathbb{C}\{z\}$  be continuous, analytic and one-to-one on an open set  $\Omega \subset \mathbb{C}^n$ . Let  $E < \mathbb{C}\{z\}$  be any subspace of infinite dimension and suppose that  $D_x F$  is of rank  $n$  for some  $x \in \Omega$ . Then there exist  $\delta$  in  $E$  and  $\varepsilon > 0$  such that for any  $0 < |t| < \varepsilon$  the germ  $F(x) + t\delta$  does not belong to  $F(\Omega)$ .*

*Proof.* Suppose the claim is false and fix  $\delta \in E \setminus \{0\}$ . There exists a sequence  $(u_n)_{n \in \mathbb{N}} \subset \mathbb{C}^n$  such that, for  $n$  large enough,  $x + u_n \in \Omega$  and

$$F(x + u_n) = F(x) + \frac{\delta}{n}.$$

Any accumulation point  $u$  of  $u_n$  satisfies  $F(x+u) = F(x)$ . Because  $F$  is one-to-one  $u$  must vanish, which in turn implies that  $u_n$  converges towards zero. Besides, the definition of differentiability we use implies

$$o(u_n) = \left\| D_x F(u_n) - \frac{\delta}{n} \right\| = \|u_n\| \left\| D_x F\left(\frac{u_n}{\|u_n\|}\right) - \frac{\delta}{n\|u_n\|} \right\|.$$

Dividing by  $\|u_n\|$  yields  $\left\| D_x F\left(\frac{u_n}{\|u_n\|}\right) - \frac{\delta}{n\|u_n\|} \right\| = o(1)$ . Now by compactness of the unit sphere of  $\mathbb{C}^n$  we can assume that  $\frac{u_n}{\|u_n\|}$  tends to some  $u \neq 0$  when  $n$  tends to infinity. Hence  $\frac{\delta}{n\|u_n\|}$  has to tend to some  $\lambda\delta$  as  $n$  tends to infinity and, according to the rank assumption,  $\lambda \neq 0$ . As a matter of consequence

$$D_x F(u) = \lambda\delta,$$

which cannot be possible for every  $\delta$  in  $E$ , for the image of the differential map  $D_x F$  is finite dimensional.  $\square$

### 2.3. Analytical Baire property of $\mathbb{C}\{z\}$ : proof of Theorem 6.

We show here that  $\mathbb{C}\{z\}$  has an analytical Baire property by supposing on the contrary that  $\mathbb{C}\{z\}$  is a countable union of analytic sets :

$$\mathbb{C}\{z\} = \bigcup_{n \in \mathbb{N}} \bigcup_{j \in \mathbb{N}} F_{j,n}(\Omega_{j,n}),$$

where  $F_{j,n}$  is a differentiable function defined on an open set  $\Omega_{j,n}$  of  $\mathbb{C}^n$ . Taking if necessary a finite covering of each  $\Omega_{j,n}$ , one can assume that  $F_{j,n}$  is of rank  $n$  on  $\Omega_{j,n}$ . Indeed the set of points where  $F_{j,n}$  is not of maximal rank is an analytical subset  $\Sigma_{j,n}$  of  $\Omega_{j,n}$  locally closed. The analytical set  $\Sigma_{j,n}$  admits a decomposition  $\Sigma_{j,n} = \bigcup C_k$  where each cell  $C_k$  is biholomorphic to an open set of some  $\mathbb{C}^p$  with  $0 \leq p < n$  [2]. Hence we get the following decomposition

$$F_{j,n}(\Omega_{j,n}) = F_{j,n}(\Omega_{j,n} \setminus \Sigma_{j,n}) \bigcup_k F_{j,n}(C_k).$$

If the rank  $p$  of  $F_{j,n}$  is strictly smaller than  $n$  on  $\Omega_{j,n} \setminus \Sigma_{j,n}$  then one can find a finer covering of  $\Omega_{j,n} \setminus \Sigma_{j,n} = \bigcup_k B_{j,n,k}$  and a family of smooth hypersurfaces  $S_{j,n,k} \subset B_{j,n,k}$  of dimension  $p$  such that the rank of  $F_{j,n}|_{S_{j,n,k}}$  is  $n$  and  $F_{j,n}|_{B_{j,n,k}}$  and  $F_{j,n}|_{S_{j,n,k}}$  has the same image. Now on each cell  $C_k$  one can seek the points where  $F_{j,n}|_{C_k}$  is not of maximal rank and do the same procedure as above. This construction stops after finitely many steps since at each stage the dimension of the open set we consider is strictly less than that of the previous stage. Finally since any open set of  $\mathbb{C}^p$  is a countable union of compact sets, we obtain the following decomposition

$$\mathbb{C}\{h\} = \bigcup_{n \in \mathbb{N}} \bigcup_{j \in \mathbb{N}} \bigcup_{q \in \mathbb{N}} R_{j,n}(K_{j,n,q}),$$

where  $\Omega_{j,n} = \bigcup_{q \in \mathbb{N}} K_{j,n,q}$  and each  $K_{j,n,q}$  is a full compact subset of some  $\mathbb{C}^p$  with  $p \leq n$ .

The set  $R_{j,n}(K_{j,n,q})$  is compact and therefore closed for the topology induced by  $(\|\cdot\|_k)_{k>0}$ . According to Lemma 9 the set  $R_{j,n}(K_{j,n,q}) \cap \mathcal{A}$  is also closed in  $\mathcal{A}$  for the norm  $\|\cdot\|_\infty$ . It is besides of empty interior : since  $\mathcal{A}$  is infinite dimensional if



$R_{j,n}(x)$  belongs to  $\mathcal{A}$  we can invoke Proposition 12 to obtain  $\delta \in \mathcal{A}$  such that for  $t$  small enough

$$R_{j,n}(x) + t\delta \notin R_{j,n}(\Omega_{j,n}),$$

which ensures that any small ball for the norm  $\|\cdot\|_\infty$  in  $\mathcal{A}$  around  $R_{j,n}(x)$  cannot be contained in  $R_{j,n}(\Omega_{j,n})$ . Finally we obtain the sought contradiction since then  $\mathcal{A}$  can be split into a countable union of closed subset with empty interior :

$$\mathcal{A} = \bigcup_{n \in \mathbb{N}} \bigcup_{j \in \mathbb{N}} \bigcup_{q \in \mathbb{N}} R_{j,n}(K_{j,n,q}) \cap \mathcal{A},$$

which is impossible since  $\mathcal{A}$  is a Banach thus Baire space.

### 3. ANALYTICITY OF MARTINET-RAMIS INVARIANTS AND PROOF OF THE MAIN THEOREM

The tool that we need is a map which to a saddle-node equation, written in the most general form (0.1), associates its invariant of orbital classification. This is the goal of this section, as well as proving that this map is actually analytic.

Firstly we need to put the general equation (0.1) in a prepared form; this is done using Dulac's prenormalization procedure in Section 3.2. We will restrict our construction to those equations whose first topological invariant equals 1. Geometrically speaking this invariant is the order of tangency between the foliation defined by the equation and the separatrix tangent to the eigenspace associated to the eigenvalue  $\lambda_2 \neq 0$ . This defines the stratum  $\mathcal{E}_1$ , studied in Section 3.3. After applying Dulac's procedure  $\mathcal{D}$  we deal with equations in the form

$$x^2 dy = (y + R(x, y)) dx.$$

Define the space  $\mathcal{M} := \mathbb{C} \times \mathbb{C} \times \text{Diff}(\mathbb{C}, 0)$  the equivalence relation on  $\mathcal{M}$  by  $(\mu, \tau, \phi) \sim (\tilde{\mu}, \tilde{\tau}, \tilde{\phi})$  if, and only if,  $\mu = \tilde{\mu}$  and there exists  $c \in \mathbb{C}_{\neq 0}$  such that  $\phi(ch) = \tilde{\phi}(h)$  and  $\tau = c\tilde{\tau}$ .

**Theorem. (Martinet-Ramis, [5])** *There exists a map  $M : \mathcal{E}_1 \rightarrow \mathcal{M}$  such that two equations  $E$  and  $\tilde{E}$  of  $\mathcal{E}_1$  are orbitally conjugate if, and only if,  $M(E) = M(\tilde{E})$ . Moreover this map is onto and if  $\mathfrak{t} \in (\mathbb{C}^n, 0) \mapsto E_{\mathfrak{t}} \in \mathcal{E}_1$  is an analytic family of equations written in Dulac's form then  $\mathfrak{t} \mapsto \mathcal{M}(E_{\mathfrak{t}})$  is an analytic family too (that is, for all  $\mathfrak{t}$  one can choose a representant of  $\mathcal{M}(E_{\mathfrak{t}})$  such that this family is analytic).*

In other words, once written in Dulac's form the germ-component of Martinet-Ramis' map, which we will write  $\phi_{MR}$ , is strongly analytic with respect to  $R$ . The aim of this section is to provide a proof for the :

**Theorem 13.** *The complete Martinet-Ramis map  $(A dy - B dx) \xrightarrow{\mathcal{D}} (x^2 dy - (y + R) dx) \xrightarrow{\phi_{MR}} \phi \in \text{Diff}(\mathbb{C}, 0)$  is a strongly analytic association.*

Thus all that remains is to show that  $\mathcal{D}$  is strongly analytic. Before investigating this result we begin with giving the proof of Theorem 1 in the upcoming section.

**3.1. Proof of the existence of countably many non-algebraizable saddle-node singularities (Theorem 1).** First we show that there exists at least one such non-algebraizable equation. Suppose on the contrary that any saddle-node singularity is algebraizable. We define  $\mathcal{P}_d$  to be the set of all equations in  $\mathcal{E}_1$  with polynomial coefficients of maximum degree  $d$  which, as we will see in Corollary 15, is an analytic space. Then, according to Theorem 13 the restriction of  $\phi_{MR} \circ \mathcal{D}$  to the space of polynomials must be onto : therefore, we would obtain the decomposition

$$\text{Diff}(\mathbb{C}, 0) \simeq \mathbb{C}\{h\} = \bigcup_{n \in \mathbb{N}} \phi_{MR} \circ \mathcal{D}(\mathcal{P}_n)$$

which would be a countable union of analytical sets. This is impossible in view of the analytic Baire property of  $\mathbb{C}\{h\}$  (Theorem 6). Hence, there exists at least one saddle-node equation which is not algebraizable.

Obviously the same argument works for countably many equations as a point of  $\mathbb{C}\{h\}$  is a compact with empty interior.

**3.2. Dulac's procedure.** Let  $\mathcal{E}$  be the set of couples  $(A, B) \in \mathbb{C}\{x, y\} \times \mathbb{C}\{x, y\}$  such that the matrix

$$\begin{pmatrix} \frac{\partial A}{\partial x} & \frac{\partial A}{\partial y} \\ \frac{\partial B}{\partial x} & \frac{\partial B}{\partial y} \end{pmatrix}$$

has exactly one non-vanishing eigenvalue. One can assume that, up to a linear change of variables, the linear part of  $X_{A,B} = -B \frac{\partial}{\partial x} + A \frac{\partial}{\partial y}$  is diagonal :

$$\begin{aligned} A(x, y) &= o(\|x, y\|) \\ B(x, y) &= y + o(\|x, y\|) . \end{aligned}$$

Notice that the latter change of variable depends rationally on the coefficients of the linear part of  $A$  and  $B$ . In all the sequel the only changes of variables we allow will be required to preserve this diagonal form. The existence of a unique analytic solution  $x = s(y)$  (a separatrix of  $X_{A,B}$ ) tangent to the eigenspace  $\{x = 0\}$  at  $(0, 0)$  is well known (see [1] for example). The other separatrix  $y = \hat{s}(x)$ , tangent to  $\{y = 0\}$ , only exists *a priori* at a formal level (and generically this series, though unique, is divergent). The reader will find in [3] the material needed to carry out the complete prenormalization procedure. What we retain from it is the following steps :

- Applying the change of coordinates  $(x, y) \mapsto (x + s(y), y)$  transforms  $X_{A,B}$  into a vector field  $X_{A_1, B_1}$  where

$$A_1(x, y) \in x\mathbb{C}\{x, y\} .$$

- It is possible to further orbitally normalize  $A_1$  to obtain a new vector field  $X_{A_D, B_D}$  such that

$$(3.1) \quad \begin{aligned} A_D(x, y) &= x^{k+1} \\ B_D(x, y) &= y + r(x) + yR(x, y) \end{aligned}$$

with  $r(0) = r'(0) = R(0, 0) = 0$ . The integer  $k \in \mathbb{N}_{>0}$  is a topological invariant (but not a complete topological invariant).

- We define the map

$$\mathcal{D} : (A, B) \in \mathcal{E} \mapsto B_D - y \in \mathbb{C}\{x, y\} .$$

At this stage this map may not be well defined. We will give a canonical way of obtaining  $\mathcal{D}(A, B)$  from the original vector field without ambiguity. We do this in Section (3.4).

**3.3. The stratum  $\mathcal{E}_1$ .** Denote by  $\mathcal{E}_1$  the stratum of  $\mathcal{E}$  consisting of equations that can be put under the previous form (3.1) with  $k = 1$ .

**Proposition 14.** *The stratum  $\mathcal{E}_1$  is constructable: it is the complementary of a dimension 1 affine subspace of  $\mathcal{E}$ .*

*Proof.* First we apply the change of coordinates  $(x, y) \mapsto (x + s(y), y)$  which brings  $X_{A,B}$  to  $X_{x\tilde{A},\tilde{B}}$ . In this situation the separatrix is straightened to  $\{x = 0\}$ . Write  $\tilde{A}(x, y) = ax + by + o(\|x, y\|)$ ; we claim that  $X_{A,B}$  belongs to  $\mathcal{E}_1$  if, and only if,  $a \neq 0$ . On the one hand suppose that there exists a local analytic change of coordinates  $\Psi(x, y) = (\alpha x + C(x, y), \beta y + D(x, y))$ , with  $C$  and  $D$  in  $\mathbb{C}\{x, y\}_1$ , defining a conjugacy between  $X_{x\tilde{A},\tilde{B}}$  and some  $UX_{x^2,\hat{B}}$  with  $\eta := U(0, 0) \neq 0$ . Then :

$$(3.2) \quad U(x + C, y + D)(\alpha x + C)^2 = x\tilde{A}\left(\alpha + \frac{\partial C}{\partial x}\right) + \tilde{B}\frac{\partial C}{\partial y}.$$

Written for the term of least homogeneous degree this equation becomes, since  $\tilde{B}(x, y) = y + o(\|x, y\|)$  :

$$\eta\alpha^2x^2 = \alpha x(ax + by) + y(\delta x + \gamma y)$$

where  $\delta = \frac{\partial^2 C}{\partial x \partial y}(0, 0)$  and  $\gamma = \frac{1}{2}\frac{\partial^2 C}{\partial y^2}(0, 0)$ . Hence  $\alpha\eta = a$ , meaning  $a \neq 0$  as requested. On the other hand we use Dulac's result : we know that there exists such a  $\Psi$  between  $X_{x\tilde{A},\tilde{B}}$  and some  $UX_{x^{k+1},\hat{B}}$ . If  $a \neq 0$  then necessarily  $k = 1$ , as can be seen for the analog of (3.2) (the term  $(\tilde{B} - y)\frac{\partial C}{\partial y}$  is indeed of homogeneous degree strictly greater than 2 and thus cannot cancel  $\alpha ax^2$  out). To complete the proof we only have to mention that the condition  $a \neq 0$  is equivalent to  $A_{2,0} \neq 0$ . But this is obviously the case : we even have  $A_{2,0} = a$  according to

$$A(x + s(y), y) = x\tilde{A}(x, y) + \tilde{B}(x, y)s'(y)$$

with  $s'(0) = s(0) = 0$ . Hence  $\mathcal{E}_1 = \mathcal{E} \setminus \{A_{2,0} = 0\}$  is constructable.  $\square$

**Corollary 15.** *Let  $\mathbb{C}[x, y]_{\leq d}$  be the space of all polynomials of degree at most  $d$  and define*

$$\mathcal{P}_d := \mathcal{E}_1 \cap \left(\mathbb{C}[x, y]_{\leq d} \times \mathbb{C}[x, y]_{\leq d}\right).$$

*Then  $\mathcal{P}_d$  is a constructable set.*

Particularly  $\mathcal{P}_d$  is a finite union of smooth analytical sets.

**3.4. Strong analyticity of Dulac's procedure.** We first begin with building the map  $\mathcal{D}$  in a canonical way.

- (1) As already stated, there exists a unique germ  $s(y)$  such that  $\{x = s(y)\}$  is a separatrix of  $X_{A,B}$ .

(2) Applying the change of coordinates  $(x, y) \mapsto (x - s(y), y)$  transforms  $X_{A,B}$  into  $X_{A_1, B_1}$  where

$$\begin{aligned} B_1(x, y) &:= B(x - s(y), y) \\ A_1(x, y) &:= A(x - s(y), y) - B_1(x, y) s'(y) =: x(a_0(y) + \alpha x A_2(x, y)) \end{aligned}$$

with  $A_2(0, 0) = 1$  and  $\alpha \neq 0$ .

(3) There exists a unique holomorphic function  $y \mapsto C(y)$  such that  $C(0) = 0$  and  $(x, y) \mapsto (\frac{x}{\alpha}(1 + C(y)), y)$  transforms  $X_{A_1, B_1}$  into  $UX_{x^2, B_D}$  where

$$\begin{aligned} U(x, y) &:= \frac{a_0(y)}{B_1(0, y)} \frac{B_1(0, y) - B_1(x, y)}{\alpha x} + A_2(x, y) \\ B_D(x, y) &:= \frac{B_1(\frac{x}{\alpha}(1 + C(y)), y)}{U(x, y)}. \end{aligned}$$

This function  $C$  is the unique holomorphic solution to the (regular) linear differential equation with  $C(0) = 0$  :

$$B_1(0, y) C'(y) = (1 + C(y)) a_0(y),$$

whose solution is given by  $C(y) = e^{\int_0^y \frac{a_0(u)}{B_1(0, u)} du} - 1$ . We have  $U \in \mathbb{C}\{x, y\}^*$  since  $U(0, 0) = A_2(0, 0) = 1$ .

**Definition 16.** We define Dulac's map as

$$\mathcal{D}(A, B) := B_D - y \in \mathbb{C}\{x, y\}_1.$$

To prove that the map  $\mathcal{D}$  is strongly analytic, it is enough to prove that each step of the above construction shares this property: it should be obvious for the second and third steps. It remains to check that it is also the case for the first step.

**Lemma 17.** *The correspondence  $(A, B) \in \mathcal{E}_1 \mapsto s(y) \in \mathbb{C}\{z\}$  is strongly analytic.*

*Proof.* It is enough to prove that one can control the disc of convergence  $s$  in terms of parameters depending on  $A$  and  $B$ . Let  $(A_\epsilon, B_\epsilon)$  be some analytic family in  $\mathcal{E}_1$  with  $\epsilon \in (\mathbb{C}^p, 0)$ . The lemma is deduced from the following formal computation. Let us write

$$s(y) = \sum_{j \geq 0} s_j y^j, \quad s_0 = s_1 = 0.$$

Then for all  $n \in \mathbb{N}$  :

$$s^n(y) = \sum_{j \geq 0} \underbrace{\left( \sum_{j_1 + \dots + j_n = j} s_{j_1} \cdots s_{j_n} \right)}_{\mathcal{S}_{n,j}} y^j$$

where for  $p \geq j$  we have  $\mathcal{S}_{p,j} = 0$ . Write

$$A_\epsilon(x, y) = \sum_{n,m} a_{n,m}^\epsilon x^n y^m, \quad a_{0,0}^\epsilon = a_{1,0}^\epsilon = a_{0,1}^\epsilon = 0$$

so that

$$A_\epsilon(s_\epsilon(y), y) = \sum_{n,m} a_{n,m}^\epsilon s_\epsilon(y)^n y^m = \sum_{p \geq 0} \underbrace{\left( \sum_{j+m=p} \sum_{n \leq j} a_{n,m}^\epsilon \mathcal{S}_{n,j}^\epsilon \right)}_{\mathcal{W}(A)_p} y^p.$$

The equation defining  $s_\epsilon$ , namely  $A_\epsilon(s_\epsilon(y), y) = B_\epsilon(s_\epsilon(y), y) s'_\epsilon(y)$ , thus becomes with a similar notation for  $B_\epsilon(x, y) = \sum b_{n,m}^\epsilon x^n y^m$  :

$$\sum_{p \geq 0} \mathcal{W}(A_\epsilon)_p y^p = \sum_{p \geq 0} p s_p^\epsilon y^p + \sum_{p \geq 0} \left( \sum_{m+n-1=p} n \mathcal{W}(B_\epsilon)_m s_n^\epsilon \right) y^p.$$

After identifying the coefficients in  $y^p$  we derive

$$(3.3) \quad p s_p^\epsilon = \mathcal{W}(A_\epsilon)_p + \sum_{m+n=p+1} n \mathcal{W}(B_\epsilon)_m s_n^\epsilon.$$

Hence, in a standard fashion, for any  $p$  we have  $|s_p^\epsilon| \leq \bar{s}_p$ , where  $\bar{s}_p$  satisfies the same recurrence equation as  $s_p^\epsilon$  except that we set  $a_{m,n} = b_{m,n} = M \rho^{m+n}$ , where  $M$  is a constant and  $\rho$  a lower bound for the radius of convergence of  $A_\epsilon$  and  $B_\epsilon$ . Thus  $|s_p^\epsilon|$  is less or equal than the coefficient  $\bar{s}_p$  of  $\bar{s}$  satisfying

$$\frac{1}{1-\rho y} \times \frac{1}{1-\rho \bar{s}(y)} - 1 - \rho y - \rho \bar{s}(y) = \bar{s}'(y) \left( \frac{y}{M} + \frac{1}{1-\rho y} \times \frac{1}{1-\rho \bar{s}(y)} - 1 - \rho y - \rho \bar{s}(y) \right)$$

Since this equation admits a convergent solution with  $\bar{s}(0) = 0$ , its radius of convergence is a lower bound for the radius of convergence of the family  $s_\epsilon$ .  $\square$

#### REFERENCES

- [1] César Camacho and Paulo Sad. Invariant varieties through singularities of holomorphic vector fields. *Ann. of Math. (2)*, 115(3):579–595, 1982.
- [2] Michel Coste. Ensembles semi-algébriques. In *Real algebraic geometry and quadratic forms (Rennes, 1981)*, volume 959 of *Lecture Notes in Math.*, pages 109–138. Springer, Berlin, 1982.
- [3] Henri Dulac. Recherches sur les points singuliers des équations différentielles. *Journal de l'École Polytechnique*, 1904.
- [4] Henri Dulac. Sur les points singuliers d'une équation différentielle. *Ann. Fac. Sci. Toulouse Sci. Math. Sci. Phys. (3)*, 1:329–379, 1909.
- [5] Jean Martinet and Jean-Pierre Ramis. Classification analytique des équations différentielles non linéaires résonnantes du premier ordre. *Ann. Sci. École Norm. Sup. (4)*, 16(4):571–621 (1984), 1983.
- [6] Waclaw Sierpiński and Kazimierz Kuratowski. *Oeuvres choisies. Tome III. Théorie des ensembles et ses applications, travaux des années 1930–1966*. PWN-Éditions Scientifiques de Pologne, Warsaw, 1976.