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Random Walks in the Quarter Plane Absorbed at the Boundary : Exact and Asymptotic

Kilian Raschel*

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Abstract

Nearest neighbor random walks in the quarter plane that are absorbed when reaching the boundary are studied. The cases of positive and zero drift are considered. Absorption probabilities at a given time and at a given site are made explicit. The following asymptotics for these random walks starting from a given point (n_0, m_0) are computed : that of probabilities of being absorbed at a given site $(i, 0)$ [resp. $(0, j)$] as $i \to \infty$ [resp. $j \to \infty$], that of the distribution's tail of absorption time at x-axis [resp. y-axis], that of the Green functions at site (i, j) when $i, j \to \infty$ and $j/i \to \tan\gamma$ for $\gamma \in [0, \pi/2]$. These results give the Martin boundary of the process and in particular the suitable Doob h-transform in order to condition the process never to reach the boundary. They also show that this h-transformed process is equal in distribution to the limit as $n \to \infty$ of the process conditioned by not being absorbed at time n. The main tool used here is complex analysis.

Keywords : random walk, Green functions, absorption probabilities, hitting times, Martin boundary, Doob h-transform, boundary value problems, integral representations, steepest descent method.
AMS 2000 *Subject Classification : primary 60G50, 60G40, 31C35 ; secondary 30E20, 30E25.*

1 Introduction

The interest in random processes in open domains of $\mathbb{Z}^2$ conditioned in the sense of Doob h-transform never to reach the boundary dates back to Dyson [Dys62]. He looked at a process version of the famous Gaussian Unitary Ensemble (GUE) and observed that the process of vectors of eigenvalues of that matrix process is equal in distribution to a family of standard Brownian motions conditioned on never colliding.

After quiet years there was renewed interest in the 90s. An important class of such processes is since then studied, the so called "non-colliding" random walks, also called "vicious walkers" or "non-intersecting paths". These walks are the processes $Z(n) = (Z_1(n), \ldots, Z_k(n))_{n\geq 0}$ composed of k independent and identically distributed random walks that never leave the Weyl chamber $W = \{z \in \mathbb{R}^k : z_1 < \cdots < z_k\}$. The distances between these random walks $(Z_2(n)-Z_1(n), \ldots, Z_k(n)-Z_{k-1}(n))_{n\geq 0}$ give a $k-1$-dimensional random process whose components are positive. It turned out that these processes appear in the eigenvalues description of interesting matrix-valued stochastic processes (see e.g. [Bru91], [KO01], [KT04], [Gra99], [HW96]) and in the analysis of corner-growth model (see [Joh00] and [Joh02]). Moreover, interesting connections between non-colliding walks, random matrices and queues in tandem are the subject of [O'C03c]. Chapter 4 of [Kön05] gives besides a survey on this topic.

It turns out that it is possible to construct these processes thanks to a suitable Doob h-transform. Paper [EK08] reveals the general mechanism of this construction : the authors find there –under rather general assumptions– a positive regular function h, namely $h(z) = \prod_{1\leq i<j\leq k}(z_j - z_i)$, such that the Doob h-transformed of Z, defined by $\hat{P}^h_u(Z(n) \in \mathrm{d}v) = \mathbb{P}_u(\tau > n, Z(n) \in \mathrm{d}v)h(v)/h(u)$, where $\tau = \inf\{n > 0 : Z(n) \notin W\}$, is equal to the conditional version of Z given never exiting the Weyl chamber W. *Prima facie*, it is not only the existence of such functions h that is far from clear, but also the fact that the corresponding process $\hat{P}^h$ has anything to do with the limit as $n \to \infty$ of the conditional version of Z given $\{\tau > n\}$. To prove these results, the authors compute the asymptotic behavior of the probabilities $\mathbb{P}_u(\tau > n)$. In their paper, they also show that the rescaled conditional process $(n^{-1/2}Z(\lfloor tn\rfloor))_{t\geq 0}$ converges towards Dyson's Brownian motion.

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Most of the previous results concern only distances between independent random walks. In [KOR02], random walks with exchangeable increments and conditioned never to exit the Weyl chamber are considered. In [OY02], the authors study a certain class of random walks, namely $(X_i(n))_{1\leq i\leq k} = (|\{1 \leq m \leq n : \xi_m = i\}|)_{1\leq i\leq k}$, where $(\xi_m, m \geq 1)$ is a sequence of independent and identically distributed random variables with common distribution on $\{1, 2, \dots, k\}$, and identify in law their conditional version with a certain path-transformation of the initial process. In [O'C03a] and [O'C03b], O'Connell relates these objects to the Robinson-Schensted algorithm.

Another important area where random processes in angles of $\mathbb{Z}^d$ conditioned never to reach the boundary appear is the domain of "quantum random walks". In [Bia92a], Biane constructs a quantum Markov chain on the von Neumann algebra of $\mathrm{SU}(n)$ and interprets the restriction of this quantum Markov chain to the algebra of a maximal torus of $\mathrm{SU}(n)$ as a random walk on the lattice of integral forms on $\mathrm{SU}(n)$ with respect to this maximal torus. He proves that the restriction of the quantum Markov chain to the center of the von Neumann algebra is a Markov chain on the same lattice obtained from the preceding by conditioning it in Doob's sense to exit a Weyl chamber at infinity. In case $n = 3$, the Weyl chamber of the corresponding Lie algebra $\mathfrak{sl}_3(\mathbb{C})$ is the domain of $(\mathbb{R}_+)^2$ delimited on the one hand by the x-axis and on the other by the axis making an angle equal to $\pi/3$ with the x-axis. One gets a spatially homogeneous random walk in the interior of the weights lattice, with three transition probabilities $1/3$ as in the left side of Figure 1 ; random walk which can be of course thought as a walk in $(\mathbb{Z}_+)^2$ with transition probabilities described in the second picture of Figure 1. Biane shows that for this walk, a proper h-transform $h(x, y)$ is the dimension of the representation of $\mathfrak{sl}_3(\mathbb{C})$ with highest weight $(x - 1, y - 1)$, equal to $xy(x + y)/2$.

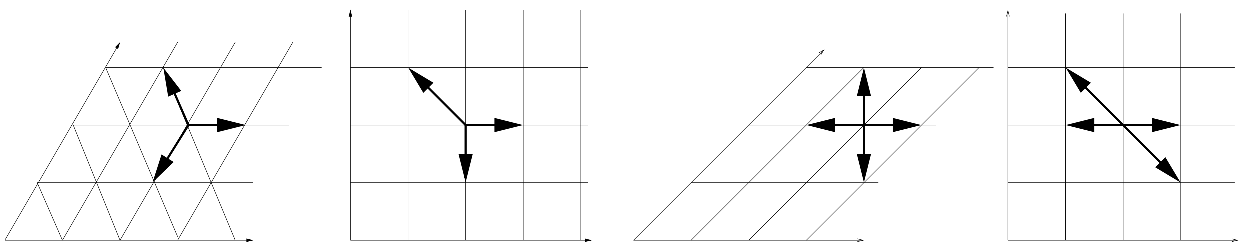

Figure 1: Walks in Weyl chambers of weights lattices of Lie algebras –above, $\mathfrak{sl}_3(\mathbb{C})$ and $\mathfrak{sp}_4(\mathbb{C})$– can be viewed as random walks on $(\mathbb{Z}_+)^d$.

Then, in [Bia92b], Biane extends these results to the case of general semi-simple connected and simply connected compact Lie groups, the basic notion being that of minuscule weight. The corresponding random walk on the weights lattice in the interior of the Weyl chamber can be obtained as follows : one draws the vector corresponding to the minuscule weight and its images under the Weyl group ; then one translates these vectors to each point of the weight lattice in the interior of the Weyl chamber and we assign to them equal probabilities $2/l$, l being the order of the Weyl group, these probabilities are the transitions of the walk. For example, in case of the Lie algebras $\mathfrak{sp}_4(\mathbb{C})$ or $\mathfrak{so}_5(\mathbb{C})$, the associated Weyl chamber and random walks are drawn in the right side of Figure 1. In [Bia92b], Biane also makes some generalizations to non-centered random walks. Nevertheless these algebraic methods do not allow the computation of the distribution of random time τ to reach the boundary ; they neither allow to relate the limit as $n \to \infty$ of the process conditioned by $\{\tau > n\}$ to a h-transformed process.

In [Bia91], Biane also computes the asymptotic of the Green functions $G_{x,y}$ for the first random walk on Figure 1, asymptotic as $x, y \to \infty$ and $y/x \to \tan(\gamma)$, for γ in $[\epsilon, \pi/2 - \epsilon]$, $\epsilon > 0$. The description of the Martin boundary for this random walk could not be completed since the asymptotic of the Green functions as $y/x \to 0$ or $y/x \to \infty$ could not be found.

Once again with a view to applying to Lie algebras, Varopoulos studies in [Var99] and [Var00] random walks in general conical domains of $\mathbb{Z}^d$, inside of which the processes are supposed to be spatially homogeneous and to have non-correlated components. He estimates the distribution of time τ to reach the boundary, he shows more precisely that $\mathbb{P}(\tau > n)$ is bounded from above and below by $n^{-\alpha}$ –up to some multiplicative constant– with a proper exponent α depending on the conical domain and on the dimension d.

In [AIM96] and [AI97], the authors are interested in passage-times moments in centered balls for reflected random walks in a quadrant homogeneous and with zero drift in the interior. They find a critical exponent, depending only on the transition probabilities, under (resp. above) which the passage-time moments are finite (resp. infinite). They also give lower bounds for the tails of the distributions of the first-passage times in centered balls for these walks, in terms of the same critical exponent as before.

As for non-homogeneous random walks in $\mathbb{Z}^d$, a recent paper by [WMM08] studies the exit time moments of cones in case of an asymptotically zero drift.

In [IR08], Ignatiuk obtains, under general assumptions and for all $d \geq 2$, the Martin boundary of some random walks in the half-space $\mathbb{Z}^{d-1} \times \mathbb{Z}_+$ killed on the boundary. Her method can unfortunately not be generalized to random walks on $(\mathbb{Z}_+)^d$, even for $d = 2$.

In this paper we restrict ourselves to spatially homogeneous random walks $(X(n), Y(n))_{n\geq 0}$ in $(\mathbb{Z}_+)^2$ with jumps at distance at most 1. We denote by $\mathbb{P}(X(n+1) = i_0 + i, Y(n+1) = j_0 + j \mid X(n) = i_0, Y(n) = j_0) = p_{(i_0,j_0),(i+i_0,j+j_0)}$ the transition probabilities. So we do the hypothesis :

(H1) *For all (i_0, j_0) such that $i_0 > 0, j_0 > 0$, $p_{(i_0,j_0),(i_0,j_0)+(i,j)}$ does not depend on (i_0, j_0) and can thus be denoted by p_{ij}.*

(H2) $p_{i,j} = 0$ *if* $|i| > 1$ *or* $|j| > 1$.

(H3) *The boundary $\{(0, 0)\} \cup \{(i, 0) : i \geq 1\} \cup \{(0, j) : j \geq 1\}$ is absorbing.*

Let

$$h_{i,n}^{n_0,m_0} = \mathbb{P}_{(n_0,m_0)} \text{(to hit } (i, 0) \text{ at time } n), \qquad \widetilde{h}_{j,n}^{n_0,m_0} = \mathbb{P}_{(n_0,m_0)} \text{(to hit } (0, j) \text{ at time } n) \tag{1}$$

be the probabilities of being absorbed at points $(i, 0)$ and $(0, j)$ at time n. Let $h(x, z)$ and $\tilde{h}(y, z)$ be their generating functions, initially defined for $|x|, |z| \leq 1$:

$$h(x, z) = \sum_{i\geq 1,n\geq 0} h_{i,n}^{n_0,m_0} x^i z^n, \qquad \widetilde{h}(y, z) = \sum_{j\geq 1,n\geq 0} \widetilde{h}_{j,n}^{n_0,m_0} y^j z^n. \tag{2}$$

Book [FIM99] studies the random walks in $(\mathbb{Z}_+)^2$ under assumptions (H1) and (H2) but not (H3) : the jump probabilities from the boundaries to the interior of $(\mathbb{Z}_+)^2$ are there not zero and the x-axis, the y-axis and $(0, 0)$ are three other domains of spatial homogeneity. Moreover, the jumps from the boundaries are supposed such that the Markov chain is ergodic. The authors elaborate in this book a profound and ingenious analytic approach to compute the generating functions of stationary probabilities of these random walks. This approach serves as a starting point for our investigation and therefore plays a crucial role : Subsections 2.1–2.3 of this paper leading to the first integral representation of the functions $h(x, z)$ and $\tilde{h}(y, z)$ are inspired from [FIM99]. Indeed, we reduce, as there, the computation of these functions to the resolve of a Riemann boundary value problem with shift. Then, we use the classical way to study this kind of problem, namely we transform it into a Riemann-Hilbert problem for which there exists a suitable and complete theory ; the conversion between Riemann problems with shift and Riemann-Hilbert problem being done thanks to the use of conformal gluing function. On closer analysis we observed that the conformal gluing function has an explicit and particularly nice form under the simplifying hypothesis

(H2') $p_{01} + p_{10} + p_{-10} + p_{0-1} = 1$.

Then we found it instructive to carry out first our analysis under this simplifying hypothesis (H2') that makes all investigations much more transparent. Therefore in this paper we restrict ourselves to the random walks under hypothesis (H1), (H2'), (H3) above and (H2"), (H4) below :

(H2") $p_{01}, p_{10}, p_{0-1}, p_{-10} \neq 0$.

(H4) The drifts are non negative :

$$M_x = p_{10} - p_{-10} \geq 0, \quad M_y = p_{01} - p_{0-1} \geq 0. \tag{3}$$

At the end of this paper we consider some extensions of the hypothesis (H2') that keep the conformal gluing function in the same nice form as for (H2').

We will be interested here in the following questions.

(1) What are $h(x, z)$ and $\tilde{h}(y, z)$ starting from (n_0, m_0) ?

(2) Let S (resp. T) be the first time of reaching the x-axis (resp. y-axis). What are the distributions' tails of S and T starting from (n_0, m_0) ? Let $\tau = T \wedge S$ be the time of absorption on the boundary –the absorption at $(0, 0)$ starting from $(n_0, m_0) \neq (0, 0)$ under hypothesis (H2') is impossible–. What is the distribution's tail of τ starting from (n_0, m_0) ?

(3) What are the absorption probabilities $h_i^{n_0,m_0} = \sum_{n=0}^{\infty} h_{i,n}^{n_0,m_0}$ and $\tilde{h}_j^{n_0,m_0} = \sum_{n=0}^{\infty} \tilde{h}_{j,n}^{n_0,m_0}$ at points $(i,0)$ and $(0,j)$? What are their asymptotic as i and j go to infinity ? What is the probability of absorption $h(1,1)+\tilde{h}(1,1) = \sum_{i\geq 1} h_i^{n_0,m_0} + \sum_{j\geq 1} \tilde{h}_j^{n_0,m_0}$?

(4) What is the suitable Doob h-transform to condition the process never to touch the boundary ? Is this Doob h-transformed process equal in distribution to the limit as $n\to\infty$ of the conditional process given $\{\tau > n\}$?

(5) What are the asymptotic of the Green functions $G_{x,y}^{n_0,m_0}$ of the mean number of visits to (x,y) starting from (n_0,m_0) as $x,y\to\infty$, $y/x\to\tan(\gamma)$ where $\gamma\in[0,\pi/2]$? What is the Martin boundary of this random walk ?

In Section 2 we find $h(x,z)$ and $\tilde{h}(y,z)$ under four different forms, all of which having an own interest and also an usefulness in the sequel ; what answers to Question (1).

The analysis of Question (2) is performed in Section 3, using $h(x,z)$ and $\tilde{h}(y,z)$ with $x=1$ and $y=1$. The behavior of the process is of course notably different in cases $M_x>0, M_y>0$ and $M_x=M_y=0$. In first case the process is not absorbed with positive probability and

$$\mathbb{P}_{(n_0,m_0)}(\tau>n)\to 1-h(1,1)-\widetilde{h}(1,1),\quad n\to\infty. \tag{4}$$

In second case the process is almost surely absorbed and we will find that :

$$\mathbb{P}_{(n_0,m_0)}(\tau>n)\sim \frac{n_0m_0}{\pi\sqrt{p_{10}p_{01}}}\frac{1}{n},\quad n\to\infty. \tag{5}$$

Question (3) is studied in Section 4 using $h(x,z)$ and $\tilde{h}(y,z)$ with $z=1$. In particular we get the probability of non absorption on the boundary that we denote by $A(n_0,m_0)$:

$$A(n_0,m_0)=1-h(1,1)-\widetilde{h}(1,1)=\left(1-\left(\frac{p_{-10}}{p_{10}}\right)^{n_0}\right)\left(1-\left(\frac{p_{0-1}}{p_{01}}\right)^{m_0}\right). \tag{6}$$

We can then reply to Question (4). If $M_x>0$ and $M_y>0$, the harmonic function in order to condition the process never to reach the boundary in Doob's sense is of course $A(n_0,m_0)$. If $M_x=M_y=0$, then the harmonic function is n_0m_0. Moreover for all $x,y>0$ and $n>m>0$,

$$\mathbb{P}_{(n_0,m_0)}((X(m),Y(m))=(x,y)\mid\tau>n)=\frac{\mathbb{P}_{(n_0,m_0)}((X(m),Y(m))=(x,y))\,\mathbb{P}_{(x,y)}(\tau>n-m)}{\mathbb{P}_{(n_0,m_0)}(\tau>n)}.$$

If $M_x>0$ and $M_y>0$, this quantity converges as $n\to\infty$ to $\mathbb{P}_{n_0,m_0}((X(m),Y(m))=(x,y))A(x,y)/A(n_0,m_0)$, thanks to (4) and (6). If $M_x=M_y=0$ it converges as $n\to\infty$ to $\mathbb{P}_{n_0,m_0}((X(m),Y(m))=(x,y))xy/(n_0m_0)$ by (5). Consequently, the Doob h-transformed process is equal in distribution to the limit as $n\to\infty$ of the process conditioned by $\{\tau>n\}$.

The harmonic function $A(n_0,m_0)$ in case $M_x>0$, $M_y>0$ (resp. n_0m_0 in case $M_x=M_y=0$) provides us with a point of the Martin boundary. To complete the study of the Martin boundary, we should find the asymptotic of the Martin kernel along all different infinite paths of the random walk. We analyze for that the asymptotic of the Green functions. In Section 4 we find the asymptotic of $h_x^{n_0,m_0}$ and of $\tilde{h}_y^{n_0,m_0}$ as $x\to\infty$ and $y\to\infty$ and in Section 5 we compute the asymptotic of $G_{x,y}^{n_0,m_0}$ as $x,y>0$, $y/x\to\tan(\gamma)$, where γ is a given angle in $[0,\pi/2]$. In [KM98], using the approach of [Mal73], it has already been done when $M_x>0$ and $M_y>0$ but in case of non-zero jump probabilities from the boundaries to the interior of $(\mathbb{Z}_+)^2$ –so that the interior of the quadrant, the x-axis, the y-axis and $(0,0)$ are four domains of spatial homogeneity– ; also, in [KS03], this approach has been successfully applied to the analysis of JS-queues. In our case of an absorbing boundary, it can be done by exactly the same methods, and even easier, using the explicit representations of functions $h(x,1)$ and $\tilde{h}(y,1)$ obtained in Section 2. In particular we will deduce that in case $M_x>0$ and $M_y>0$, all angles $\gamma\in[0,\pi/2]$ will correspond to different points of the Martin boundary, which will be therefore homeomorphic to the segment $[0,\pi/2]$. Note that for the angle γ of the drift (i.e. $\tan(\gamma)=M_y/M_x$) the asymptotic of the Martin kernel is proportional to $A(n_0,m_0)$. As for case $M_x=M_y=0$, we prove in Section 5 that $G_{x,y}^{n_0,m_0}\sim 4\sqrt{p_{10}p_{01}}n_0m_0xy/(\pi(p_{01}x^2+p_{10}y^2)^2)$ for any angle $\gamma\in[0,\pi/2]$. In other words, the function n_0m_0 is in this case the unique point of the Martin boundary.

In a next work, we will answer Questions (1)–(5) without making hypothesis (H2') but only under (H2) and supposing that $M_x=\sum_{i,j} ip_{ij}>0$ and $M_y=\sum_{i,j} jp_{ij}>0$. Furthermore, motivated by Biane's works on "quantum random walks", we will also, in an other work, analyze these questions for the walks with transition probabilities drawn in Figure 1, that is to say random walks in the Weyl chambers of $\mathfrak{sl}_3(\mathbb{C})$ and $\mathfrak{sp}_4(\mathbb{C})$ verifying $M_x = M_y = 0$. One of the main difference between these walks and those studied here is the fact that the underlying conformal gluing functions become quite elaborated.

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2 Generating functions of absorption probabilities

2.1 A functional equation

We start here by establishing a functional equation that the generating functions of the absorption probabilities verify. Let :

$$G(x,y,z) = \sum_{i,j\geq 1, n\geq 0} \mathbb{P}_{(n_0,m_0)}\left((X(n),Y(n)) = (i,j)\right) x^{i-1} y^{j-1} z^n. \tag{7}$$

We write now the following functional equation (8), on which all our study is based :

$$Q(x,y,z)\, G(x,y,z) = h(x,z) + \widetilde{h}(y,z) - x^{n_0} y^{m_0}, \tag{8}$$

where h and $\tilde{h}$ are defined in (2) and Q is the following polynomial, depending only on the walk's transition probabilities :

$$Q(x,y,z) = xyz\Big(p_{10}x + p_{-10}x^{-1} + p_{01}y + p_{0-1}y^{-1} - z^{-1}\Big). \tag{9}$$

The functions $h(x,z)$, $\tilde{h}(y,z)$ for $|x|,|y|,|z| \leq 1$, and $G(x,y,z)$ for $|x|,|y| < 1$, $|z| \leq 1$, are unknown. Equation (8) has a meaning in (at least) $\{x,y \in \mathbb{C} : |x| < 1, |y| < 1, |z| \leq 1\}$. Note that the proof of (8) comes from writing that for $k,l,p \in (\mathbb{Z}_+)^2$:

$$\begin{aligned}\mathbb{P}_{(n_0,m_0)}\left((X(p+1),Y(p+1)) = (k,l)\right) &= \sum_{i,j\geq 1} \mathbb{P}_{(n_0,m_0)}\left((X(p),Y(p)) = (i,j)\right) p_{(i,j),(k,l)} \\ + \quad \sum_{i\geq 1} \mathbb{P}_{(n_0,m_0)}\left((X(p),Y(p)) = (i,0)\right) \delta_{(k,l)}^{(i,0)} &+ \sum_{j\geq 1} \mathbb{P}_{(n_0,m_0)}\left((X(p),Y(p)) = (0,j)\right) \delta_{(k,l)}^{(0,j)},\end{aligned}$$

where $\delta_{(a,b)}^{(c,d)} = 1$ if $a = c$ and $b = d$, otherwise 0. It remains to multiply by $x^k y^l z^{p+1}$ and then to sum with respect to k,l,p.

2.2 The algebraic curve $Q(x,y,z) = 0$.

The polynomial (9) can be written alternatively :

$$Q(x,y,z) = a(x,z)\,y^2 + b(x,z)\,y + c(x,z) = \widetilde{a}(y,z)\,x^2 + \widetilde{b}(y,z)\,x + \widetilde{c}(y,z),$$

where

$$\begin{array}{rclrcl} a(x,z) &=& zp_{01}x, & \widetilde{a}(y,z) &=& zp_{10}y, \\ b(x,z) &=& zp_{10}x^2 - x + zp_{-10}, & \widetilde{b}(y,z) &=& zp_{01}y^2 - y + zp_{0-1}, \\ c(x,z) &=& zp_{0-1}x, & \widetilde{c}(y,z) &=& zp_{-10}y. \end{array}$$

With these notations, building the algebraic function $Y(x,z)$ defined by $Q(x,y,z) = 0$ is tantamount to the construction of the square root of the four degree polynomial $d(x,z) = b(x,z)^2 - 4a(x,z)\,c(x,z)$. Indeed, $Q(x,y,z) = 0$ is equivalent to $(b(x,z) + 2a(x,z)\,y)^2 = d(x,z)$. As for any non zero polynomial, there are two branches of the square root of d. Each determination leads to a well defined (i.e. single valued) and meromorphic function on the complex plane $\mathbb{C}$ appropriately cut. As usual, these cuts are constructed using the roots of d, called the branched points. In our case, we have an explicit expression for these branched points, some properties of which are collected in the following :

Lemma 1. *Define $z_1 = 1/(2(p_{10}p_{-10})^{1/2} + 2(p_{01}p_{0-1})^{1/2})$ (and note that $z_1 \geq 1$, with equality if and only if the two drifts are equal to zero). For $z \in [0, z_1]$, the four roots of $d(x, z) = 0$ are real and non negative. For $z \in]0, z_1[$, these roots are mutually different and positive. We call them in such a way that for $z \in]0, z_1[$, $0 < x_1(z) < x_2(z) < x_3(z) < x_4(z)$. Their explicit expressions are :*

$$x_{2,3}(z) = \frac{z^{-1} - 2\sqrt{p_{01}p_{0-1}}}{2p_{10}} \pm \sqrt{\left(\frac{z^{-1} - 2\sqrt{p_{01}p_{0-1}}}{2p_{10}}\right)^2 - \frac{p_{-10}}{p_{10}}},$$

$$x_{1,4}(z) = \frac{z^{-1} + 2\sqrt{p_{01}p_{0-1}}}{2p_{10}} \pm \sqrt{\left(\frac{z^{-1} + 2\sqrt{p_{01}p_{0-1}}}{2p_{10}}\right)^2 - \frac{p_{-10}}{p_{10}}}.$$

They are tied together by $x_1(z)x_4(z) = x_2(z)x_3(z) = p_{-10}/p_{10}$ and verify $x_1(z), x_2(z) \in]0, (p_{-10}/p_{10})^{1/2}[$, $x_3(z), x_4(z) \in](p_{-10}/p_{10})^{1/2}, +\infty[$ for $z \in]0, z_1[$. Also, $x_1(0) = x_2(0) = 0$ and $x_3(0) = x_4(0) = \infty$. Moreover, if $M_y > 0$ then $0 < x_1(1) < x_2(1) < 1 < x_3(1) < x_4(1)$ and if $M_y = 0$ then $0 < x_1(1) < x_2(1) = 1 = x_3(1) < x_4(1)$. At last, the x_i vary continuously and monotonously with respect to z.

Proof. All the facts described in Lemma 1 are based on the explicit expression of the branched points that we get by solving $d(x, z) = 0$. Here d is a four degree polynomial that we can split in two polynomials of degree two : $d(x, z) = (b(x, z) - 2zx(p_{01}p_{0-1})^{1/2})(b(x, z) + 2zx(p_{01}p_{0-1})^{1/2})$, since a and c are proportional. □

This precise knowledge of the branched points allows us to complete the construction of the algebraic function Y : this function has two branches, each of them being well defined and meromorphic on $\mathbb{C} \setminus [x_1(z), x_2(z)] \cup [x_3(z), x_4(z)]$. We can write the analytic expression of these two branches Y_0 and Y_1 of Y : $Y_0(x, z) = Y_-(x, z)$ and $Y_1(x, z) = Y_+(x, z)$ where :

$$Y_{\pm}(x, z) = \frac{-b(x, z) \pm \sqrt{d(x, z)}}{2a(x, z)}.$$

Note that just above, and in fact throughout the whole paper, we chose the principal determination of the logarithm as soon as we use the complex logarithm ; in this case to define the square root.

For more details about the construction of algebraic functions, see for instance Book [SG69].

In a similar way, the functional equation (8) defines also an algebraic function $X(y, z)$. All the results concerning $X(y, z)$ can be deduced from those obtained for $Y(x, z)$ after a proper change of the parameters, namely p_{-10} (resp. p_{10}, p_{0-1}, p_{01}) in p_{0-1} (resp. p_{01}, p_{-10}, p_{10}).

To conclude this part, we give a lemma that clarifies some properties of the functions X and Y, that will be useful in the sequel. This lemma is an adaptation of results of [FIM99], so we refer to this book for the proof.

Lemma 2. *The two following equalities hold : $Y_1(1, 1) = 1$ and $Y_0(1, 1) = p_{0-1}/p_{01}$. Moreover, the next properties are valid for all $z \in]0, z_1]$.*

(i) *For all $x \in \mathbb{C} \setminus [x_1(z), x_2(z)] \cup [x_3(z), x_4(z)]$, $Y_0(x, z)Y_1(x, z) = p_{0-1}/p_{01}$, $|Y_0(x, z)| \leq (p_{0-1}/p_{01})^{1/2} \leq |Y_1(x, z)|$.*

(ii) *If $|x| < (p_{-10}/p_{10})^{1/2}$, then $X_0(Y_0(x, z), z) = X_0(Y_1(x, z), z) = x$ and $X_1(Y_0(x, z), z) = X_1(Y_1(x, z), z) = p_{-10}/(p_{10}x)$.*

(iii) *If $|x| > (p_{-10}/p_{10})^{1/2}$, then $X_0(Y_0(x, z), z) = X_0(Y_1(x, z), z) = p_{-10}/(p_{10}x)$ and $X_1(Y_0(x, z), z) = X_1(Y_1(x, z), z) = x$.*

(iv) *As $x \to \infty$, $Y_0(x, z) = -p_{0-1}/(p_{10}x) - p_{0-1}/(p_{10}x)^2 + \mathcal{O}(1/x^3)$ and $Y_1(x, z) = -p_{10}x/p_{01} + 1/p_{01} + \mathcal{O}(1/x)$.*

2.3 Riemann boundary problem and conformal gluing functions

Throughout all Subsection 2.3, z lies in $]0, z_1]$, unless otherwise specified. Using the notations of Subsections 2.1 and 2.2, we define the two following curves :

$$\mathcal{L}_z = Y_0\left(\left[\overleftrightarrow{x_1(z), x_2(z)}\right], z\right), \quad \mathcal{M}_z = X_0\left(\left[\overleftrightarrow{y_1(z), y_2(z)}\right], z\right). \tag{10}$$

Just above, we use the notation $[\overleftarrow{\overrightarrow{u,v}}]$ for the contour $[u,v]$ traversed from u to v along the upper edge of the slit $[u,v]$ and then back to u along the lower edge of the slit.

A worthwhile sight is that under the hypothesis (H2'), these two curves are quite simple since they are in fact just two circles, centered at the origin and of radius $\tilde{r}=(p_{0-1}/p_{01})^{1/2}\leq 1$ and $r=(p_{-10}/p_{10})^{1/2}\leq 1$ respectively. One verifies these facts directly : if $t\in[x_1(z),x_2(z)]$, then $d(t,z)\in\mathbb{R}_-$ and so $|-b(t,z)\pm d(t,z)^{1/2}|^2=4a(t,z)c(t,z)$. Thus, $|Y_{0,1}(t,z)|^2=c(t,z)/a(t,z)=p_{0-1}/p_{01}$ and $\mathcal{L}_z=\mathcal{C}(0,\tilde{r})$; likewise, we prove that $\mathcal{M}_z=\mathcal{C}(0,r)$.

The reason why we have introduced these curves appears now : the functions h (of the argument x) and $\tilde{h}$ (of the argument y), defined in (2), verify the following boundary conditions on $\mathcal{M}_z=\mathcal{C}(0,r)$ and $\mathcal{L}_z=\mathcal{C}(0,\tilde{r})$:

$$\begin{aligned}
\forall t\in\mathcal{C}(0,r):\quad h(t,z)-h(\bar{t},z) &= t^{n_0}Y_0(t,z)^{m_0}-\bar{t}^{n_0}Y_0(\bar{t},z)^{m_0}, \\
\forall t\in\mathcal{C}(0,\tilde{r}):\quad \tilde{h}(t,z)-\tilde{h}(\bar{t},z) &= X_0(t,z)^{n_0}t^{m_0}-X_0(\bar{t},z)^{n_0}\bar{t}^{m_0}.
\end{aligned}\tag{11}$$

The way to obtain (11) and the analogue for $\tilde{h}$ is described in [FIM99], so we refer to this book for the details ; nevertheless, we recall here briefly the explanations : taking $|y|\leq 1$ and $x=X_0(y,z)$ (whose modulus is less than one thanks to Lemma 2) in (8) leads to :

$$h(X_0(y,z),z)+\tilde{h}(y,z)-X_0(y,z)^{n_0}y^{m_0}=0.$$

We let now y go successively to the upper and lower edge of $[y_1(z),y_2(z)]$ and we make the difference of these two equations so obtained. Since the slit $[y_1(z),y_2(z)]$ is included in the unit disc where $\tilde{h}$ is holomorphic, $\tilde{h}$ vanishes and we find that for $y\in[y_1(z),y_2(z)]$,

$$h(X_0(y,z),z)-h(X_1(y,z),z)=X_0(y,z)^{n_0}y^{m_0}-X_1(y,z)^{n_0}y^{m_0}.$$

According to Lemma 2, for any $y\in[y_1(z),y_2(z)]$ we have $Y_0(X_0(y,z),z)=y$; so we obtain that for any $t\in\mathcal{M}_z$, $h(t,z)-h(\bar{t},z)=Y_0(t,z)^{m_0}(t^{n_0}-\bar{t}^{n_0})$. To complete the proof of (11), it remains to show that $Y_0(t,z)=Y_0(\bar{t},z)$ for $t\in\mathcal{M}_z$; but this is once again a consequence of Lemma 2 since we proved there that for $y\in[y_1(z),y_2(z)]$, $Y_0(X_1(y,z),z)=y$.

For any $z\in[0,1]$, the function h of the argument x, as a generating function of probabilities, is well defined on the closed unit disc $|x|\leq 1$, holomorphic inside it and continuous up to its boundary. With Lemma 2, the curve $\mathcal{M}_z$ is included in the closed unit disc. Now we have the problem *to find h holomorphic inside $\mathcal{M}_z$, continuous up to the boundary and verifying the boundary condition (11). In addition $h(0,z)=0$ for all $z\in[0,1]$.*

Problems with boundary conditions like (11) are called Riemann boundary value problems with shift. The classical way to study this kind of problems is to reduce them to Riemann-Hilbert problems, for which there exists a suitable and complete theory. The conversion between Riemann problems with shift and Riemann-Hilbert problems is done thanks to the use of conformal gluing functions, notion defined just below. For details about boundary value problems, we refer to [Lu93].

Definition 3. *Let $\mathcal{C}$ be a simple closed curve, symmetrical with respect to the real axis. Denote by $\mathcal{G}_\mathcal{C}$ the interior of the bounded domain delimited by $\mathcal{C}$. w is called a conformal gluing function (CGF) for the curve $\mathcal{C}$ if* (i) *w is meromorphic in $\mathcal{G}_\mathcal{C}$, continuous up to its boundary* (ii) *w establishes a conformal mapping of $\mathcal{G}_\mathcal{C}$ onto the complex plane cut along a smooth arc U* (iii) *for all $t\in\mathcal{C}$, $w(t)=w(\bar{t})$.*

In the general case, finding a CGF associated to some curve without strong hypothesis on this curve is a quite difficult problem, we will besides discuss this fact in Section 6 ; however, in our case, the curves $\mathcal{M}_z$ and $\mathcal{L}_z$ are circles and we have therefore an explicit expression of possible CGF :

Proposition 4. *CGF for the curves $\mathcal{M}_z=\mathcal{C}(0,r)$ and $\mathcal{L}_z=\mathcal{C}(0,\tilde{r})$ are equal to :*

$$w(t)=\frac{1}{2}\left(t+\frac{r^2}{t}\right),\qquad \tilde{w}(t)=\frac{1}{2}\left(t+\frac{\tilde{r}^2}{t}\right).$$

Proof. We verify easily that w and $\tilde{w}$ are indeed CGF, following the different *item* of Definition 3. First, w is manifestly holomorphic on $\mathbb{C}\setminus\{0\}$ and has a simple pole at 0, that proves (i). Moreover, w is a conformal mapping from $\mathcal{D}(0,r)$ onto $\mathbb{C}\setminus U$, where U is the segment $[-r,r]$, hence (ii). At last, (iii) comes from remarking that $w(re^{i\theta})=r\cos(\theta)=w(re^{-i\theta})$. Of course, the proof is similar for $\tilde{w}$, so we omit it. □

2.4 Integral representations of the generating functions

In Subsection 2.3, we have showed that h verifies a Riemann problem with shift with boundary condition (11) ; we will now see how we can deduce from this an explicit expression for h. In fact, we will obtain four different explicit formulations for h, in Propositions 5, 6 , 9 and 10 ; each of these expressions will have an own interest and will serve in the sequel –for instance Proposition 9 will be the starting point of Section 4, Proposition 10 the one of Section 3–.

Proposition 5. *Let $\mathcal{C}(0,r)$ be the circle of the radius $r = (p_{-10}/p_{10})^{1/2}$. The function h admits the following integral expression :*

$$h\left(x,z\right)=\frac{x}{2\pi i}\int_{\mathcal{C}(0,r)}t^{n_0}Y_0\left(t,z\right)^{m_0}\left(\frac{1}{t\left(t-x\right)}+\frac{1}{xt-r^2}\right)\mathrm{d}t.$$

Above, x belongs to the open centered disc of radius r, and z to $]0,z_1]$, z_1 being defined in Lemma 1.

Proof. This proposition corresponds to the standard way to obtain a Riemann-Hilbert problem starting from a Riemann problem with shift.

We recall from Definition 3 and Proposition 4 that w is a conformal mapping from the open disc $D(0,r)$ onto $\mathbb{C}\setminus U$, where U is the segment $[w(X(y_1(z),z)), w(X(y_2(z),z))] = [-r,r]$. Therefore, the function w admits an inverse from $\mathbb{C}\setminus[-r,r]$ onto $D(0,r)$, inverse that we call v. We can here give the explicit expression of v : it is equal to $v(w) = w-(w^2-r^2)^{1/2}$.

If we denote by $v^+(w)$ (resp. $v^-(w)$) the limit value of $v(y)$ when $y\to w$ from the upper half plane $\{s\in\mathbb{C} : \mathrm{Im}(s)>0\}$ (resp. lower half plane $\{s\in\mathbb{C} : \mathrm{Im}(s)<0\}$), then $v^+(U) = \mathcal{C}(0,r)\cap\{t\in\mathbb{C} : \mathrm{Im}(t)<0\}$ and $v^-(U) = \mathcal{C}(0,r)\cap\{t\in\mathbb{C} : \mathrm{Im}(t)>0\}$. This is why we can, thanks to the function v, rewrite the boundary condition (11) in terms of $\phi = h\circ v$ as follows :

$$\phi^+\left(w\right)-\phi^-\left(w\right)=v^+\left(w\right)^{n_0}Y_0\left(v^+\left(w\right),z\right)^{m_0}-v^-\left(w\right)^{n_0}Y_0\left(v^-\left(w\right),z\right)^{m_0},\quad w\in U, \tag{12}$$

the advantage of this new formulation being that we have now to solve a more classical Riemann-Hilbert problem. The properties of h (as a generating function of probabilities) and v are such that ϕ has to be sought among the functions holomorphic on $\mathbb{C}\setminus U$ with a finite limit at infinity and bounded near the ends of U. Thus, the index (see e.g. [Lu93]) of this Riemann-Hilbert problem is equal to zero, what in concrete terms means that two solutions of the boundary value problem with boundary condition (12), or equivalently (11), differ by a constant ; the constant will be fixed by using the fact that $h(0,z) = 0$.

Using the theory of Riemann-Hilbert problems developed for instance in [Lu93], we obtain that h is equal, up to an additive constant, to :

$$\frac{1}{2\pi i}\int_U\left(v^+\left(w\right)^{n_0}Y_0\left(v^+\left(w\right),z\right)^{m_0}-v^-\left(w\right)^{n_0}Y_0\left(v^-\left(w\right),z\right)^{m_0}\right)\frac{1}{w-w\left(x\right)}\mathrm{d}w.$$

So that, taking account of the equality $h(0,z) = 0$, we obtain that h is equal to :

$$\frac{1}{2\pi i}\int_U\left(v^+\left(w\right)^{n_0}Y_0\left(v^+\left(w\right),z\right)^{m_0}-v^-\left(w\right)^{n_0}Y_0\left(v^-\left(w\right),z\right)^{m_0}\right)\left(\frac{1}{w-w\left(x\right)}-\frac{1}{w-w\left(0\right)}\right)\mathrm{d}w. \tag{13}$$

Then, we take the notation $\phi(t,x) = w'(t)/(w(t)-w(x)) - w'(t)/(w(t)-w(0))$ and we make the change of variable $w = w(t)$ in (13) :

$$\begin{aligned}
&\frac{1}{2\pi i}\int_U\left(v^+\left(w\right)^{n_0}Y_0\left(v^+\left(w\right),z\right)^{m_0}-v^-\left(w\right)^{n_0}Y_0\left(v^-\left(w\right),z\right)^{m_0}\right)\left(\frac{1}{w-w\left(x\right)}-\frac{1}{w-w\left(0\right)}\right)\mathrm{d}w\\
=\;&\frac{1}{2\pi i}\int_{v^+(U)}t^{n_0}Y_0\left(t,z\right)^{m_0}\phi\left(t,x\right)\mathrm{d}t-\left(-\frac{1}{2\pi i}\int_{v^-(U)}t^{n_0}Y_0\left(t,z\right)^{m_0}\phi\left(t,x\right)\mathrm{d}t\right)\\
=\;&\frac{1}{2\pi i}\int_{\mathcal{C}(0,r)}t^{n_0}Y_0\left(t,z\right)^{m_0}\phi\left(t,x\right)\mathrm{d}t,
\end{aligned}$$

since $v^+(U)\cup v^-(U) = \mathcal{C}(0,r)$, as written at the beginning of the proof. To close the proof of Proposition 5, it suffices to write the partial fraction expansion of ϕ, namely $\phi(t,x) = x/(t(t-x)) + x/(tx-r^2)$. ☐

We transform now the integral on $\mathcal{M}_z = \mathcal{C}(0,r)$ obtained in Proposition 5 into an integral on the cut $[x_1(z), x_2(z)]$. We start by giving the definition :

$$\mu_{m_0}(t,z) = \frac{1}{(2a(t,z))^{m_0}} \sum_{k=0}^{\lfloor (m_0-1)/2 \rfloor} \binom{m_0}{2k+1} d(t,z)^k \left(-b(t,z)\right)^{m_0-(2k+1)}. \tag{14}$$

The function μ_{m_0} is such that for all t in $[x_1(z), x_2(z)] \pm 0 \cdot i$, $Y_0(t,z)^{m_0} - \overline{Y_0(t,z)}^{m_0} = \mp 2i(-d(t,z))^{1/2}\mu_{m_0}(t,z)$.

An application of residue theorem in Proposition 5 and the use of the definition (14) of μ_{m_0} allow to obtain :

Proposition 6. *The function h admits the following integral expression :*

$$h(x,z) = x^{n_0} Y_0(x,z)^{m_0} + \frac{x}{\pi} \int_{x_1(z)}^{x_2(z)} t^{n_0} \left(\frac{1}{t(t-x)} + \frac{1}{xt - r^2} \right) \mu_{m_0}(t,z) \sqrt{-d(t,z)} \mathrm{d}t. \tag{15}$$

Above, x belongs to the open centered disc of radius r, and $z \in]0, z_1]$, z_1 being defined in Lemma 1.

Proof. We take here, as in the proof of Proposition 5, the notation $\phi(t,x) = x/(t(t-x)) + x/(tx - r^2)$. Consider the contour $\mathcal{H}_\epsilon = \mathcal{M}_\epsilon \cup \mathcal{S}_\epsilon^1 \cup \mathcal{S}_\epsilon^2 \cup \mathcal{C}_\epsilon^1 \cup \mathcal{C}_\epsilon^2 \cup \mathcal{D}_\epsilon^1 \cup \mathcal{D}_\epsilon^2$, drawn in Figure 2. The following facts hold :

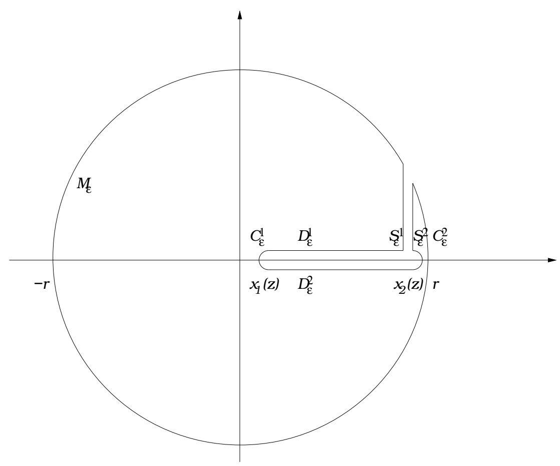

Figure 2: Contour of integration

(i) $\int_{\mathcal{C}(0,r)} t^{n_0} Y_0(t,z)^{m_0} \phi(t,x) \mathrm{d}t = \lim_{\epsilon \to 0} \int_{\mathcal{M}_\epsilon} t^{n_0} Y_0(t,z)^{m_0} \phi(t,x) \mathrm{d}t$, thanks to the continuity of the integrand on the circle $\mathcal{C}(0,r)$,

(ii) the residue theorem gives that for all $\epsilon > 0$ sufficiently small and all $x \in D(0,r) \setminus [x_1(z), x_2(z)]$: $\int_{\mathcal{H}_\epsilon} (t^{n_0-1} Y_0(t,z)^{m_0})/(t-x) \mathrm{d}t = 2\pi i x^{n_0-1} Y_0(x,z)^{m_0}$,

(iii) the integral on $\mathcal{S}_\epsilon^1 \cup \mathcal{S}_\epsilon^2$ goes to zero as ϵ goes to zero. Indeed, the integrand is holomorphic in the neighborhood of $\mathcal{S}_\epsilon^1 \cup \mathcal{S}_\epsilon^2$ and for this reason, $\lim_{\epsilon \to 0} \int_{\mathcal{S}_\epsilon^1} t^{n_0} Y_0(t,z)^{m_0} \phi(t,x) \mathrm{d}t = -\lim_{\epsilon \to 0} \int_{\mathcal{S}_\epsilon^2} t^{n_0} Y_0(t,z)^{m_0} \phi(t,x) \mathrm{d}t$. Also, for $k = 1, 2$, $\lim_{\epsilon \to 0} \int_{\mathcal{C}_\epsilon^k} t^{n_0} Y_0(t,z)^{m_0} \phi(t,x) \mathrm{d}t = 0$ since the integrand is integrable in the neighborhood of the branched points $x_1(z)$ and $x_2(z)$.

(iv) $\lim_{\epsilon \to 0} \int_{\mathcal{D}_\epsilon^1 \cup \mathcal{D}_\epsilon^2} t^{n_0} Y_0(t,z)^{m_0} \phi(t,x) \mathrm{d}t = \int_{x_1(z)}^{x_2(z)} t^{n_0} (Y_0(t,z)^{m_0} - \overline{Y_0(t,z)}^{m_0}) \phi(t,x) \mathrm{d}t$, thanks to the algebricity of the function Y_0.

If we bring together all these facts, we obtain the equality :

$$\int_{\mathcal{C}(0,r)} t^{n_0} Y_0(t,z)^{m_0} \phi(t,x) \,\mathrm{d}t = x^{n_0} Y_0(x,z)^{m_0} - \frac{1}{2\pi i} \int_{x_1(z)}^{x_2(z)} t^{n_0} \left(Y_0(t,z)^{m_0} - \overline{Y_0(t,z)}^{m_0} \right) \phi(t,x) \,\mathrm{d}t,$$

from which Proposition 6 follows immediately, using the definitions of μ_{m_0} and ϕ. □

We carry on with the simplifications of the explicit expression of the function h. The formulation (15) is nearly satisfactory but has yet a defect : h is a function holomorphic in the neighborhood of $[x_1(z), x_2(z)]$, but is written in (15) as the sum of two functions which are not holomorphic but algebraic in the neighborhood of $[x_1(z), x_2(z)]$. The next lemma overcomes this fact :

Lemma 7. *For $x \in \mathbb{C} \setminus [x_1(z), x_2(z)] \cup [x_3(z), x_4(z)]$ and $z \in]0, z_1]$, the following equality holds :*

$$\begin{aligned} x^{n_0} Y_0(x,z)^{m_0} + \frac{x}{\pi} \int_{x_1(z)}^{x_2(z)} \frac{t^{n_0-1}\mu_{m_0}(t,z)}{t-x} \sqrt{-d(t,z)} \mathrm{d}t &= \frac{x}{\pi} \int_{x_3(z)}^{x_4(z)} \frac{t^{n_0-1}\mu_{m_0}(t,z)\sqrt{-d(t,z)}}{t-x} \mathrm{d}t \\ &+ xP_\infty\left(x \mapsto x^{n_0-1}Y_0(x,z)^{m_0}\right)(x). \end{aligned}$$

Above, $P_\infty(x \mapsto x^{n_0-1}Y_0(x,z)^{m_0})$ denotes the principal part at infinity of the meromorphic function at infinity $x \mapsto x^{n_0-1}Y_0(x,z)^{m_0}$; in other words, the polynomial part of the Laurent expansion at infinity of this function. In particular, $(x,z) \mapsto xP_\infty(x \mapsto x^{n_0-1}Y_0(x,z)^{m_0})(x)$ is a polynomial in the two variables (x,z). For more comments about this quantity, see Remark 8.

Proof. Consider the contour $\mathcal{C}_\epsilon$ drawn in Figure 3 and apply on it the residue theorem at infinity (a precise

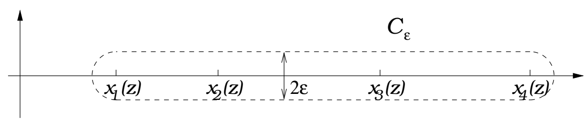

Figure 3: Contour of residue theorem

statement of this theorem can be found e.g. in [Cha90]) to the function $t \mapsto t^{n_0-1}Y_0(t,z)^{m_0}$; we obtain that for all x in the unbounded domain delimited by $\mathcal{C}_\epsilon$,

$$\frac{x}{2\pi i} \int_{\mathcal{C}_\epsilon} \frac{t^{n_0-1}Y_0(t,z)^{m_0}}{t-x} \mathrm{d}t = x^{n_0}Y_0(x,z)^{m_0} - xP_\infty\left(x \mapsto x^{n_0-1}Y_0(x,z)^{m_0}\right)(x), \tag{16}$$

where, if f is a function meromorphic at infinity, $P_\infty(f)$ denotes its principal part at infinity. On the other hand, using the algebricity of Y_0 and the definition of μ_{m_0}, we get :

$$\begin{aligned} \lim_{\epsilon\to 0} \frac{x}{2\pi i} \int_{\mathcal{C}_\epsilon} \frac{t^{n_0-1}Y_0(t,z)^{m_0}}{t-x} \mathrm{d}t = & - \frac{x}{\pi} \int_{x_1(z)}^{x_2(z)} \frac{t^{n_0-1}\mu_{m_0}(t,z)\sqrt{-d(t,z)}}{t-x} \mathrm{d}t \\ & + \frac{x}{\pi} \int_{x_3(z)}^{x_4(z)} \frac{t^{n_0-1}\mu_{m_0}(t,z)\sqrt{-d(t,z)}}{t-x} \mathrm{d}t. \end{aligned}$$

We conclude by taking the limit when ϵ goes to zero in (16). □

Remark 8. Lemma 2 gives that $x^{n_0-1}Y_0(x,z)^{m_0} \sim cx^{n_0-m_0-1}$ as $x \to \infty$, where c is a non zero constant. Thus, if $n_0 \leq m_0$, then the principal part is equal to zero, whereas if $n_0 > m_0$, then the degree of the principal part is $n_0 - m_0 - 1$ and the degree of $xP_\infty(x \mapsto x^{n_0-1}Y_0(x,z)^{m_0})(x)$ equal to $n_0 - m_0$. For briefness, we set $P(x,z) = xP_\infty(x \mapsto x^{n_0-1}Y_0(x,z)^{m_0})(x)$.

The Lemma 7 allows to write, in accordance with (15),

$$h(x,z) = \frac{x}{\pi}\left(\int_{x_3(z)}^{x_4(z)} \frac{t^{n_0}\mu_{m_0}(t,z)\sqrt{-d(t,z)}}{t(t-x)} \mathrm{d}t - \int_{x_1(z)}^{x_2(z)} \frac{t^{n_0}\mu_{m_0}(t,z)\sqrt{-d(t,z)}}{r^2-tx} \mathrm{d}t\right) + P(x,z),$$

where the polynomial $P(x,z)$ is defined in Remark 8. Instead of the two integrals above, we can write just one, making a simple change of variable based on the two properties described below :

(i) In conformity with Lemma 1 , the branched points verify $x_2(z)x_3(z) = x_1(z)x_4(z) = r^2$. Thus, by the change of variable r^2/t in the second integral above, we can express the integral part of h as a single integral between $x_3(z)$ and $x_4(z)$.

(ii) The polynomials a, b, c, d verify an interesting relationship with respect to the transformation $t \mapsto r^2/t$. Indeed, if f stands for a, b or c, we easily verify that $f(r^2/t, z) = (r^2/t^2) f(t, z)$, so that we also have $d(r^2/t, z) = (r^4/t^4) d(t, z)$. In particular, we get $\mu_{m_0}(r^2/t, z) = (t^2/r^2)\mu_{m_0}(t, z)$.

With these remarks we obtain :

Proposition 9. *The function h admits the following integral expression :*

$$h\left(x,z\right)=\frac{x}{\pi}\int_{x_3(z)}^{x_4(z)}\left(t^{n_0}-\left(\frac{r^2}{t}\right)^{n_0}\right)\frac{\mu_{m_0}\left(t,z\right)\sqrt{-d\left(t,z\right)}}{t\left(t-x\right)}\mathrm{d}t+xP_\infty\left(x\mapsto x^{n_0-1}Y_0\left(x,z\right)^{m_0}\right)\left(x\right).$$

Above, x belongs to the open centered disc of radius r, z to $]0, z_1]$, z_1 being defined in Lemma 1, and P_∞ is the principal part at infinity, defined in Lemma 7 and Remark 8.

2.5 Chebyshev polynomials

We close the study of the explicit expressions of h by making a –last but– quite natural change of variable in the integral (15). Define $\hat{b}(t, z) = b(t, z)/(4a(t, z)c(t, z))^{1/2}$. Then $t \mapsto \hat{b}(t, z)$ is clearly a diffeomorphism between $]x_1(z), x_2(z)[$ (resp. $]x_3(z), x_4(z)[$) and $]-1, 1[$. Moreover, μ_{m_0} expresses oneself with a more natural way in the variable $\hat{b}$ since the following equality holds :

$$\mu_{m_0}\left(t,z\right)\sqrt{-d\left(t,z\right)}=\left(\frac{c\left(t,z\right)}{a\left(t,z\right)}\right)^{m_0/2}U_{m_0-1}\left(-\widehat{b}\left(t,z\right)\right)\sqrt{1-\widehat{b}\left(t,z\right)^2}, \tag{17}$$

where the U_n, $n \in \mathbb{N}$, are the Chebyshev polynomials of the second kind. We recall that they are the orthogonal polynomials associated to the weight $t \mapsto (1 - t^2)^{1/2} 1_{]-1,1[}(t)$ and that their explicit expression is :

$$U_n\left(t\right)=\frac{\left(t+\sqrt{t^2-1}\right)^{n+1}-\left(t-\sqrt{t^2-1}\right)^{n+1}}{2\sqrt{t^2-1}}=\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n+1}{2k+1}\left(t^2-1\right)^k t^{n-2k},\quad n\in\mathbb{N}. \tag{18}$$

We also recall two properties of the Chebyshev polynomials of the second kind that we will especially use : first that they have the parity of their order : for all n in $\mathbb{N}$ and all u in $\mathbb{C}$, $U_n(-u) = (-1)^n U_n(u)$; second their expansion in the neighborhood of 1 : $U_n(u) = (n+1)(1+n(n+2)(u-1)/3+\mathcal{O}((u-1)^2))$. For all the facts concerning the Chebyshev polynomials used here, we refer to [Sze75].

We are now ready to make the change of variable mentioned above : to set $\hat{b}(t, z) = u$. But $\hat{b}(t, z) = u$ if and only if $b(t, z) - 2urp_{01}zt = 0$, in other words if and only if $t = t_1(u, z)$ or $t = t_2(u, z)$ where $t_1 = t_-$, $t_2 = t_+$ and :

$$t_\pm\left(u,z\right)=\frac{1+2\sqrt{p_{01}p_{0-1}}uz\pm\sqrt{\left(1+2\sqrt{p_{01}p_{0-1}}uz\right)^2-4p_{10}p_{-10}z^2}}{2p_{10}z}. \tag{19}$$

Of course, we find again the explicit expression of the branched points as the values of t_1 and t_2 at $u = \pm 1$, more precisely $t_1(1, z) = x_1(z)$, $t_1(-1, z) = x_2(z)$, $t_2(-1, z) = x_3(z)$ and $t_2(1, z) = x_4(z)$, in accordance with Lemma 1. This change of variable allows us to write the following integral representation for the function h :

Proposition 10. *The function h admits the following integral expression :*

$$\begin{aligned} h\left(x,z\right) &= xP_\infty\left(x\mapsto x^{n_0-1}Y_0\left(x,z\right)\right)^{m_0}\left(x\right) \\ &+ \frac{x}{\pi}\left(\frac{p_{0-1}}{p_{01}}\right)^{m_0/2}\int_{-1}^{1}\left(t_2\left(u,z\right)^{n_0}-\left(\frac{r^2}{t_2\left(u,z\right)}\right)^{n_0}\right)\frac{\partial_u t_2\left(u,z\right)U_{m_0-1}\left(-u\right)}{t_2\left(u,z\right)\left(t_2\left(u,z\right)-x\right)}\sqrt{1-u^2}\mathrm{d}u. \end{aligned} \tag{20}$$

Above, x belongs to the open centered disc of radius r, and $z \in]0, z_1]$, z_1 being defined in Lemma 1.

We take now some notations that will be useful in the sequel, notably in Section 3 : we set $k_1(u) = -2(p_{01}p_{0-1})^{1/2}u + 2(p_{10}p_{-10})^{1/2}$ and $k_2(u) = -2(p_{01}p_{0-1})^{1/2}u - 2(p_{10}p_{-10})^{1/2}$; so that for instance $t_\pm(u, z) = (1 + 2(p_{01}p_{0-1})^{1/2}uz \pm ((1 - k_1(u)z)(1 - k_2(u)z))^{1/2})/(2p_{10}z)$. Moreover, we easily show the two following facts :

$$\left\{\begin{array}{rcl} \partial_u t_\pm\left(u,z\right) &=& \pm t_\pm\left(u,z\right)\dfrac{2\sqrt{p_{01}p_{0-1}}z}{\sqrt{\left(1-k_1\left(u\right)z\right)\left(1-k_2\left(u\right)z\right)}}, \\ \dfrac{1}{t_2\left(u,z\right)-x} &=& \dfrac{1}{2}\dfrac{\sqrt{\left(1-k_1\left(u\right)z\right)\left(1-k_2\left(u\right)z\right)}-\left(1+\left(2\sqrt{p_{01}p_{0-1}}u-2p_{10}x\right)z\right)}{x+\left(2\sqrt{p_{01}p_{0-1}}ux-\left(p_{10}x^2+p_{-10}\right)\right)z}. \end{array}\right. \tag{21}$$

Remark 11 (the gambler ruin)**.** Consider the explicit expression (20), in which we do $p_{10}, p_{-10} \to 0$, $x = 1$. Moreover, we take $n_0 = 1$ to lighten the technical details. This expression becomes :

$$h\left(1,z\right) = \frac{1}{\pi}\left(\frac{p_{0-1}}{p_{01}}\right)^{m_0/2} \int_{-1}^{1} \frac{2\sqrt{p_{01}p_{0-1}}z}{1+2\sqrt{p_{01}p_{0-1}}zu} U_{m_0-1}\left(-u\right)\sqrt{1-u^2}\mathrm{d}u. \tag{22}$$

Not unexpectedly this quantity is equal to :

$$\lambda_{m_0}\left(z\right) = \left(\frac{1-\sqrt{1-4p_{01}p_{0-1}z^2}}{2p_{01}z}\right)^{m_0},$$

which is the generating function of the ruin probabilities for the gambler ruin problem : $\lambda_{m_0}(z) = \sum_{k=0}^{+\infty} \mathbb{P}_{m_0}(\text{to be ruined in a time } k)z^k$, in accordance with [Fel57]. Let us sketch the proof of this fact. We start by remarking that for $n \in \mathbb{N}$ and $t \in \mathbb{C} \setminus [-1,1]$, we have :

$$\frac{1}{\pi}\int_{-1}^{1} \frac{u^n\sqrt{1-u^2}}{u-t}\mathrm{d}u = \left(t^k\sqrt{t^2-1} - P_\infty\left(t \mapsto t^k\sqrt{t^2-1}\right)(t)\right), \tag{23}$$

where, if f is a function meromorphic at infinity, $P_\infty(f)$ denotes its principal part at infinity. To get (23), we consider the function $u^n(1-u^2)^{1/2}/(u-t)$ and we integer it on a closed contour that surrounds at a distance equal to ϵ the segment $[-1,1]$, like the contour drawn in Figure 3 of Subsection 2.4, then we apply the residue theorem at infinity and at last we do ϵ going to zero.

Next, we apply (23) at every monomial that composes the polynomial U_{m_0-1}. Using the linearity of the principal part, integral (22) becomes :

$$h\left(1,z\right) = -\left(\frac{p_{0-1}}{p_{01}}\right)^{m_0/2}\left(U_{m_0-1}\left(u\right)\sqrt{u^2-1} - P_\infty\left(u \mapsto U_{m_0-1}\left(u\right)\sqrt{u^2-1}\right)\left(u\right)\right)\Big|_{u=\frac{1}{2\sqrt{p_{01}p_{0-1}}z}}. \tag{24}$$

Introduce now $(T_n)_{n\in\mathbb{N}}$, the Chebyshev polynomials of the first kind ; we recall that they are the orthogonal polynomials associated to the weight $t \mapsto 1_{]-1,1[}\left(t\right)/(1-t^2)^{1/2}$. As for the Chebyshev polynomials of the second kind U_n (defined in (18)), there exists an explicit formulation for the Chebyshev polynomials of the first kind :

$$T_n\left(t\right) = \frac{1}{2}\left(\left(t+\sqrt{t^2-1}\right)^n + \left(t-\sqrt{t^2-1}\right)^n\right), \quad n \in \mathbb{N}. \tag{25}$$

Moreover, as it is proved in [Sze75], there exists –among others– the following link between the Chebyshev polynomials of the first and second kind :

$$P_\infty\left(t \mapsto U_n\left(t\right)\sqrt{t^2-1}\right) = T_{n+1}, \qquad P_\infty\left(t \mapsto T_{n+1}\left(t\right)/\sqrt{t^2-1}\right) = U_n, \qquad n \in \mathbb{N}.$$

These relations allow to simplify (24). We find :

$$h\left(1,z\right) = -\left(\frac{p_{0-1}}{p_{01}}\right)^{m_0/2}\left(U_{m_0-1}\left(u\right)\sqrt{u^2-1} - T_{m_0}\left(u\right)\right)\Big|_{u=1/\left(2\sqrt{p_{01}p_{0-1}}z\right)}.$$

But with (18) and (25), $U_{m_0-1}\left(u\right)(u^2-1)^{1/2} - T_{m_0}\left(u\right) = -(u-(u^2-1)^{1/2})^{m_0}$ so that we find that $h(1,z) = \lambda_{m_0}(z)$.

2.6 Analytic continuation

We recall from Equation (2) of Section 1 that h is initially defined on the closed unit disc, inside of which it is holomorphic.

Proposition 12. *The function h admits an analytic continuation on $\mathbb{C} \setminus [x_3(z), x_4(z)]$. Moreover, the set $\mathbb{C} \setminus [x_3(z), x_4(z)]$ is the biggest one where h can be continued into a single valued function.*

Proof. We could equally use one or the other of the formulations obtained in Propositions 5, 6, 9 and 10, but the simplest is perhaps to make the use of Proposition 9, as well as the analytic properties of Cauchy integrals, that can be found for instance in [Lu93]. □

Remark 13. A nice property peculiar to the random walks we are studying is that the curve $\mathcal{M}_z = \mathcal{C}(0, r)$ belongs to the closed unit disc. We can thus solve the boundary value problem with boundary condition (11) and then continue h using its explicit expression. In Section 6, we will see that for walks under general hypothesis (H2) of the introduction, it is quite possible that a part –or even the whole– of the associated curve $\mathcal{M}_z$ belongs to the exterior of the closed unit disc. There we will have *first* to continue h as a holomorphic function up to $\mathcal{M}_z$ and *after* to solve the boundary value problem. We will do this continuation using Galois automorphisms (notion that will be defined here in the proof of Proposition 36), following the procedure which in the heart of Book [FIM99].

3 Probability of being absorbed with a fixed time

3.1 Absorption probabilities in the case of a drift zero

Proposition 14. *We suppose here that the two drifts (3) are equal to zero, in other words that* $p_{-10} = p_{10}$ *and* $p_{0-1} = p_{01}$. *Define* $S = \inf\{n \in \mathbb{N} : (X(n), Y(n)) \text{ hits the } x\text{-axis}\}$ *the hitting time of the x-axis. The following asymptotic holds :*

$$\mathbb{P}_{(n_0,m_0)}(S = k) \sim \frac{n_0 m_0}{2\pi\sqrt{p_{10}p_{01}}}\frac{1}{k^2}, \quad k \to \infty. \tag{26}$$

Proof. Setting $x = 1$, $p_{-10} = p_{10}$, $p_{0-1} = p_{01}$ in (20) leads to :

$$h(1, z) = \frac{p_{01}z}{\pi}\int_{-1}^{1}\frac{t_2(u, z)^{n_0} - t_1(u, z)^{n_0}}{\sqrt{(1 - k_1(u)z)(1 - k_1(u)z)}}U_{m_0-1}(-u)\left(\sqrt{\frac{1 - k_2(u)z}{1 - k_1(u)z}} - 1\right)\sqrt{1 - u^2}\mathrm{d}u.$$

Using the explicit expressions of t_1 and t_2 given in (19), we immediately notice that the function

$$F(u, z) = \frac{p_{01}z}{\pi}\frac{t_2(u, z)^{n_0} - t_1(u, z)^{n_0}}{\sqrt{(1 - k_1(u)z)(1 - k_1(u)z)}}U_{m_0-1}(-u)$$

is a polynomial in the two variables (u, z). So we can write F as the following finite sum : $F(u, z) = \sum_{i,j} F_{ij}(u + 1)^i(z - 1)^j$ with coefficients F_{ij} that can of course be computed explicitly, for example $F_{00} = n_0 m_0 p_{01}/(\pi p_{10})$. Since adding a polynomial does not change the asymptotic of the function's coefficients, the coefficients of $h(1, z)$ have the same asymptotic as those of the following function :

$$l(z) = \int_{-1}^{1} F(u, z)\sqrt{\frac{1 - k_2(u)z}{1 - k_1(u)z}}\sqrt{1 - u^2}\mathrm{d}u.$$

Consider now the function $G(u, z) = F(u, z)(1 - k_2(u)z)^{1/2}$. Since $k_2(-1) = 2(p_{01} - p_{10}) < 1$, the function of two variables G is holomorphic in $\mathcal{D}(0, 1 + \epsilon)^2$, where $\epsilon > 0$ is sufficiently small. For this reason, G can be expanded according to the powers $(u + 1)^i(z - 1)^j$: $G(u, z) = \sum_{i,j} G_{ij}(u + 1)^i(z - 1)^j$. As for F, all the coefficients G_{ij} can be explicited ; for instance $G_{00} = 2n_0 m_0 p_{01}/(\pi\sqrt{p_{10}})$. With these notations, the function l becomes

$$l(z) = \sum_{i,j} G_{ij}(z - 1)^j \int_{-1}^{1}(1 - u)^i\frac{\left(1 - u^2\right)^{1/2}}{(1 - k_1(-u)z)^{1/2}}\mathrm{d}u.$$

Thanks to Lemma 15 below, we obtain that $l(z)$ (and therefore $h(1, z)$) is a function (i) holomorphic in the unit disc (ii) having a holomorphic continuation up to every point of the unit circle except 1 (iii) having a logarithmic singularity at 1.

As concerns the logarithmic singularity, we can be more precise : Lemma 15 asserts the existence of $f(z) = \sum_{i,j} G_{ij} f_i(z)(z - 1)^{i+j}$ and $g(z) = \sum_{i,j} G_{ij} g_i(z)(z - 1)^j$ such that $l(z) = f(z)(z - 1)\ln(1 - z) + g(z)$.

Moreover, using the fact that $G_{00} = 2n_0 m_0 p_{01}/(\pi\sqrt{p_{10}})$ and once again with Lemma 15, we find $f(1) = -n_0 m_0/(2\pi\sqrt{p_{01}p_{10}})$.

We can now easily find the asymptotic of the coefficients of the Taylor series at 0 of $l(z)$, thus also of $h(1, z)$, following the principle explained hereunder : if $F(z) = \sum_k c_k z^k$ is a function (i) holomorphic in the open unit disc (ii) having a holomorphic continuation at every point of the unit circle except 1 (iii) having at 1 a logarithmic singularity in the sense that in the neighborhood of 1, F can be written as $F(z) = F_1(z) + F_2(z)\ln(1-z)$ where F_1 and F_2 are holomorphic functions at 1, then the asymptotic of the coefficients of the Taylor series can easily be calculated : if $q = \inf\{p \in \mathbb{N} : F_2{}^{(p)}(1) \neq 0\}$, then $c_k \sim (-1)^q F_2{}^{(q)}(1)/k^{q+1}$ as $k \to +\infty$.

We use this result with $q = 1$ and $F_2'(1) = f(1) = -n_0 m_0/(2\pi\sqrt{p_{01}p_{10}})$, the asymptotic (26) comes immediately.

□

Lemma 15. *Let i be a non negative integer. The function F_i, defined by*

$$F_i(z) = \int_{-1}^{1} (1-u)^i \frac{\left(1-u^2\right)^{1/2}}{\left(1-k_1(-u)z\right)^{1/2}} \mathrm{d}u, \tag{27}$$

is holomorphic in the open unit disc. Moreover, it can be continued into a holomorphic function in the neighborhood of any point of the unit circle except 1. At $z = 1$, the function has a logarithmic singularity ; more precisely there exist two functions f_i and g_i holomorphic at $z = 1$, $f_i(1) \neq 0$, such that $F_i(z) = (z-1)^{i+1}\ln(1-z)f_i(z) + g_i(z)$. Moreover, $f_0(1) = -1/(4p_{01}{}^{3/2})$.

Proof. The two facts that the integrals considered in Lemma 15 are holomorphic in the unit disc and also that they can be continued into holomorphic functions trough every point of the unit circle except 1 come immediately from the theory of integrals with parameters. We will therefore concentrate the proof on the logarithmic singularity.

First, we replace the lower bound -1 in the integrals (27) by $-p_{10}/p_{01}$, which does not change the singularity in the neighborhood of 1 of the functions $F_i(z)$ since doing this is equivalent to add to $F_i(z)$ a function with a radius of convergence strictly larger than 1. Then, the change of variable $v^2 = k_1(-u) = 2(p_{01}u + p_{10})$ gives

$$\int_{-p_{10}/p_{01}}^{1} \frac{(1-u)^i\left(1-u^2\right)^{1/2}}{\left(1-k_1(-u)z\right)^{1/2}} \mathrm{d}u = \frac{2}{\left(2p_{01}\right)^{2+i}} \int_0^1 \frac{\left(1-v^2\right)^{1/2+i}}{\left(1-zv^2\right)^{1/2}} v\sqrt{v^2 + 2\left(p_{01}-p_{10}\right)}\mathrm{d}v.$$

Next, using the expansion of $\sqrt{v}$ at $v = 1$, we can expand the function $v(v^2 + 2\left(p_{01} - p_{10}\right))^{1/2}$ according to the powers of $(1-v^2)$: $v(v^2 + 2(p_{01} - p_{10}))^{1/2} = \sum_i c_i(1-v^2)^i$ with $c_0 = 2\sqrt{p_{01}}$, $c_1 = (1+4p_{01})/(4\sqrt{p_{01}})$, etc.

We will now explain why there exist functions ϕ_k and ψ_k holomorphic in the neighborhood of 1, $\phi_k(1) \neq 0$ such that

$$\int_0^1 \frac{\left(1-v^2\right)^{1/2+k}}{\left(1-zv^2\right)^{1/2}} \mathrm{d}v = (z-1)^{k+1}\ln(1-z)\,\phi_k(z) + \psi_k(z). \tag{28}$$

But before, we show how (28) allows to complete the proof of Lemma 15 : with the notations of (28) we set $\tilde{g}_i(z) = 2/(2p_{01})^{i+2}\sum_k c_k\psi_{k+i}(z)$ and $f_i(z) = 2/(2p_{01})^{i+2}\sum_k c_k\phi_{k+i}(z)(z-1)^k$, we obtain that :

$$\int_{-p_{10}/p_{01}}^{1} \frac{(1-u)^i\left(1-u^2\right)^{1/2}}{\left(1-k_1(-u)z\right)^{1/2}} \mathrm{d}u = (z-1)^{i+1}\ln(1-z)\,f_i(z) + \tilde{g}_i(z). \tag{29}$$

Then, we replace the lower bound $-p_{10}/p_{01}$ by -1, what changes $\tilde{g}_i$ in a new function holomorphic in the neighborhood of 1, that we call g_i, but what does not change the function f_i, for the reasons already explained at the beginning of the proof.

So, it remains to prove (28). The proof consists in expressing the integrals (28) in terms of K and E, the two classical Legendre's complete elliptic integrals of the first and second kind, defined by :

$$K(z) = \int_0^1 \frac{\mathrm{d}v}{\left(\left(1-v^2\right)\left(1-zv^2\right)\right)^{1/2}}, \qquad E(z) = \int_0^1 \frac{\left(1-zv^2\right)^{1/2}}{\left(1-v^2\right)^{1/2}} \mathrm{d}v,$$

and next in using the well known results concerning these elliptic integrals, notably their behavior in the neighborhood of 1 and in particular their –logarithmic– singularity at 1 ; all these properties can be found e.g. in [SG69].

The functions K and E are manifestly holomorphic in the open unit disc, continuable trough any point of the unit circle except 1, and from the so called Abel's identity (see [SG69]) it can be deduced that the functions K and E have at 1 a logarithmic singularity as follows : $K(z) = \rho_K(z) + \sigma_K(z)\ln(1-z)$ and $E(z) = \rho_E(z) + \sigma_E(z)(z-1)\ln(1-z)$, where the functions ρ and σ are holomorphic in the neighborhood of 1 and $\sigma_K(1) = -1/2$ and $\sigma_E(1) = 1/4$.

Moreover, for any non negative integer k, we can find two polynomials P_k and Q_k such that :

$$\int_0^1 \frac{\left(1-v^2\right)^{1/2+k}}{\left(1-zv^2\right)^{1/2}}\mathrm{d}v = \frac{P_k\left(z\right)E\left(z\right)+Q_k\left(z\right)K\left(z\right)}{z^{k+1}}.$$

These polynomials could be explicitly calculated, for instance, $P_0(z) = 1$ and $Q_0(z) = z-1$. Therefore, setting $\psi_k(z) = (P_k(z)\rho_E(z) + Q_k(z)\rho_K(z))/z^{k+1}$ and $\tilde{\phi}_k(z) = (P_k(z)\sigma_E(z)(z-1) + Q_k(z)\sigma_K(z))/z^{k+1}$, we obtain that

$$\int_0^1 \frac{\left(1-v^2\right)^{1/2+k}}{\left(1-zv^2\right)^{1/2}}\mathrm{d}v = \ln\left(1-z\right)\widetilde{\phi}_k\left(z\right) + \psi_k\left(z\right).$$

Then, to prove (28) it suffices to verify that we can write $\tilde{\phi}_k(z)$ as $(z-1)^{k+1}\phi_k(z)$, where ϕ is holomorphic at 1. We don't make this verification in the general case, because the calculations are somewhat tedious –the expressions of the polynomials P_k and Q_k are rather complicated–, but do it in case $k=0$: thanks to the explicit expression of P_0 and Q_0 given above we have $\tilde{\phi}_0 = (z-1)(\sigma_E(z)+\sigma_K(z))/z$, hence the result by setting $\phi_0(z) = (\sigma_E(z)+\sigma_K(z))/z$.

To prove the last fact claimed in Lemma 15, namely that $f_0(1) = -1/(4{p_{01}}^{3/2})$, we use the fact that $f_0(1) = 2c_0(\sigma_E(1)+\sigma_K(1))/(2p_{01})^2 = -1/(4{p_{01}}^{3/2})$. □

Corollary 16. *Take the following notations :*

$$\begin{cases} S &= \inf\left\{n \in \mathbb{N} : \left(X\left(n\right), Y\left(n\right)\right) \text{ hits the } x\text{-axis}\right\}, \\ T &= \inf\left\{n \in \mathbb{N} : \left(X\left(n\right), Y\left(n\right)\right) \text{ hits the } y\text{-axis}\right\}, \\ \tau &= \inf\left\{n \in \mathbb{N} : \left(X\left(n\right), Y\left(n\right)\right) \text{ hits the boundary}\right\} = S \wedge T. \end{cases} \tag{30}$$

Then the following equivalents hold for $\mathbb{P}_{(n_0,m_0)}(T=k)$ and for the probability of not being absorbed at time k :

$$\mathbb{P}_{(n_0,m_0)}\left(T=k\right) \sim \frac{n_0 m_0}{2\pi\sqrt{p_{10}p_{01}}}\frac{1}{k^2}, \qquad \mathbb{P}_{(n_0,m_0)}\left(\tau \geq k\right) \sim \frac{n_0 m_0}{\pi\sqrt{p_{10}p_{01}}}\frac{1}{k}, \qquad k \to \infty.$$

Proof. We immediately obtain the first part of Corollary 16 from Proposition 14 by exchanging the parameters p_{10}, p_{-10} and p_{01}, p_{0-1}. Moreover, since the random walk can be absorbed by at most one of the axes, we get

$$\mathbb{P}_{(n_0,m_0)}\left(\tau \geq k\right) = \mathbb{P}_{(n_0,m_0)}\left(k \leq S < \infty\right) + \mathbb{P}_{(n_0,m_0)}\left(k \leq T < \infty\right) + \mathbb{P}_{(n_0,m_0)}\left(\left(S=\infty\right)\cap\left(T=\infty\right)\right). \tag{31}$$

This random walk being absorbed almost surely (we recall that we have supposed $p_{-10} = p_{10}$ and $p_{0-1} = p_{01}$), we have $\mathbb{P}_{(n_0,m_0)}\left(\left(S=\infty\right)\cap\left(T=\infty\right)\right) = 0$ and Corollary 16 is immediate. □

3.2 Absorption probabilities in the case of a non zero drift

In Subsection 3.1, we were interested in the hitting time of the boundary of $(\mathbb{Z}_+)^2$ in the case of the two drifts M_x and M_y equal to zero. Now, we state analogous results when one (Proposition 17) or two (Proposition 19) of M_x and M_y are not zero.

Proposition 17. *Suppose that $M_x > 0$, $M_y > 0$ and let S be the hitting time of the x-axis, defined in (30). Then $\mathbb{P}_{(n_0,m_0)}(S=k)$, the probability of being absorbed in the x-axis at time k, admits the asymptotic as k goes to infinity :*

$$\frac{m_0}{2\sqrt{\pi}}\sqrt{\frac{p_{10}+p_{-10}+2\sqrt{p_{01}p_{0-1}}}{\sqrt{p_{01}p_{0-1}}}}\left(\frac{p_{0-1}}{p_{01}}\right)^{m_0/2}\left(1-\left(\frac{p_{-10}}{p_{10}}\right)^{n_0}\right)\frac{\left(p_{10}+p_{-10}+2\sqrt{p_{01}p_{0-1}}\right)^k}{k^{3/2}}. \tag{32}$$

Proof. Lemma 10 gives that $h\left(1,z\right)$ is, up to a polynomial, equal to :

$$\frac{1}{\pi}\left(\frac{p_{0-1}}{p_{01}}\right)^{m_0/2}\int_{-1}^{1}\left(t_2\left(u,z\right)^{n_0} - \left(\frac{r^2}{t_2\left(u,z\right)}\right)^{n_0}\right)\frac{2\sqrt{p_{01}p_{0-1}}z}{\sqrt{\left(1-k_1\left(u\right)z\right)\left(1-k_2\left(u\right)z\right)}} \times$$
$$\times\frac{1}{2}\frac{\sqrt{\left(1-k_1\left(u\right)z\right)\left(1-k_2\left(u\right)z\right)} - \left(1-k_4\left(u\right)z\right)}{1-k_3\left(u\right)z}U_{m_0-1}\left(-u\right)\sqrt{1-u^2}\mathrm{d}u, \tag{33}$$

where we have set $k_1(u) = -2(p_{01}p_{0-1})^{1/2}u + 2(p_{10}p_{-10})^{1/2}$, $k_2(u) = -2(p_{01}p_{0-1})^{1/2}u - 2(p_{10}p_{-10})^{1/2}$, $k_3(u) = -2(p_{01}p_{0-1})^{1/2}u + p_{10} + p_{-10}$ and $k_4(u) = -2(p_{01}p_{0-1})^{1/2}u + 2p_{10}$. Due to the inequalities $2(p_{10}p_{-10})^{1/2} < p_{10} + p_{-10} < 2p_{10}$, the integral (33) is holomorphic in the open disc $\mathcal{D}(0, k_3(-1)^{-1})$, continuable at every point of the boundary $\mathcal{C}(0, k_3(-1)^{-1})$ except at $k_3(-1)^{-1}$. Now we set $F(u,z) = (t_2(u,z)^{n_0} - (r^2/t_2(u,z))^{n_0})(p_{01}p_{0-1})^{1/2}zU_{m_0-1}(-u)(((1-k_1u)z)((1-k_2u)z))^{1/2} - (1-k_4(u)z))/((1-k_1u)z)((1-k_2u)z))^{1/2}$ in such a way that the function (33) can be expressed as the integral :

$$\frac{1}{\pi}\left(\frac{p_{0-1}}{p_{01}}\right)^{m_0/2}\int_{-1}^{1}\frac{F\left(u,z\right)}{1-k_3\left(u\right)z}\sqrt{1-u^2}\mathrm{d}u. \tag{34}$$

The function of two variables F is certainly not holomorphic on the whole $\mathbb{C}^2$ but is holomorphic on $\mathcal{D}(0, k_3(-1)^{-1}+\epsilon) \times \mathcal{D}(0, 1+\epsilon)$, where ϵ, which depends on the p_{ij}, is sufficiently small : indeed, thanks to –once again– the obvious inequalities $2(p_{10}p_{-10})^{1/2} < p_{10} + p_{-10} < 2p_{10}$, we immediately notice that $(u,z) \mapsto (1-k_i(u)z)^{1/2}$ is, for $i \in \{1,2,4\}$, holomorphic in $\mathcal{D}(0, k_3(-1)^{-1}+\epsilon) \times \mathcal{D}(0, 1+\epsilon)$, for sufficiently small values of ϵ.

Therefore, we can write the expansion of $F(u,z)$ in the neighborhood of $(-1, k_3(-1)^{-1})$, say $F(u,z) = \sum_{i,j} F_{ij}(1+u)^i(1-k_3(-1)^{-1}z)^j$. The coefficients of this expansion could be explicitly calculated, for instance, using that $(k_3(-1)-k_1(-1))(k_3(-1)-k_2(-1)) = (p_{10}-p_{-10})^2$ and $t_2(-1, k_3(-1)) = 1$ we find $F_{00} = 2m_0(1-(p_{-10}/p_{10})^{n_0})(p_{01}p_{0-1})^{1/2}/k_3(-1)$.

Then, in accordance with Lemma 18 below we set $f(z) = \sum_{i,j} F_{ij}(1-k_3(-1)^{-1})^{i+j}f_i(z)\,(p_{0-1}/p_{01})^{m_0/2}/\pi$ and $g(z) = \sum_{i,j} F_{ij}(1-k_3(-1)^{-1})^j g_i(z)(p_{0-1}/p_{01})^{m_0/2}/\pi$. With these notations, the function defined in (34) is equal to $g(z) + f(z)(1-k_3(-1)^{-1})^{1/2}$.

We can now easily find the asymptotic of the coefficients of the Taylor series at 0 of function (34), or equivalently of $h(1,z)$, following a similar principle as the one explained in the proof of Proposition 14, and summarized below : if $F(z) = \sum_k c_k z^k$ is a function (i) holomorphic in the open disc of radius r (ii) having a holomorphic continuation at every point of the circle of radius r except r (iii) having at r an algebraic singularity in the sense that in the neighborhood of r, F can be written as $F(z) = F_0(z) + \sum_{i=1}^d F_i(z)(1-z/r)^{\theta_i}$ where the F_i, $i \geq 0$, are holomorphic functions in the neighborhood of r, not vanishing at r for $i \geq 1$, the $\theta_1 < \cdots < \theta_d$ are rational but not integer, then the asymptotic of the coefficients of the Taylor series at 0 can easily be calculated : $c_k \sim F_1(r)r^k/(\Gamma(-\theta_1)k^{\theta_1+1})$ as $k \to +\infty$. This principle is known as Pringsheim theorem.

With the last part of Lemma 18, we obtain

$$F_1(r) = F_{00}f_0\left(k_3\left(-1\right)^{-1}\right) = -m_0\sqrt{\frac{p_{10}+p_{-10}+2\sqrt{p_{01}p_{0-1}}}{\sqrt{p_{01}p_{0-1}}}}\left(\frac{p_{0-1}}{p_{01}}\right)^{m_0/2}\left(1-\left(\frac{p_{-10}}{p_{10}}\right)^{n_0}\right),$$

so that, using Pringsheim result with this value of $F_1(r)$, $\theta_1 = 1/2$, $r = k_3(-1)^{-1}$ and using the fact that $\Gamma(-1/2) = -2\sqrt{\pi}$, we get immediately the announced asymptotic (32). □

Lemma 18. *Let i be a non negative integer. The function G_i, defined by*

$$G_i\left(z\right) = \int_{-1}^{1}\left(1-u\right)^i\frac{\left(1-u^2\right)^{1/2}}{1-k_3\left(-u\right)z}\mathrm{d}u,$$

where $k_3(u) = -2(p_{01}p_{0-1})^{1/2}u + p_{10} + p_{-10}$, is holomorphic in the open disc $\mathcal{D}(0, k_3(-1)^{-1})$. Moreover, it can be continued into a holomorphic function in the neighborhood of any point of the circle $\mathcal{C}(0, k_3(-1)^{-1})$, except $k_3(-1)^{-1}$. At $z = k_3(-1)^{-1}$, the function has an algebraic singularity ; more precisely there exist two functions f_i and g_i holomorphic at $z = k_3(-1)^{-1}$, $f_i(k_3(-1)^{-1}) \neq 0$, such that $G_i(z) = (1-k_3(-1)z)^{i+1/2}f_i(z) + g_i(z)$. Moreover, $f_0(k_3(-1)^{-1}) = -(\pi/2)(k_3(-1)/(p_{01}p_{0-1})^{1/2})^{3/2}$.

Proof. The proofs of all assertions of Lemma 18 are based on the fact that the functions G_i can be explicitly calculated :

$$G_i\left(z\right) = \frac{-\pi}{2\sqrt{p_{01}p_{0-1}}z}\left(\left(1-Z\right)^i\sqrt{Z^2-1} - P_\infty\left(\left(1-Z\right)^i\sqrt{Z^2-1}\right)\right)\Big|_{Z=\frac{1-z(p_{10}+p_{-10})}{2\sqrt{p_{01}p_{0-1}}z}}, \tag{35}$$

where P_∞ is the principal part defined in Lemma 7.

To prove (35), we start by remarking that $1 - k_3(-u)z = -2(p_{01}p_{0-1})^{1/2}(u - Z)$, where $Z = (1 - z(p_{10} + p_{-10}))/(2(p_{01}p_{0-1})^{1/2}z)$. Then, we consider the function $(1 - u)^i(u^2 - 1)^{1/2}/(u - Z)$, well defined on $\mathbb{C}\setminus]-1, 1[\cup\{Z\}$, at which we apply the residue theorem at infinity, on the same contour as the one used in the Remark 11, namely a closed contour that surrounds at a distance equal to ϵ the segment $[-1, 1]$. After that ϵ has gone to zero, we get :

$$\int_{-1}^{1} (1-u)^i \frac{(1-u^2)^{1/2}}{u-Z} \mathrm{d}u = \left((1-Z)^i \sqrt{Z^2-1} - P_\infty\left((1-Z)^i \sqrt{Z^2-1}\right)\right),$$

from which (35) and thus Lemma 18 are immediate consequences. ☐

Proposition 19. *Suppose that $M_x = 0$, $M_y > 0$ and let S and T be the hitting times of the x and y-axis, defined in (30). Then $\mathbb{P}_{(n_0,m_0)}(S = k)$ and $\mathbb{P}_{(n_0,m_0)}(T = k)$ admit the following asymptotic as k goes to infinity :*

$$\mathbb{P}_{(n_0,m_0)}(S = k) \sim \frac{n_0 m_0}{2\pi\sqrt{p_{10}}\,(p_{01}p_{0-1})^{1/4}} \left(\frac{p_{0-1}}{p_{01}}\right)^{m_0/2} \frac{\left(2\left(p_{10} + \sqrt{p_{01}p_{0-1}}\right)\right)^k}{k^2}, \tag{36}$$

$$\mathbb{P}_{(n_0,m_0)}(T = k) \sim \frac{n_0}{\sqrt{\pi p_{10}}} \left(1 - \left(\frac{p_{0-1}}{p_{01}}\right)^{m_0}\right) \frac{1}{(2p_{01})^{m_0}} \frac{1}{k^{3/2}}. \tag{37}$$

Proof. The proof of (36) is quite similar to the one of (26) and the proof of (37) is quite similar to the one of (32). We omit the details. ☐

Remark 20. Note that equation (36) formally implies (26). Also, (37) formally follows from (32) after a proper change of the parameters. But one can not obtain (36) starting from (32) and then making the drift go to zero.

Remark 21. Let $\tau = \inf\{n \in \mathbb{N} : (X(n), Y(n)) \text{ hits the boundary}\}$ be the hitting time of the boundary of $(\mathbb{Z}_+)^2$. To find the tail's asymptotic of τ, we can now apply (31). If at least one of the two drifts (3) is zero, then the last term in (31) is zero and the result comes immediately. If both drifts are positive, then we have to compute the probability of non absorption, that will be done in Proposition 28 of Subsection 4.3.

4 Probability of being absorbed in a fixed site

4.1 Explicit form and asymptotic

We recall from the very beginning of this paper that taking $z = 1$ in $h(x, z)$ (see (2)), leads to $h(x, 1) = \sum_{i=1}^{+\infty} \mathbb{P}_{(n_0,m_0)}(\text{to be absorbed at } (i, 0))x^i$. In addition, putting $z = 1$ in the explicit expression of $h(x, z)$ obtained in Proposition 9 yields

$$h(x,1) = \frac{x}{\pi}\int_{x_3(1)}^{x_4(1)} \left(t^{n_0} - \left(\frac{r^2}{t}\right)^{n_0}\right) \frac{\mu_{m_0}(t,1)\sqrt{-d(t,1)}}{t(t-x)}\mathrm{d}t + xP_\infty\left(x \mapsto x^{n_0-1}Y_0(x,1)^{m_0}\right)(x). \tag{38}$$

Above, $x_3(1)$ and $x_4(1)$ are defined in Lemma 1, μ_{m_0} in (14) and P_∞ in Lemma 7. We recall about $xP_\infty(x \mapsto x^{n_0-1}Y_0(x, 1)^{m_0})(x)$ that it is simply a polynomial, the null polynomial if $n_0 \leq m_0$, of degree $n_0 - m_0$ if $n_0 > m_0$. Note that the equality (38) is viable equally in the cases $M_x > 0$, $M_x = 0$, $M_y > 0$, $M_y = 0$. In particular, we immediately deduce the explicit expression of the coefficients $h_i = \mathbb{P}_{(n_0,m_0)}(\text{to be absorbed at } (i, 0))$:

Proposition 22. *Suppose that $M_x \geq 0$, $M_y \geq 0$. Then, for $i \geq \max(n_0 - m_0, 1)$, the following equality holds :*

$$h_i = \frac{1}{\pi}\int_{x_3(1)}^{x_4(1)} \left(t^{n_0} - \left(\frac{r^2}{t}\right)^{n_0}\right) \frac{\mu_{m_0}(t,1)\sqrt{-d(t,1)}}{t^{i+1}}\mathrm{d}t. \tag{39}$$

For $i \in \{1, \max(n_0 - m_0, 0)\}$, the equality (39) is still true if we add the contribution of the polynomial $xP_\infty(x \mapsto x^{n_0-1}Y_0(x, 1)^{m_0})(x)$, defined in Lemma 7.

We will now study the asymptotic of h_i, first in case of a zero drift, then in case of a non zero drift. We will see that the decrease of these probabilities is respectively polynomial and exponential, with an exponential rate equal to $1/x_3(1)$, what we would have been able to anticipate from Proposition 12 of Subsection 2.6, where we have seen that $x_3(1)$ is the first positive singularity of $h(x,1)$.

Among other things, we will see that the asymptotic of h_i in case of a drift zero is not the limit, when the drift goes to zero, of the asymptotic in case of a non zero drift, thought $x_3(1)=1$.

Of course, the calculation of the asymptotic can be deduced from the explicit expression (39), using e.g. Laplace's method. However, and since it will be useful later, we prefer, like in Section 3, deduce this asymptotic from the study of singularities of the function h ; singularities that will be of two different types, namely logarithmic and algebraic, according to the drift is zero or positive, see Propositions 23 and 24.

Proposition 23. *Suppose that $M_y=0$ and $M_x\geq 0$. The function $h(x,1)$ admits a singularity of a logarithmic type at $x=1$, where its development is :*

$$h\left(x,1\right)=h\left(1,1\right)+n_0\left(x-1\right)\left(1+\left(x-1\right)f\left(x\right)\right)-\frac{2n_0m_0}{\pi}\sqrt{\frac{p_{10}}{p_{01}}}\left(x-1\right)^2\ln\left(1-x\right)\left(1+\left(x-1\right)g\left(x\right)\right),$$

where f and g are holomorphic in the neighborhood of 1, and could be made explicit from the proof.

Proof. The proof is lightly different according to $n_0\leq m_0$ or $n_0>m_0$; indeed, as said in Remark 8, in first case the polynomial $xP_\infty(x\mapsto x^{n_0-1}Y_0(x,1)^{m_0})(x)$ is zero, whereas in second it is of degree n_0-m_0. We choose to do the proof in case $n_0\leq m_0$, knowing that in the other case, it suffices to do an induction on n_0-m_0 to show that the Proposition 23 is still valid.

Under this assumption the expression of $h(x,1)$ written in (38), Subsection 4.1, becomes

$$h\left(x,1\right)=\frac{x}{\pi}\int_1^{x_4(1)}\frac{t^{n_0}-t^{-n_0}}{t\left(t-x\right)}\mu_{m_0}\left(t,1\right)\sqrt{-d\left(t,1\right)}\mathrm{d}t, \tag{40}$$

so that, using twice that $1/(t-x)=1/(t-1)+(x-1)/((t-x)(t-1))$, we get $h(x,1)/x=h(1,1)+(x-1)H_1+(x-1)^2H_2(x)$, where

$$\left\{\begin{array}{lcl} H_1 & = & \dfrac{1}{\pi}\displaystyle\int_1^{x_4(1)}\left(t^{n_0}-\frac{1}{t^{n_0}}\right)\frac{\mu_{m_0}\left(t,1\right)}{t\left(t-1\right)^2}\sqrt{-d\left(t,1\right)}\mathrm{d}t,\\ H_2\left(x\right) & = & \dfrac{1}{\pi}\displaystyle\int_1^{x_4(1)}\left(t^{n_0}-\frac{1}{t^{n_0}}\right)\frac{\mu_{m_0}\left(t,1\right)}{t\left(t-1\right)^2\left(t-x\right)}\sqrt{-d\left(t,1\right)}\mathrm{d}t.\end{array}\right.$$

The function $l(t)$, that we define by $l(t)=(t^{n_0}-t^{-n_0})\mu_{m_0}(t,1)(-d(t,1))^{1/2}/(t(t-1)^2)$, which appears in H_1 and $H_2(x)$, is continuable into a holomorphic function in the neighborhood of 1. Indeed, we recall from Lemma 1 that $x_2(1)=x_3(1)=1$, since $M_y=0$. We still note $l(t)$ this continuation and write $l(t)=\sum_{k=0}^{+\infty}l_k(t-1)^k$. The l_k could of course be calculated, for instance $l_0=2n_0\mu_{m_0}(1,1)p_{10}((x_4(1)-1)(1-x_1(1)))^{1/2}$, that we can simplify by using that $\mu_{m_0}(1,1)=m_0/(2p_{01})$ and $(x_4(1)-1)(1-x_1(1))=4p_{01}/p_{10}$, we finally find $l_0=2n_0m_0(p_{10}/p_{01})^{1/2}$.

We will now study successively $H_2(x)$ and H_1, start with $H_2(x)$. We split the integral $H_2(x)$ in two terms : $\int_1^{1+\epsilon}l(t)/(t-x)\mathrm{d}t+\int_{1+\epsilon}^{x_4(1)}l(t)/(t-x)\mathrm{d}t$, where $\epsilon\in[0,x_4(1)-1]$. The fact that the second term in the last sum is, as a function of x, holomorphic on the open disc $\mathcal{D}(0,1+\epsilon)$ is clear. In addition, it is easily shown that

$$\int_1^{1+\epsilon}\frac{\left(t-1\right)^k}{t-x}\mathrm{d}t=P_k\left(x\right)+\left(x-1\right)^k\ln\left(\frac{1+\epsilon-x}{1-x}\right), \tag{41}$$

where P_0 is the null polynomial, and for $k\geq 1$, $\deg(P_k)=k-1$ –of course, P_k could be calculated in an explicit way–. This leads to

$$\int_1^{1+\epsilon}\frac{l\left(t\right)}{t-x}\mathrm{d}t=\sum_{k=0}^{+\infty}l_kP_k\left(x\right)+\ln\left(\frac{1+\epsilon-x}{1-x}\right)l\left(x\right). \tag{42}$$

This is here that having split the integral in two terms turns out to be useful : if we had left $x_4(1)$ as the upper bound of the integral, it would have been quite possible that the function $\sum_k l_kP_k$ does not exist : indeed, the radius of convergence of l is equal to $\inf\{1-x_1(1),x_4(1)-1\}$ and for $k\geq 1$, $P_k(1)=\epsilon^k/k$ –as we show by taking $x = 1$ in (41) for $k \geq 1-$. However, for sufficiently small values of ϵ, the function $\sum_k l_k P_k$ exists well and truly.

We have thus showed that $H_2(x)$ is the sum of a function holomorphic at 1 and of a function having at 1 a logarithmic singularity, see (41) and (42).

To complete the proof of Lemma 23, it remains to study the term H_1, and in particular to show that $H_1 + h(1,1) = n_0$. We recall that we have supposed $n_0 \leq m_0$, so that differentiating (40) and taking $x = 1$ yields :

$$\partial_x h(1,1) = H_1 + h(1,1) = \frac{1}{\pi}\int_1^{x_4(1)} \frac{t^{n_0} - t^{-n_0}}{(t-1)^2} \mu_{m_0}(t,1)\sqrt{-d(t,1)}\mathrm{d}t.$$

After having made the change of variable $t = t_2(u,1)$, see (17), (19) and (21), and after some simplifications, we find :

$$\partial_x h(1,1) = \frac{p_{10}}{\pi}\int_{-1}^{1} \frac{t_2(u,1)^{n_0} - t_1(u,1)^{n_0}}{\sqrt{(1-k_1(u))(1-k_2(u))}} U_{m_0-1}(-u)\sqrt{\frac{1-u}{1+u}}\mathrm{d}u.$$

Using the explicit expressions of t_1 and $t_2 = 1/t_1$ written in (19), we notice that $(t_2(u,1)^{n_0} - t_1(u,1)^{n_0})/((1-k_1(u))(1-k_2(u)))^{1/2}$ is in fact a polynomial of degree $n_0 - 1$, that we note $P_{n_0-1}(u)$. Moreover, it turns out that $P_{n_0-1}(-1) = n_0/p_{10}$. Define now $Q_{n_0-2}(u)$ the $n_0 - 2$ degree polynomial defined by $P_{n_0-1}(u) = P_{n_0-1}(-1) + (u+1)Q_{n_0-2}(u)$. With these notations,

$$\partial_x h(1,1) = \frac{n_0}{\pi}\int_{-1}^{1} U_{m_0-1}(-u)\sqrt{\frac{1-u}{1+u}}\mathrm{d}u + \frac{p_{10}}{\pi}\int_{-1}^{1} Q_{n_0-2}(u)\,U_{m_0-1}(-u)\sqrt{1-u^2}\mathrm{d}u.$$

The second term in the sum above is null. Indeed, being the (m_0-1)-th orthogonal polynomial associated to the weight $1_{]-1,1[}(u)(1-u^2)^{1/2}$, U_{m_0-1} is such that $\int_{-1}^{1} U_{m_0-1}(u)P(u)(1-u^2)^{1/2}\mathrm{d}u = 0$ for all polynomial P whose the degree is less or equal than $m_0 - 2$, that is actually the case for Q_{n_0-2} since we have supposed that $n_0 \leq m_0$.

As for the first term in the sum above, we show, using induction and the recurrence relationship verified by the Chebyshev polynomials, namely $U_{m_0+1}(u) = 2uU_{m_0}(u) - U_{m_0-1}(u)$, see [Sze75], that for all $m_0 \in \mathbb{N}^*$, $\int_{-1}^{1} U_{m_0-1}(-u)((1-u)/(1+u))^{1/2}\mathrm{d}u = \pi$. □

Proposition 24. *Suppose that* $M_y > 0$ *and* $M_x \geq 0$. *The function* $h(x,1)$ *admits a singularity of an algebraic type at* $x = x_3(1)$, *where its development is :*

$$h(x,1) = f(x) + \sqrt{1 - x/x_3(1)}\,g(x),$$

where f *and* g *are holomorphic in the neighborhood of* $x_3(1)$, *and could be made explicit from the proof ; in particular,*

$$g(x_3(1)) = -\left(x_3(1)^{n_0} - x_2(1)^{n_0}\right)\left(p_{10}\left(x_3(1) - x_2(1)\right)\right)^{1/2} m_0 \left(\frac{p_{0-1}}{p_{01}}\right)^{m_0/2} \frac{1}{\left(p_{01}p_{0-1}\right)^{1/4}}.$$

Proof. Using the equality $1/(t-x) = 1/(t-x_3(1)) + (x-x_3(1))/((t-x)(t-x_3(1)))$ in (38) and setting temporarily $l(t) = (t^{n_0} - (r^2/t)^{n_0})\mu_{m_0}(t,1)(-p_{10}^2(t-x_1(1))(t-x_2(1))(t-x_3(1)))^{1/2}/t$ we obtain :

$$h(x,1) = \frac{x}{x_3(1)}h(x_3(1),1) + \frac{x(x-x_3(1))}{\pi}\int_{x_3(1)}^{x_4(1)} \frac{l(t)}{(t-x)\sqrt{t-x_3(1)}}\mathrm{d}t + P(x),$$

where $P(x)$ is a polynomial, null at $x_3(1)$, obtained from $xP_\infty(x \mapsto x^{n_0-1}Y_0(x,1)^{m_0})(x)$. But we can easily find the singularities of the following Cauchy type integral, see [Lu93] :

$$\int_{x_3(1)}^{x_4(1)} \frac{1}{(t-x)\sqrt{t-x_3(1)}}\mathrm{d}t = \frac{\pi}{\sqrt{x_3(1)-x}}\left(1 + (x-x_3(1))\,u(x)\right),$$

where u is a function holomorphic in the neighborhood of $x_3(1)$. Making an expansion of $l(t) - l(x_3(1))$ in the neighborhood of $x_3(1)$ and with a repeated use of $1/(t-x) = 1/(t-x_3(1)) + (x-x_3(1))/((t-x)(t-x_3(1)))$, we get :

$$\int_{x_3(1)}^{x_4(1)} \frac{l(t) - l(x_3(1))}{(t-x)\sqrt{t-x_3(1)}}\mathrm{d}t = c + v(x)\sqrt{x_3(1)-x},$$

where c is some constant, v some function holomorphic at $x_3(1)$. Thus, Proposition 24 will be proved as soon as we will have made explicit $g(x_3(1))$. Before any simplifications, we have $g(x_3(1)) = l(x_3(1))x_3(1)^{3/2}$. To simplify this quantity, note that $(x_3(1) - x_1(1))(x_4(1) - x_3(1)) = 4(p_{01}p_{0-1})^{1/2}x_3(1)/p_{10}$ and that $\mu_{m_0}(x_3(1), 1) = (p_{0-1}/p_{01})^{(m_0-1)/2}m_0/(2p_{01}x_3(1))$, where the announced value of $g(x_3(1))$ comes from. □

Propositions 23 and 24 allow to derive easily the asymptotic of the absorption probabilities :

Proposition 25. *Suppose that $M_y = p_{01} - p_{0-1} = 0$. The probability of being absorbed at $(i, 0)$ admits the following asymptotic as $i \to +\infty$:*

$$\mathbb{P}_{(n_0,m_0)}\,(\text{to be absorbed at}\,(i,0)) \sim \frac{4}{\pi}\sqrt{\frac{p_{10}}{p_{01}}}n_0 m_0 \frac{1}{i^3}.$$

Proposition 26. *Suppose that $M_y = p_{01} - p_{0-1} > 0$. The probability of being absorbed at $(i, 0)$ admits the following asymptotic as $i \to +\infty$:*

$$\mathbb{P}_{(n_0,m_0)}\,(\text{to be absorbed at}\,(i,0)) \sim \frac{\sqrt{p_{10}\,(x_3\,(1) - x_2\,(1))}}{2\sqrt{\pi}\,(p_{01}p_{0-1})^{1/4}} m_0 \left(\frac{p_{0-1}}{p_{01}}\right)^{m_0/2} \left(x_3^{n_0} - x_2^{n_0}\right) \frac{1}{i^{3/2} x_3\,(1)^i}.$$

Proof. These two propositions are corollaries from Propositions 23 and 24, following the principles giving the way to obtain the asymptotic of the coefficients of a Taylor series at zero starting from the knowledge of the first singularity, principles explained in the proof of Proposition 14 for a logarithmic singularity, in the one of Proposition 17 for an algebraic singularity. □

4.2 Green functions associated to some sets in the case of a drift zero

In this part, we suppose that the two drifts M_x and M_y are equal to zero. Define, for $a, k \in \mathbb{Z}_+$, $\Gamma_{a,k} = \{(i, j) \in (\mathbb{Z}_+)^2 : i - 1 + a(j - 1) = k\}$ and denote by $G_{\Gamma_{a,k}}$ the Green function associated to $\Gamma_{a,k}$, in other words the mean number of visits of the walk in $\Gamma_{a,k}$. Note that $G_{\Gamma_{a,k}}$ is connected with the Green functions $G_{i,j}$ via $G_{\Gamma_{a,k}} = \sum_{i-1+a(j-1)=k} G_{i,j}$

Proposition 27. *The following asymptotic holds as $k \to +\infty$:*

$$G_{\Gamma_{a,k}} \sim \frac{2n_0 m_0}{\sqrt{p_{10}p_{01}}k}.$$

Proof. We start by remarking that for $a \in \mathbb{Z}_+$,

$$G\,(x, x^a, 1) = \sum_{i,j\geq 1} G_{i,j} x^{i-1+a(j-1)} = \sum_{k=0}^{+\infty} x^k \sum_{\Gamma_{a,k}} G_{i,j}.$$

Besides, Equation (8) gives $G(x, x^a, 1) = (h(x, 1) + \tilde{h}(x^a, 1) - x^{n_0+am_0})/Q(x, x^a, 1)$. Also, applied to h and $\tilde{h}$, Proposition 23 leads to :

$$h\,(x, 1) + \widetilde{h}\,(x^a, 1) - x^{n_0+am_0} = \frac{-2n_0 m_0}{\pi}\left(\sqrt{\frac{p_{10}}{p_{01}}} + a^2\sqrt{\frac{p_{01}}{p_{10}}}\right)\ln\,(1 - x)\,(1 + (x - 1)\,l_1\,(x)),$$

where l_1 is holomorphic at $x = 1$. Moreover, an easy calculation yields $Q\,(x, x, 1) = x\,(x - 1)^2/2$; more generally, for any $a > 0$, $Q\,(x, x^a, 1) = (x - 1)^2\,P_a\,(x)$ where

$$P_a\,(x) = p_{01}x\left(\sum_{k=1}^{a-1} k\left(x^{k-1} + x^{2a-1-k}\right) + \left(a + \frac{p_{10}}{p_{01}}\right)x^{a-1}\right),$$

In particular, $P_a\,(1) = p_{01}a^2 + p_{10}$. Thus, we obtain that $G(x, x^a, 1) = -c\ln(1 - x)(1 + (x - 1)l_2(x))$ where l_1 is holomorphic at $x = 1$ and :

$$c = \frac{2n_0 m_0}{\pi P_a\,(1)}\left(\sqrt{\frac{p_{10}}{p_{01}}} + a^2\sqrt{\frac{p_{01}}{p_{10}}}\right) = \frac{2n_0 m_0}{\sqrt{p_{10}p_{01}}}.$$

Then, Proposition 27 follows from the from now on usual way to obtain the asymptotic of coefficients of a function starting from the study of its singularities, see the proof of Proposition 14 for more details. □

We can remark two facts about this asymptotic. First, as the result of Proposition 27 shows, the asymptotic of $G_{\Gamma_{a,k}}$ does not depend on $a > 0$. Secondly, this result is in fact also true for $a = 0$. To show this fact, we have to adapt a little the proof since the explicit expression of $P_a(x)$ is no more valid ; to overcome that, we just have to use the equality $Q(x, 1, 1) = p_{10}(x-1)^2$, the proof is then the same as before.

4.3 Probability of being absorbed

In this subsection, we give a nice explicit expression of the probability for the walk to be absorbed on the boundary. This explicit formulation, obtained in Proposition 28, allows us to find again the well known fact that when at least one of the two drifts (3) is zero, then the walk is almost surely absorbed.

Proposition 28. *The probability of being absorbed is equal to :*

$$h(1,1) + \widetilde{h}(1,1) = 1 - \left(1 - \left(\frac{p_{-10}}{p_{10}}\right)^{n_0}\right)\left(1 - \left(\frac{p_{0-1}}{p_{01}}\right)^{m_0}\right).$$

Proof. We will use the equality (45) of Lemma 29 below. In fact, the right member of (45) can be simplified, but according to the location of y in the complex plane, the simplification will not be the same. Suppose for instance that $|y| > (p_{0-1}/p_{01})^{1/2} (= \tilde{r})$, then Lemma 2 gives that $Y_0(X_0(y,z),z) = Y_0(X_1(y,z),z) = p_{0-1}/(p_{01}y)$, so that (45) becomes :

$$h(X_1(y,z), z) + \widetilde{h}(y,z) - X_1(y,z)^{n_0} y^{m_0} = \left(y^{m_0} - \left(\frac{\tilde{r}^2}{y}\right)^{m_0}\right)\left(X_0(y,z)^{n_0} - \left(\frac{r^2}{X_0(y,z)}\right)^{n_0}\right). \quad (43)$$

Then, Proposition 28 follows immediately from (43), taking $y = 1 \geq \tilde{r}$ and using the facts that $X_0(1,1) = r^2$ and $X_1(1,1) = 1$, seen in Lemma 2. □

Lemma 29. *Suppose that $z \in]0, z_1]$ and $y \in \mathbb{C} \setminus [x_1(z), x_2(z)] \cup [x_3(z), x_4(z)]$. The functions h and $\tilde{h}$ are connected by :*

$$\begin{aligned} \widetilde{h}(y,z) &= X_0(y,z)^{n_0} y^{m_0} - h(X_0(y,z), z), & (44)\\ \widetilde{h}(y,z) &= X_0(y,z)^{n_0} y^{m_0} + X_1(y,z)^{n_0} Y_0(X_1(y,z), z)^{m_0} & (45)\\ &- X_0(y,z)^{n_0} Y_0(X_0(y,z), z)^{n_0} - h(X_1(y,z), z). \end{aligned}$$

Remark 30. (i) The equalities (44) and (45) could be obtained with the procedure of continuation of the functions h and $\tilde{h}$ explained in [FIM99], procedure briefly recalled in the proof of Proposition 36. Here, we have chosen to not firm up all the details of this procedure, and we show how we can find again these equalities using only the explicit expressions of h and $\tilde{h}$. (ii) Equality (44) could be used to continue $\tilde{h}$, since, as it is proved in Lemma 2, $|X_0(y,z)| \leq (p_{-10}/p_{10})^{1/2} \leq 1$, and thus $h(X_0(y,z),z)$ is well defined. (iii) On the other hand, it is quite possible that $|X_1(y,z)| \geq 1$, see once again Lemma 2, so that in (45), $h(X_1(y,z),z)$ has to be defined using its analytic continuation, established here in Proposition 12. (iv) If we make the difference of equations (44) and (45), we find again the boundary condition that verifies h, see (11) of Subsection 2.3.

Proof of Lemma 29. Lemma 29 will follow from a suitable change of variable in the integral expressions of h and $\tilde{h}$ obtained in Proposition 6. The change of variable $t = Y_0(u,z)$, or, equivalently here, $u = X_0(t,z)$, gives

$$\begin{aligned} &\frac{y}{\pi}\int_{y_1(z)}^{y_2(z)} t^{m_0}\widetilde{\mu}_{n_0}(t,z)\left(\frac{1}{t(t-y)} + \frac{1}{ty - \tilde{r}^2}\right)\sqrt{-\widetilde{d}(t,z)}\mathrm{d}t = \\ &\frac{y}{2\pi i}\int_{\mathcal{C}(0,r)} u^{n_0} Y_0(u,z)^{m_0-1}\left(\frac{1}{Y_0(u,z) - y} + \frac{Y_0(u,z)}{Y_0(u,z)y - \tilde{r}^2}\right)\partial_u Y_0(u,z)\,\mathrm{d}u, \end{aligned} \quad (46)$$

since $X_0(\overrightarrow{\overleftarrow{[y_1(z), y_2(z)]}}, z) = \mathcal{C}(0,r)$, as proved in Subsection 2.3. In addition, an immediate consequence of the definition of $Q(u,y,z)$ is the equality $(Y_0(u,z)-y)(Y_1(u,z)-y) = (u-X_0(y,z))(u-X_1(y,z))p_{10}y/(p_{01}u)$; also, since $Y_0(u,z)Y_1(u,z) = \tilde{r}^2$, $(Y_0(u,z)y - \tilde{r}^2)(Y_1(u,z)y - \tilde{r}^2) = \tilde{r}^2(Y_0(u,z) - y)(Y_1(u,z) - y)$. Then, using that $\partial_u Y_0(u,z) = zp_{10}(r^2 - u^2)Y_0(u,z)/(ud(u,z)^{1/2})$, we find that (46) is equal to :

$$\frac{y}{2\pi i}\int_{\mathcal{C}(0,r)} \frac{u^{n_0} Y_0(u,z)^{m_0}\left(r^2 - u^2\right)}{u(u - X_0(y,z))(u - X_1(y,z))}\mathrm{d}u,$$

which, in turn, thanks to the equality $(r^2-u^2)/(u(u-x)(u-r^2/x)) = -x(1/(ux-r^2)+1/(u(u-x)))$, is equal to :

$$\begin{aligned}
&\frac{-X_0\left(y,z\right)}{2\pi i}\int_{\mathcal{C}(0,r)} u^{n_0}Y_0\left(u,z\right)^{m_0}\left(\frac{1}{uX_0\left(y,z\right)-r^2}+\frac{1}{u\left(u-X_0\left(y,z\right)\right)}\right)\mathrm{d}u \\
=\;&\frac{-X_1\left(y,z\right)}{2\pi i}\int_{\mathcal{C}(0,r)} u^{n_0}Y_0\left(u,z\right)^{m_0}\left(\frac{1}{uX_1\left(y,z\right)-r^2}+\frac{1}{u\left(u-X_1\left(y,z\right)\right)}\right)\mathrm{d}u.
\end{aligned}$$

The first equality above will lead to (44), the second to (45). Indeed, we use the residue theorem on the contour drawn in Figure 2 for both integrals above, we obtain a residue part, equal in both cases to $X_0(y,z)^{n_0}y^{m_0}$ and also a algebraic part, that is to say an integral on $[x_1(z),x_2(z)]$. Then, using the explicit expressions of h and $\tilde{h}$ written in Proposition 6, we conclude. We have omitted some details because they are similar to those present in the proofs of the Propositions 5 and 6. □

5 Asymptotic of Green functions. Martin boundary.

In this section, we will be interested in the asymptotic of Green functions $G_{i,j}^{n_0,m_0}$; we recall that

$$G_{i,j}^{n_0,m_0} = \mathbb{E}_{(n_0,m_0)}\left[\sum_{k=0}^{+\infty} 1_{\{(X(k),Y(k))=(i,j)\}}\right].$$

Proposition 31. *Suppose that $M_x = M_y = 0$. The Green functions $G_{i,j}^{n_0,m_0}$ admit the following asymptotic when $i,j\to+\infty$, $j/i\to\tan(\gamma)\in[0,+\infty]$:*

$$G_{i,j}^{n_0,m_0} \sim \frac{4\sqrt{p_{01}p_{10}}}{\pi} n_0 m_0 \frac{ij}{\left(p_{01}i^2+p_{10}j^2\right)^2}.$$

Proof. We will prove Proposition 31 in the case of $\gamma\in[0,\pi/2[$. The result remains the same if we exchange i in j and simultaneously p_{01} in p_{10}, so that from the result corresponding to $j/i\to 0$ we easily deduce the one for $j/i\to\infty$.

In Subsection 2.1 we have already seen that $(x,y)\mapsto G(x,y,1)$ is holomorphic in $\mathcal{D}(0,1)^2$; as a consequence the Cauchy formulas allow to write $G_{i,j}^{n_0,m_0}$ as the following double integrals :

$$G_{i,j}^{n_0,m_0} = \frac{1}{\left(2\pi\imath\right)^2}\iint_{\substack{|x|=1-\epsilon\\|y|=1-\epsilon}} \frac{G\left(x,y,1\right)}{x^iy^j}\mathrm{d}x\mathrm{d}y = \frac{1}{\left(2\pi\imath\right)^2}\iint_{\substack{|x|=1-\epsilon\\|y|=1-\epsilon}} \frac{h\left(x,1\right)+\widetilde{h}\left(y,1\right)-x^{n_0}y^{m_0}}{Q\left(x,y,1\right)x^iy^j}\mathrm{d}x\mathrm{d}y,$$

where $\epsilon\in]0,1[$ and $\imath^2=-1$. The second equality above comes from Equation (8) where we have taken $z=1$. In this way, we can write $G_{i,j}$ as the sum $G_{i,j}=G_{i,j,1}(\epsilon)+G_{i,j,2}(\epsilon)+G_{i,j,3}(\epsilon)$ where :

$$\begin{aligned}
G_{i,j,1}(\epsilon) &= \frac{1}{\left(2\pi\imath\right)^2}\iint_{|x|=|y|=1-\epsilon}\frac{h\left(x,1\right)-h\left(X_1\left(y,1\right),1\right)}{Q\left(x,y,1\right)x^iy^j}\mathrm{d}x\mathrm{d}y,\\
G_{i,j,2}(\epsilon) &= \frac{1}{\left(2\pi\imath\right)^2}\iint_{|x|=|y|=1-\epsilon}\frac{X_1\left(y,1\right)^{n_0}y^{m_0}-x^{n_0}y^{m_0}}{Q\left(x,y,1\right)x^iy^j}\mathrm{d}x\mathrm{d}y,\\
G_{i,j,3}(\epsilon) &= \frac{1}{\left(2\pi\imath\right)^2}\int_{|y|=1-\epsilon}\frac{h\left(X_1\left(y,1\right),1\right)+\widetilde{h}\left(y,1\right)-X_1\left(y,1\right)^{n_0}y^{m_0}}{y^j}\int_{|x|=1-\epsilon}\frac{1}{x^iQ\left(x,y,1\right)}\mathrm{d}x\mathrm{d}y,
\end{aligned}$$

and we will successively study these three integrals. But before going into details, let us explain the ideas of the proof and also make two technical remarks.

We will first transform the double integrals defining each of the $G_{i,j,k}(\epsilon)$ into single integrals, to that purpose we will apply at the $G_{i,j,k}(\epsilon)$ the residue theorem and use the remark (i) below. Then, we will take the limit of these quantities as ϵ goes to zero ; in this way we will obtain that for all i and j, $G_{i,j,1}(\epsilon)\to G_{i,j,1}$, $G_{i,j,2}(\epsilon)\to 0$ and $G_{i,j,3}(\epsilon)\to G_{i,j,3}$ as $\epsilon\to 0$, $G_{i,j,1}$ and $G_{i,j,3}$ being defined in (49) and (50).

It will therefore remain to study $G_{i,j,1}$ and $G_{i,j,3}$, that are integrals on the unit circle. It can be easily shown that the modulus of functions $x^iY_1(x,1)^j$ and $X_1(y,1)^iy^j$, that appear in (49) and (50), have a strict maximum value, equal to 1 and reached for $x=1$ and $y=1$ respectively. Moreover, the exponents i

and j will go to infinity, so that applying a steepest descent method could be interesting ; not exactly the usual steepest descent method –that can be found e.g. in [Fed86]– for the notable reason that the functions considered in integrals (49) and (50) are not holomorphic at the "saddle point" 1 but have there singularities. Indeed, we recall from Subsections 2.2 and 4.1 that X, Y, h and $\tilde{h}$ are not holomorphic at 1 : X (resp. Y) is not holomorphic in the neighborhood of $[y_1(1), y_4(1)]$ (resp. $[x_1(1), x_4(1)]$) and h, $\tilde{h}$ have a logarithmic singularity at 1.

Nevertheless, we will be able to find a steepest descent path for the function $\chi_{j/i}(x) = \ln(xY_1(x,1)^{j/i})$ on both sides of the x-axis, which means that we will construct a function $x_{j/i}$, defined on a neighborhood of 0 and having a singularity at 0, such that $\chi_{j/i}(x_{j/i}(t)) = |t|$; likewise we will find a steepest descent path for the function $\tilde{\chi}_{j/i}(y) = \ln(X_1(y,1)y^{j/i})$ on both sides of the y-axis.

Next, using Cauchy theorem, we will move the contours of integrals (49) and (50), equal initially to the unit circle, into contours that coincide in the neighborhood of 1 with the steepest descent path and that elsewhere remain outside some proper level lines of $\chi_{j/i}$ and $\tilde{\chi}_{j/i}$.

The way to find the asymptotic of $G_{i,j,1}$ and $G_{i,j,3}$ is then classical. We start by splitting the integrals defining them into two parts, one on the steepest descent path, the other one on the remaining part of the contour and next we show that the contribution of this remaining part is, for $G_{i,j,1}$ and for $G_{i,j,3}$, exponentially negligible, which means that we can find two positive constants, say c and ρ, such that the quantity $c\exp(-\rho i)$ is an upper bound of these integrals ; we will even show that we can take c and ρ independent of j/i, provided that j/i remains in some compact. As for the integral on the steepest descent path, we will show that it is identically zero in case of $G_{i,j,1}$, thanks to an interesting relationship between the steepest descent contours proved in remark (ii) ; in case of $G_{i,j,3}$ we will prove that its contribution leads to the result announced in Proposition 31, making a precise study of the behavior of the integrand in the neighborhood of 1.

Before beginning the study of the $G_{i,j,k}(\epsilon)$, $k = 1, 2, 3$, we make the two remarks mentioned above and that will be useful there.

(i) For any $\epsilon > 0$ (and $< 1 - x_1(1)$) and any $|y| = 1 - \epsilon$, $|X_1(y,1)| \geq 1$ and $|X_0(y,1)| \leq 1$, as we already know from Lemma 2. Moreover, there exists a function $\theta_0 = \theta_0(\epsilon)$, continuous and going to zero as ϵ goes to zero, such that :

- if $y = (1-\epsilon)\exp(\imath\theta)$ with $-\theta_0(\epsilon) < \theta < \theta_0(\epsilon)$ then $|X_0(y,1)| > 1 - \epsilon$,
- if $y = (1-\epsilon)\exp(\imath\theta)$ with $\theta \in]\theta_0(\epsilon), 2\pi - \theta_0(\epsilon)[$ then $|X_0(y,1)| < 1 - \epsilon$.

Of course, we can also define a function $\tilde{\theta}(\epsilon)$ that is the analogous of $\theta(\epsilon)$ for the function Y.

(ii) Consider the two functions $\chi_{j/i}(x) = \ln(xY_1(x,1)^{j/i})$ and $\tilde{\chi}_{j/i}(y) = \ln(X_1(y)y^{j/i})$. One can find $\eta > 0$, independent of $j/i \in [0, M]$, $M > 0$ being fixed but as large as wished, such that $\chi_{j/i}$ and $\tilde{\chi}_{j/i}$ are holomorphic in $\mathcal{D}(1,\eta) \cap \{s \in \mathbb{C} : \mathrm{Im}(s) > 0\}$ and in $\mathcal{D}(1,\eta) \cap \{s \in \mathbb{C} : \mathrm{Im}(s) < 0\}$; moreover, they are continuable at 1, where they take the value 0. These continuations are, at 1, continuous, but not differentiable, let alone holomorphic. Consider now the functions $x_{j/i}(t)$ and $y_{j/i}(t)$ defined by

$$\chi_{j/i}\left(x_{j/i}(t)\right) = |t|, \qquad \widetilde{\chi}_{j/i}\left(y_{j/i}(t)\right) = |t|, \tag{47}$$

and by $\mathrm{sign}(\mathrm{Im}(x_{j/i}(t))) = \mathrm{sign}(t)$ and $\mathrm{sign}(\mathrm{Im}(y_{j/i}(t))) = \mathrm{sign}(t)$. These last conditions concerning the sign of the imaginary parts allow in fact to define $x_{j/i}$ and $y_{j/i}$ not ambiguously, indeed, Equation (47) doesn't suffice to determine them, since $|t| = |-t|$.

We will now be interested in the properties of $x_{j/i}$ and $y_{j/i}$.

As a function of a real variable, $t \mapsto |t|$ has clearly no series expansion in the neighborhood of 0, but is equal, on both sides of 0, to functions which have a series expansion at 0, namely $t \mapsto t$ for $t > 0$ and $t \mapsto -t$ for $t < 0$. The same happens for $x_{j/i}$ and $y_{j/i}$: they are not holomorphic at 0 but are equal to holomorphic functions on either side of 0. We will write, for $t > 0$ (resp. $t < 0$), $x_{j/i}(t) = x^+_{j/i}(t) = x_{j/i}(0^+) + x'_{j/i}(0^+)t + \cdots$ and $y_{j/i}(t) = y^+_{j/i}(t) = y_{j/i}(0^+) + y'_{j/i}(0^+)t + \cdots$ (resp. $x_{j/i}(t) = x^-_{j/i}(t) = x_{j/i}(0^-) + x'_{j/i}(0^-)t + \cdots$ and $y_{j/i}(t) = y^-_{j/i}(t) = y_{j/i}(0^-) + y'_{j/i}(0^-)t + \cdots$) to emphasize the fact that the equalities are true only on the right of 0 (resp. on the left of 0). Of course, we find the coefficients of these expansions $x^{(k)}_{j/i}(0^\pm)$ and $y^{(k)}_{j/i}(0^\pm)$ by inverting the relationships (47). For instance,

$$x'_{j/i}\left(0^\pm\right) = \frac{\pm 1 + \frac{j}{i}\sqrt{\frac{p_{10}}{p_{01}}}\,\imath}{1 + \left(\frac{j}{i}\right)^2 \frac{p_{10}}{p_{01}}}, \qquad y'_{j/i}\left(0^\pm\right) = \frac{\pm\frac{j}{i} + \sqrt{\frac{p_{01}}{p_{10}}}\,\imath}{\left(\frac{j}{i}\right)^2 + \frac{p_{10}}{p_{01}}}. \tag{48}$$

A priori, (47) is true on $]-\delta_{j/i},\delta_{j/i}[$, where $\delta_{j/i}>0$ depends on j/i. An important fact is that we can chose δ sufficiently small such that (47) is true for all $t\in]-\delta,\delta[$ and for all $j/i\in[0,M]$. This uniformity property is quite important since it allows to make $j/i\to\tan(\gamma)$ without requiring $j/i=\tan(\gamma)$, what would be too much restrictive ; in particular this is partly that property that allows the calculation of the asymptotic of the Green functions in case $j/i\to 0$. To prove it, we remark that it suffices to find a lower bound, positive and independent of $j/i\in[0,M]$, of the radius of convergence of functions $x_{j/i}^{\pm}$ and $y_{j/i}^{\pm}$; this can be done by finding an upper bound independent of $j/i\in[0,M]$ of the coefficients of the Taylor series at 0 of $x_{j/i}^{\pm}$ and $y_{j/i}^{\pm}$. To find this upper bound, we can use the so called Bürman-Lagrange formula (see e.g. [Cha90]), which allows to write the coefficients of the Taylor series of an inverse function as integrals in terms of the direct function. Then a tedious calculation allows to find $\delta>0$ such that $\sup_{k\in\mathbb{N}}\sup_{j/i\in[0,M]}|x_{j/i}^{(k)}(0^{\pm})|\delta^k<\infty$ and $\sup_{k\in\mathbb{N}}\sup_{j/i\in[0,M]}|y_{j/i}^{(k)}(0^{\pm})|\delta^k<\infty$, from which the uniformity property comes.

In addition, $x_{j/i}$ and $y_{j/i}$ are strongly connected since these two parameterizations are tied together by $Y_1(x_{j/i}(t),1)=y_{j/i}(-t)$ and $X_1(y_{j/i}(t),1)=x_{j/i}(-t)$. To prove this fact, note that a consequence of the definition (47) and also of the relationships concerning the composed functions $X_k\circ Y_l$, where $k,l\in\{0,1\}$, given in Lemma 2 is that $Y_1(x_{j/i}(t),1)\in\{y_{j/i}(-t),y_{j/i}(t)\}$. Then, it suffices to calculate the sign of imaginary part of the previous quantities to identify which of $y_{j/i}(-t),y_{j/i}(t)$ is equal to $Y_1(x_{j/i}(t),1)$; that is, in this case, $y_{j/i}(-t)$.

To sum up, we have built, for all $j/i\in[0,M]$, two contours that coincide in the neighborhood of 1 with the steepest descent paths $x_{j/i}(]-\delta,\delta[)$ and $y_{j/i}(]-\delta,\delta[)$, that elsewhere remain outside some suitable level lines, that are symmetrical w.r.t. the x-axis and that are smooths, except at 1. As an example we have represented in Figure 4 four of these contours. The first two correspond to $j/i\to 0$, the last two are associated to some j/i fixed in $]0,\infty[$. Note that to draw the first two contours, we have used the equalities (which are consequences of (47)) :

$$x_0(]-\delta,\delta[)=\left[\underrightarrow{\overleftarrow{1,\exp(\delta)}}\right[,\qquad y_0(]-\delta,\delta[)=Y_1\left(\left[\underleftarrow{\overrightarrow{1,\exp(\delta)}}\right[,1\right),$$

$y_0(]-\delta,\delta[)$ is thus an arc of circle (recall the definition (10) of the curve $\mathcal{L}_z$ and the fact that it is just a circle).

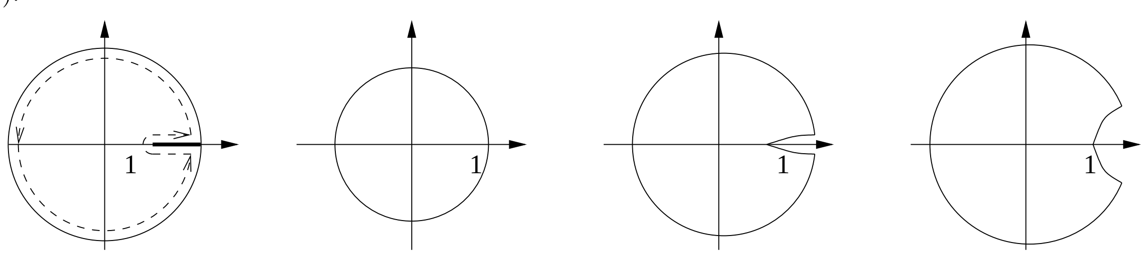

Figure 4: From left to right, the contours associated to χ_0, $\tilde{\chi}_0$, $\chi_{j/i}$ and $\tilde{\chi}_{j/i}$ for some j/i in $]0,+\infty[$.

Study of $G_{i,j,1}(\epsilon)$. The definition of $G_{i,j,1}(\epsilon)$ immediately leads to :

$$\begin{aligned}G_{i,j,1}(\epsilon) &= \frac{1}{(2\pi\imath)^2}\int_{|x|=1-\epsilon}\frac{h(x,1)-h(1,1)}{x^i}\left(\int_{|y|=1-\epsilon}\frac{\mathrm{d}y}{Q(x,y,1)\,y^j}\right)\mathrm{d}x\\ &- \frac{1}{(2\pi\imath)^2}\int_{|y|=1-\epsilon}\frac{h(X_1(y,1),1)-h(1,1)}{y^j}\left(\int_{|x|=1-\epsilon}\frac{\mathrm{d}x}{Q(x,y,1)\,x^i}\right)\mathrm{d}y.\end{aligned}$$

Then, we apply the residue theorem at infinity to the two integrals $\int\mathrm{d}x/(Q(x,y,1)x^i)$ and $\int\mathrm{d}y/(Q(x,y,1)y^j)$ on the contours $|x|=1-\epsilon$ and $|y|=1-\epsilon$. Since $Q(x,y,1)=\tilde{a}(y,1)(x-X_0(y,1))(x-X_1(y,1))=a(x,1)(y-Y_0(x,1))(y-Y_1(x,1))$ we have to know the position of $X_i(y,1)$ and $Y_i(x,1)$ w.r.t. the circle

$\mathcal{C}(0,1-\epsilon)$, but this is exactly the object of the remark (i), so we can write :

$$\begin{aligned} G_{i,j,1}(\epsilon) = & - \frac{1}{2\pi\imath}\int_{\substack{|\theta|<\widetilde{\theta}_0(\epsilon)\\ x=(1-\epsilon)\exp(\imath\theta)}} \frac{h(x,1)-h(1,1)}{a(x,1)(Y_0(x,1)-Y_1(x,1))x^iY_0(x,1)^j}\mathrm{d}x \\ & + \frac{1}{2\pi\imath}\int_{\substack{|\theta|<\theta_0(\epsilon)\\ y=(1-\epsilon)\exp(\imath\theta)}} \frac{h(X_1(y,1),1)-h(1,1)}{\widetilde{a}(y,1)(X_0(y,1)-X_1(y,1))X_0(y,1)^i y^j}\mathrm{d}y \\ & - \frac{1}{2\pi\imath}\int_{|x|=1-\epsilon} \frac{h(x,1)-h(1,1)}{a(x,1)(Y_0(x,1)-Y_1(x,1))x^iY_1(x,1)^j}\mathrm{d}x \\ & + \frac{1}{2\pi\imath}\int_{|y|=1-\epsilon} \frac{h(X_1(y,1),1)-h(1,1)}{\widetilde{a}(y,1)(X_0(y,1)-X_1(y,1))X_1(y,1)^i y^j}\mathrm{d}y. \end{aligned}$$

Due to the occurrence of $h(1,1)$, all the integrands above are integrable in the neighborhood of 1. Moreover, $\theta_0(0)=\widetilde{\theta}_0(0)=0$ so that the first two integrals have a limit equal to zero as ϵ goes to zero. Thus, after ϵ has gone to zero, we obtain that $G_{i,j,1}=\lim_{\epsilon\to 0}G_{i,j,1}(\epsilon)$ is equal to :

$$\frac{1}{2\pi\imath}\int_{|y|=1} \frac{(h(X_1(y,1),1)-h(1,1))\,\mathrm{d}y}{\widetilde{a}(y,1)(X_0(y,1)-X_1(y,1))X_1(y,1)^i y^j} - \frac{1}{2\pi\imath}\int_{|x|=1} \frac{(h(x,1)-h(1,1))\,\mathrm{d}x}{a(x,1)(Y_0(x,1)-Y_1(x,1))x^iY_1(x,1)^j}. \quad (49)$$

We will now use the Cauchy theorem to move the contours $|x|=1$ and $|y|=1$. Since $x\mapsto Y_i(x,1)$ and $y\mapsto X_i(y,1)$, $i=0,1$, are holomorphic on $\mathbb{C}\setminus[x_1(1),x_4(1)]$ and $\mathbb{C}\setminus[y_1(1),y_4(1)]$ and $x\mapsto h(x,1)$ on $\mathbb{C}\setminus[1,x_4(1)]$, we can move the contours $\mathcal{C}(0,1)$ into new ones, that coincide in the neighborhood of 1 with the steepest descent paths constructed previously, and whose remaining part lies in $|\chi_{j/i}(x)|>\rho>0$. In other words, these new contours –that we call $\mathcal{C}_{j/i,x}$ and $\mathcal{C}_{j/i,y}$– verify :

- $\mathcal{C}_{j/i,x}$ (resp. $\mathcal{C}_{j/i,y}$) contains $x_{j/i}(]-\delta,\delta[)$ (resp. $y_{j/i}(]-\delta,\delta[)$) where $x_{j/i}$, $y_{j/i}$ and δ are defined in (ii),
- for all $x\in\mathcal{C}_{j/i,x}\setminus x_{j/i}(]-\delta,\delta[)$ (resp. $y\in\mathcal{C}_{j/i,y}\setminus y_{j/i}(]-\delta,\delta[)$), $|\chi_{j/i}(x)|>\rho(\delta)$ (resp. $|\widetilde{\chi}_{j/i}(x)|>\widetilde{\rho}(\delta)$) where ρ and $\widetilde{\rho}$ are two positive functions.

The fact that ρ and $\widetilde{\rho}$ can be chosen independently of $j/i\in[0,M]$ follows from the continuity of the modulus of $\chi_{j/i}$ and $\widetilde{\chi}_{j/i}$ relatively to $j/i\in[0,M]$.

So, by a standard argument, the integrals on $\mathcal{C}_{j/i,x}\setminus x_{j/i}(]-\delta,\delta[)$ and $\mathcal{C}_{j/i,y}\setminus y_{j/i}(]-\delta,\delta[)$ are exponentially negligible . Indeed, setting $E_1=\{x\in\mathbb{C}:|x-x_1(1)|>\alpha,|x-x_4(1)|>\alpha,|x|<1/\alpha\}$ and $E_2=\{y\in\mathbb{C}:|y-y_1(1)|>\alpha,|y-y_4(1)|>\alpha,|y|<1/\alpha\}$, a consequence of expressions of h and $\tilde{h}$ obtained in Proposition 9 and of the expressions of Y_i and X_i (see Subsection 2.2) is that

$$K=\sup_{x\in E_1}\left|\frac{h(x,1)-h(1,1)}{a(x,1)(Y_0(x,1)-Y_1(x,1))}\right|+\sup_{y\in E_2}\left|\frac{h(X_1(y,1),1)-h(1,1)}{\widetilde{a}(y,1)(X_0(y,1)-X_1(y,1))}\right|<\infty.$$

In addition, the contours $\mathcal{C}_{j/i,x}$ and $\mathcal{C}_{j/i,y}$ can clearly be chosen such that $\cup_{j/i\in[0,M]}\mathcal{C}_{j/i,x}\subset E_1$ and $\cup_{j/i\in[0,M]}\mathcal{C}_{j/i,y}\subset E_2$ and also such that $l=\sup_{j/i\in[0,M]}(\text{length}(\mathcal{C}_{j/i,x})+\text{length}(\mathcal{C}_{j/i,y}))<\infty$. So,

$$\begin{aligned} &\left|\frac{1}{2\pi\imath}\int_{\mathcal{C}_{j/i,y}\setminus y_{j/i}(]-\delta,\delta[)} \frac{h(X_1(y,1),1)-h(1,1)}{\widetilde{a}(y,1)(X_0(y,1)-X_1(y,1))X_1(y,1)^i y^j}\mathrm{d}y\right. \\ &\left.-\frac{1}{2\pi\imath}\int_{\mathcal{C}_{j/i,x}\setminus x_{j/i}(]-\delta,\delta[)} \frac{h(x,1)-h(1,1)}{a(x,1)(Y_0(x,1)-Y_1(x,1))x^iY_1(x,1)^j}\mathrm{d}x\right| \leq \frac{Kl}{2\pi}e^{-i\min(\rho(\delta),\widetilde{\rho}(\delta))}. \end{aligned}$$

It remains to consider :

$$\begin{aligned} & \frac{1}{2\pi\imath}\int_{-\delta}^{\delta} \frac{h\left(X_1\left(y_{j/i}(t),1\right),1\right)-h(1,1)}{\{\widetilde{a}(y,1)(X_0(y,1)-X_1(y,1))\}|_{y=y_{j/i}(t)}}\exp(-i|t|)\,y_{j/i}'(t)\,\mathrm{d}t \\ - & \frac{1}{2\pi\imath}\int_{-\delta}^{\delta} \frac{h\left(x_{j/i}(t),1\right)-h(1,1)}{\{a(x,1)(Y_0(x,1)-Y_1(x,1))\}|_{x=x_{j/i}(t)}}\exp(-i|t|)\,x_{j/i}'(t)\,\mathrm{d}t. \end{aligned}$$

Actually, this quantity is equal to zero. To prove that, we proceed to the three following manipulations in the first integral above : (1) we do the change of variable $t\mapsto -t$ (2) we use the fact (mentioned in (ii)) that $X_1(y_{j/i}(-t),1) = x_{j/i}(t)$ (3) we use the equality $\{\tilde{a}(y,1)(X_0(y,1) - X_1(y,1))\}|_{y=y_{j/i}t)} = -Y_1'(x_{j/i}(t),1)\{a(x,1)(Y_0(x,1) - Y_1(x,1))\}|_{x=x_{j/i}(t)}$. Then, we immediately obtain that the difference of the two integrals is equal to zero. It remains to prove (3) : if we differentiate the equality $Q(x,Y_1(x),1) = 0$, we obtain :

$$\left(2\widetilde{a}\left(y,1\right)x+\widetilde{b}\left(y,1\right)\right)\Big|_{y=Y_1(x,1)}+Y_1'\left(x,1\right)\left(2a\left(x,1\right)Y_1\left(x,1\right)+b\left(x,1\right)\right)=0.$$

Then, taking $x = x_{j/i}(t)$ and using that $X_1(y_{j/i}(-t),1) = x_{j/i}(t)$ and $Y_1(x_{j/i}(-t),1) = y_{j/i}(t)$, we obtain (3).

Study of $G_{i,j,2}(\epsilon)$. As for $G_{i,j,1}(\epsilon)$ we start by splitting $G_{i,j,2}(\epsilon)$ in two terms :

$$\begin{aligned} G_{i,j,2}\left(\epsilon\right) &= \frac{1}{\left(2\pi\imath\right)^2}\int_{|y|=1-\epsilon}\frac{X_1\left(y,1\right)^{n_0}}{y^{j-m_0}}\left(\int_{|x|=1-\epsilon}\frac{\mathrm{d}x}{Q\left(x,y,1\right)x^i}\right)\mathrm{d}y \\ &- \frac{1}{\left(2\pi\imath\right)^2}\int_{|y|=1-\epsilon}\frac{1}{y^{j-m_0}}\left(\int_{|x|=1-\epsilon}\frac{\mathrm{d}x}{Q\left(x,y,1\right)x^{i-n_0}}\right)\mathrm{d}y. \end{aligned}$$

Then, we use that $Q(x,y,1) = \tilde{a}(y,1)(x - X_0(y,1))(x - X_1(y,1))$ and we apply the residue theorem at infinity to the integrals $\int \mathrm{d}x/(Q(x,y,1)x^i)$ and $\int \mathrm{d}x/(Q(x,y,1)x^{i-n_0})$. Using the properties of the modulus of X_0 and X_1 described in (i), we find :

$$G_{i,j,2}\left(\epsilon\right)=\frac{1}{2\pi\imath}\int_{\substack{|\theta|<\theta_0(\epsilon)\\ y=(1-\epsilon)\exp(\imath\theta)}}\frac{1}{\widetilde{a}\left(y,1\right)X_0\left(y,1\right)^i y^{j-m_0}}\frac{X_1\left(y,1\right)^{n_0}-X_0\left(y,1\right)^{n_0}}{X_1\left(y,1\right)-X_0\left(y,1\right)}\mathrm{d}y.$$

First, the integrand is integrable on the contour considered, and secondly, once again thanks (i), $\theta_0(\epsilon) \to 0$ when $\epsilon \to 0$; so $G_{i,j,2}(\epsilon) \to 0$ as $\epsilon \to 0$.

Study of $G_{i,j,3}(\epsilon)$. As for $G_{i,j,1}(\epsilon)$ and $G_{i,j,2}(\epsilon)$ we write $Q(x,y,1) = \tilde{a}(y,1)(x - X_0(y,1))(x - X_1(y,1))$ and we apply the residue theorem at infinity. Next, we let $\epsilon \to 0$; we obtain :

$$G_{i,j,3}=\lim_{\epsilon\to 0}G_{i,j,3}(\epsilon)=\frac{1}{2\pi\imath}\int_{|y|=1}\frac{h\left(X_1\left(y,1\right),1\right)+\widetilde{h}\left(y,1\right)-X_1\left(y,1\right)^{n_0}y^{m_0}}{\widetilde{a}\left(y,1\right)\left(X_0\left(y,1\right)-X_1\left(y,1\right)\right)X_1\left(y,1\right)^i y^j}\mathrm{d}y. \tag{50}$$

We now move the contour from the unit circle into $\mathcal{C}_{j/i,y}$, using the same arguments as in the paragraph concerning the study of $G_{i,j,1}$. For the same reasons as before, the integral on $\mathcal{C}_{j/i,y} \setminus y_{j/i}(]-\delta,\delta[)$ is exponentially negligible, so that from now we consider that the integral defining $G_{i,j,3}$ is on the contour $y_{j/i}(]-\delta,\delta[)$.

Now, we notice that from remark (ii) we can deduce that $y_{j/i}(]-\delta,\delta[) \subset \{s \in \mathbb{C} : |s| > 1\}$, so that in accordance with (43) we can write, for all $y \in y_{j/i}(]-\delta,\delta[)$:

$$h\left(X_1\left(y,1\right),1\right)+\widetilde{h}\left(y,1\right)-X_1\left(y,1\right)^{n_0}y^{m_0}=-\left(y^{m_0}-y^{-m_0}\right)\left(X_1\left(y,1\right)^{n_0}-X_1\left(y,1\right)^{-n_0}\right).$$

For this reason,

$$\frac{h\left(X_1\left(y,1\right),1\right)+\widetilde{h}\left(y,1\right)-X_1\left(y,1\right)^{n_0}y^{m_0}}{\widetilde{a}\left(y,1\right)\left(X_0\left(y,1\right)-X_1\left(y,1\right)\right)}=\frac{\left(y-1\right)\left(X_1\left(y,1\right)-1\right)}{\sqrt{\widetilde{d}\left(y,1\right)}X_1\left(y,1\right)^{n_0}y^{m_0}}\sum_{i=0}^{2m_0-1}y^i\sum_{i=0}^{2n_0-1}X_1\left(y,1\right)^i.$$

Since $X_1(y,1) - 1 = \tilde{d}(y,1)^{1/2}/(2\tilde{a}(y,1)) - p_{01}(y-1)^2/(2\tilde{a}(y,1))$, we have the following expansion :

$$\frac{\left(y-1\right)\left(X_1\left(y,1\right)-1\right)}{\sqrt{\widetilde{d}\left(y,1\right)}X_1\left(y,1\right)^{n_0}y^{m_0}}\sum_{i=0}^{2m_0-1}y^i\sum_{i=0}^{2m_0-1}X_1\left(y,1\right)^i=\frac{2n_0m_0}{p_{10}}\left(y-1\right)+c_2\left(y-1\right)^2+c_3\left(y-1\right)^3+\cdots,$$

where the coefficients $c_2, c_3, \dots$ depend on the half plane (upper or lower) where the previous expansion is written. We could of course male explicit these coefficients $c_2, c_3, \dots$ but we don't need it. Indeed, using the expansion of $y_{j/i}$ at the left and the right of 0, and Laplace's method, we see that the integrals

$$\int_{-\delta}^{\delta}\left(y_{j/i}\left(t\right)-1\right)^k\exp\left(-i\left|t\right|\right)y_{j/i}'\left(t\right)\mathrm{d}t$$

are for $k \geq 2$ polynomially negligible with respect to the same integral where $k=1$. Therefore, to find the asymptotic, we have only to consider the above integral for $k=1$. We find that $\int_{-\delta}^{\delta}(y_{j/i}(t)-1)\exp(-i|t|)y'_{j/i}(t)\mathrm{d}t \sim (y'_{j/i}(0+)^2 - y'_{j/i}(0-)^2)\int_0^{\delta} t\exp(-it)\mathrm{d}t$ as i goes to infinity ; then with (48) we get :

$$y'_{j/i}(0+)^2 - y'_{j/i}(0-)^2 = 4\imath p_{01}^{1/2}p_{10}^{3/2}\frac{ij}{(p_{10}j^2+p_{01}i^2)^2},$$

from which Proposition 31 follows immediately. □

Let us now turn to the case $M_x>0$ and $M_y>0$. In [KM98], the authors obtain the Green functions' asymptotic for the random walks in $(\mathbb{Z}_+)^2$ with the same jump probabilities in the interior of the quadrant as ours but with non zero jumps from the axes to the interior. The analysis of this problem in our case can be carried out by the same methods. Therefore we just claim

Proposition 32. *Denote by $(s_3(\gamma), t_3(\gamma))$ the unique critical point of $(x,y)\mapsto xy^{\tan(\gamma)}$ on $\{(x,y)\in \mathbb{C}^2, Q(x,y,1)=0, x\in\mathbb{R}_+, y\in\mathbb{R}_+\}$. The Green functions admit the following asymptotic when $i,j\to\infty$ and $j/i\to\tan(\gamma)\in]0,+\infty[$:*

$$G_{i,j}^{n_0,m_0} \sim \frac{\sqrt{s_3(\gamma)^{\frac{1}{\tan(\gamma)}}t_3(\gamma)}\left(s_3(\gamma)^{n_0}t_3(\gamma)^{m_0} - h(s_3(\gamma),1) - \widetilde{h}(t_3(\gamma),1)\right)}{\sqrt{2\pi}\big(2a(s_3(\gamma))t_3(\gamma)+b(s_3(\gamma))\big)\sqrt{j\frac{\mathrm{d}}{\mathrm{d}x^2}\left(x^{1/\tan(\gamma)}Y_1(x,1)\right)\Big|_{x=s_3(\gamma)}}s_3(\gamma)^i t_3(\gamma)^j}. \tag{51}$$

In cases $j/i\to 0$ or $+\infty$, the Green functions admit the asymptotic :

$$\begin{aligned} G_{i,j}^{n_0,m_0} &\sim \left(\left(1-2\sqrt{p_{0-1}p_{01}}\right)^2 - 4p_{-10}p_{10}\right)^{1/2}\frac{m_0\widetilde{r}^{m_0-1}\left(x_3(1)^{n_0}-x_2(1)^{n_0}\right)j}{\sqrt{\pi p_{01}\sqrt{\widetilde{d}(\widetilde{r},1)}}i^{3/2}x_3(1)^i\,\widetilde{r}^j},\ j/i\to 0,\\ G_{i,j}^{n_0,m_0} &\sim \left(\left(1-2\sqrt{p_{-10}p_{10}}\right)^2 - 4p_{0-1}p_{01}\right)^{1/2}\frac{n_0 r^{n_0-1}\left(y_3(1)^{m_0}-y_2(1)^{m_0}\right)i}{\sqrt{\pi p_{10}\sqrt{d(r,1)}}j^{3/2}r^i y_3(1)^j},\ j/i\to+\infty. \end{aligned} \tag{52}$$

Our previous remarks allow us to be more specific about this result concerning two things. First, as it was already remarked in [Mal73] and in [KM98], $t_3(\gamma)\in[(p_{0-1}/p_{01})^{1/2}, y_3(1)]$ and $s_3(\gamma)=X_1(t_3(\gamma),1)$. So with (43) we obtain that the constant $s_3(\gamma)^{n_0}t_3(\gamma)^{m_0}-h(s_3(\gamma),1)-\widetilde{h}(t_3(\gamma),1)$ is equal to $(t_3(\gamma)^{m_0} - (\tilde{r}^2/t_3(\gamma))^{m_0})(s_3(\gamma)^{n_0}-(r^2/s_3(\gamma))^{n_0})$, which is a simpler expression. In fact, we can simplify longer, and that is the second thing that we wanted to add, the critical point $(s_3(\gamma), t_3(\gamma))$ has the following explicit expression :

$$s_3(\gamma) = -B(\gamma)/2 + \sqrt{(B(\gamma)/2)^2-1}, \qquad t_3(\gamma) = -\widetilde{B}(\gamma)/2 + \sqrt{\left(\widetilde{B}(\gamma)/2\right)^2-1},$$

where $B(\gamma) = (1-(1-(1-\tan(\gamma)^2)(1-4p_{0-1}p_{01}+4p_{-10}p_{10}\tan(\gamma)^2))^{1/2})/(\tan(\gamma)^2-1)$ and $\tilde{B}(\gamma) = (1-(1-(1-\tan(\gamma)^{-2})(1-4p_{-10}p_{10}+4p_{0-1}p_{01}\tan(\gamma)^{-2}))^{1/2})/(\tan(\gamma)^{-2}-1)$. In particular, note that $s_3(0)=x_3(1)$, $s_3(\pi/2)=r$, $t_3(0)=\tilde{r}$ and $t_3(\pi/2)=y_3(1)$. Note that we obtain the explicit expressions of $s_3(\gamma)$ and $t_3(\gamma)$ by solving $(xY_1(x,1)^{\tan(\gamma)})'=0$ and $(X_1(y,1)y^{\tan(\gamma)})'=0$. We will discuss this fact again in a next work.

Results of Sections 4 and 5 allow to describe completely the Martin boundary of the process, both in case of a positive drift and in case of a zero drift. For definitions, details and applications of Martin boundary theory, we refer to [Dyn69] and [Rev84].

Remark 33. Let us now fix any reference state in the interior of the quadrant, e.g. $(1,1)$, and consider the Martin kernel $k_{(n_0,m_0)}(i,j)$. It equals to $G_{(i,j)}^{(n_0,m_0)}/G_{(i,j)}^{(1,1)}$ and as well to $\mathbb{P}_{(n_0,m_0)}(\text{to hit }(i,j))/\mathbb{P}_{(1,1)}(\text{to hit }(i,j))$. We will use the first form in the case $i,j>0$ and the second one in the case $i=0$ or $j=0$. At last we set, for $\gamma\in[0,\pi/2]$, $k_{(n_0,m_0)}(\gamma)=\lim_{i,j\to+\infty, j/i\to\tan(\gamma)}k_{(n_0,m_0)}(i,j)$.

It follows from Proposition 31 that in the case $M_x=M_y=0$, for any sequence (i,j) where both coordinates are positive and j/i converges to $\tan(\gamma)\in[0,+\infty]$, the Martin kernel $k_{(n_0,m_0)}(i,j)$ converges to n_0m_0. Moreover, we can deduce from Proposition 26 that in the case $M_x=M_y=0$, for any sequence of pairs (i,j) where one of the coordinates i or j goes to infinity, the other being 0, the Martin kernel $k_{(n_0,m_0)}(i,j)$ converges also to n_0m_0. So if the two drifts are equal to zero, n_0m_0 is the unique point of the Martin boundary.

Suppose now that the drifts M_x and M_y are positive and show that in this case, the Martin boundary is homeomorphic to $[0,\pi/2]$. It follows from (51) of Proposition 32 that for any sequence (i,j) where both coordinates are positive and j/i converges to $\tan(\gamma)\in]0,+\infty[$, the Martin kernel $k_{(n_0,m_0)}(i,j)$ converges to $k_{(n_0,m_0)}(\gamma)=(t_3(\gamma)^{m_0}-(\tilde{r}^2/t_3(\gamma))^{m_0})(s_3(\gamma)^{n_0}-(r^2/s_3(\gamma))^{n_0})/((t_3(\gamma)-\tilde{r}^2/t_3(\gamma))(s_3(\gamma)-r^2/s_3(\gamma)))$. If now (i,j) is a sequence whose both coordinates are positive and such that j/i goes to 0 or ∞, Proposition 32 gives that the Martin kernel $k_{(n_0,m_0)}(i,j)$ converges to $k_{(n_0,m_0)}(0)=m_0\tilde{r}^{m_0/2-1}(x_3(1)^{n_0}-x_2(1)^{n_0})/(x_3(1)-x_2(1))$ and $k_{(n_0,m_0)}(\pi/2)=n_0r^{n_0/2-1}(y_3(1)^{m_0}-y_2(1)^{m_0})/(y_3(1)-y_2(1))$ respectively. A consequence of Proposition 25 is that the Martin kernels $k_{(n_0,m_0)}(i,0)$ and $k_{(n_0,m_0)}(0,j)$ converge too, respectively to $m_0\tilde{r}^{m_0/2-1}(x_3(1)^{n_0}-x_2(1)^{n_0})/(x_3(1)-x_2(1))$ and $n_0r^{n_0/2-1}(y_3(1)^{m_0}-y_2(1)^{m_0})/(y_3(1)-y_2(1))$ as i and j go to infinity, respectively. In particular, the Martin kernel is the same depending on whether $j/i\to 0$ with $j>0$ or $j/i\to 0$ with $j=0$; also on whether $i/j\to 0$ with $i>0$ or $i/j\to 0$ with $i=0$. At last, using the explicit expression of the critical point $(s_3(\gamma),t_3(\gamma))$ given just above Remark 33, we notice that the function $k_{(n_0,m_0)}(\gamma)$ is continuous on $[0,\pi/2]$. Therefore the Martin boundary is homeomorphic to a segment $[0,\pi/2]$.

6 Extension of the results

Suppose now that in addition to p_{10}, p_{-10}, p_{01}, p_{0-1}, we permit the transition probabilities p_{11}, p_{-11}, p_{-1-1}, p_{1-1} to be non zero as in the hypothesis (H2) of Section 1, and suppose that the two drifts $M_x=\sum_{i,j}ip_{ij}$ and $M_y=\sum_{i,j}jp_{ij}$ are non negative. We make the following additional hypothesis : in the list $p_{-1-1},p_{-10},p_{-11},p_{01},p_{11},p_{10},p_{1-1},p_{0-1},p_{-1-1},p_{-10}$, there are no three consecutive zeros. This technical assumption allows to avoid studying degenerated random walks.

We ask us the same questions as before : can we still find the asymptotic of $\mathbb{P}_{(n_0,m_0)}(\text{to be absorbed at }(i,0))$, that of $\mathbb{P}_{(n_0,m_0)}(\tau=k)$, that of $G_{i,j}^{n_0,m_0}$?

To answer these questions we take back the analytic aspects considered at the beginning of this article and we try to generalize them : we can define an analogous of the polynomial Q presented in (9) : $Q(x,y,z)=xyz(\sum_{i,j}p_{ij}x^iy^j-z^{-1})$, and Equation (8) is still true, if we add to the right member the function $h_{00}(z)$, equal to the generating function of the probabilities of being absorbed at $(0,0)$ at time n : $h_{00}(z)=\sum_{n=0}^{+\infty}\mathbb{P}_{(n_0,m_0)}(\text{to hit }(0,0)\text{ at time }n)z^n$. Next, we can also define the functions $X(y,z)$ and $Y(x,z)$, that verify properties closed to those described in Lemma 2. We can too, as in (10), define the curves $\mathcal{L}_z$ and $\mathcal{M}_z$, on which the functions h and $\tilde{h}$ verify again boundary conditions, like (11). There is however here a striking difference between the walks studied in the previous sections and the more general walks : for the first, the curves $\mathcal{L}_z$ and $\mathcal{M}_z$ are included in the closed unit disc (indeed, they are circles centered at zero and with radius less than one, as we have seen in Subsection 2.3), what is *prima facie* no more true for the second. In addition, in both cases, the functions h and $\tilde{h}$ are holomorphic in the open unit disc, continuous up to its boundary. So, for general walks, h and $\tilde{h}$ could be not defined on the curves on which they satisfy formally the boundary condition (11) ; this is why for such walks, we have first to continue h and $\tilde{h}$ into functions holomorphic on sets whose adherence contains respectively $\mathcal{L}_z$ and $\mathcal{M}_z$. We have already mentioned in Subsection 4.3 that there exists in [FIM99] a nice method to construct this continuation elaborated by using Galois automorphisms, notion that we will explain briefly in the proof of Proposition 36.

But suppose now that we did this continuation and also that somehow or other we have found the CGF w_z ; then it is possible, following the method developed in Subsection 2.4, to find an integral representation of h :

$$h\left(x,z\right)=x^{n_0}Y_0\left(x,z\right)^{m_0}+\frac{1}{\pi}\int_{x_1(z)}^{x_2(z)}t^{n_0}\mu_{m_0}\left(t,z\right)\phi\left(t,x,z\right)\sqrt{-d\left(t,z\right)}\mathrm{d}t,$$

where $\phi(t,x,z)=w_z'(t)/(w_z(t)-w_z(x))-w_z'(t)/(w_z(t)-w_z(0))$. Therefore, it is definitely the search and the study of the CGF that constitute the key points of the generalization, and that is here that there is an other fundamental difference between the walks verifying $p_{10}+p_{-10}+p_{01}+p_{0-1}=1$ and the more general walks : as we will see in a next work, it is still possible, adapting an idea present in [FIM99], to prove in the general case the existence of the CGF, we will even obtain the explicit expression of the CGF for the curves $\mathcal{L}_z$ and $\mathcal{M}_z$; but these explicit expressions are strongly complicated and hardly usable. For all that, in this next work, we will be able to find the asymptotic of $\mathbb{P}_{(n_0,m_0)}(\text{to be absorbed at }(i,0))$ for any walk, as for the one $\mathbb{P}_{(n_0,m_0)}(\tau=k)$, we will have to concentrate us on a few particular cases.

But go back to our study and search how to generalize our results relatively easily. For the walks such that $p_{10}+p_{-10}+p_{01}+p_{0-1}=1$, Proposition 4 was quite advantageous, since the expression of the CGF could not really be more simple ; one can think that for the other walks for which the curves $\mathcal{L}_z$ and $\mathcal{M}_z$ are equal to circles, we will be able to generalize our main results without real difficulties. So we ask us which are the walks such that $\mathcal{L}_z$ and $\mathcal{M}_z$ are certainly circles. To answer this question, we have to introduce the quantity $\Delta(z)$, equal to :

$$\Delta(z)=\begin{vmatrix} p_{11} & p_{10} & p_{1-1} \\ p_{01} & -1/z & p_{0-1} \\ p_{-11} & p_{-10} & p_{-1-1} \end{vmatrix}.$$

The answer is then given by the following result, whose proof is postponed to Subsection 6.3.

Lemma 34. *Define $z_1=\inf\{z\geq 0 : x_2(z)=x_3(z)\}$ and let z be in $]0,z_1]$. Suppose that $\Delta(z)=0$; then the curves $\mathcal{L}_z$ and $\mathcal{M}_z$, defined in (10), are circles, eventually degenerated in straight lines.*

This lemma will allow us to make the suitable hypothesis in Subsections 6.1 and 6.2.

We will first of all, in Part 6.1, be interested in the asymptotic of $\mathbb{P}_{(n_0,m_0)}$(to be absorbed at $(i,0)$) ; to begin with a drift zero, then with a positive drift. The study of these quantities is based on the analysis of the singularities of $h(x,1)$ and $\tilde{h}(y,1)$, as in Subsection 4.1. It makes not appear the time z, which is in fact equal there to 1, so that we will suppose in all Part 6.1 that $\Delta(1)=0$. In concrete terms, this hypothesis means that the three polynomials $a(x,1)=p_{11}x^2+p_{01}x+p_{-11}$, $b(x,1)=p_{10}x^2-x+p_{-10}$ and $c(x,1)=p_{1-1}x^2+p_{0-1}x+p_{-1-1}$ are linearly dependent.

Then, in Subsection 6.2, we will be interested in the asymptotic of $\mathbb{P}_{(n_0,m_0)}(S=k)$ and $\mathbb{P}_{(n_0,m_0)}(T=k)$ in case of a zero drift, S and T being the hitting times (30) ; that will be derived from the study of the functions $h(1,z)$ and $\tilde{h}(1,z)$ of the variable z. Therefore, we will there suppose that for all $z\in]0,1]$ –if the drift is zero then $z_1=1$–, $\Delta(z)=0$, or equivalently that $\Delta(1)=\Delta'(1)=0$. Here is an interpretation of this hypothesis :

Lemma 35. *Suppose that $M_x=M_y=0$. Then $\Delta(1)=0$ is equivalent to $p_{11}+p_{-1-1}=p_{1-1}+p_{-11}$, which means that the process has a covariance equal to zero. Suppose still $M_x=M_y=0$ and that $\Delta(1)=0$ and make the additional assumption $\Delta'(1)=0$. Then $a=c$ or $\tilde{a}=\tilde{c}$, which means that either $p_{ij}=p_{i-j}$ for all i,j or $p_{ij}=p_{-ij}$ for all i,j.*

This lemma and the forthcoming Proposition 36 will be proved in Subsection 6.3.

We close this introductory part by stating a result generalizing Proposition 12 : suppose that the drifts M_x and M_y are non negative and that for some $z\in]0,z_1]$, $\Delta(z)=0$. Then it is still possible to continue $x\mapsto h(x,z)$ and $y\mapsto\tilde{h}(y,z)$ on $\mathbb{C}\setminus[x_3(z),x_4(z)]$ and $\mathbb{C}\setminus[y_3(z),y_4(z)]$.

Proposition 36. *Suppose that for some $z\in]0,z_1]$, $\Delta(z)=0$. Then the functions $x\mapsto h(x,z)$ and $y\mapsto\tilde{h}(y,z)$ are continuable into functions holomorphic on $\mathbb{C}\setminus[x_3(z),x_4(z)]$ and $\mathbb{C}\setminus[y_3(z),y_4(z)]$ respectively.*

In fact the hypothesis $\Delta(z)=0$ is not necessary but we do it for two reasons : first because all the walks we are studying here verify this assumption for at least one $z\in]0,z_1]$, second because the proof (done in Subsection 6.3) is quite simpler in this case.

6.1 Asymptotic in the case $\Delta(1)=0$

We have already defined, in the discussion beginning the Section 6, the polynomials $a(x,1)=p_{11}x^2+p_{01}x+p_{-11}$, $b(x,1)=p_{10}x^2-x+p_{-10}$ and $c(x,1)=p_{1-1}x^2+p_{0-1}x+p_{-1-1}$. Likewise, we define $d(x,1)=b(x,1)^2-4a(x,1)c(x,1)$ and $x_i(1)$, $i\in\{1,\cdots,4\}$, to be the (real) roots of $d(x,1)$, say that they are enumerated such that $|x_1(1)|<|x_2(1)|\leq 1\leq|x_3(1)|<|x_4(1)|$. Note that in [FIM99] the authors show that if $M_y=0$ then $x_2(1)=1=x_3(1)$ whereas if $M_y>0$ then $0<x_2(1)<1<x_3(1)$. We recall that $h_i=\mathbb{P}_{(n_0,m_0)}$(to be absorbed at $(i,0)$).

Proposition 37. *We suppose here that $M_y=\sum_{i,j}jp_{ij}=0$. The probability of being absorbed at $(i,0)$ admits the following asymptotic :*

$$h_i\sim\frac{2n_0m_0}{\pi}\sqrt{\frac{\left(p_{10}^2-4p_{11}p_{1-1}\right)\left(x_4(1)-1\right)\left(1-x_1(1)\right)}{a(1,1)\,c(1,1)}}\,\frac{1}{i^3},\qquad i\to\infty.$$

Proposition 38. *We suppose here that $M_y = \sum_{i,j} ip_{ij} > 0$. The probability of being absorbed at $(i,0)$ admits the following asymptotic :*

$$h_i \sim \sqrt{\frac{-x_3(1)\,d'(x_3(1),1)}{a(x_3(1),1)\,c(x_3(1),1)}}\frac{(x_3(1)^{n_0} - x_2(1)^{n_0})}{4\sqrt{\pi}} m_0 \left(\frac{c(x_3(1),1)}{a(x_3(1),1)}\right)^{\frac{m_0}{2}} \frac{1}{x_3(1)^i}\frac{1}{i^{3/2}}, \qquad i \to \infty.$$

Proof. Propositions 37 and 38 are generalizations of Propositions 25 and 26, which are themselves corollaries of the writing of $h(x,1)$ in the neighborhood of $x_3(1)$, its first singularity. This expansion of $h(x,1)$ at $x_3(1)$ was the object of Propositions 23 and 24, which are consequences of the explicit expression of h, written in Proposition 9. We could follow the same development here : *first* finding a generalization of the explicit expression of $h(x,1)$, *then* studying this integral function and its singularity at $x_3(1)$, *at last* deducing the asymptotic of the coefficients of the Taylor series at 0. The technical details look like those already outlined, notably in the proofs of Propositions 23–26, so we omit them ; except the generalization of the integral representation of h, which is quite interesting. According to Lemma 34 and since $\Delta(1) = 0$, $\mathcal{M}_1$ is a circle –suppose non degenerated–, of center γ and radius ρ say. With these notations, define $\sigma(t) = \gamma + \rho^2/(t-\gamma)$ and suppose that $x_4(1) > 0$. Then, the following equality holds :

$$h(x,1) = \frac{x}{\pi}\int_{x_3(1)}^{x_4(1)} (t^{n_0} - \sigma(t)^{n_0}) \frac{\mu_{m_0}(t,1)}{t(t-x)}\sqrt{-d(t,1)}\mathrm{d}t + xP_\infty\left(x \mapsto x^{n_0-1}Y_0(x,z)^{m_0}\right)(x). \tag{53}$$

If $x_4(1) < 0$ or if the circle $\mathcal{M}_1$ is degenerated, we could even so find an explicit formulation like (53). In any case, it would be useful next, starting from (53) or an equivalent, to study the singularity of $h(x,1)$ at $x_3(1)$; as already said we don't write the details and refer to the proofs of Propositions 23–26. □

6.2 Asymptotic in the case $\Delta(1) = \Delta'(1) = 0$

We have already explained in Lemma 35 that the hypothesis $\Delta(1) = \Delta'(1) = 0$ for all z in $]0, z_1]$ is equivalent, in case of two zero drifts, to the fact that $a = c$ or $\tilde{a} = \tilde{c}$. A particular case of random walks verifying these assumptions is the set of walks such that $p_{10} + p_{-10} + p_{01} + p_{0-1} = 1$, $p_{-10} = p_{10}$, $p_{0-1} = p_{01}$, studied in the previous sections. The next proposition consists in generalizing Proposition 14 in the case of all walks with drifts zero and verifying in addition $\Delta(1) = \Delta'(1) = 0$. We recall from (30) that we denote by S and T the hitting times of the x and y-axis.

Proposition 39. *We suppose here that $M_x = M_y = 0$ and that $\Delta(1) = \Delta'(1) = 0$. Then the probability of being absorbed at time k on the x-axis admits the following asymptotic :*

$$\mathbb{P}_{(n_0,m_0)}(S = k) \sim \frac{n_0 m_0}{2\pi\left((p_{11} + p_{10} + p_{1-1})(p_{11} + p_{01} + p_{-11})\right)^{1/2} k^2} \tag{54}$$

The same asymptotic holds for $\mathbb{P}_{(n_0,m_0)}(T = k)$, the probability of being absorbed at time k on the y-axis.

Proof. We will now give a sketch of the proof of Proposition 39, in the case $\tilde{a} = \tilde{c}$, the case $a = c$ being of course symmetrical.

First, we are interested in the asymptotic of $\mathbb{P}_{(n_0,m_0)}(S = k)$. Since $\Delta(z) = 0$ for all $z \in]0, z_1]$, Lemma 34 gives that for all $z \in]0, z_1]$ the curve $\mathcal{M}_z$ is a circle ; in fact simply the unit circle $\mathcal{C}(0,1)$, because $\tilde{a} = \tilde{c}$. The CGF associated to this curve is thus equal to $t \mapsto t + 1/t$ and the –fundamental– Proposition 9, which gives the explicit expression of $h(x,z)$, is still valid. Then, we can adapt the change of variable $t = t_2(u,z)$, see (19), made in Part 2.5, and, with some additional technical details, we can follow the Subsection 3.1 and finally obtain the asymptotic (54).

Now, we are interested in the asymptotic of $\mathbb{P}_{(n_0,m_0)}(T = k)$. Once again thanks to Lemma 34, we find that $\mathcal{L}_z$ is also a circle –suppose non degenerated–, but this time with a center $\tilde{\gamma}(z)$ and a radius $\tilde{\rho}(z)$ that depend on z and that can be not equal to zero and one respectively. In particular, we will perhaps have first to continue $\tilde{h}$ into a holomorphic function up to $\mathcal{L}_z$, using Proposition 36. $\tilde{\gamma}$ and $\tilde{\rho}$ are defined in substance in the proof of Lemma 34 : indeed, we give there an explicit expression of the circle $\mathcal{L}_z$. In particular, the CGF associated to $\mathcal{L}_z$ is now equal to $\tilde{w}_z(t) = t + \tilde{\rho}(z)^2/(t - \tilde{\gamma}(z))$. A consequence of these facts is that we have to adapt the Proposition 9, giving the explicit expression of $\tilde{h}(y,z)$, which, as it is, is no more true. Skipping over the details, we obtain the following integral formulation for $\tilde{h}(y,z)$:

$$\tilde{h}(y,z) = \frac{y}{\pi}\int_{y_3(z)}^{y_4(z)} (t^{m_0} - \tilde{\sigma}(t,z)^{m_0}) \frac{\tilde{\mu}_{n_0}(t,z)}{t(t-y)}\sqrt{-\tilde{d}(t,z)}\mathrm{d}t + yP_\infty\left(y \mapsto X_0(y,z)^{n_0}y^{m_0-1}\right)(y),$$

where P_∞ is the principal part, defined in Lemma 7 and $\tilde{\sigma}(t,z)=\tilde{\gamma}(z)+\tilde{\rho}(z)^2/(t-\tilde{\gamma}(z))$. The function $\tilde{\sigma}$ satisfies many noteworthy relationships ; for instance, $\tilde{\sigma}$ leaves $\tilde{w}_z$ invariant : $\tilde{w}_z(\tilde{\sigma}(t,z))=\tilde{w}_z(t)$. Then, we can one more time adapt the change of variable $t=t_2(u,z)$, see (19), proposed in Part 2.5, and we finally get the asymptotic of $\mathbb{P}_{(n_0,m_0)}(T=k)$. $\square$

6.3 Proofs of lemmas

Proof of Lemma 34. The proof is based on the explicit expressions of $\mathcal{L}_z$ and $\mathcal{M}_z$: in [FIM99], the authors give the way to find the explicit expressions of these curves in the particular case $z=1$. We can adapt this argument and obtain the expressions of the curves for all $z\in]0,z_1]$; what is new here is the possibility of expressing the curves $\mathcal{L}_z$ and $\mathcal{M}_z$ in terms of three determinants : $\mathcal{L}_z$ is equal to $\{u+iv\in\mathbb{C}:q_z(u,v)^2-q_{1,z}(u,v)q_{2,z}(u,v)=0\}$ where q_z, $q_{1,z}$ and $q_{2,z}$ are respectively equal to :

$$\begin{vmatrix} p_{11} & p_{10} & p_{1-1}\\ 1 & -2u & u^2+v^2\\ p_{-11} & p_{-10} & p_{-1-1}\end{vmatrix},\quad \begin{vmatrix} 1 & -2u & u^2+v^2\\ p_{01} & -1/z & p_{0-1}\\ p_{-11} & p_{-10} & p_{-1-1}\end{vmatrix},\quad \begin{vmatrix} p_{11} & p_{10} & p_{1-1}\\ p_{01} & -1/z & p_{0-1}\\ 1 & -2u & u^2+v^2\end{vmatrix},$$

and, of course, we could write a similar expression for $\mathcal{M}_z$. Now that we have the expression of these polynomials, we will establish some relationships between their coefficients ; but before take the notations : for $i=1,2$, α_i, β_i and γ_i will stand for the coefficients –depending on z– of $q_{i,z}$: $q_{i,z}(u,v)=\alpha_i-2\beta_i u+\gamma_i(u^2+v^2)$; one obtains obviously the explicit expression of these coefficients by expanding the determinants above. Then, by a simple calculation, we verify the three following facts : first $\alpha_1\gamma_2-\alpha_2\gamma_1=-\Delta(z)/z$, then $\alpha_1\beta_2-\alpha_2\beta_1=-p_{0-1}\Delta(z)$ and at last $\gamma_1\beta_2-\gamma_2\beta_1=p_{01}\Delta(z)$. In particular, if $\Delta(z)=0$, then the polynomials $q_{1,z}$ and $q_{2,z}$ are proportional. In addition to that, Cramer's formulas give $z^{-1}q_z(u,v)=p_{10}q_{1,z}(u,v)+p_{-10}q_{2,z}(u,v)+2u\Delta(z)$; so that if $\Delta(z)=0$, q_z, $q_{1,z}$ and $q_{2,z}$ are multiple of a same polynomial. *A priori*, it may quite happen that one or even several of q_z, $q_{1,z}$, $q_{2,z}$ are zero ; in fact, we could show that at most one of these three polynomials can be equal to zero, otherwise the walk would be degenerated (we recall from the very beginning of Section 6 that we say that a walk is degenerated if there are three consecutive zeros in the list $p_{-1-1},p_{-10},p_{-11},p_{01},p_{11},p_{10},p_{1-1},p_{0-1},p_{-1-1},p_{-10}$).

So in any non degenerated case, we can write that $\mathcal{L}_z=\{u+iv\in\mathbb{C}:r_z(u,v)=0\}$, where r_z stands for one of the non zero polynomials in the list q_z, $q_{1,z}$, $q_{2,z}$. But the curve $\{u+iv\in\mathbb{C}:r_z(u,v)=0\}$ is clearly a circle, eventually degenerated in a straight line, for which we could easily write the center and the radius, the proof of Lemma 34 is completed.

To be exhaustive, we give here the single possibilities for $\mathcal{L}_z$ and $\mathcal{M}_z$ to be straight lines : (i) $\mathcal{L}_z$ is a straight line if and only if $p_{01}+p_{1-1}+p_{0-1}+p_{-1-1}=1$, in that case $\mathcal{L}_z=\{u+iv:2p_{01}zu=1\}$ and $\mathcal{M}_z=\mathcal{C}(0,(p_{-1-1}/p_{1-1})^{1/2})$, (ii) $\mathcal{M}_z$ is a straight line if and only if $p_{10}+p_{-11}+p_{-10}+p_{-1-1}=1$, in that case $\mathcal{M}_z=\{u+iv:2p_{10}zu=1\}$ and $\mathcal{L}_z=\mathcal{C}(0,(p_{-1-1}/p_{-11})^{1/2})$. The proof of these facts consists simply in a play with the parameters, so we omit it. $\square$

Proof of Lemma 35. Start by showing that $\Delta(1)=0$ is equivalent to $p_{11}+p_{-1-1}=p_{1-1}+p_{-11}$, and suppose first that $p_{11}+p_{-1-1}=p_{1-1}+p_{-11}$. We have the system of equations (1) $M_x=0$ (2) $M_y=0$ (3) $p_{11}+p_{-1-1}=p_{1-1}+p_{-11}$ (4) $\sum p_{ij}=1$ (5) $p_{ij}\geq 0$. This system has no single solutions but implies some relationships between the parameters, which in turn imply, by a direct calculation, that $\Delta(1)=1$. Likewise, we prove that $\Delta(1)$ implies that $p_{11}+p_{-1-1}=p_{1-1}+p_{-11}$.

Suppose now that $\Delta(1)=\Delta'(1)=0$ and consider the system composed from the equations : (1) $M_x=0$ (2) $M_y=0$ (3) $p_{11}p_{-1-1}-p_{1-1}p_{-11}=0$ (4) $p_{11}+p_{-1-1}=p_{1-1}+p_{-11}$ (5) $\sum p_{ij}=1$ (6) $p_{ij}\geq 0$. It turns out that this system implies either $p_{11}=p_{1-1}$, $p_{01}=p_{0-1}$ and $p_{-11}=p_{-1-1}$ (in other words $a=c$) or $p_{11}=p_{-11}$, $p_{10}=p_{-10}$ and $p_{1-1}=p_{-1-1}$ (that means $\tilde{a}=\tilde{c}$). $\square$

Proof of Proposition 36. First, we lift $x\mapsto h(x,z)$ and $y\mapsto\tilde{h}(y,z)$, initially defined on the unit disc, on the algebraic curve $\{(x,y)\in\mathbb{C}^2:Q(x,y,z)=0\}$: we obtain so two functions, say g and $\tilde{g}$, defined on $\{(x,y)\in\mathbb{C}^2:Q(x,y,z)=0,|x|<1\}$ and $\{(x,y)\in\mathbb{C}^2:Q(x,y,z)=0,|y|<1\}$ respectively. Then, we will continue g and $\tilde{g}$ on the whole $Q(x,y,z)=0$, into functions again denoted by g and $\tilde{g}$, and verifying $g(\xi(x,y),z)=g(x,y,z)$ and $\tilde{g}(\eta(x,y),z)=\tilde{g}(x,y,z)$; we will explain how to get this continuation in a few lines. Suppose before that we did successfully this continuation, and see how to conclude : we will continue h and $\tilde{h}$ by setting $h(x,z)=g(x,y,z)$ and $\tilde{h}(x,z)=\tilde{g}(x,y,z)$. The two relationships $g(\xi(x,y),z)=g(x,y)$ and $\tilde{g}(\eta(x,y),z)=\tilde{g}(x,y,z)$ allow to the continuation to be not ambiguous.

It remains to see how continue g and $\tilde{g}$ from $\{(x,y)\in\mathbb{C}^2 : Q(x,y,z)=0, |x|<1\}$ and $\{(x,y)\in\mathbb{C}^2 : Q(x,y,z)=0, |y|<1\}$ to the whole $\{(x,y)\in\mathbb{C}^2 : Q(x,y,z)=0\}$. Define $\xi(x,y)=(x, c(x,z)/(a(x,z)y))$, $\eta(x,y)=(\tilde{c}(y,z)/(\tilde{a}(y,z)x), y)$; they are the Galois automorphisms attached to the algebraic curve $Q(x,y,z)$, see [Mal72] and [FIM99]. They are such that if $Q(x,y,z)=0$ then $Q(\xi(x,y),z)=0$ and $Q(\eta(x,y),z)=0$.

The key step is the following : a worthwhile fact of having supposed $\Delta(z)=0$ is that the group $\mathcal{H}$, generated by ξ and η is of order four ; we can prove this fact by a direct calculation.

Consider next the following sub-domains of $\{(x,y)\in\mathbb{C}^2 : Q(x,y,z)=0\}$: $\mathcal{D}_{z,-,-}=\{(x,y)\in\mathbb{C} : Q(x,y,z)=0, |x|\leq 1, |y|\leq 1\}$, $\mathcal{D}_{z,-,+}=\{(x,y)\in\mathbb{C} : Q(x,y,z)=0, |x|\leq 1, |y|\geq 1\}$, $\mathcal{D}_{z,+,-}=\{(x,y)\in\mathbb{C} : Q(x,y,z)=0, |x|\geq 1, |y|\leq 1\}$ and $\mathcal{D}_{z,+,+}=\{(x,y)\in\mathbb{C} : Q(x,y,z)=0, |x|\geq 1, |y|\geq 1\}$, whose the union equals $\{(x,y)\in\mathbb{C}^2 : Q(x,y,z)=0\}$. *A priori*, g and $\tilde{g}$ are well defined only in $\mathcal{D}_{z,-,-}\cup\mathcal{D}_{z,-,+}$ and $\mathcal{D}_{z,-,-}\cup\mathcal{D}_{z,+,-}$ respectively. Then, we continue g in $\mathcal{D}_{z,+,-}$ using the functional equation (8) : we set $g(x,y)=x^{n_0}y^{m_0}-\tilde{g}(x,y)-h_{00}(z)$. Likewise, in $\mathcal{D}_{z,-,+}$, we continue $\tilde{g}$ by setting $\tilde{g}(x,y)=x^{n_0}y^{m_0}-g(x,y)-h_{00}(z)$. In $\mathcal{D}_{z,+,+}$, we set $g(x,y)=g\circ\xi(x,y)$ and $\tilde{g}(x,y)=\tilde{g}\circ\eta(x,y)$.

Thanks to the properties of the automorphisms ξ and η described above, we get the sub-domains $\mathcal{D}_{z,\pm,\pm}$ as the successive ranges of $\mathcal{D}_{z,-,-}$ by the automorphisms ξ and η : for instance $\xi(\mathcal{D}_{z,+,+})=\mathcal{D}_{z,+,-}$ and $\eta(\mathcal{D}_{z,+,+})=\mathcal{D}_{z,-,+}$, so that the continuation is done on the whole $\{(x,y)\in\mathbb{C}^2 : Q(x,y,z)=0\}$ and is not ambiguous. $\square$

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