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# Stochastic Homogenization of Reflected Diffusion Processes

Rémi Rhodes\*

February 16, 2009

## Abstract

We investigate stochastic homogenization for Reflected Stochastic Differential Equations on a half-plane when it is necessary to analyze both the homogenized equation and the boundary condition. We prove that the limiting process is a reflected non-standard Brownian motion.

## 1 Introduction

**Statement of the problem.** This paper is concerned with homogenization of Reflected Stochastic Differential Equations (RSDE for short) evolving in a random medium, that is (see e.g. [12])

**Definition 1.1. (Random medium)** Let  $(\Omega, \mathcal{G}, \mu)$  be a probability space and  $\{\tau_x; x \in \mathbb{R}^d\}$  a group of measure preserving transformations acting ergodically on  $\Omega$ :

- 1)  $\forall A \in \mathcal{G}, \forall x \in \mathbb{R}^d, \mu(\tau_x A) = \mu(A)$ ,
- 2) If for any  $x \in \mathbb{R}^d, \tau_x A = A$  then  $\mu(A) = 0$  or 1,
- 3) For any measurable function  $\mathbf{g}$  on  $(\Omega, \mathcal{G}, \mu)$ , the function  $(x, \omega) \mapsto \mathbf{g}(\tau_x \omega)$  is measurable on  $(\mathbb{R}^d \times \Omega, \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{G})$ .

More precisely, we suppose that we are given some coefficients  $\sigma : \Omega \rightarrow \mathbb{R}^{d \times d}$  and  $\mathbf{b}, \gamma : \Omega \rightarrow \mathbb{R}^d$  and a  $d$ -dimensional Brownian motion  $B$  defined on a complete probability space  $(\Omega', \mathcal{F}, \mathbb{P})$  (the Brownian motion and the random medium are independant). We want to describe the limit in law, as  $\varepsilon$  goes to 0, of RSDEs of the type

$$(1) \quad dX_t^\varepsilon = \varepsilon^{-1} \mathbf{b}(\tau_{X_t^\varepsilon/\varepsilon} \omega) dt + \sigma(\tau_{X_t^\varepsilon/\varepsilon} \omega) dB_t + \gamma(\tau_{X_t^\varepsilon/\varepsilon} \omega) dK_t^\varepsilon,$$

where  $X^\varepsilon, K^\varepsilon$  are  $(\mathcal{F}_t)_t$ -adapted processes ( $\mathcal{F}_t$  is the  $\sigma$ -field generated by  $B$  up to time  $t$ ) with constraint  $X_t^\varepsilon \in \bar{D}$ , where  $D \subset \mathbb{R}^d$  is the half-plane  $\{(x_1, \dots, x_d) \in \mathbb{R}^d; x_1 > 0\}$ ,  $K^\varepsilon$  is the so-called local time of the process  $X^\varepsilon$ , namely a continuous nondecreasing process, which only increases on the set  $\{t; X_t^\varepsilon \in \partial D\}$ . The reader is referred to [15] (see e.g [22] for the weak existence) for strong existence and uniqueness results of (1), in particular under the assumptions on the coefficients  $\sigma, \mathbf{b}$  and  $\gamma$  listed below. Those stochastic processes are involved in the probabilistic representation of second order partial differential equations in half-space with oblique derivative boundary conditions (see [19] for an insight of the topic). In particular, we are interested in homogenization problems for which it is necessary to identify both the homogenized equation and the homogenized boundary conditions.

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\* Address: Université Paris-Dauphine, Ceremade, Place du maréchal De Lattre de Tassigny, 75775 Paris Cedex 16, France. Phone: (33)(0)1 44 05 48 51. E-mail: rhodes@ceremade.dauphine.fr

Without the reflection term  $\gamma(X_t^\varepsilon/\varepsilon) dK_t^\varepsilon$ , the issue of determining the limit in (1) is the subject of an extensive literature in the case when the coefficients  $\mathbf{b}, \boldsymbol{\sigma}$  are periodic, quasi-periodic and, more recently, evolving in a stationary ergodic random medium. Quoting all references is beyond the scope of this paper. Concerning the homogenization of RSDEs, there are only a few works dealing with periodic coefficients (see [1, 2, 3, 21]). As pointed out in [2], homogenizing (1) in a random medium is a well-known problem which remains unsolved yet. There are several difficulties in this framework that make the classical machinery of diffusions in random media (i.e. without reflection) fall short of determining the limit in (1). In particular, the reflection term breaks the stationarity properties of the process  $X^\varepsilon$  so that the method of the *environment as seen from the particle* (see [17] for an insight of the subject) is inefficient. Moreover, the lack of compactness of a random medium prevents from using compactness methods. The main resulting difficulty is the lack of invariant probability measure for the process  $X^\varepsilon$ . The aim of this paper is precisely to investigate the random case and prove the convergence of the process  $X^\varepsilon$  towards a reflected Brownian motion. The convergence is established in probability with respect to the random medium and the starting point  $x$ .

We should also point out that the problem of determining the limit in (1) could be expressed in terms of reflected random walks in random environment, and remains quite open as well. Though we don't treat explicitly that case, our proofs may be adapted to the discrete framework.

**Structure of the coefficients.** Throughout the paper, we use the convention of summation over repeated indices  $\sum_{i=1}^d a_i b_i = a_i b_i$  and we use the superscript  $*$  to denote the transpose  $A^*$  of some given matrix  $A$ . If a random function  $\varphi : \Omega \rightarrow \mathbb{R}$  possesses smooth trajectories, i.e.  $\forall \omega \in \Omega$  the mapping  $x \in \mathbb{R}^d \mapsto \varphi(\tau_x \omega)$  is smooth with bounded derivatives, we can consider its partial derivatives at 0 denoted by  $D_i \varphi$ , that is  $D_i \varphi(\omega) = \partial_{x_i}(x \mapsto \varphi(\tau_x \omega))|_{x=0}$ .

We define  $\mathbf{a} = \boldsymbol{\sigma} \boldsymbol{\sigma}^*$ . We assume that we are given an antisymmetric matrix-valued function  $\mathbf{H} : \Omega \rightarrow \mathbb{R}^{d \times d}$  such that  $\mathbf{H}_{ij} = 0$  whenever  $i = 1$  or  $j = 1$ . For the sake of simplicity, we assume that  $\forall \omega \in \Omega$  the mappings  $x \in \mathbb{R}^d \mapsto \boldsymbol{\sigma}(\tau_x \omega)$ ,  $x \mapsto \mathbf{H}(\tau_x \omega)$  are bounded and smooth with bounded derivatives of all orders. We further impose these bounds do not depend on  $\omega$ . Then, it makes sense to consider the following coefficients defined on  $\Omega$ :

$$(2) \quad \forall j = 1, \dots, d, \quad \mathbf{b}_j = \frac{1}{2} D_i \mathbf{a}_{ij}, \quad \gamma_j = \mathbf{a}_{j1} + D_i \mathbf{H}_{ji}.$$

The entries of the  $d$ -dimensional vector  $\mathbf{b}$  (resp.  $\boldsymbol{\gamma}$ ) are  $(\mathbf{b}_j)_{1 \leq j \leq d}$  (resp.  $(\gamma_j)_{1 \leq j \leq d}$ ). Under these assumptions, the generator of the Markov process  $X^\varepsilon$  coincides (for a sufficiently smooth function  $f$  on  $\bar{D}$ ) with

$$(3) \quad \mathcal{L}^\varepsilon f = \frac{1}{2} \partial_{x_i} (\mathbf{a}_{ij}(\tau_{x/\varepsilon} \omega) \partial_{x_j} f)$$

with boundary condition  $\gamma_i(\tau_{x/\varepsilon} \omega) \partial_{x_i} f = 0$  on  $\partial D$ . Note that, in the case  $\mathbf{H}$  is chosen equal to 0, the reflection term  $\gamma$  coincides with the so-called conormal field and the associated PDE problem is said to be of Neumann type. From the physical point of view, the conormal field is the "canonical" assumption that makes valid the mass conservation law since the relation  $\mathbf{a}_{j1}(\tau_{x/\varepsilon} \omega) \partial_{x_j} f = 0$  on  $\partial D$  means that the flux through the boundary must vanish. It is then natural to wonder if we can consider a larger class of reflection coefficients  $\gamma$  while preserving the mass conservation law. The answer is positive if  $\gamma$  has the structure described in (2) and this remark motivates our definition of  $\gamma$ .

We assume that  $\mathbf{a}$  is uniformly elliptic, i.e. there exists a constant  $\Lambda > 0$  such that

$$(4) \quad \forall \omega \in \Omega, \quad \Lambda \mathbf{I} \leq \mathbf{a}(\omega) \leq \Lambda^{-1} \mathbf{I},$$

and, without loss of generality, we assume that  $a_{11} = 1$ .

**Main Result.** In what follows, we indicate by  $\mathbb{P}_x^\varepsilon$  the law of the process  $X^\varepsilon$  starting from  $x \in \bar{D}$  (keep in mind that this probability measure also depends on  $\omega$  though it does not appear through the notations). Let us consider a nonnegative function  $\chi : \bar{D} \rightarrow \mathbb{R}_+$  such that  $\int_{\bar{D}} \chi(x) dx = 1$ . Such a function defines a probability measure denoted by  $\chi(dx) = \chi(x)dx$  on  $\bar{D}$ . We are now in position to state the main result of the paper:

**Theorem 1.2.** *The couple of processes  $(X^\varepsilon, K^\varepsilon)_\varepsilon$  converges weakly, in  $\mu \otimes \chi$  probability, towards the solution  $(\bar{X}, \bar{K})$  of the RSDE*

$$\bar{X}_t = x + \bar{A}^{1/2} B_t + \bar{\Gamma} \bar{K}_t,$$

with constraint  $\bar{X}_t \in \bar{D}$ ,  $\bar{K}$  is the local time associated to  $\bar{X}$ . In other words, for each bounded continuous function  $F$  on  $C(\bar{D} \times \mathbb{R}_+; \mathbb{R})$  and  $\delta > 0$ , we have

$$\lim_{\varepsilon \rightarrow 0} \mu \otimes \chi \{(\omega, x) \in \Omega \times \bar{D}; |\mathbb{E}_x^\varepsilon(F(X^\varepsilon, K^\varepsilon)) - \mathbb{E}_x(F(\bar{X}, \bar{K}))| \geq \delta\} = 0.$$

The so-called homogenized (or effective) coefficients  $\bar{A}$  and  $\bar{\Gamma}$  are constant. Moreover  $\bar{A}$  is invertible, obeys a variational formula (see section 4 for the meaning of the various terms)

$$\bar{A} = \inf_{\varphi \in \mathcal{C}} \mathbb{M}[(\mathbf{I} + D\varphi)^* \mathbf{a}(\mathbf{I} + D\varphi)],$$

and  $\bar{\Gamma}$  is the conormal field associated to  $\bar{A}$ , that is  $\bar{\Gamma}_i = \bar{A}_{1i}$  for  $i = 1, \dots, d$ .

**Remark 1.3.** *As explained in [2], the periodic framework remains not completely understood for general bounded domains. Indeed, this latter situation gives rise to the issue of determining the limit in (1) when the ratio between the period of the coefficients  $\mathbf{b}, \boldsymbol{\sigma}$  and the period of  $\gamma$  is irrational. That situation breaks the periodicity properties of the model but can be understood with the theory of random media. Let us explain why. Consider  $\sigma : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$ , 1-periodic in each direction of  $\mathbb{R}^d$ , and  $\gamma : \{0\} \times \mathbb{R}^{d-1}$ ,  $\alpha$ -periodic in each direction with  $\alpha \notin \mathbb{Q}$ . That situation boils down to the random case in the following way: the random medium is the  $2d - 1$  dimensional torus  $\mathbb{T} = (\mathbb{R}/\mathbb{Z}) \times (\mathbb{R}/\mathbb{Z} \times \mathbb{R}/\alpha\mathbb{Z})^{d-1}$  equipped with the induced Lebesgue measure  $dx$ . For any  $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ , the translation  $\tau_x : \mathbb{T} \rightarrow \mathbb{T}$  is defined by*

$$\tau_x(y_1, (y_2, z_2), \dots, (y_d, z_d)) = (y_1 + x_1, (y_2 + x_2, z_2 + x_2), \dots, (y_d + x_d, z_d + x_d))$$

for any  $(y_1, (y_2, z_2), \dots, (y_d, z_d)) \in \mathbb{T}$ . 1) and 3) of Definition 1.1 are obvious. 2) results from the equipartition Weyl theorem ( $\alpha \notin \mathbb{Q}$ ). Of course, the coefficients on the random medium are defined by  $\boldsymbol{\sigma} : \mathbb{T} \ni (y_1, (y_2, z_2), \dots, (y_d, z_d)) \rightarrow \boldsymbol{\sigma}(y_1, \dots, y_d)$  and  $\gamma : \mathbb{T} \ni (y_1, (y_2, z_2), \dots, (y_d, z_d)) \rightarrow \gamma(z_2, \dots, z_d)$ . So Theorem 1.2 applies.

The organization of this note is the following. In Section 2, we gather the main notations and detail the preliminary background necessary for the reading of the paper. Section 3 explains how to reduce the problem by considering a function  $\chi$  with a particular structure that allows to use the Girsanov transform to modify equation (1). The main objective of this step is to deal with a process admitting an invariant distribution. Section 4 describes the ergodic properties of the problem and the construction of the so-called correctors. The resulting ergodic theorems for the process  $X^\varepsilon$  are stated in Section 5. The tightness of  $X^\varepsilon$  is proved in Section 6. Finally, we carry through the homogenization procedure in Section 7. The appendix gathers the proofs of several auxiliary results.

## 2 Notations and preliminary background

Before proceeding with the proofs of our main result, let us first gather a few notations to which the reader may refer throughout the reading of the paper.

**Classical spaces.** Given an open domain  $\mathcal{O} \subset \mathbb{R}^n$ , For  $k \in \mathbb{N} \cup \{\infty\}$ ,  $C^k(\mathcal{O})$  (resp.  $C^k(\bar{\mathcal{O}})$ ), resp.  $C_b^k(\bar{\mathcal{O}})$  denotes the space of functions admitting derivatives up to order  $k$  over  $\mathcal{O}$  (resp. with continuous derivatives over  $\bar{\mathcal{O}}$ , resp. with continuous bounded derivatives over  $\bar{D}$ ). The spaces  $C_c^k(\mathcal{O})$  and  $C_c^k(\bar{\mathcal{O}})$  denote the subspaces of  $C^k(\bar{\mathcal{O}})$  whose functions respectively have a compact support in  $\mathcal{O}$  or have a compact support in  $\bar{\mathcal{O}}$ .

Let  $C_b^{1,2}$  denote the space of bounded functions  $f : [0, T] \times \bar{D}$  admitting bounded and continuous derivatives  $\partial_t f$ ,  $\partial_x f$ ,  $\partial_{tx}^2 f$  and  $\partial_{xx}^2 f$  on  $[0, T] \times \bar{D}$ .

**Random medium.** The expectation with respect to the random medium is denoted by  $\mathbb{M}$ . In what follows we will use the bold type to denote a random function  $\mathbf{g}$  from  $\Omega \times \mathbb{R}^p$  into  $\mathbb{R}^n$  ( $n \geq 1$  and  $p \geq 0$ ). The space of square integrable (resp. integrable, resp. essentially bounded) functions on  $(\Omega, \mathcal{G}, \mu)$  is denoted by  $L^2(\Omega)$  (resp.  $L^1(\Omega)$ , resp.  $L^\infty(\Omega)$ ), the usual norm by  $\|\cdot\|_2$  (resp.  $\|\cdot\|_1$ , resp.  $\|\cdot\|_\infty$ ) and the corresponding inner product by  $(\cdot, \cdot)_2$ . The operators on  $L^2(\Omega)$  defined by  $T_x \mathbf{g}(\omega) = \mathbf{g}(\tau_x \omega)$  form a strongly continuous group of unitary maps in  $L^2(\Omega)$ . Each function  $\mathbf{g}$  in  $L^2(\Omega)$  defines in this way a stationary ergodic random field on  $\mathbb{R}^d$ . The group possesses  $d$  generators defined, for  $i = 1, \dots, d$ , by  $D_i \mathbf{g} = \lim_{h \in \mathbb{R} \rightarrow 0} h^{-1} (T_{he_i} \mathbf{g} - \mathbf{g})$  whenever the limit exists in the  $L^2(\Omega)$ -sense ( $(e_1, \dots, e_d)$  stands for the canonical basis of  $\mathbb{R}^d$ ), which are closed and densely defined. Given  $\varphi \in \bigcap_{i=1}^d \text{Dom}(D_i)$ ,  $D\varphi$  stands for the  $d$ -dimensional vector whose entries are  $D_i \varphi$  for  $i = 1, \dots, d$ . We distinguish  $D_i$  from the usual differential operator  $\partial_{x_i}$  acting on sufficiently smooth functions  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  (more generally, for  $k \geq 2$ ,  $\partial_{x_{i_1} \dots x_{i_k}}^k$  denotes the iterated operator  $\partial_{x_{i_1}} \dots \partial_{x_{i_k}}$ ). However, we point out that, whenever a function  $\varphi \in \text{Dom}(D)$  possesses differentiable trajectories (i.e.  $\mu$  a.s. the mapping  $x \mapsto \varphi(\tau_x \omega)$  is differentiable in the classical sense), we have  $D_i \varphi(\tau_x \omega) = \partial_{x_i} \varphi(\tau_x \omega)$ .

Denote by  $\mathcal{C}$  the dense subspace of  $L^2(\Omega)$  defined by

$$\mathcal{C} = \text{Span} \left\{ \mathbf{g} \star \varphi; \mathbf{g} \in L^\infty(\Omega), \varphi \in C_c^\infty(\mathbb{R}^d) \right\}$$

with  $\mathbf{g} \star \varphi(\omega) = \int_{\mathbb{R}^d} \mathbf{g}(\tau_x \omega) \varphi(x) dx$ . We point out that  $\mathcal{C} \subset \text{Dom}(D_i)$ , for all  $1 \leq i \leq d$ , and  $D_i(\mathbf{g} \star \varphi) = -\mathbf{g} \star \partial_{x_i} \varphi$ . This last quantity is also equal to  $D_i \mathbf{g} \star \varphi$  if  $\mathbf{g} \in \text{Dom}(D_i)$ .

We introduce the  $\sigma$ -field  $\mathcal{G}^* \subset \mathcal{G}$  generated by the subsets of  $\Omega$  that are invariant under the operators  $\{T_x; x \in \{0\} \times \mathbb{R}^{d-1}\}$ , and  $\mathbb{M}_1$  the conditional expectation with respect to  $\mathcal{G}^*$ , which coincides with the orthogonal projection from  $L^2(\Omega, \mathcal{G}, \mu)$  into the closed subspace  $L^2(\Omega, \mathcal{G}^*, \mu)$ .

We define the bounded antisymmetric matrix-valued function  $\bar{H} : \Omega \rightarrow \mathbb{R}^{d \times d}$  by  $\bar{H}_{i1} = D_j \mathbf{H}_{ij}$  for  $i = 1, \dots, d$  and  $\bar{H}_{ij} = 0$  if  $i \geq 2$  and  $j \geq 2$ .

**Green's formula.** We remind the reader of the Green formula (see [16, eq. (6.5)]). Given a smooth matrix-valued function  $c : \bar{D} \rightarrow \mathbb{R}^{d \times d}$ , and  $(\varphi, \psi) \in C^2(\bar{D}) \times C_c^1(\bar{D})$ :

$$(5) \quad \int_D \partial_{x_i} (c_{ij} \partial_{x_j} \varphi) \psi(x) dx = - \int_{\partial D} c_{1j} \partial_{x_j} \varphi \psi(x) dx - \int_D c_{ij} \partial_{x_j} \varphi \partial_{x_i} \psi(x) dx.$$

As a consequence, by considering the following operator acting on  $C^2(\bar{D})$

$$(6) \quad \mathcal{L}_V^\varepsilon = \frac{e^{2V(x)}}{2} \sum_{i,j=1}^d \partial_{x_i} (e^{-2V(x)} \mathbf{a}_{ij}(\tau_{x/\varepsilon} \omega) \partial_{x_j}),$$

where  $V : \bar{D} \rightarrow \mathbb{R}$  is smooth, we have

$$(7) \quad \int_D \mathcal{L}_V^\varepsilon \varphi(x) \psi(x) e^{-2V(x)} dx + \frac{1}{2} \int_D (\mathbf{a} + \bar{\mathbf{H}})_{ij}(\tau_{x/\varepsilon} \omega) \partial_{x_i} \psi(x) \partial_{x_j} \varphi(x) e^{-2V(x)} dx \\ = - \int_{\partial D} \gamma_i(\tau_{x/\varepsilon} \omega) \partial_{x_i} \psi(x) \varphi(x) e^{-2V(x)} dx.$$

Note that the Lebesgue measure on  $\bar{D}$  or  $\partial D$  is indistinctly denoted by  $dx$  since the domain of integration avoids confusion.

### 3 Reduction of the problem

**Reduction of the problem.** Suppose for a while that we can prove Theorem 1.2 for any function  $\chi$  that can be rewritten as  $\chi(x) = e^{-2V(x)}$ , where  $V : \bar{D} \rightarrow \mathbb{R}$  is a smooth function of the form

$$(8) \quad V(x_1, \dots, x_d) = Ax_1 + A(1 + x_2^2 + \dots + x_d^2)^{1/2} + c,$$

for some positive constant  $A$  and some renormalization constant  $c$  such that  $\int_{\bar{D}} e^{-2V(x)} dx = 1$ .

It is then plain to see that Theorem 1.2 holds for any nonnegative function  $\chi$  not greater than  $Ce^{-2V(x)}$ , for some positive constant  $C$  and some function  $V$  obeying the above conditions. Indeed, each measurable subset  $A \subset \Omega \times \bar{D}$  satisfies  $\mathbb{M} \int_{\bar{D}} \mathbb{1}_A \chi(x) dx \leq C \mathbb{M} \int_{\bar{D}} \mathbb{1}_A e^{-2V(x)} dx$ . In particular, Theorem 1.2 holds for any continuous function  $\chi$  with compact support over  $\bar{D}$ .

Consider now a general function  $\chi : \bar{D} \rightarrow \mathbb{R}_+$  satisfying  $\int_{\bar{D}} \chi(x) dx = 1$  and  $\chi' : \bar{D} \rightarrow \mathbb{R}_+$  with compact support in  $\bar{D}$ . Let  $A^\varepsilon \subset \Omega \times \bar{D}$  be defined as

$$A^\varepsilon = \{(\omega, x) \in \Omega \times \bar{D}; |\mathbb{E}_x^\varepsilon(F(X^\varepsilon, K^\varepsilon)) - \mathbb{E}_x(F(\bar{X}, \bar{K}))| \geq \delta\}.$$

From the relation  $\mathbb{M} \int_{\bar{D}} \mathbb{1}_{A^\varepsilon} \chi(x) dx \leq \mathbb{M} \int_{\bar{D}} |\chi(x) - \chi'(x)| dx + \mathbb{M} \int_{\bar{D}} \mathbb{1}_{A^\varepsilon} \chi'(x) dx$ , we deduce

$$\limsup_{\varepsilon \rightarrow 0} \mathbb{M} \int_{\bar{D}} \mathbb{1}_{A^\varepsilon} \chi(x) dx \leq \mathbb{M} \int_{\bar{D}} |\chi(x) - \chi'(x)| dx,$$

in such a way that the Theorem 1.2 holds for  $\chi$  by density arguments.

To sum up, it suffices to prove Theorem 1.2 in the case of a function  $\chi(x) = e^{-2V(x)}$  for some function  $V$  of the form (8). We gathered below the main properties of such a function  $V$ :

$$(9) \quad \int_{\bar{D}} e^{-2V(x)} dx = 1, \quad \text{and} \quad \int_{\bar{D}} x_1 e^{-2V(x)} dx = M < +\infty,$$

$$(10) \quad V \text{ is smooth and } \partial_x V \text{ is bounded over } \bar{D}, \text{ as well as the derivatives of } \partial_x V.$$

$$(11) \quad \sup_{x_1 \geq 0} \int_{\mathbb{R}^{d-1}} e^{-2V(x_1, y)} dy = M' < +\infty.$$

**Girsanov's transform.** From now on, keep in mind that  $V$  is of the form (8) and thus satisfies (9), (10) and (11). Let us fix  $T > 0$  and define the following probability measure on the filtered space  $(\Omega'; \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T})$

$$d\mathbb{P}_x^{\varepsilon*} = \exp \left( - \int_0^T \partial_{x_i} V(X_r^\varepsilon) \sigma_{ij}(\tau_{X_r^\varepsilon/\varepsilon} \omega) dB_r^j - \frac{1}{2} \int_0^T \partial_{x_i} V(X_r^\varepsilon) \mathbf{a}_{ij}(\tau_{X_r^\varepsilon/\varepsilon} \omega) \partial_{x_j} V(X_r^\varepsilon) dr \right) d\mathbb{P}_x^\varepsilon.$$

Under  $\mathbb{P}_x^{\varepsilon*}$ , the process  $B_t^* = B_t + \int_0^t \boldsymbol{\sigma}(\tau_{X_r^\varepsilon/\varepsilon}\omega) \partial_x V(X_r^\varepsilon) dr$  ( $0 \leq t \leq T$ ) is a Brownian motion and the process  $X^\varepsilon$  solves the RSDE

$$(12) \quad dX_t^\varepsilon = \varepsilon^{-1} \mathbf{b}(\tau_{X_t^\varepsilon/\varepsilon}\omega) dt - \mathbf{a}(\tau_{X_t^\varepsilon/\varepsilon}\omega) \partial_x V(X_t^\varepsilon) dt + \boldsymbol{\sigma}(\tau_{X_t^\varepsilon/\varepsilon}\omega) dB_t^* + \gamma(\tau_{X_t^\varepsilon/\varepsilon}\omega) dK_t^\varepsilon$$

starting from  $X_0^\varepsilon = x$ , where  $K^\varepsilon$  is the local time of  $X^\varepsilon$ .

The guiding line of the remaining part of the paper is to establish estimates under  $\mathbb{P}_x^{\varepsilon*}$  and switch the estimates to  $\mathbb{P}_x^\varepsilon$  thanks to the inequality

$$(13) \quad \forall A \text{ } \mathcal{F}_T\text{-measurable subset, } \mathbb{P}_x^\varepsilon(A) \leq C(\mathbb{P}_x^{\varepsilon*}(A))^{1/2}$$

where  $C$  is a constant only depending on  $T$ ,  $|\mathbf{a}|_\infty$  and  $\sup_{\bar{D}} |\partial_x V|$  (not on  $\varepsilon$ ). The proof of such a relation follows from the Cauchy-Schwarz inequality and the boundedness of  $\mathbf{a}$  and  $\partial_x V$ .

**Notations for measures.** Finally,  $\mathbb{P}^\varepsilon$  (resp.  $\mathbb{P}^{\varepsilon*}$ ) stands for the probability measure  $\mathbb{M} \int_{\bar{D}} \mathbb{P}_x^\varepsilon(\cdot) e^{-2V(x)} dx$  (resp.  $\mathbb{M} \int_{\bar{D}} \mathbb{P}_x^{\varepsilon*}(\cdot) e^{-2V(x)} dx$ ), and  $\bar{\mathbb{E}}^\varepsilon$  (resp.  $\bar{\mathbb{E}}^{\varepsilon*}$ ) for the corresponding expectation.  $\mathbb{P}_D^*$  (resp.  $\mathbb{P}_{\partial D}^*$ ) denotes the probability measure  $e^{-2V(x)} dx \otimes d\mu$  on  $\bar{D} \times \Omega$  (resp. the nonnegative finite measure  $e^{-2V(x)} dx \otimes d\mu$  on  $\partial D \times \Omega$ ). We further define  $\mathbb{M}_D^*$  and  $\mathbb{M}_{\partial D}^*$  as the respective expectations.

## 4 Ergodic problems and correctors

Our first objective is to determine the coefficients of the homogenized equation both on the boundary and inside the domain  $D$ .

**Generator on the random medium associated to the diffusion process inside  $D$ .** The construction of the correctors inside the domain  $D$  is the same as in the stationary case (i.e. without reflection term) and is therefore well known. So we just sum up the main tools and give references for further details.

We associate to the operator  $\mathcal{L}^\varepsilon$  (Eq. (3)) an unbounded operator acting on  $\mathcal{C} \subset L^2(\Omega)$

$$(14) \quad \mathbf{L} = \frac{1}{2} D_i (\mathbf{a}_{ij} D_j \cdot).$$

Following [7, Ch. 3, Sect 3.] (see also [20, Sect. 4]), we can consider its Friedrich extension, still denoted by  $\mathbf{L}$ , which is a self-adjoint operator on  $L^2(\Omega)$ . The domain  $\mathbb{H}$  of the corresponding Dirichlet form can be described as the closure of  $\mathcal{C}$  with respect to the norm  $\|\varphi\|_{\mathbb{H}}^2 = |\varphi|_2^2 + |D\varphi|_2^2$ . Since  $\mathbf{L}$  is self-adjoint, it also defines a resolvent family  $(U_\lambda)_{\lambda>0}$ . For each  $\mathbf{f} \in L^2(\Omega)$ , the function  $\mathbf{w}_\lambda = U_\lambda(\mathbf{f}) \in \mathbb{H} \cap \text{Dom}(\mathbf{L})$  equivalently solves the  $L^2(\Omega)$ -sense equation

$$(15) \quad \lambda \mathbf{w}_\lambda - \mathbf{L} \mathbf{w}_\lambda = \mathbf{f}$$

or the weak formulation equation

$$(16) \quad \forall \varphi \in \mathbb{H}, \quad \lambda (\mathbf{w}_\lambda, \varphi)_2 + (1/2) (\mathbf{a}_{ij} D_i \mathbf{w}_\lambda, D_j \varphi)_2 = (\mathbf{f}, \varphi)_2.$$

Moreover, the resolvent operator  $U_\lambda$  satisfies the maximum principle (see Appendix A):

**Lemma 4.1.** *For any function  $\mathbf{f} \in L^\infty(\Omega)$ , the function  $U_\lambda(\mathbf{f})$  belongs to  $L^\infty(\Omega)$  and satisfies*

$$|U_\lambda(\mathbf{f})|_\infty \leq |\mathbf{f}|_\infty / \lambda.$$

One can prove

**Proposition 4.2.** Given  $\mathbf{f} \in L^2(\Omega)$ , the solution  $\mathbf{w}_\lambda$  of the resolvent equation  $\lambda\mathbf{w}_\lambda - \mathbf{L}\mathbf{w}_\lambda = \mathbf{f}$  ( $\lambda > 0$ ) satisfies

$$|\lambda\mathbf{w}_\lambda - \mathbb{M}[\mathbf{f}]|_2 \rightarrow 0 \text{ as } \lambda \rightarrow 0, \quad \text{and } \forall \lambda > 0, \quad |\lambda^{1/2}D\mathbf{w}_\lambda|_2 \leq \Lambda^{-1/2}|\mathbf{f}|_2.$$

**Proof.** The first statement is a particular case, for instance, of [20, Lemma 6.2]. To follow the proof in [20], forget the dependency on the parameter  $y$ , take  $\mathbf{H} = 0$  and  $\Psi = \mathbf{f}$ . To prove the second statement, choose  $\varphi = \mathbf{w}_\lambda$  in (16) and plug the relation

$$(\mathbf{f}, \mathbf{w}_\lambda)_2 \leq |\mathbf{f}|_2 |\mathbf{w}_\lambda|_2 \leq 1/(2\lambda)|\mathbf{f}|_2^2 + (\lambda/2)|\mathbf{w}_\lambda|_2^2$$

into the right-hand side to obtain  $\lambda|\mathbf{w}_\lambda|_2^2 + (\mathbf{a}_{ij}D_i\mathbf{w}_\lambda, D_j\mathbf{w}_\lambda)_2 \leq |\mathbf{f}|_2^2/\lambda$ . From (4), we deduce  $\Lambda|D\mathbf{w}_\lambda|_2^2 \leq |\mathbf{f}|_2^2/\lambda$  and the result follows.  $\square$

Concerning the drift term  $\mathbf{b}$ , the convergence is stronger

**Proposition 4.3.** For  $i = 1, \dots, d$ , let  $\mathbf{u}_\lambda^i$  denote the solution of the resolvent equation ( $\lambda > 0$ )

$$(17) \quad \lambda\mathbf{u}_\lambda^i - \mathbf{L}\mathbf{u}_\lambda^i = \mathbf{b}_i.$$

Then there exists  $\zeta^i \in (L^2(\Omega))^d$  such that

$$(18) \quad \lambda|\mathbf{u}_\lambda^i|_2^2 + |D\mathbf{u}_\lambda^i - \zeta^i|_2 \rightarrow 0, \quad \text{as } \lambda \rightarrow 0.$$

Define the random matrix-valued function  $\zeta \in L^2(\Omega; \mathbb{R}^{d \times d})$  by its entries  $\zeta_{ij} = \zeta_i^j = \lim_{\lambda \rightarrow 0} D_i\mathbf{u}_\lambda^j$ , the matrix

$$(19) \quad \bar{A} = \mathbb{M}[(\mathbf{I} + \zeta^*)\mathbf{a}(\mathbf{I} + \zeta)] \text{ which also matches } \mathbb{M}[(\mathbf{I} + \zeta^*)\mathbf{a}]$$

( $\mathbf{I}$  denotes the  $d$ -dimensional identity matrix) and the  $d$ -dimensional vector

$$(20) \quad \bar{\Gamma} = \mathbb{M}[(\mathbf{I} + \zeta^*)\boldsymbol{\gamma}] \in \mathbb{R}^d.$$

Then  $\bar{A}$  obeys the variational formula:

$$(21) \quad \forall X \in \mathbb{R}^d, \quad X^* \bar{A} X = \inf_{\varphi \in \mathcal{C}} \mathbb{M}[(X + D\varphi)^* \mathbf{a}(X + D\varphi)].$$

Moreover, we have  $\bar{A} \geq \Lambda \mathbf{I}$  (in the sense of symmetric nonnegative matrices) and the first component  $\bar{\Gamma}_1$  of  $\bar{\Gamma}$  satisfies  $\bar{\Gamma}_1 \geq \Lambda$ . Finally,  $\bar{\Gamma}$  coincides with the orthogonal projection  $\mathbb{M}_1[(\mathbf{I} + \zeta^*)\boldsymbol{\gamma}]$ .

**Proof.** The first part of statement (18) is quite classical. The reader is referred to [17, Ch. 2] for an insight of the method and to [20, Prop. 4.3] for a proof in a more general context. So we just have to prove (19), (21) and the properties mentioned below. In what follows, for each  $i = 1, \dots, d$ ,  $(\varphi_n^i)_n$  stands for a sequence in  $\mathcal{C}$  such that  $D\varphi_n^i \rightarrow \zeta^i$  in  $L^2(\Omega)^d$  as  $n \rightarrow +\infty$ .

Let us first focus on (21). Fix  $X \in \mathbb{R}^d$  whose entries are denoted by  $(X_i)_{1 \leq i \leq d}$ . We have:  $D(X_i\varphi_n^i) = X_i D\varphi_n^i \rightarrow X_i \zeta_i = \zeta X$  in  $L^2(\Omega)^d$  as  $n \rightarrow +\infty$  and

$$\begin{aligned} X^* \bar{A} X &= \mathbb{M}[(X + \zeta X)^* \mathbf{a}(X + \zeta X)] = \lim_{n \rightarrow +\infty} \mathbb{M}[(X + D(X_i\varphi_n^i))^* \mathbf{a}(X + D(X_i\varphi_n^i))] \\ &\geq \inf_{\varphi \in \mathcal{C}} \mathbb{M}[(X + D\varphi)^* \mathbf{a}(X + D\varphi)]. \end{aligned}$$



Conversely, from Lemma 4.4 below, we have:

$$(22) \quad \forall Y \in \mathbb{R}^d, \quad \mathbb{M}[(Y + \zeta Y)^* \mathbf{a} \zeta X] = \lim_{n \rightarrow +\infty} \mathbb{M}[(Y + \zeta Y)^* \mathbf{a} D(X_i \varphi_n^i)] = 0.$$

We deduce that  $\mathbb{M}[(X + \zeta X)^* \mathbf{a} (D\varphi - \zeta X)] = 0$  for any  $\varphi \in \mathcal{C}$  in such a way that, for  $\varphi \in \mathcal{C}$ :

$$\begin{aligned} \mathbb{M}[(X + D\varphi)^* \mathbf{a} (X + D\varphi)] &= \mathbb{M}[(X + \zeta X + D\varphi - \zeta X)^* \mathbf{a} (X + \zeta X + D\varphi - \zeta X)] \\ &= \mathbb{M}[(X + \zeta X)^* \mathbf{a} (X + \zeta X)] + 2\mathbb{M}[(X + \zeta X)^* \mathbf{a} (D\varphi - \zeta X)] \\ &\quad + \mathbb{M}[(D\varphi - \zeta X)^* \mathbf{a} (D\varphi - \zeta X)] \\ &\geq \mathbb{M}[(X + \zeta X)^* \mathbf{a} (X + \zeta X)] \end{aligned}$$

so that (21) follows. By the way, (22) also proves that  $\bar{A} = \mathbb{M}[(I + \zeta^*) \mathbf{a}]$ .

Now we prove  $\Lambda I \leq \bar{A}$ . Fix  $X \in \mathbb{R}^d$ . From (4) and Cauchy-Schwarz's inequality, we get

$$X^* \bar{A} X = \mathbb{M}[(X + \zeta X)^* \mathbf{a} (X + \zeta X)] \geq \Lambda \mathbb{M}[|X + \zeta X|^2] \geq \Lambda |\mathbb{M}[X + \zeta X]|^2 = \Lambda |X|^2,$$

since  $\mathbb{M}[\zeta X] = 0$ . The estimate  $\Lambda I \leq \bar{A}$  follows.

We now prove that  $\bar{\Gamma} = \mathbb{M}[(I + \zeta^*) \gamma]$  coincides with the orthogonal projection  $\mathbb{M}_1[(I + \zeta^*) \gamma]$ . Note that  $\gamma$  can be rewritten as  $\gamma = \mathbf{a} e_1 + \bar{\mathbf{H}} e_1$  (see Section 2 for the definition of  $\bar{\mathbf{H}}$  and (2)). Clearly, the result follows from the two relations listed below

$$i) \quad \mathbb{M}_1[(I + \zeta^*) \bar{\mathbf{H}} e_1] = 0, \quad ii) \quad \mathbb{M}_1[(I + \zeta^*) \mathbf{a} e_1] = \mathbb{M}[(I + \zeta^*) \mathbf{a} e_1].$$

*Proof of i).* The  $i$ -th entry  $\mathbb{M}_1[(e_i + \zeta e_i)^* \bar{\mathbf{H}} e_1]$  of  $\mathbb{M}_1[(I + \zeta^*) \bar{\mathbf{H}} e_1]$  satisfies ( $\delta$  denotes the Kronecker symbol)

$$\mathbb{M}_1[(e_i + \zeta e_i)^* \bar{\mathbf{H}} e_1] = \mathbb{M}_1[(I + \zeta^*)_{ij} D_k \mathbf{H}_{jk}] = \lim_{n \rightarrow \infty} \mathbb{M}_1[(\delta_{ij} + D_j \varphi_n^i) D_k \mathbf{H}_{jk}]$$

Since  $\mathbf{H}_{ij} = 0$  if  $i = 1$  or  $j = 1$ , the index  $k$  varies from 2 to  $d$  in the latter expression. By using Lemma 4.5 iii) below, we deduce

$$\lim_{n \rightarrow \infty} \mathbb{M}_1[(\delta_{ij} + D_j \varphi_n^i) D_k \mathbf{H}_{jk}] = - \lim_{n \rightarrow \infty} \mathbb{M}_1[D_{jk}^2 \varphi_n^i \mathbf{H}_{jk}] = 0$$

because of the antisymmetry of  $\mathbf{H}$ . i) follows.

*Proof of ii).* Because of the ergodicity of the measure  $\mu$  (2. of Definition 1.1), we stress that a function  $\psi \in L^2(\Omega, \mathcal{G}^*, \mu)$  invariant under the translations  $\{\tau_x; x \in \mathbb{R} \times \{0\}^{d-1}\}$  must be constant and therefore satisfies  $\mathbb{M}_1[\psi] = \mathbb{M}[\psi]$ . So we just have to prove that the entries  $\mathbb{M}_1[(e_i + \zeta e_i)^* \mathbf{a} e_1]$  are invariant under the translations  $\{\tau_x; x \in \mathbb{R} \times \{0\}^{d-1}\}$ . To that purpose, we only need to check that

$$\mathbb{M}[\mathbb{M}_1[(e_i + \zeta e_i)^* \mathbf{a} e_1] D_1 \varphi] = 0$$

for any  $i = 1, \dots, d$  and  $\varphi \in \mathcal{C}$ . By using Lemma 4.5 ii), we get:

$$\mathbb{M}[\mathbb{M}_1[(e_i + \zeta e_i)^* \mathbf{a} e_1] D_1 \varphi] = \mathbb{M}[(e_i + \zeta e_i)^* \mathbf{a} e_1 \mathbb{M}_1[D_1 \varphi]] = \mathbb{M}[(e_i + \zeta e_i)^* \mathbf{a} e_1 D_1 \mathbb{M}_1[\varphi]].$$

Since  $D_k \mathbb{M}_1[\varphi] = 0$  for  $k = 2, \dots, d$  (see Lemma 4.5 i), we have  $e_1 D_1 \mathbb{M}_1[\varphi] = D \mathbb{M}_1[\varphi]$ . We deduce

$$\mathbb{M}[\mathbb{M}_1[(e_i + \zeta e_i)^* \mathbf{a} e_1] D_1 \varphi] = \mathbb{M}[(e_i + \zeta e_i)^* \mathbf{a} D \mathbb{M}_1[\varphi]].$$

Since  $\mathbb{M}_1[\varphi] \in \mathcal{C}$  (Lemma 4.5 ii), the latter quantity is equal to 0 (Lemma 4.4). We complete the proof of ii). Note that the above computations also prove that  $\bar{\Gamma}_1 = \mathbb{M}[(e_1 + \zeta e_1)^* \mathbf{a} e_1] = \bar{A}_{11} \geq \Lambda$ .  $\square$

**Lemma 4.4.** *The following relation holds:*

$$(23) \quad \forall X \in \mathbb{R}^d, \quad \forall \psi \in \mathbb{H}, \quad \mathbb{M}[(X + \zeta X)^* \mathbf{a} D \psi] = 0.$$

**Proof.** Since  $\mathbf{b}_i = \frac{1}{2} D_k \mathbf{a}_{ik}$ , the weak form of the resolvent equation (16) reads, for any  $\psi \in \mathbb{H}$ :  $\lambda(\mathbf{u}_\lambda^i, \psi)_2 + (1/2)(\mathbf{a}_{jk} D_j \mathbf{u}_\lambda^i, D_k \psi)_2 = (1/2)(D_k \mathbf{a}_{ik}, \psi)_2 = -(1/2)(\mathbf{a}_{ik}, D_k \psi)_2$ . By letting  $\lambda$  go to 0, we get from (18):  $(1/2)(\mathbf{a}_{jk} \zeta_j^i, D_k \psi)_2 = -(1/2)(\mathbf{a}_{ik}, D_k \psi)_2$ , that is:

$$(24) \quad 0 = \mathbb{M}[(\zeta_j^i + \delta_{ij}) \mathbf{a}_{jk} D_k \psi] = \mathbb{M}[(e_i + \zeta e_i)^* \mathbf{a} D \psi].$$

The result follows by linearity.  $\square$

**Lemma 4.5.** *The projection operator  $\mathbb{M}_1$  satisfies the following elementary properties:*

- i)  $\forall k = 2, \dots, d$  and  $\forall \varphi \in \text{Dom}(D_k)$ ,  $D_k \mathbb{M}_1[\varphi] = \mathbb{M}_1[D_k \varphi] = 0$ ,
- ii)  $\forall \varphi \in \mathcal{C}$ ,  $\mathbb{M}_1[\varphi] \in \mathcal{C}$  and  $\mathbb{M}_1[D_1 \varphi] = D_1 \mathbb{M}_1[\varphi]$ ,
- iii)  $\forall k = 2, \dots, d$  and  $\forall \varphi, \psi \in \text{Dom}(D_k)$ ,  $\mathbb{M}_1[D_k \varphi \psi] = -\mathbb{M}_1[\varphi D_k \psi]$ .

**Proof.** The properties i) and ii) are easily derived from the identities  $\mathbb{M}_1[T_x \varphi] = \mathbb{M}_1[\varphi]$  for any  $x \in \{0\} \times \mathbb{R}^{d-1}$ ,  $T_x \mathbb{M}_1 = \mathbb{M}_1 T_x$  for any  $x \in \mathbb{R} \times \{0\}^{d-1}$ , and  $\mathbb{M}_1[\psi * \rho] = \mathbb{M}_1[\psi] * \rho$  for any  $\psi \in L^\infty(\Omega)$  and  $\rho \in C_c^\infty(\mathbb{R}^d)$ . iii) results from i). Details are left to the reader.  $\square$

**Generator on the random medium associated to the boundary reflection.** We now focus on the asymptotic behavior of the process  $X^\varepsilon$  near the boundary. Generalizing the arguments in [21], it seems natural to look at the unbounded operator in random medium  $H_\gamma$ , whose construction is formally the following: given  $\omega \in \Omega$  and a smooth function  $\varphi \in \mathcal{C}$ , let us denote by  $\tilde{u}_\omega : \bar{D} \rightarrow \mathbb{R}$  the solution of the problem

$$(25) \quad \begin{cases} L^\omega \tilde{u}_\omega(x) = 0, & x \in D, \\ \tilde{u}_\omega(x) = \varphi(\tau_x \omega), & x \in \partial D. \end{cases}$$

where

$$(26) \quad L^\omega f(x) = (1/2) \mathbf{a}_{ij}(\tau_x \omega) \partial_{x_i x_j}^2 f(x) + \mathbf{b}_i(\tau_x \omega) \partial_{x_i} f(x)$$

whenever  $f \in C_b^2(\bar{D})$ . Then we define

$$(27) \quad H_\gamma \varphi(\omega) = \gamma_i(\omega) \partial_{x_i} \tilde{u}_\omega(0).$$

Though we don't need (and thus don't prove) this fact, the operator  $H_\gamma$  is the generator of the  $\Omega$ -valued Markov process  $Z_t(\omega) = \tau_{Y_t(\omega)} \omega$ , where  $Y_t(\omega) = X_{K^{-1}(t)}^1$  and the function  $K^{-1}$  stands for the left inverse of  $K^1$ :  $K^{-1}(t) = \inf\{s > 0; K_s^1 \geq t\}$ . The process  $Y$  describes the points on the boundary hit by the process  $X^1$  (the superscript 1 just means that you take  $\varepsilon = 1$  in (1)).

The main difficulty lies in constructing a solution of Problem (25) with suitable growth and integrability properties because of the lack of compactness of  $D$ . To get round this difficulty, we develop a set of tools on the random medium that makes possible to solve the "third boundary value problem on the random medium" and, as a byproduct, to construct  $H_\gamma$  through its associated resolvent family.

Now we give a few notations to pursue the analysis of  $H_\gamma$ . In what follows, the notation  $(x_1, y)$  stands for a  $d$ -dimensional vector, where the first component  $x_1$  belongs to  $\mathbb{R}$  (eventually  $\mathbb{R}_+ = [0; +\infty[$ ) and the second component  $y$  belongs to  $\mathbb{R}^{d-1}$ .

We denote by  $\Omega^+$  the product space  $\mathbb{R}_+ \times \Omega$  equipped with the measure  $d\mu^+ \stackrel{def}{=} dx_1 \otimes d\mu$  where  $dx_1$  is the Lebesgue measure on  $\mathbb{R}_+$ . Thus it makes sense to consider the standard spaces  $L^p(\Omega^+)$  for  $p \in [1; +\infty]$ .

For each random function  $\varphi$  defined on  $\Omega$ , we associate a function  $\varphi^+$  defined on  $\Omega^+$  by

$$\forall (x_1, \omega) \in \Omega^+, \quad \varphi^+(x_1, \omega) = \varphi(\tau_{x_1}\omega).$$

Hence, we can associate to the random matrices  $\mathbf{a}$  and  $\bar{\mathbf{H}}$  (defined in Sections 1 and 2) the corresponding matrix-valued functions  $\mathbf{a}^+$  and  $\bar{\mathbf{H}}^+$  defined on  $\Omega^+$ .

From now on, keep in mind that our main purpose is to define the Dirichlet form associated to  $H_\gamma$ . Given  $\varphi \in L^\infty(\Omega)$  and  $\rho \in C_c^\infty(\bar{D})$ , the function  $\varphi \odot \rho : \Omega^+ \rightarrow \mathbb{R}$  is defined by:

$$\varphi \odot \rho(x_1, \omega) = \int_{\mathbb{R}^{d-1}} \varphi(\tau_{(0,y)}\omega) \rho(x_1, y) dy.$$

The subspaces  $\mathbb{C}(\Omega^+)$  and  $\mathbb{C}_c(\Omega^+)$  of  $L^\infty(\Omega^+)$  are defined as the spaces respectively spanned by

$$\begin{aligned} \mathbb{C}(\Omega^+) &= \text{Span}\{\varphi \odot \rho; \varphi \in L^\infty(\Omega), \rho \in C_c^\infty(\bar{D})\} \\ \mathbb{C}_c(\Omega^+) &= \text{Span}\{\varphi \odot \rho; \varphi \in L^\infty(\Omega), \rho \in C_c^\infty(D)\} \end{aligned}$$

and stand, in a way, for the spaces of test functions on  $\Omega^+$ . For any  $\mathbf{g} \in \mathbb{C}(\Omega^+)$ , we introduce a sort of gradient  $\partial \mathbf{g}$  of  $\mathbf{g}$ , which entries are defined by

$$(28) \quad \begin{aligned} \partial_1 \mathbf{g}(x_1, \omega) &= \lim_{h>0, h \rightarrow 0} \frac{\mathbf{g}(x_1 + h, \omega) - \mathbf{g}(x_1, \omega)}{h}, \\ \partial_i \mathbf{g}(x_1, \omega) &= \lim_{h \rightarrow 0} \frac{\mathbf{g}(x_1, \tau_{he_i}\omega) - \mathbf{g}(x_1, \omega)}{h}, \quad i = 2, \dots, d. \end{aligned}$$

In other words, if we fix  $\omega \in \Omega$ ,  $\partial_1 \mathbf{g}$  is defined as the classical derivative of the function  $x_1 \in \mathbb{R}_+ \mapsto \mathbf{g}(x_1, \omega)$  whereas  $\partial_i \mathbf{g}$  (for  $i = 2, \dots, d$ ) coincides with the random partial derivative  $D_i$  of the random function  $\omega \in \Omega \mapsto \mathbf{g}(x_1, \omega)$ , where  $x_1$  is fixed. We further stress that the relation  $\partial \mathbf{g} = 0$  does not imply the expected relation " $\mathbf{g} = 0$  is constant". However, we will make use of the fact:  $\partial \mathbf{g} = 0 \Rightarrow \mathbf{g}(0, \cdot)$  is  $\mathcal{G}^*$ -measurable and  $x_1 \mapsto \mathbf{g}(x_1, \cdot)$  is constant.

We define on  $\mathbb{C}(\Omega^+)$  the norm

$$(29) \quad N(\mathbf{g})^2 = |\mathbf{g}(0, \cdot)|_2^2 + \int_{\Omega^+} |\partial \mathbf{g}|_2^2 d\mu^+,$$

and  $\mathbb{W}^1$  as the closure of  $\mathbb{C}(\Omega^+)$  with respect to the norm  $N$ . Obviously, the mapping

$$P : \mathbb{W}^1 \ni \mathbf{g} \mapsto \mathbf{g}(0, \cdot) \in L^2(\Omega)$$

is continuous (with norm equal to 1) and stands, in a way, for the trace operator on the random medium. Equip the topological dual space  $(\mathbb{W}^1)'$  of  $\mathbb{W}^1$  with the dual norm  $N'$ . The adjoint  $P^*$  of  $P$  is given by  $P^* : \varphi \in L^2(\Omega) \mapsto P^*(\varphi) \in (\mathbb{W}^1)'$  where the mapping  $P^*(\varphi)$  exactly matches

$$P^*(\varphi) : \mathbf{g} \in \mathbb{W}^1 \mapsto (\mathbf{g}, P^*(\varphi)) = (\varphi, \mathbf{g}(0, \cdot))_2.$$

For any  $\lambda > 0$ , we further define on  $\mathbb{W}^1 \times \mathbb{W}^1$  the following bilinear forms

$$(30) \quad B_\lambda(\mathbf{g}, \mathbf{h}) = \lambda(P\mathbf{g}, P\mathbf{h})_2 + \frac{1}{2} \int_{\Omega^+} (\mathbf{a}^+ + \bar{\mathbf{H}}^+)_{ij} \partial_i \mathbf{g} \partial_j \mathbf{h} d\mu^+$$

$$(31) \quad B_\lambda^*(\mathbf{g}, \mathbf{h}) = \lambda(P\mathbf{g}, P\mathbf{h})_2 + \frac{1}{2} \int_{\Omega^+} (\mathbf{a}^+ - \bar{\mathbf{H}}^+)_{ij} \partial_i \mathbf{g} \partial_j \mathbf{h} d\mu^+$$

From (4), the boundedness and the antisymmetry of  $\bar{\mathbf{H}}$ , it is readily seen that they are continuous and coercive on  $\mathbb{W}^1 \times \mathbb{W}^1$ . From the Lax-Milgram theorem, we can then define two continuous resolvent families  $G_\lambda : (\mathbb{W}^1)' \rightarrow \mathbb{W}^1$  and  $G_\lambda^* : (\mathbb{W}^1)' \rightarrow \mathbb{W}^1$  such that:

$$(32) \quad \forall F \in (\mathbb{W}^1)', \forall \mathbf{g} \in \mathbb{W}^1, \quad B_\lambda(G_\lambda F, \mathbf{g}) = F(\mathbf{g}) \quad \text{and} \quad B_\lambda^*(G_\lambda^* F, \mathbf{g}) = F(\mathbf{g}).$$

We now tackle the construction of the resolvent family associated to  $H_\gamma$ . For any  $\lambda > 0$  and  $\varphi \in L^2(\Omega)$ , we can then define the operators

$$(33) \quad \begin{array}{ccc} R_\lambda : L^2(\Omega) & \rightarrow & L^2(\Omega) & \text{and} & R_\lambda^* : L^2(\Omega) & \rightarrow & L^2(\Omega) \\ \varphi & \mapsto & PG_\lambda P^*(\varphi) & & \varphi & \mapsto & PG_\lambda^* P^*(\varphi) \end{array}$$

Given  $\varphi \in L^2(\Omega)$ , we can plug  $\mathbf{F} = P^*\varphi$  into (32) and we get

$$(34) \quad \forall \mathbf{g} \in \mathbb{W}^1, \quad B_\lambda(G_\lambda P^*\varphi, \mathbf{g}) = (\mathbf{g}, P^*\varphi) \quad \text{and} \quad B_\lambda^*(G_\lambda^* P^*\varphi, \mathbf{g}) = (\mathbf{g}, P^*\varphi),$$

that is, by using (30) and (31):

$$(35) \quad \forall \mathbf{g} \in \mathbb{W}^1, \quad \lambda(R_\lambda \varphi, P\mathbf{g})_2 + \frac{1}{2} \int_{\Omega^+} (\mathbf{a}^+ + \bar{\mathbf{H}}^+)_{ij} \partial_i (G_\lambda P^*\varphi) \partial_j \mathbf{g} d\mu^+ = (\mathbf{g}, P^*\varphi),$$

$$(36) \quad \forall \mathbf{g} \in \mathbb{W}^1, \quad \lambda(R_\lambda^* \varphi, P\mathbf{g})_2 + \frac{1}{2} \int_{\Omega^+} (\mathbf{a}^+ - \bar{\mathbf{H}}^+)_{ij} \partial_i (G_\lambda^* P^*\varphi) \partial_j \mathbf{g} d\mu^+ = (\mathbf{g}, P^*\varphi).$$

**Proposition 4.6.** *The families  $(R_\lambda)_\lambda$  and  $(R_\lambda^*)_\lambda$  are both strongly continuous resolvent families, and:*

1) *the operator  $R_\lambda$  is the adjoint of  $R_\lambda^*$  in  $L^2(\Omega)$ .*

2) *given  $\varphi \in L^2(\Omega)$ , we have:*

$$\varphi \in \bigcup_{\lambda > 0} \text{Ker}(\lambda R_\lambda - \text{I}) \Leftrightarrow \varphi = \mathbb{M}_1[\varphi] \Leftrightarrow \varphi \in \bigcup_{\lambda > 0} \text{Ker}(\lambda R_\lambda^* - \text{I}).$$

3) *for each function  $\varphi \in L^2(\Omega)$ ,  $|\lambda R_\lambda \varphi - \mathbb{M}_1[\varphi]|_2 \rightarrow 0$  as  $\lambda \rightarrow 0$ .*

**Proof.** The resolvent properties of both families can be deduced from the resolvent properties of the operators  $G_\lambda$  and  $G_\lambda^*$ . Details are left to the reader.

So we first prove 1). Consider  $\varphi, \psi \in L^2(\Omega)$ . Then, using (34), we obtain

$$\begin{aligned} (R_\lambda \varphi, \psi)_2 &= (PG_\lambda P^*\varphi, \psi)_2 = (G_\lambda P^*\varphi, P^*\psi) = B_\lambda^*(G_\lambda^* P^*\psi, G_\lambda P^*\varphi) \\ &= B_\lambda(G_\lambda P^*\varphi, G_\lambda^* P^*\psi) = (P^*\varphi, G_\lambda^* P^*\psi) = (\varphi, R_\lambda^* \psi)_2 \end{aligned}$$

so that  $R_\lambda$  and  $R_\lambda^*$  are adjoint in  $L^2(\Omega)$ .

We now prove 2). Let us consider a function  $\varphi \in L^2(\Omega)$  satisfying  $\lambda R_\lambda \varphi = \varphi$  for some  $\lambda > 0$ . By plugging  $\mathbf{g} = G_\lambda P^*\varphi \in \mathbb{W}^1$  into (35), we obtain

$$(37) \quad \lambda |R_\lambda \varphi|_2^2 + \frac{1}{2} \int_{\Omega^+} \mathbf{a}_{ij}^+ \partial_i (G_\lambda P^*\varphi) \partial_j (G_\lambda P^*\varphi) d\mu^+ = (G_\lambda P^*\varphi, P^*\varphi) = (R_\lambda \varphi, \varphi)_2.$$

Since  $\lambda R_\lambda \varphi = \varphi$ , the right-hand side matches  $(R_\lambda \varphi, \varphi)_2 = \lambda |R_\lambda \varphi|_2^2$  so that the integral term in (37) must vanish, that is  $\int_{\Omega^+} \mathbf{a}_{ij}^+ \partial_i (G_\lambda P^* \varphi) \partial_j (G_\lambda P^* \varphi) d\mu^+ = 0$ . From (4),  $\partial (G_\lambda P^* \varphi) = 0$  and  $G_\lambda P^* \varphi(0, \cdot)$  is thus  $\mathcal{G}^*$ -measurable. Moreover, we have  $\lambda G_\lambda P^* \varphi(0, \cdot) = \lambda P G_\lambda P^* \varphi = \lambda R_\lambda \varphi = \varphi$  so that  $\varphi$  is  $\mathcal{G}^*$ -measurable. Hence  $\varphi = \mathbb{M}_1[\varphi]$ .

Conversely, we assume  $\varphi = \mathbb{M}_1[\varphi]$ , which equivalently means  $\varphi$  is  $\mathcal{G}^*$  measurable. We define the function  $\mathbf{u} : \Omega^+ \rightarrow \mathbb{R}$  by  $\mathbf{u}(x_1, \omega) = \varphi(\omega)$ . It is obvious to check that  $\mathbf{u}$  belongs to  $W^1$  and satisfies  $\partial \mathbf{u} = 0$ . So  $B_\lambda^*(\mathbf{u}, \cdot) = \lambda P^* \varphi(\cdot)$  for any  $\lambda > 0$ . This means  $\mathbf{u} = G_\lambda^* P^* \varphi$  in such a way that  $\lambda R_\lambda \varphi = P \mathbf{u} = \varphi$ . The same argument holds to prove:  $\varphi = \mathbb{M}_1[\varphi] \Leftrightarrow \varphi \in \bigcup_{\lambda > 0} \text{Ker}(\lambda R_\lambda^* - I)$ .

We prove 3). Consider  $\varphi \in L^2(\Omega)$ . Note that (37) remains valid for such a function  $\varphi$ . Since the integral term in (37) is nonnegative, we deduce  $\lambda |R_\lambda \varphi|_2^2 \leq (R_\lambda \varphi, \varphi)_2 \leq |R_\lambda \varphi|_2 |\varphi|_2$ . Hence  $|\lambda R_\lambda \varphi|_2 \leq |\varphi|_2$  and we can extract a subsequence, still indexed by  $\lambda > 0$ , such that  $(\lambda R_\lambda \varphi)_\lambda$  weakly converges in  $L^2(\Omega)$  towards a function  $\hat{\varphi}$ . Now we want to establish that there is a unique possible weak limit for the family  $(\lambda R_\lambda \varphi)_\lambda$  by proving  $\hat{\varphi} = \mathbb{M}_1[\varphi]$ .

By multiplying the resolvent relation  $(\lambda - \mu) R_\lambda R_\mu \varphi = R_\mu \varphi - R_\lambda \varphi$  by  $\mu$  and passing to the limit as  $\mu \rightarrow 0$ , we get  $\lambda R_\lambda \hat{\varphi} = \hat{\varphi}$ . Because of 2),  $\hat{\varphi}$  is  $\mathcal{G}^*$ -measurable. Consider  $\psi \in L^2(\Omega)$  that is  $\mathcal{G}^*$ -measurable (i.e.  $\mathbb{M}_1[\psi] = \psi$ ). Because of 2), we have  $\lambda R_\lambda^* \psi = \psi$ . We deduce

$$(\varphi, \psi)_2 = (\varphi, \lambda R_\lambda^* \psi)_2 = \lim_{\lambda \rightarrow 0} (\lambda R_\lambda \varphi, \psi)_2 = (\hat{\varphi}, \psi)_2$$

and this proves that  $\hat{\varphi} = \mathbb{M}_1[\varphi]$ . As a consequence, there is a unique possible limit for each weakly converging subsequence of the family  $(\lambda R_\lambda \varphi)_\lambda$ . The whole family is therefore weakly converging in  $L^2(\Omega)$ .

To establish the strong convergence, it suffices to prove the convergence of the norms. As a weak limit,  $\hat{\varphi}$  satisfies the property  $|\hat{\varphi}|_2 \leq \liminf_{\lambda \rightarrow 0} |\lambda R_\lambda \varphi|_2$ . Conversely, (37) yields

$$\limsup_{\lambda \rightarrow 0} |\lambda R_\lambda \varphi|_2^2 \leq \limsup_{\lambda \rightarrow 0} (\lambda R_\lambda \varphi, \varphi)_2 = (\hat{\varphi}, \varphi)_2 = |\hat{\varphi}|_2^2$$

and the strong convergence follows.  $\square$

The remaining part of this section is concerned with the regularity properties of  $G_\lambda P^* \varphi$ . The proofs, which may be omitted upon the first reading, are gathered in the appendix. Given  $\mathbf{u} \in L^2(\Omega^+)$ , we will say that  $\mathbf{u}$  is a weakly differentiable if, for  $i = 1, \dots, d$ , we can find some function  $\partial_i \mathbf{u} \in L^2(\Omega^+)$  such that, for any  $\mathbf{g} \in \mathcal{C}_c(\Omega^+)$ , we have

$$\int_{\Omega^+} \mathbf{u} \partial_i \mathbf{g} d\mu^+ = - \int_{\Omega^+} \partial_i \mathbf{u} \mathbf{g} d\mu^+.$$

For  $k \geq 2$ , the space  $W^k$  is recursively defined as the set of functions  $\mathbf{u} \in W^1$  such that  $\partial_i \mathbf{u}$  is  $k - 1$  times weakly differentiable for  $i = 1, \dots, d$ .

**Remark 4.7.** *The above notion of differentiability on  $\Omega^+$  extends that introduced in (28) for smooth functions. That is the reason why we keep the same notation. Furthermore, we distinguish the operator  $\partial_i$  acting on a functions defined on  $\Omega^+$  from the classical operator  $\partial_{x_i}$  acting on functions defined on a domain of  $\mathbb{R}^d$ .*

**Proposition 4.8.** *If  $\varphi$  belongs to  $\mathcal{C}$ , then  $G_\lambda P^* \varphi \in \bigcap_{k=1}^{\infty} W^k$ .*

**Proposition 4.9.** *Given  $\varphi \in \mathcal{C}$ , the trajectories of  $G_\lambda P^* \varphi$  are smooth. More precisely, we can find  $N \subset \Omega$  satisfying  $\mu(N) = 0$  and such that  $\forall \omega \in \Omega \setminus N$ , the function*

$$\tilde{u}_\omega : x = (x_1, y) \in \bar{D} = \mathbb{R}_+ \times \mathbb{R}^{d-1} \mapsto G_\lambda P^* \varphi(x_1, \tau_{(0,y)} \omega)$$

*belongs to  $C^\infty(\bar{D})$ . Furthermore, it is a classical solution to the problem:*

$$(38) \quad \begin{cases} L^\omega \tilde{u}_\omega(x) = 0, & x \in D, \\ \lambda \tilde{u}_\omega(x) - \gamma_i(\tau_x \omega) \partial_{x_i} \tilde{u}_\omega(x) = \varphi(\tau_x \omega), & x \in \partial D. \end{cases}$$

**Proposition 4.10. (Maximum principle)** *Given  $\varphi \in \mathcal{C}$ , we have  $|G_\lambda P^* \varphi|_{L^\infty(\Omega^+)} \leq \lambda^{-1} |\varphi|_\infty$ .*

## 5 Ergodic theorems for the diffusion process

We now explain the connection between the operators in random medium and the asymptotic behavior of additive functionals of the process  $X^\varepsilon$  and  $K^\varepsilon$ . The main advantage of considering the processes  $X^\varepsilon, K^\varepsilon$  under  $\bar{\mathbb{P}}^{\varepsilon*}$  is that we can exhibit an invariant distribution for the process  $X^\varepsilon$  (remind the definitions of  $\bar{\mathbb{P}}^{\varepsilon*}, \bar{\mathbb{P}}^\varepsilon, \mathbb{P}_D^*, \mathbb{P}_{\partial D}^*$  at the very end of Section 3). More precisely

**Lemma 5.1.** *The process  $X^\varepsilon$  satisfies:*

1) *For each function  $\mathbf{f} \in L^1(\bar{D} \times \Omega; \mathbb{P}_D^*)$  and  $t \geq 0$ :*

$$(39) \quad \bar{\mathbb{E}}^{\varepsilon*}[\mathbf{f}(X_t^\varepsilon, \tau_{X_t^\varepsilon/\varepsilon} \omega)] dx = \mathbb{M}_D^*[\mathbf{f}].$$

2) *For each function  $\mathbf{f} \in L^1(\partial D \times \Omega; \mathbb{P}_{\partial D}^*)$  and  $t \geq 0$ :*

$$(40) \quad \bar{\mathbb{E}}^{\varepsilon*} \left[ \int_0^t \mathbf{f}(X_r^\varepsilon, \tau_{X_r^\varepsilon/\varepsilon} \omega) dK_r^\varepsilon \right] = t \mathbb{M}_{\partial D}^*[\mathbf{f}]$$

**Proof.** 1) We first suppose that  $\mathbf{f}$  can be rewritten as  $\mathbf{f}(x, \omega) = \Psi(\omega) \varrho(x)$ , where  $\Psi \in \mathcal{C}$  and  $\varrho \in C_c^\infty(D)$ . Fix  $T > t$ . From Lemma B.1, there exists a classical bounded solution  $w_\varepsilon \in C^\infty([0, T] \times \bar{D}) \cap C_b^{1,2}$  to the problem

$$\partial_t w_\varepsilon = \mathcal{L}_V^\varepsilon w_\varepsilon \text{ on } [0, T] \times D, \quad \gamma_i(\tau_{\cdot/\varepsilon} \omega) \partial_{x_i} w_\varepsilon = 0 \text{ on } [0, T] \times \partial D, \quad \text{and } w_\varepsilon(0, \cdot) = \mathbf{f}(\cdot, \tau_{\cdot/\varepsilon} \omega),$$

where  $\mathcal{L}_V^\varepsilon$  is defined in (6). Moreover, Lemma B.2 provides the probabilistic interpretation:

$$w_\varepsilon(t, x) = \mathbb{E}_x^{\varepsilon*}[\mathbf{f}(X_t^\varepsilon, \tau_{X_t^\varepsilon/\varepsilon} \omega)].$$

The Green formula (7) then yields

$$\begin{aligned} \partial_t \int_D w_\varepsilon(t, x) e^{-2V(x)} dx &= \int_D \mathcal{L}_V^\varepsilon w_\varepsilon(t, x) e^{-2V(x)} dx \\ &= - \int_{\partial D} \gamma_i(\tau_{x/\varepsilon} \omega) \partial_{x_i} w_\varepsilon(t, x) e^{-2V(x)} dx = 0 \end{aligned}$$

so that  $\int_{\bar{D}} \bar{\mathbb{E}}_x^{\varepsilon*}[\mathbf{f}(X_t^\varepsilon, \tau_{X_t^\varepsilon/\varepsilon} \omega)] e^{-2V(x)} dx = \int_{\bar{D}} \mathbf{f}(x, \tau_{x/\varepsilon} \omega) e^{-2V(x)} dx$ . It just remains to integrate with respect to the measure  $\mu$  and use the invariance under translations. The first point then follows from the density of  $C_c^\infty(D) \times \mathcal{C}$  in  $L^1(\bar{D} \times \Omega; \mathbb{P}_D^*)$ .

Let us now focus on the second assertion. As previously, it is sufficient to consider a smooth function  $\mathbf{f}(x, \omega) = \varrho(x)\boldsymbol{\psi}(\omega)$ ,  $\varrho \in C_c^\infty(\bar{D})$  and  $\boldsymbol{\psi} \in \mathcal{C}$ . Recall that the local time  $K_t^\varepsilon$  is the density of occupation time at  $\partial D$  (see [4, Prop. 1.19] with  $\boldsymbol{\psi}(x) = x_1$ ,  $V_0 = \gamma$  and  $a^2(x) = 1$ ). As a consequence,

$$\begin{aligned} \bar{\mathbb{E}}^{\varepsilon*} \left[ \int_0^t \mathbf{f}(X_r^\varepsilon, \tau_{X_r^\varepsilon/\varepsilon}\omega) dK_r^\varepsilon \right] &= \bar{\mathbb{E}}^{\varepsilon*} \left[ \lim_{\delta \rightarrow 0} \delta^{-1} \int_0^t \mathbf{f}(X_r^\varepsilon, \tau_{X_r^\varepsilon/\varepsilon}\omega) \mathbb{1}_{[0,\delta]}(X_r^{1,\varepsilon}) dr \right] \\ &= \lim_{\delta \rightarrow 0} \bar{\mathbb{E}}^{\varepsilon*} \left[ \delta^{-1} \int_0^t \mathbf{f}(X_r^\varepsilon, \tau_{X_r^\varepsilon/\varepsilon}\omega) \mathbb{1}_{[0,\delta]}(X_r^{1,\varepsilon}) dr \right] \\ &= t \lim_{\delta \rightarrow 0} \delta^{-1} \mathbb{M}_D^* [\mathbf{f}(x, \tau_{x/\varepsilon}\omega) \mathbb{1}_{[0,\delta]}(x_1)] \\ &= t \mathbb{M}_{\partial D}^* [\mathbf{f}(x, \tau_{x/\varepsilon}\omega)]. \end{aligned}$$

We complete the proof with the invariance of  $\mu$  under the translations  $\{\tau_x; x \in \mathbb{R}^d\}$ .  $\square$

We are now in position to state an ergodic theorem for the process  $X^\varepsilon$  inside the domain  $D$

**Theorem 5.2.** *For each function  $\mathbf{f} \in L^1(\Omega)$  and  $T > 0$ , we have*

$$(41) \quad \lim_{\varepsilon \rightarrow 0} \bar{\mathbb{E}}^{\varepsilon*} \left[ \sup_{0 \leq t \leq T} \left| \int_0^t \mathbf{f}(\tau_{X_r^\varepsilon/\varepsilon}\omega) dr - t \mathbb{M}[\mathbf{f}] \right| \right] = 0.$$

**Proof.** We first suppose that  $\mathbf{f}$  belongs to  $\mathcal{C}$ . Even if it means replacing  $\mathbf{f}$  by  $\mathbf{f} - \mathbb{M}[\mathbf{f}]$ , we assume  $\mathbb{M}[\mathbf{f}] = 0$ . Consider the solution  $\mathbf{v}_\lambda \in L^2(\Omega) \cap \text{Dom}(\mathbf{L})$  to the resolvent equation

$$(42) \quad \lambda \mathbf{v}_\lambda - \mathbf{L} \mathbf{v}_\lambda = \mathbf{f}.$$

We let the reader check (or adapt the proof of Proposition 4.9) that  $\mu$  a.s. the function  $\vartheta : x \in \mathbb{R}^d \mapsto \mathbf{v}_\lambda(\tau_x \omega)$  satisfies  $\lambda \vartheta(x) - L^\omega \vartheta(x) = \mathbf{f}(\tau_x \omega)$   $x \in \mathbb{R}^d$ . So  $\vartheta$  is smooth [8, Th. 6.17]. Applying the Itô formula to the function  $x \mapsto \mathbf{v}_\lambda(\tau_x \omega)$  then yields

$$\begin{aligned} d\mathbf{v}_\lambda(\tau_{X_t^\varepsilon/\varepsilon}\omega) &= \varepsilon^{-1} D_i \mathbf{v}_\lambda \boldsymbol{\sigma}_{ij}(\tau_{X_t^\varepsilon/\varepsilon}\omega) dB_t^{*j} - \varepsilon^{-1} \partial_{x_i} V(X_t^\varepsilon) \mathbf{a}_{ij} D_j \mathbf{v}_\lambda(\tau_{X_t^\varepsilon/\varepsilon}\omega) dt \\ &\quad + \varepsilon^{-2} \mathbf{L} \mathbf{v}_\lambda(\tau_{X_t^\varepsilon/\varepsilon}\omega) dt + \varepsilon^{-1} D_i \mathbf{v}_\lambda \gamma_i(\tau_{X_t^\varepsilon/\varepsilon}\omega) dK_t^\varepsilon. \end{aligned}$$

In the above expression, we replace  $\mathbf{L} \mathbf{v}_\lambda$  by  $\lambda \mathbf{v}_\lambda - \mathbf{f}$ , multiply both sides of the equality by  $\varepsilon^2$  and isolate the term  $\mathbf{f}(\tau_{X_t^\varepsilon/\varepsilon}\omega) dt$ . We obtain

$$\begin{aligned} \int_0^t \mathbf{f}(\tau_{X_r^\varepsilon/\varepsilon}\omega) dr &= \varepsilon \int_0^t D_i \mathbf{v}_\lambda \boldsymbol{\sigma}_{ij}(\tau_{X_r^\varepsilon/\varepsilon}\omega) dB_r^{*j} - \varepsilon^2 (\mathbf{v}_\lambda(\tau_{X_t^\varepsilon/\varepsilon}\omega) - \mathbf{v}_\lambda(\tau_{X_0^\varepsilon/\varepsilon}\omega)) + \int_0^t \lambda \mathbf{v}_\lambda(\tau_{X_r^\varepsilon/\varepsilon}\omega) dr \\ (43) \quad &\quad + \varepsilon \int_0^t D_i \mathbf{v}_\lambda \gamma_i(\tau_{X_t^\varepsilon/\varepsilon}\omega) dK_r^\varepsilon - \varepsilon \int_0^t \partial_{x_i} V(X_r^\varepsilon) \mathbf{a}_{ij} D_j \mathbf{v}_\lambda(\tau_{X_r^\varepsilon/\varepsilon}\omega) dr \\ &\stackrel{\text{def}}{=} \Delta_t^{1,\varepsilon,\lambda} + \Delta_t^{2,\varepsilon,\lambda} + \Delta_t^{3,\varepsilon,\lambda} + \Delta_t^{4,\varepsilon,\lambda} + \Delta_t^{5,\varepsilon,\lambda}. \end{aligned}$$

Let us investigate the quantities  $\Delta^{1,\varepsilon,\lambda}$ ,  $\Delta^{2,\varepsilon,\lambda}$ ,  $\Delta^{3,\varepsilon,\lambda}$ ,  $\Delta^{4,\varepsilon,\lambda}$  and  $\Delta^{5,\varepsilon,\lambda}$ . Using the Doob inequality and Lemma 5.1, we have:

$$\bar{\mathbb{E}}^{\varepsilon*} \left[ \sup_{0 \leq t \leq T} |\Delta_t^{1,\varepsilon,\lambda}|^2 \right] = \varepsilon^2 T \mathbb{M}_D^* [ |D_i \mathbf{v}_\lambda \boldsymbol{\sigma}_{ij}|^2 ] \leq C \varepsilon^2 |D \mathbf{v}_\lambda|_2^2$$

for some positive constant  $C$  only depending on  $T$  and  $|\sigma|_\infty$ . Hence  $\bar{\mathbb{E}}^{\varepsilon*} [\sup_{0 \leq t \leq T} |\Delta_t^{1,\varepsilon,\lambda}|^2] \rightarrow 0$  as  $\varepsilon \rightarrow 0$ , for each fixed  $\lambda > 0$ . Similarly, we can prove

$$\bar{\mathbb{E}}^{\varepsilon*} \left[ \sup_{0 \leq t \leq T} |\Delta_t^{4,\varepsilon,\lambda}| + \sup_{0 \leq t \leq T} |\Delta_t^{5,\varepsilon,\lambda}|^2 \right] \rightarrow 0, \quad \text{as } \varepsilon \rightarrow 0.$$

By using Lemma 4.1, we have

$$\bar{\mathbb{E}}^{\varepsilon*} \left[ \sup_{0 \leq t \leq T} |\Delta_t^{2,\varepsilon,\lambda}|^2 \right] \leq 4\varepsilon^4 |\mathbf{f}|_\infty^2 \lambda^{-2} \rightarrow 0, \quad \text{as } \varepsilon \rightarrow 0.$$

From (43), we deduce

$$\limsup_{\varepsilon \rightarrow 0} \bar{\mathbb{E}}^{\varepsilon*} \left[ \sup_{0 \leq t \leq T} \left| \int_0^t \mathbf{f}(\tau_{X_r^\varepsilon/\varepsilon} \omega) dr \right|^2 \right] \leq \limsup_{\varepsilon \rightarrow 0} \bar{\mathbb{E}}^{\varepsilon*} \left[ \sup_{0 \leq t \leq T} |\Delta_t^{3,\varepsilon,\lambda}|^2 \right] \leq T |\lambda \mathbf{v}_\lambda|_2^2.$$

From Proposition 4.2, we have  $|\lambda \mathbf{v}_\lambda|_2 \rightarrow 0$  as  $\lambda$  goes to 0. So it just remains to choose  $\lambda$  small enough to complete the proof in the case of a smooth function  $\mathbf{f} \in \mathcal{C}$ . The general case follows from the density of  $\mathcal{C}$  in  $L^1(\Omega)$  and Lemma 5.1.  $\square$

**Ergodic theorem for the local time.** We now investigate the ergodic theorem for the local time  $K^\varepsilon$

**Theorem 5.3.** *If  $\mathbf{f} \in L^2(\Omega)$ , the following convergence holds*

$$(44) \quad \lim_{\varepsilon \rightarrow 0} \bar{\mathbb{E}}^{\varepsilon*} \left[ \sup_{0 \leq t \leq T} \left| \int_0^t \mathbf{f}(\tau_{X_r^\varepsilon/\varepsilon} \omega) dK_r^\varepsilon - \mathbb{M}_1[\mathbf{f}](\omega) K_t^\varepsilon \right| \right] = 0.$$

**Proof.** Once again, from Lemma 5.1 and density arguments, it is sufficient to consider the case of a smooth function  $\mathbf{f} \in \mathcal{C}$ . Even if it means replacing  $\mathbf{f}$  with  $\mathbf{f} - \mathbb{M}_1[\mathbf{f}]$ , we assume  $\mathbb{M}_1[\mathbf{f}] = 0$ . Let us define, for any  $\lambda > 0$ ,  $\mathbf{u}_\lambda = G_\lambda P^* \mathbf{f}$  and  $\mathbf{f}_\lambda = R_\lambda \mathbf{f}$ . We still use the notation  $\tilde{u}_\omega^\lambda(x) = \mathbf{u}_\lambda(x_1, \tau_{(0,y)} \omega)$  for any  $x = (x_1, y) \in \mathbb{R}_+ \times \mathbb{R}^{d-1}$ . We remind the reader that the main regularity properties of the function  $\tilde{u}_\omega^\lambda$  are summarized in Proposition 4.9. In particular,  $\mu$  a.s., the mapping  $x \mapsto \tilde{u}_\omega^\lambda(x)$  is smooth and we can apply the Itô formula:

$$\begin{aligned} d(\varepsilon \tilde{u}_\omega^\lambda(X_t^\varepsilon/\varepsilon)) &= [\varepsilon^{-1} L^\omega \tilde{u}_\omega^\lambda(X_t^\varepsilon/\varepsilon) - \partial_{x_j} V(X_t^\varepsilon) \mathbf{a}_{ij}(\tau_{X_t^\varepsilon/\varepsilon} \omega) \partial_{x_i} \tilde{u}_\omega^\lambda(X_t^\varepsilon/\varepsilon)] dt \\ &\quad + \partial_{x_i} \tilde{u}_\omega^\lambda(X_t^\varepsilon/\varepsilon) \sigma_{ij}(\tau_{X_t^\varepsilon/\varepsilon} \omega) dB_t^{*j} + \gamma_i(\tau_{X_t^\varepsilon/\varepsilon} \omega) \partial_{x_i} \tilde{u}_\omega^\lambda(X_t^\varepsilon/\varepsilon) dK_t^\varepsilon \end{aligned}$$

Since  $L^\omega \tilde{u}_\omega^\lambda = 0$  inside  $D$  and  $\gamma_i \partial_{x_i} u_\omega^\lambda = \lambda \mathbf{f}_\lambda - \mathbf{f}$  on  $\partial D$ , we deduce

$$\begin{aligned} \int_0^t \mathbf{f}(\tau_{X_r^\varepsilon/\varepsilon} \omega) dK_r^\varepsilon &= - \left( \varepsilon \tilde{u}_\omega^\lambda(X_t^\varepsilon/\varepsilon) - \varepsilon \tilde{u}_\omega^\lambda(X_0^\varepsilon/\varepsilon) \right) - \int_0^t \partial_{x_j} V(X_r^\varepsilon) \mathbf{a}_{ij}(\tau_{X_r^\varepsilon/\varepsilon} \omega) \partial_{x_i} \tilde{u}_\omega^\lambda(X_r^\varepsilon/\varepsilon) dr \\ &\quad + \int_0^t \partial_{x_i} \tilde{u}_\omega^\lambda(X_r^\varepsilon/\varepsilon) \sigma_{ij}(\tau_{X_r^\varepsilon/\varepsilon} \omega) dB_r^{*j} + \int_0^t \lambda \mathbf{f}_\lambda(\tau_{X_r^\varepsilon/\varepsilon} \omega) dK_r^\varepsilon \\ (45) \quad &\equiv \Delta_t^{1,\varepsilon} + \Delta_t^{2,\varepsilon} + \Delta_t^{3,\varepsilon} + \Delta_t^{4,\varepsilon}. \end{aligned}$$

The next step of the proof is to prove that  $\Delta_t^{1,\varepsilon}, \Delta_t^{2,\varepsilon}, \Delta_t^{3,\varepsilon}$  converge to 0 as  $\varepsilon$  goes to 0 for each fixed  $\lambda > 0$ . Clearly, from Proposition 4.10, we have  $\bar{\mathbb{E}}^{\varepsilon*} [\sup_{0 \leq t \leq T} |\Delta_t^{1,\varepsilon}|^2] \leq 4\varepsilon^2 |\mathbf{u}_\lambda|_{L^\infty(\Omega^+)} \xrightarrow{\varepsilon \rightarrow 0} 0$ .

Let us now focus on  $\Delta_t^{2,\varepsilon}$ . We use the boundedness of  $\partial_{x_j} V, \mathbf{a}_{ij}$  ( $1 \leq i, j \leq d$ ) and Lemma 5.1 (the constant  $C$  below may change from line to line but does not depend on  $\lambda, \varepsilon$ )

$$\bar{\mathbb{E}}^{\varepsilon*} \left[ \sup_{0 \leq t \leq T} |\Delta_t^{2,\varepsilon}|^2 \right] \leq CT \mathbb{M}_D^* [|\partial_x \tilde{u}_\omega^\lambda(\cdot/\varepsilon)|^2] = CT \int_{\mathbb{R}_+} \mathbb{M} |\partial \mathbf{u}_\lambda(\frac{x_1}{\varepsilon}, \cdot)|^2 \left( \int_{\mathbb{R}^{d-1}} e^{-2V(x_1,y)} dy \right) dx_1$$



By using (11) and by making the change of variables  $u = x_1/\varepsilon$ , the latter term is not greater than  $CM'\varepsilon T \int_{\Omega^+} |\partial \mathbf{u}_\lambda|^2 d\mu^+$  and therefore converges to 0 as  $\varepsilon \rightarrow 0$ . By combining the same argument with the Doob inequality, we prove that  $\bar{\mathbb{E}}^{\varepsilon*} [\sup_{0 \leq t \leq T} |\Delta_t^{3,\varepsilon}|^2] \rightarrow 0$  as  $\varepsilon \rightarrow 0$ . As a consequence (see (45))

$$\limsup_{\varepsilon \rightarrow 0} \bar{\mathbb{E}}^{\varepsilon*} \left[ \sup_{0 \leq t \leq T} \left| \int_0^t \mathbf{f}(\tau_{X_r^\varepsilon/\varepsilon} \omega) dK_r^\varepsilon \right| \right] \leq \limsup_{\varepsilon \rightarrow 0} \bar{\mathbb{E}}^{\varepsilon*} \left[ \int_0^T |\lambda \mathbf{f}_\lambda(\tau_{X_r^\varepsilon/\varepsilon} \omega)| dK_r^\varepsilon \right].$$

By using Lemma 5.1 in the right-hand side of the previous inequality, we deduce, for any  $\lambda > 0$ ,

$$\limsup_{\varepsilon \rightarrow 0} \bar{\mathbb{E}}^{\varepsilon*} \left[ \sup_{0 \leq t \leq T} \left| \int_0^t \mathbf{f}(\tau_{X_r^\varepsilon/\varepsilon} \omega) dK_r^\varepsilon \right| \right] \leq |\lambda \mathbf{f}_\lambda|_{2T} \int_{\partial D} e^{-2V(x)} dx \leq M'T |\lambda \mathbf{f}_\lambda|_2.$$

From Proposition 4.6 point 3), we can choose  $\lambda$  small enough so as to make the latter term arbitrarily small. So we complete the proof.  $\square$

**Ergodic theorems under  $\bar{\mathbb{P}}^\varepsilon$ .** With the help of (13), we reformulate theorems 5.2 and 5.3:

**Theorem 5.4.** 1) Let  $(\mathbf{f}_\varepsilon)_\varepsilon$  be a family converging towards  $\mathbf{f}$  in  $L^1(\Omega)$ . For each fixed  $\delta > 0$  and  $T > 0$ , the following convergence holds

$$(46) \quad \lim_{\varepsilon \rightarrow 0} \bar{\mathbb{P}}^\varepsilon \left[ \sup_{0 \leq t \leq T} \left| \int_0^t \mathbf{f}_\varepsilon(\tau_{X_r^\varepsilon/\varepsilon} \omega) dr - t\mathbb{M}[\mathbf{f}] \right| \geq \delta \right] = 0.$$

2) Let  $(\mathbf{f}_\varepsilon)_\varepsilon$  be a family converging towards  $\mathbf{f}$  in  $L^2(\Omega)$ . For each fixed  $\delta > 0$  and  $T > 0$ , the following convergence holds

$$(47) \quad \lim_{\varepsilon \rightarrow 0} \bar{\mathbb{P}}^\varepsilon \left[ \sup_{0 \leq t \leq T} \left| \int_0^t \mathbf{f}_\varepsilon(\tau_{X_r^\varepsilon/\varepsilon} \omega) dK_r^\varepsilon - \mathbb{M}_1[\mathbf{f}] K_t^\varepsilon \right| \geq \delta \right] = 0.$$

**Proof.** 1) From (13), we only have to check that (46) holds under  $\bar{\mathbb{P}}^{\varepsilon*}$ . It follows from Theorem 5.2 and the estimate (obtained with Lemma 5.1)

$$\lim_{\varepsilon \rightarrow 0} \bar{\mathbb{E}}^{\varepsilon*} \left[ \sup_{0 \leq t \leq T} \left| \int_0^t [\mathbf{f}_\varepsilon - \mathbf{f}](X_r^\varepsilon, \tau_{X_r^\varepsilon/\varepsilon} \omega) dr \right| \right] \leq T |\mathbf{f}_\varepsilon - \mathbf{f}|_1.$$

The same argument holds for (47).  $\square$

## 6 Tightness of the process $X^\varepsilon$

We now investigate the tightness of the process  $X^\varepsilon$  (and  $K^\varepsilon$ ). To that purpose, it is sufficient to establish the tightness of the process  $\frac{1}{\varepsilon} \int_0^t \mathbf{b}(\tau_{X_r^\varepsilon/\varepsilon} \omega) dr + \int_0^t \gamma(\tau_{X_r^\varepsilon/\varepsilon} \omega) dK_r^\varepsilon$ . Roughly speaking, the guiding line of our proof may be expressed as follows: on the first hand, we estimate the functional when  $X^\varepsilon$  evolves far from the boundary and then we study the limit behaviour of these estimates when we relax the constraint of being far from the boundary. The first part of the proof is inspired by [17, Chap. 3] and is based on the Garsia-Rodemich-Rumsey inequality:

**Proposition 6.1. (Garsia-Rodemich-Rumsey's inequality).** Let  $p$  and  $\Psi$  be strictly increasing continuous functions on  $[0, +\infty[$  satisfying  $p(0) = \Psi(0) = 0$  and  $\lim_{t \rightarrow \infty} \Psi(t) = +\infty$ . For given  $T > 0$  and  $f \in C([0, T]; \mathbb{R}^d)$ , suppose that there exists a finite  $B$  such that:

$$(48) \quad \int_0^T \int_0^T \Psi \left( \frac{|g(t) - g(s)|}{p(|t - s|)} \right) ds dt \leq B < \infty.$$

Then, for all  $0 \leq s \leq t \leq T$ :  $|g(t) - g(s)| \leq 8 \int_0^{t-s} \Psi^{-1}(4B/u^2) dp(u)$ .

Fix  $\omega \in \Omega$  and a smooth function  $\rho \in C_b^\infty(\bar{D})$  satisfying  $\rho(x) = 0$  whenever  $x_1 \leq \theta$  for some  $\theta > 0$ . For any  $\varepsilon > 0$  and  $j = 1, \dots, d$ , define the "truncated" drift

$$(49) \quad b_{\rho,j}^\varepsilon(x, \omega) = \frac{e^{2V(x)}}{2} \partial_{x_i} (e^{-2V(x)} \mathbf{a}_{ij}(\tau_{x/\varepsilon} \omega) \rho(x))$$

that belongs to  $C_b^\infty(\bar{D})$  too. We additionally define the set

$$C_{\gamma}^{2,\varepsilon} = \{f \in C_b^2(\bar{D}); \gamma_i(\tau_{x/\varepsilon} \omega) \partial_{x_i} f(x) = 0 \text{ for } x \in \partial D\}$$

and the norm  $|\phi|_D^2 = \int_D \phi^2(x) e^{-2V(x)} dx$  on  $L^2(\bar{D}; e^{-2V(x)} dx)$  (and  $(\cdot, \cdot)_D$  the associated inner product).

For  $\kappa > 0$  and  $\omega \in \Omega$ , let  $\psi_\omega^{\varepsilon,\kappa} \in C^\infty([0, T] \times \bar{D}) \cap C_b^{1,2}$  be the bounded solution of

$$\partial_t \psi_\omega^{\varepsilon,\kappa} = \mathcal{L}_V^\varepsilon \psi_\omega^{\varepsilon,\kappa} + \kappa b_{\rho,j}^\varepsilon (\psi_\omega^{\varepsilon,\kappa} + 1) \text{ on } [0, T] \times D, \quad \gamma_i(\tau_{x/\varepsilon} \omega) \partial_{x_i} \psi_\omega^{\varepsilon,\kappa} = 0 \text{ on } [0, T] \times \partial D$$

with initial condition  $\psi_\omega^{\varepsilon,\kappa}(0, \cdot) = 0$  on  $\bar{D}$  (see Lemma B.1). Then  $u_{\varepsilon,\kappa} = \psi_\omega^{\varepsilon,\kappa} + 1 \in C_b^{1,2}$  is a bounded classical solution of the problem

$$(50) \quad \partial_t u_\omega^{\varepsilon,\kappa} = \mathcal{L}_V^\varepsilon u_\omega^{\varepsilon,\kappa} + \kappa b_{\rho,j}^\varepsilon u_\omega^{\varepsilon,\kappa} \text{ on } [0, T] \times D, \quad \gamma_i(\tau_{x/\varepsilon} \omega) \partial_{x_i} u_\omega^{\varepsilon,\kappa} = 0 \text{ on } [0, T] \times \partial D,$$

with initial condition  $u_\omega^{\varepsilon,\kappa}(0, \cdot) = 1$  on  $\bar{D}$ . Lemma B.2 and a straightforward calculation provide the probabilistic representation

$$\begin{aligned} u_\omega^{\varepsilon,\kappa}(t, x) &= \mathbb{E}_x^{\varepsilon*} \left[ \int_0^t \kappa b_{\rho,j}^\varepsilon(X_r^\varepsilon, \omega) \exp \left( \int_0^r \kappa b_{\rho,j}^\varepsilon(X_u^\varepsilon, \omega) du \right) dr \right] + 1 \\ &= \mathbb{E}_x^{\varepsilon*} \left[ \exp \left( \kappa \int_0^t b_{\rho,j}^\varepsilon(X_r^\varepsilon, \omega) dr \right) \right]. \end{aligned}$$

**Lemma 6.2.** *For each  $\omega \in \Omega$ , we have the estimate  $|u_\omega^{\varepsilon,\kappa}(t, \cdot)|_D^2 \leq e^{2t\pi_\omega^{\varepsilon,\kappa}}$  ( $0 \leq t \leq T$ ), where  $\pi_\omega^{\varepsilon,\kappa} = \sup(\phi, \mathcal{L}_V^\varepsilon \phi + \kappa b_{\rho,j}^\varepsilon \phi)_D$  and the sup is taken over  $\{\phi \in C_{\gamma}^{2,\varepsilon}, |\phi|_D^2 = 1\}$ .*

**Proof.** We have:

$$\begin{aligned} \partial_t |u_\omega^{\varepsilon,\kappa}(t, \cdot)|_D^2 &= 2(u_\omega^{\varepsilon,\kappa}, \partial_t u_\omega^{\varepsilon,\kappa}(t, \cdot))_D \\ &= 2(u_\omega^{\varepsilon,\kappa}, \mathcal{L}_V^\varepsilon u_\omega^{\varepsilon,\kappa} + \kappa b_{\rho,j}^\varepsilon u_\omega^{\varepsilon,\kappa}(t, \cdot))_D \leq 2\pi_\omega^{\varepsilon,\kappa} |u_\omega^{\varepsilon,\kappa}(t, \cdot)|_D^2. \end{aligned}$$

Since  $|u_\omega^{\varepsilon,\kappa}(0, \cdot)|_D^2 = 1$ , we complete the proof with the Gronwall lemma.  $\square$

**Proposition 6.3.** *For any  $\kappa > 0$ ,  $\varepsilon > 0$  and  $0 \leq s, t \leq T$*

$$\mathbb{E}^{\varepsilon*} \left[ \exp \left( \left| \kappa \int_s^t b_{\rho,j}^\varepsilon(X_r^\varepsilon, \omega) dr \right| \right) \right] \leq 2 \exp(C\kappa^2(t-s)),$$

for some constant  $C$  that only depends on  $\Lambda$  and  $\sup_{x \in \bar{D}} |\rho(x)|$ .

**Proof.** By stationarity (Lemma 5.1) and Lemma 6.2, we have

$$\begin{aligned} \bar{\mathbb{E}}^{\varepsilon*} \left[ \exp \left( \kappa \int_s^t b_{\rho,j}^\varepsilon(X_r^\varepsilon, \omega) dr \right) \right] &\leq \bar{\mathbb{E}}^{\varepsilon*} \left[ \exp \left( \kappa \int_0^{t-s} b_{\rho,j}^\varepsilon(X_r^\varepsilon, \omega) dr \right) \right] \\ &= \mathbb{M}_D^* [u_\omega^{\varepsilon,\kappa}(t-s, x)] \leq \mathbb{M} |u_\omega^{\varepsilon,\kappa}(t-s, \cdot)|_D \leq \mathbb{M} [\exp((t-s)\pi_\omega^{\varepsilon,\kappa})]. \end{aligned}$$

It remains to estimate  $\pi_\omega^{\varepsilon, \kappa}$ . For any function  $\phi \in C_\gamma^{2, \varepsilon}$  such that  $|\phi|_D^2 = 1$ , we have

$$(b_{\rho, j}^\varepsilon(\cdot, \omega), \phi^2)_D = -(\mathbf{a}_{ij}(\tau_{./\varepsilon}\omega)\rho\phi, \partial_{x_i}\phi)_D \leq \Lambda^{-1} \sup_{x \in \bar{D}} |\rho(x)| |\partial_x \phi|_D = C |\partial_x \phi|_D$$

where we have set  $C = \Lambda^{-1} \sup_{x \in \bar{D}} |\rho(x)|$ . As a consequence (the sup below are taken over  $\{\phi \in C_\gamma^{2, \varepsilon}, |\phi|_D^2 = 1\}$ )

$$\begin{aligned} \pi_\omega^{\varepsilon, \kappa} &= \sup(\phi, \mathcal{L}_V^\varepsilon \phi + \kappa b_{\rho, j}^\varepsilon \phi)_D \\ &\leq \sup \left\{ - (1/2) (\mathbf{a}_{ij}(\tau_{./\varepsilon}\omega) \partial_{x_i} \phi, \partial_{x_j} \phi)_D + \kappa (b_{\rho, j}^\varepsilon(\cdot, \omega), \phi^2)_D \right\} \\ &\leq \sup \left\{ - (\Lambda/2) |\partial_x \phi|_D^2 + \kappa C |\partial_x \phi|_D \right\} \leq \kappa^2 C^2 / (2\Lambda). \end{aligned}$$

Gathering the previous inequalities yields  $\bar{\mathbb{E}}^{\varepsilon*} [\exp(\kappa \int_s^t b_{\rho, j}^\varepsilon(X_r^\varepsilon, \omega) dr)] \leq \exp(C' \kappa^2(t-s))$  where  $C' = \sup_{x \in \bar{D}} |\rho(x)|^2 / (2\Lambda^3)$ . We complete the proof by repeating the argument for  $-b_{\rho, j}^\varepsilon$  and using the inequality  $\exp(|x|) \leq \exp(-x) + \exp(x)$ .  $\square$

Let us now make a specific choice of  $\rho$ . For each  $n \in \mathbb{N}^*$ , let us consider the bounded continuous function  $\rho_n : \bar{D} \rightarrow \mathbb{R}$  defined by  $\rho_n(x) = 0$  if  $x_1 \leq n^{-1}$ ,  $\rho_n(x) = n(x_1 - n^{-1})$  if  $n^{-1} \leq x_1 \leq 2n^{-1}$  and 1 otherwise. With the help of a regularization procedure, Lemma 5.1 and the inequality  $\sup_{x \in \bar{D}} |\rho_n(x)| \leq 1$ , one can prove that Proposition 6.3 remains valid for  $\rho_n$ . Since  $\sup_{x \in \bar{D}} |\rho_n(x)| = 1$  for each  $n$ , we deduce

$$(51) \quad \forall n \in \mathbb{N}, \forall 0 \leq s, t \leq T, \quad \bar{\mathbb{E}}^{\varepsilon*} \left[ \exp \left( \left| \kappa \int_s^t b_{\rho_n, j}^\varepsilon(X_r^\varepsilon, \omega) dr \right| \right) \right] \leq 2 \exp(C \kappa^2(t-s))$$

for some constant  $C$  only depending on  $\Lambda$ .

Our objective is now to determine the limit of  $\int_s^t b_{\rho_n, j}^\varepsilon(X_r^\varepsilon, \omega) dr$  to pass to the limit as  $n \rightarrow \infty$  in (51). So we expand (49) with respect to the operator  $\partial_{x_i}$  and, by using the relation  $\partial_{x_i} \rho_n(x) = n \mathbb{I}_{[\frac{1}{n}, \frac{2}{n}]}(x)$ , we obtain

$$(52) \quad \begin{aligned} \int_0^t b_{\rho_n, j}^\varepsilon(X_r^\varepsilon, \omega) dr &= \int_0^t \left[ \frac{1}{\varepsilon} \mathbf{b}_j(\tau_{X_r^\varepsilon/\varepsilon}\omega) - \partial_{x_i} V(X_r^\varepsilon) \mathbf{a}_{ij}(\tau_{X_r^\varepsilon/\varepsilon}\omega) \right] \rho_n(X_r^\varepsilon) dr \\ &\quad + \int_0^t \mathbf{a}_{ij}(\tau_{X_r^\varepsilon/\varepsilon}\omega) n \mathbb{I}_{[\frac{1}{n}, \frac{2}{n}]}(X_r^\varepsilon) dr \end{aligned}$$

From [4, Prop 1.19],  $\int_0^t \mathbf{a}_{ij}(\tau_{X_r^\varepsilon/\varepsilon}\omega) n \mathbb{I}_{[\frac{1}{n}, \frac{2}{n}]}(X_r^\varepsilon) dr$  converges almost surely towards  $\int_0^t \mathbf{a}_{1j}(\tau_{X_r^\varepsilon/\varepsilon}\omega) dK_r^\varepsilon$  as  $n \rightarrow \infty$ . in such a way that applying the Fatou Lemma as  $n \rightarrow +\infty$  in (51) yields

$$(53) \quad \forall 0 \leq s, t \leq T, \quad \bar{\mathbb{E}}^{\varepsilon*} \left[ \exp \left( \left| \kappa (A_t^\varepsilon - A_s^\varepsilon) + \kappa \int_s^t \mathbf{a}_{1j}(\tau_{X_r^\varepsilon/\varepsilon}\omega) dK_r^\varepsilon \right| \right) \right] \leq 2 \exp(C \kappa^2(t-s))$$

where we have set  $A_t^{\varepsilon, j} = \int_0^t \left[ \frac{1}{\varepsilon} \mathbf{b}_j(\tau_{X_r^\varepsilon/\varepsilon}\omega) - \partial_{x_i} V(X_r^\varepsilon) \mathbf{a}_{ij}(\tau_{X_r^\varepsilon/\varepsilon}\omega) \right] dr$ .

**Proposition 6.4.** *We have the following estimate of the modulus of continuity*

$$(54) \quad \bar{\mathbb{E}}^{\varepsilon*} \left( \sup_{|t-s| \leq \delta; 0 \leq s, t \leq T} \left| A_t^{\varepsilon, j} - A_s^{\varepsilon, j} + \int_s^t \mathbf{a}_{1j}(\tau_{X_r^\varepsilon/\varepsilon}\omega) dK_r^\varepsilon \right| \right) \leq C \sqrt{\delta} \ln(\delta^{-1}),$$

for some constant  $C$  that only depends on  $T, \Lambda$ .

**Proof.** The result follows from Propositions (53) and 6.1 (with  $p(t) = \sqrt{t}$ ,  $\psi(t) = e^t - 1$  and  $\psi^{-1}(t) = \ln(t + 1)$ ). Since that step of the proof does not differ from [17, Ch. 3, Th 3.5], the reader is referred to this paper for further details.  $\square$

Let us now introduce the space  $D([0, T]; \mathbb{R}_+)$  of nonnegative right-continuous functions with left limits on  $[0, T]$  equipped with the S-topology of Jakubowski (see Appendix C). The space  $C([0, T]; \bar{D})$  is equipped with the sup-norm topology.

**Proposition 6.5.** *Under the law  $\bar{\mathbb{P}}^{\varepsilon^*}$ , the family of processes  $(X^\varepsilon)_\varepsilon$  is tight in  $C([0, T]; \bar{D})$ , and the family of processes  $(K^\varepsilon)_\varepsilon$  is tight in  $D([0, T]; \mathbb{R}_+)$ .*

**Proof.** Let us first investigate the tightness of  $X^\varepsilon$ . The tightness of the process  $A_t^{j,\varepsilon} + \int_0^t \mathbf{a}_{1j}(\tau_{X_r^\varepsilon/\varepsilon}\omega) dK_r^\varepsilon$  in  $C([0, T]; \mathbb{R})$  results from Proposition 6.4. Then, observe that

$$X_t^{j,\varepsilon} = x_j + A_t^{j,\varepsilon} + \int_0^t \mathbf{a}_{1j}(\tau_{X_r^\varepsilon/\varepsilon}\omega) dK_r^\varepsilon + \int_0^t (\gamma_j - \mathbf{a}_{1j})(\tau_{X_r^\varepsilon/\varepsilon}\omega) dK_r^\varepsilon + \int_0^t \boldsymbol{\sigma}_{ji}(\tau_{X_r^\varepsilon/\varepsilon}\omega) dB_r^{*i}.$$

Hence, the tightness of  $X^\varepsilon$  results from the tightness of the martingale part, which follows from the boundedness of  $\boldsymbol{\sigma}$  and the Kolmogorov criterion, and from the tightness of  $\int_0^t (\gamma_j - \mathbf{a}_{1j})(\tau_{X_r^\varepsilon/\varepsilon}\omega) dK_r^\varepsilon$ , which follows from Theorem 5.3 (recall that  $\mathbb{M}_1[\gamma_j - \mathbf{a}_{1j}] = \mathbb{M}_1[D_i \mathbf{H}_{ji}] = 0$ ).

Let us now investigate the tightness of the family  $(K^\varepsilon)_\varepsilon$ . From Lemma 5.1, we have  $\bar{\mathbb{E}}^{\varepsilon^*}[K_T^\varepsilon] = T \int_{\partial D} e^{-2V(x)} dx$ . Theorem C.2 ensures that  $(K^\varepsilon)_\varepsilon$  is tight in  $D([0, T]; \mathbb{R}_+)$  (remind that  $K^\varepsilon$  is increasing).  $\square$

To sum up, under  $\bar{\mathbb{P}}^{\varepsilon^*}$ , the family  $(X^\varepsilon, K^\varepsilon)_\varepsilon$  is tight in  $C([0, T]; \bar{D}) \times D([0, T]; \mathbb{R}_+)$  equipped with the product topology. From (13), the family is tight in  $C([0, T]; \bar{D}) \times D([0, T]; \mathbb{R}_+)$  under  $\bar{\mathbb{P}}^\varepsilon$ . The purpose of the next section is to identify the limit of each converging subsequence.

## 7 Homogenization

We now focus on the proof of Theorem 1.2 in the case  $\chi(x) = e^{-2V(x)}$ . The first step consists in introducing the function  $\mathbf{u}_\lambda \in L^2(\Omega; \mathbb{R}^d)$ , the entries of which are given, for  $j = 1, \dots, d$ , by the solution  $\mathbf{u}_\lambda^j$  to the resolvent equation  $\lambda \mathbf{u}_\lambda^j - \mathbf{L} \mathbf{u}_\lambda^j = \mathbf{b}_j$ . Let  $\boldsymbol{\zeta} \in L^2(\Omega; \mathbb{R}^{d \times d})$  be defined by  $\zeta_{ij} = \lim_{\lambda \rightarrow 0} D_i \mathbf{u}_\lambda^j$  (see Proposition 4.3).

Since,  $\mu$ -almost surely, the function  $\phi : x \mapsto \mathbf{u}_\lambda(\tau_x \omega)$  satisfies  $\lambda \phi - L^\omega \phi = \mathbf{b}(\tau_x \omega)$  on  $\mathbb{R}^d$ , the function  $u_\lambda(\cdot, \omega)$  is smooth (see [8, Th. 6.17]). Hence, we can apply the Itô formula to the function  $x \mapsto x + \mathbf{u}_\lambda(\tau_x \omega)$  and obtain

$$\begin{aligned} (55) \quad X_t^\varepsilon &= x - \varepsilon (\mathbf{u}_{\varepsilon^2}(\tau_{X_t^\varepsilon/\varepsilon}\omega) - \mathbf{u}_{\varepsilon^2}(\tau_{X_0^\varepsilon/\varepsilon}\omega)) + \varepsilon \int_0^t \mathbf{u}_{\varepsilon^2}(\tau_{X_r^\varepsilon/\varepsilon}\omega) dr \\ &\quad + \int_0^t (\mathbf{I} + D \mathbf{u}_{\varepsilon^2}^*) \boldsymbol{\gamma}(\tau_{X_r^\varepsilon/\varepsilon}\omega) dK_r^\varepsilon + \int_0^t (\mathbf{I} + D \mathbf{u}_{\varepsilon^2}^*) \boldsymbol{\sigma}(\tau_{X_r^\varepsilon/\varepsilon}\omega) dB_r. \\ &\equiv x - G_t^{1,\varepsilon} + G_t^{2,\varepsilon} + G_t^{3,\varepsilon} + M_t^\varepsilon. \end{aligned}$$

Proposition 7.1 below investigates the convergence of the various terms involved in (55) (see the proof at the end of this section).

**Proposition 7.1.** *For each subsequence of the family  $(X^\varepsilon, K^\varepsilon)_\varepsilon$ , we can extract a subsequence (still indexed with  $\varepsilon > 0$ ) such that:*

1) under  $\bar{\mathbb{P}}^\varepsilon$ , the family of processes  $(X^\varepsilon, M^\varepsilon, K^\varepsilon)_\varepsilon$  converges in law in  $C([0, T]; \bar{D}) \times C([0, T]; \mathbb{R}^d) \times D([0, T]; \mathbb{R}_+)$  towards  $(\bar{X}, \bar{M}, \bar{K})$ , where  $\bar{M}$  is a centered non-standard  $d$ -dimensional Brownian motion with covariance

$$\bar{A} = \mathbb{M}[(\mathbf{I} + \zeta^*)\mathbf{a}(\mathbf{I} + \zeta)]$$

and  $\bar{K}$  is a right-continuous increasing process.

2) the finite-dimensional distributions of the families  $(G_t^{1,\varepsilon})_\varepsilon$ ,  $(G_t^{2,\varepsilon})_\varepsilon$  and  $(G_t^{3,\varepsilon} - \bar{\Gamma}K_t^\varepsilon)_\varepsilon$  converge towards 0 in  $\bar{\mathbb{P}}^\varepsilon$ -probability, that is for each  $t \in [0, T]$

$$\forall \delta > 0, \quad \lim_{\varepsilon \rightarrow 0} \bar{\mathbb{P}}^\varepsilon \left( |G_t^{i,\varepsilon}| > \delta \right) = 0 \quad (i = 1, 2), \quad \lim_{\varepsilon \rightarrow 0} \bar{\mathbb{P}}^\varepsilon \left( |G_t^{3,\varepsilon} - \bar{\Gamma}K_t^\varepsilon| > \delta \right) = 0.$$

**Conclusion.** We can find a countable subset  $\mathcal{S} \subset [0, T[$  (see Theorem C.2) such that the finite-dimensional distributions of the process  $(X^\varepsilon, M^\varepsilon, K^\varepsilon)_\varepsilon$  converge along  $[0, T] \setminus \mathcal{S}$ . So we can pass to the limit in (55) along  $s, t \in [0, T] \setminus \mathcal{S}$  ( $s < t$ ), and this leads to

$$(56) \quad \bar{X}_t = \bar{X}_s + \bar{A}^{1/2}(\bar{B}_t - \bar{B}_s) + \bar{\Gamma}(\bar{K}_t - \bar{K}_s).$$

Since (56) is valid for  $s, t \in [0, T] \setminus \mathcal{S}$  (note that this set is dense and contains  $T$ ) and since the processes are at least right continuous, (56) remains valid on the whole interval  $[0, T]$ . As a by-product,  $\bar{K}$  is continuous and the convergence of  $(X^\varepsilon, M^\varepsilon, K^\varepsilon)_\varepsilon$  actually holds in the space  $C([0, T]; \bar{D}) \times C([0, T]; \mathbb{R}^d) \times C([0, T]; \mathbb{R}_+)$  (see Lemma C.3).

It remains to prove that  $\bar{K}$  is associated to  $\bar{X}$  in the sense of the Skorokhod problem, that is to establish that  $\{\text{Points of increase of } \bar{K}\} \subset \{t; \bar{X}_t^1 = 0\}$  or  $\int_0^T \bar{X}_r^1 d\bar{K}_r = 0$ . This results from the fact that  $\forall \varepsilon > 0 \int_0^T X_r^{1,\varepsilon} dK_r^\varepsilon = 0$  and Lemma C.4. Since uniqueness in law holds for the solution  $(\bar{X}, \bar{K})$  of Equation (56) (see [22]), we have proved that each converging subsequence of the family  $(X^\varepsilon, K^\varepsilon)_\varepsilon$  converges in law in  $C([0, T]; \bar{D} \times \mathbb{R}_+)$  as  $\varepsilon \rightarrow 0$  towards the same limit (the unique solution  $(\bar{X}, \bar{K})$  of (56)). As a consequence, the whole sequence  $(X^\varepsilon, K^\varepsilon)_\varepsilon$  is converging.  $\square$

**Proof of Proposition 7.1.** 1) The tightness of  $(X^\varepsilon, K^\varepsilon)$  results from Section 6. To prove the tightness of the martingales  $(M^\varepsilon)_\varepsilon$ , it suffices to prove the tightness of the brackets  $\langle M^\varepsilon \rangle_\varepsilon$ , which are given by

$$\langle M^\varepsilon \rangle_t = \int_0^t (\mathbf{I} + D\mathbf{u}_{\varepsilon^2})\mathbf{a}(\mathbf{I} + D\mathbf{u}_{\varepsilon^2})^*(\tau_{X_{\bar{r}/\varepsilon}\omega}) dr.$$

Proposition 4.3 and Theorem 5.4 lead to  $\langle M^\varepsilon \rangle_t \rightarrow \bar{A}t$  in probability in  $C([0, T]; \mathbb{R}^{d \times d})$  where  $\bar{A} = \mathbb{M}[(\mathbf{I} + D\mathbf{u}_{\varepsilon^2})\mathbf{a}(\mathbf{I} + D\mathbf{u}_{\varepsilon^2})^*]$ . The martingales  $(M^\varepsilon)_\varepsilon$  thus converge in law in  $C([0, T]; \mathbb{R}^d)$  towards a centered Brownian motion with covariance matrix  $\bar{A}$  (see [9]).

2) the convergence of  $(G^{i,\varepsilon})_\varepsilon$  ( $i = 1, 2$ ) follows (18), Lemma 5.1 Theorem 5.4:

$$\lim_{\varepsilon \rightarrow 0} \bar{\mathbb{E}}^{\varepsilon^*} \left[ |\varepsilon \mathbf{u}_{\varepsilon^2}(\tau_{X_{\bar{r}/\varepsilon}\omega})|^2 + \left| \int_0^t \varepsilon \mathbf{u}_{\varepsilon^2}(\tau_{X_{\bar{r}/\varepsilon}\omega}) dr \right|^2 \right] \leq (1+t) \lim_{\varepsilon \rightarrow 0} (\varepsilon^2 |\mathbf{u}_{\varepsilon^2}|_2^2) = 0.$$

From (13), we deduce  $\bar{\mathbb{P}}_x^\varepsilon \left( |G_t^{1,\varepsilon}| + |G_t^{2,\varepsilon}| > \delta \right) = 0$  for any  $\delta > 0$ . The convergence of  $(G^{3,\varepsilon})_\varepsilon$  results from Theorem 5.4 and (20).  $\square$

## Replication method

Let us use the shorthands  $C_D$  and  $C_+$  to denote respectively the spaces  $C([0, T], \bar{D})$  and  $C([0, T], \mathbb{R}_+)$ .

**Remark 7.2.** For the sake of clarity, we sum up the results obtained previously. We have proved the convergence, as  $\varepsilon \rightarrow 0$ , of  $\mathbb{E}^\varepsilon[F(X^\varepsilon, K^\varepsilon)]$  towards  $\mathbb{E}[F(\bar{X}, \bar{K})]$ , for each continuous bounded function  $F : C_D \times C_+ \rightarrow \mathbb{R}$ . Here  $\mathbb{E}$  denotes the expectation with respect to the law  $\mathbb{P}$  of the process  $(\bar{X}, \bar{K})$ , solving the RSDE (56) with initial distribution  $\mathbb{P}(\bar{X}_0 \in dx) = e^{-2V(x)} dx$ . From [22], the law  $\mathbb{P}$  coincides with  $\int_{\bar{D}} \bar{\mathbb{P}}_x(\cdot) e^{-2V(x)} dx$  where  $\bar{\mathbb{P}}_x$  denotes the law of  $(\bar{X}, \bar{K})$  solving (56) and starting from  $x \in \bar{D}$ .

**Remark 7.3.** In the classical framework of Brownian motion driven SDE (i.e. without reflection term in (1)), it is plain to see that, under the invariant measure, the convergence in law of the process  $X^\varepsilon$  towards a Brownian motion (the so-called annealed convergence) implies the convergence in probability of the law of  $X^\varepsilon$ , as stated in Theorem 1.2. Indeed, the convergence in law towards 0 of the correctors (the terms  $G^{1,\varepsilon}, G^{2,\varepsilon}$  in (55)) implies their convergence in probability towards 0. Moreover the convergence in probability of the law of the martingale term  $M^\varepsilon$  in (55) is obvious since the brackets converge in probability (it does not cost much more to prove that they converge almost surely). In our case, we have proved that the additional term  $G^{3,\varepsilon}$  corresponding to the local time converge, under the annealed law  $\mathbb{P}^\varepsilon$ , towards a random variable  $\bar{\Gamma} \bar{K}$ , but there is no obvious way to switch from convergence in law to convergence in probability. That is the purpose of the computations below.

So we have to establish the convergence in  $\mu \otimes \chi$ -probability of  $\mathbb{E}_x^\varepsilon[F(X^\varepsilon, K^\varepsilon)]$  towards  $\mathbb{E}_x[F(\bar{X}, \bar{K})]$  for each continuous bounded function  $F : C_D \times C_+ \rightarrow \mathbb{R}$ . To that purpose, we will prove

$$(57) \quad \lim_{\varepsilon \rightarrow 0} \mathbb{M}_D^* \left[ \left( \mathbb{E}_x^\varepsilon[F(X^\varepsilon, K^\varepsilon)] - \mathbb{E}_x[F(\bar{X}, \bar{K})] \right)^2 \right] = 0.$$

The expression  $\mathbb{M}_D^* \left[ \left( \mathbb{E}_x^\varepsilon[F(X^\varepsilon, K^\varepsilon)] - \mathbb{E}_x[F(\bar{X}, \bar{K})] \right)^2 \right]$  expands as

$$\mathbb{M}_D^* \left[ \left( \mathbb{E}_x^\varepsilon[F(X^\varepsilon, K^\varepsilon)] \right)^2 \right] - 2\mathbb{M}_D^* \left[ \mathbb{E}_x^\varepsilon[F(X^\varepsilon, K^\varepsilon)] \mathbb{E}_x[F(\bar{X}, \bar{K})] \right] + \mathbb{M}_D^* \left[ \left( \mathbb{E}_x[F(\bar{X}, \bar{K})] \right)^2 \right].$$

Clearly, the mapping  $x \mapsto \mathbb{E}_x[F(\bar{X}, \bar{K})]$  is bounded and continuous. So

$$G : (x, k) \in C_D \times C_+ \mapsto F(x, k) \mathbb{E}_{x(0)}[F(\bar{X}, \bar{K})]$$

is bounded and continuous. From Remark 7.2, the following convergence holds as  $\varepsilon \rightarrow 0$

$$\mathbb{M}_D^* \left[ \mathbb{E}_x^\varepsilon[F(X^\varepsilon, K^\varepsilon)] \mathbb{E}_x[F(\bar{X}, \bar{K})] \right] = \mathbb{E}^\varepsilon[G(X^\varepsilon, K^\varepsilon)] \rightarrow \mathbb{E}[G(\bar{X}, \bar{K})] = \mathbb{M}_D^* \left[ \left( \mathbb{E}_x[F(\bar{X}, \bar{K})] \right)^2 \right]$$

So, we just have to prove that

$$(58) \quad \mathbb{M}_D^* \left[ \left( \mathbb{E}_x^\varepsilon[F(X^\varepsilon, K^\varepsilon)] \right)^2 \right] \rightarrow \mathbb{M}_D^* \left[ \left( \mathbb{E}_x[F(\bar{X}, \bar{K})] \right)^2 \right] \quad \text{as } \varepsilon \rightarrow 0.$$

We consider 2 independant Brownian motions  $(B^1, B^2)$  and solve (1) for each Brownian motion. This provides two independant couple of processes (with respect to the randomness of the Brownian motion)  $(X^{\varepsilon,1}, K^{\varepsilon,1})$  and  $(X^{\varepsilon,2}, K^{\varepsilon,2})$ . Furthermore, we have

$$\mathbb{M}_D^* \left[ \left( \mathbb{E}_x^\varepsilon[F(X^\varepsilon, K^\varepsilon)] \right)^2 \right] = \mathbb{M}_D^* \left[ \mathbb{E}_{xx}^\varepsilon [F(X^{\varepsilon,1}, K^{\varepsilon,1}) F(X^{\varepsilon,2}, K^{\varepsilon,2})] \right]$$

where  $\mathbb{P}_{xx}^\varepsilon$  denotes the law of  $(X^{\varepsilon,1}, K^{\varepsilon,1}, X^{\varepsilon,2}, K^{\varepsilon,2})$  when both  $X^{\varepsilon,1}$  and  $X^{\varepsilon,2}$  start from  $x \in \bar{D}$ , and  $\mathbb{E}_{xx}^\varepsilon$  the corresponding expectation. Under  $\mathbb{M}_D^* \mathbb{P}_{xx}^\varepsilon$ , the results of Section 5 and 6 remain valid since the marginal laws of each couple of processes coincide with  $\bar{\mathbb{P}}_x$ . So we can repeat the arguments of Section

7 and prove that the processes  $(X^{\varepsilon,1}, K^{\varepsilon,1}, X^{\varepsilon,2}, K^{\varepsilon,2})_\varepsilon$  converge in law in  $C_D \times C_+ \times C_D \times D_+$ , under  $\mathbb{M}_D^* \mathbb{E}_{xx}^\varepsilon$ , towards a process  $(\bar{X}^1, \bar{K}^1, \bar{X}^2, \bar{K}^2)$  satisfying:

$$(59) \quad \forall t \in [0, T], \quad \bar{X}_t^1 = \bar{X}_0^1 + A^{1/2} \bar{B}_t^1 + \bar{K}_t^1, \quad \bar{X}_t^2 = \bar{X}_0^2 + A^{1/2} \bar{B}_t^2 + \bar{K}_t^2,$$

where  $(\bar{B}^1, \bar{B}^2)$  is a standard  $2d$ -dimensional Brownian motion and  $\bar{K}^1, \bar{K}^2$  are the local times respectively associated to  $\bar{X}^1, \bar{X}^2$ . Let  $\bar{\mathbb{P}}$  denote the law of  $(\bar{X}^1, \bar{K}^1, \bar{X}^2, \bar{K}^2)$  and  $\bar{\mathbb{P}}_{xx}$  the law of  $(\bar{X}^1, \bar{K}^1, \bar{X}^2, \bar{K}^2)$  solution of (59) where both  $\bar{X}^1$  and  $\bar{X}^2$  start from  $x \in \bar{D}$ . Under  $\bar{\mathbb{P}}$ , the initial distribution of the limiting process is given by  $\bar{P}(\bar{X}_0^1 \in dx, \bar{X}_0^2 \in dy) = \delta_x(dy) e^{-2V(x)} dx$ . To obtain (58), it just remains to remark that

$$\begin{aligned} \bar{\mathbb{E}}[F(\bar{X}^1, \bar{K}^1)F(\bar{X}^2, \bar{K}^2)] &= \int_{\bar{D}} \bar{\mathbb{E}}_{xx}[F(\bar{X}^1, \bar{K}^1)F(\bar{X}^2, \bar{K}^2)] e^{-2V(x)} dx \\ &= \int_{\bar{D}} \bar{\mathbb{E}}_x[F(\bar{X}^1, \bar{K}^1)] \bar{\mathbb{E}}_x[F(\bar{X}^2, \bar{K}^2)] e^{-2V(x)} dx, \end{aligned}$$

since, under  $\bar{\mathbb{P}}_{xx}$ , the couples  $(\bar{X}^1, \bar{K}^1)$  and  $(\bar{X}^2, \bar{K}^2)$  are adapted to the filtrations generated respectively by  $\bar{B}^1$  and  $\bar{B}^2$  and are therefore independent.  $\square$

## Appendix

### A Proofs of section 4

We stick to the notations introduced in Section 4.

**Proof of Proposition 4.8.** The strategy is based on the well-known method of difference quotients (see [8, Sect. 7.11 & Th. 8.8] for instance, on which the proof below is based). The main properties of difference quotients in random media are summarized below (see e.g. [20, Sect. 5]):

i) for  $j = 2, \dots, d, r \in \mathbb{R} \setminus \{0\}$  and  $\mathbf{g} \in \mathbb{C}_c(\Omega_+)$ , we define

$$\Delta_r^j \mathbf{g}(x_1, \omega) = \frac{1}{r} (\mathbf{g}(x_1, \tau_{re_j} \omega) - \mathbf{g}(x_1, \omega)).$$

ii) for each  $r \in \mathbb{R} \setminus \{0\}$  and  $\mathbf{g} \in \mathbb{C}_c(\Omega_+)$ , we define

$$\Delta_r^1 \mathbf{g} = \frac{1}{r} (\mathbf{g}(x_1 + r, \omega) - \mathbf{g}(x_1, \omega)).$$

iii) for any  $j = 1, \dots, d, r \in \mathbb{R} \setminus \{0\}$  and  $\mathbf{g}, \mathbf{h} \in \mathbb{C}_c(\Omega_+)$ , the discrete integration by parts holds

$$\int_{\Omega^+} \Delta_r^j \mathbf{g} \mathbf{h} d\mu^+ = - \int_{\Omega^+} \mathbf{g} \Delta_{-r}^j \mathbf{h} d\mu^+$$

provided that  $r$  is small enough to ensure  $\Delta_r^j \mathbf{g}, \Delta_r^j \mathbf{h} \in \mathbb{C}_c(\Omega_+)$ .

iv) for any  $j = 1, \dots, d, r \in \mathbb{R} \setminus \{0\}$  and  $\mathbf{g} \in \mathbb{C}_c(\Omega_+)$  such that  $\Delta_r^j \mathbf{g} \in \mathbb{C}_c(\Omega_+)$ , we have

$$\int_{\Omega^+} |\Delta_r^j \mathbf{g}|^2 d\mu^+ \leq \int_{\Omega^+} |\partial_j \mathbf{g}|^2 d\mu^+.$$

Up to the end of the proof, we omit the parameter  $\lambda$  from the notations and note  $\mathbf{u}$  for  $G_\lambda P^* \varphi$ . The strategy consists in differentiating the resolvent equation  $B_\lambda(\mathbf{u}, \cdot) = P^* \varphi(\cdot)$  to prove that the derivatives

satisfy some equation of the same type. For  $p = 2, \dots, d$ , it raises no difficulty to adapt the method explained in [20, Sect. 5] and prove that the "tangential derivatives"  $\partial_p \mathbf{u}$  belongs to  $\mathbb{W}^1$  and solves the equation

$$(60) \quad B_\lambda(\partial_p \mathbf{u}, \cdot) = P^* D_p \varphi(\cdot) - F_p(\cdot),$$

where  $F_p : \mathbb{W}^1 \rightarrow \mathbb{R}$  is defined by

$$F_p(\mathbf{g}) = (1/2) \int_{\Omega^+} (D_p \mathbf{a}^+ + D_p \mathbf{H}^+)_{ij} \partial_i \mathbf{u} \partial_j \mathbf{g} \, d\mu^+.$$

In particular,  $\partial_{ij} \mathbf{u} \in L^2(\Omega^+; \mu^+)$  for  $(i, j) \neq (1, 1)$ . We let the reader check the details.

The main difficulty lies in the "normal derivative"  $\partial_1 \mathbf{u}$ : we have to prove that  $\partial_1 \mathbf{u}$  is weakly differentiable. Actually, it just remains to prove that there exists a function  $\partial_{11}^2 \mathbf{u} \in L^2(\Omega^+; \mu^+)$  such that  $\forall \mathbf{g} \in \mathbb{C}_c(\Omega_+)$ :

$$(61) \quad \int_{\Omega^+} \partial_{11}^2 \mathbf{u} \mathbf{g} \, d\mu^+ = - \int_{\Omega^+} \partial_1 \mathbf{u} \partial_1 \mathbf{g} \, d\mu^+.$$

To that purpose, we have to plug a generic function  $\mathbf{g} \in \mathbb{C}_c(\Omega_+)$  into the resolvent equation (35). The boundary terms  $(P^* \varphi, \mathbf{g}) = (\varphi, \mathbf{g}(0, \cdot))_2$  and  $\lambda(P\mathbf{u}, P\mathbf{g})_2$  vanish. We obtain (remind that  $\mathbf{a}_{11} = 1$  and  $\bar{\mathbf{H}}_{11} = 0$ )

$$\begin{aligned} \int_{\Omega^+} \partial_1 \mathbf{u} \partial_1 \mathbf{g} \, d\mu^+ &= - \sum_{(i,j) \neq (1,1)} \int_{\Omega^+} (\mathbf{a}^+ + \bar{\mathbf{H}}^+)_{ij} \partial_i \mathbf{u} \partial_j \mathbf{g} \, d\mu^+ \\ &= \sum_{(i,j) \neq (1,1)} \int_{\Omega^+} \partial_j (\mathbf{a}^+ + \bar{\mathbf{H}}^+)_{ij} \partial_i \mathbf{u} \mathbf{g} \, d\mu^+ + \sum_{(i,j) \neq (1,1)} \int_{\Omega^+} (\mathbf{a} + \bar{\mathbf{H}})_{ij} \partial_{ij}^2 \mathbf{u} \mathbf{g} \, d\mu^+ \end{aligned}$$

Since  $\partial_{ij} \mathbf{u} \in L^2(\Omega^+; \mu^+)$  for  $(i, j) \neq (1, 1)$ , we deduce that

$$\int_{\Omega^+} \partial_1 \mathbf{u} \partial_1 \mathbf{g} \, d\mu^+ \leq C \left( \int_{\Omega^+} \mathbf{g}^2 \, d\mu^+ \right)^{1/2}$$

for some positive constant  $C$ . So the mapping  $\mathbf{g} \in \mathbb{C}_c(\Omega_+) \mapsto \int_{\Omega^+} \partial_1 \mathbf{u} \partial_1 \mathbf{g} \, d\mu^+$  is  $L^2(\Omega^+; \mu^+)$ -continuous and there exists a unique function denoted by  $\partial_{11}^2 \mathbf{u}$  such that (61) holds. As a consequence,  $\partial_1 \mathbf{u}$  is weakly differentiable, that is  $\mathbf{u} \in \mathbb{W}^2$ . Note that (60) only involves the functions  $\mathbf{a}, \bar{\mathbf{H}}, \varphi$  and their derivatives in such a way that we can iterate the argument in differentiating (60) and so on. So we complete the proof recursively.  $\square$

**Proof of Proposition 4.9.** The function  $\mathbf{u}$  still stands for  $G_\lambda P^* \varphi$ . To begin with, we state the following lemma (see the proof thereafter)

**Lemma A.1.** *For each function  $\varrho \in C_c^\infty(\bar{D})$ ,  $\chi \in \mathcal{C}$  and  $\mathbf{v} \in \mathbb{W}^1$ , we define  $\tilde{v}_\omega : (x_1, y) \in \bar{D} \mapsto \mathbf{v}(x_1, \tau_{(0,y)} \omega)$ . We have:*

$$\begin{aligned} 1) \quad \mathbb{M}[\chi(\omega) \int_{\bar{D}} \tilde{v}_\omega \varrho(x) \, dx] &= \int_{\Omega^+} \mathbf{v}(\chi \odot \check{\varrho}) \, d\mu^+, \\ 2) \quad \mathbb{M}[\chi(\omega) \int_{\bar{D}} (\mathbf{a} + \bar{\mathbf{H}})_{ij} (\tau_x \omega) \partial_{x_i} \tilde{v}_\omega(x) \partial_{x_j} \varrho(x) \, dx] &= \int_{\Omega^+} (\mathbf{a}^+ + \bar{\mathbf{H}}^+)_{ij} \partial_i \mathbf{v} \partial_j (\chi \odot \check{\varrho}) \, d\mu^+ \end{aligned}$$

where  $\check{\varrho}(x_1, y) = \varrho(x_1, -y)$  for each  $(x_1, y) \in \bar{D}$ .



We first prove that  $\mu$  a.s., the function  $x \mapsto \tilde{u}_\omega(x)$  admits weak derivatives in the classical sense on  $\bar{D}$ . So we consider  $\varrho \in C_c^\infty(D)$  and we want to compute  $\int_D \tilde{u}_\omega \partial_{x_i} \varrho(x) dx$  for  $i = 1, \dots, d$ . To that purpose, we consider  $\chi \in \mathcal{C}$ . By using Lemma A.1 above and the relation  $\partial_i(\chi \odot \check{\varrho}) = \chi \odot (\partial_{x_i} \check{\varrho})$ , we have

$$\begin{aligned} \mathbb{M}\left[\chi(\omega) \int_{\bar{D}} \tilde{u}_\omega \partial_{x_i} \varrho(x) dx\right] &= \int_{\Omega^+} \mathbf{u}(\chi \odot (\partial_{x_i} \check{\varrho})) d\mu^+ = \int_{\Omega^+} \mathbf{u} \partial_i(\chi \odot \check{\varrho}) d\mu^+ \\ &= - \int_{\Omega^+} \partial_i \mathbf{u}(\chi \odot \check{\varrho}) d\mu^+ = -\mathbb{M}\left[\chi(\omega) \int_{(x_1, y) \in \bar{D}} \partial_i \mathbf{u}(x_1, \tau_{(0, y)} \omega) \varrho(x_1, y) dx_1 dy\right] \end{aligned}$$

It follows that  $\mu$  a.s.

$$\int_{\bar{D}} \tilde{u}_\omega(x) \partial_{x_i} \varrho(x) dx = - \int_{(x_1, y) \in \bar{D}} \partial_i \mathbf{u}(x_1, \tau_{(0, y)} \omega) \varrho(x_1, y) dx_1 dy.$$

Hence,  $\mu$  a.s.,  $\tilde{u}_\omega$  admits weak derivatives (in the classical sense) of first order and they are given by  $\partial_{x_i} \tilde{u}_\omega(x) = \partial_i \mathbf{u}(x_1, \tau_{(0, y)} \omega)$  for  $x = (x_1, y) \in \bar{D}$ . Recursively, we can prove that,  $\mu$  a.s.,  $\tilde{u}_\omega$  admits weak derivatives (in the classical sense) of all orders and they are given by  $\partial_{x_{i_1} \dots x_{i_k}}^k \tilde{u}_\omega(x) = \partial_{i_1 \dots i_k}^k \mathbf{u}(x_1, \tau_{(0, y)} \omega)$ , for  $x = (x_1, y) \in \bar{D}$ . Since those weak derivatives are locally square-integrable, we deduce that,  $\mu$  a.s.,  $\tilde{u}_\omega$  belongs to the classical Sobolev space  $H_{loc}^{\infty, 2}(\bar{D})$  (the space of functions having weak derivatives of all orders that are locally square-integrable). From the Sobolev embeddings,  $\tilde{u}_\omega$  belongs to  $C^\infty(\bar{D})$ .

It remains to prove that  $\tilde{u}_\omega$  solves (38). We first prove that  $L^\omega \tilde{u}_\omega = 0$  on  $D$ . To that purpose, let us consider  $\varrho \in C_c^\infty(D)$  and prove that  $\int_{\bar{D}} L^\omega \tilde{u}_\omega(x) \varrho(x) dx = 0$   $\mu$  a.s. by establishing

$$\forall \chi \in \mathcal{C}, \quad \mathbb{M}\left[\chi(\omega) \int_{\bar{D}} L^\omega \tilde{u}_\omega(x) \varrho(x) dx\right] = 0.$$

By integrating by parts and using Lemma A.1, we obtain:

$$\begin{aligned} -\mathbb{M}\left[\chi(\omega) \int_{\bar{D}} L^\omega \tilde{u}_\omega(x) \varrho(x) dx\right] &= (1/2) \mathbb{M}\left[\chi(\omega) \int_{\bar{D}} (\mathbf{a} + \bar{\mathbf{H}})_{ij}(\tau_x \omega) \partial_{x_i} \tilde{u}_\omega(x) \partial_{x_j} \varrho(x) dx\right] \\ &= (1/2) \int_{\Omega^+} (\mathbf{a}^+ + \bar{\mathbf{H}}^+)_{ij} \partial_i \mathbf{u} \partial_j(\chi \odot \check{\varrho}) d\mu^+ \\ &= B_\lambda(\mathbf{u}, \chi \odot \check{\varrho}) = (\chi \odot \check{\varrho}, P^* \varphi) = 0, \end{aligned}$$

the last equality resulting from the relation  $\chi \odot \check{\varrho} \in \mathbb{C}_c(\Omega^+)$ .

Finally, we identify the boundary condition. Let us consider  $\varrho \in C_c^\infty(\bar{D})$ ,  $\chi \in \mathcal{C}$ . By using the Green formula (5) and Lemma A.1, we have

$$\begin{aligned} \mathbb{M}\left[\chi(\omega) \int_{\partial D} (\lambda \tilde{u}_\omega(y) - \gamma_i(\tau_y \omega) \partial_{x_i} \tilde{u}_\omega(y)) \varrho(y) dy\right] \\ &= \mathbb{M}\left[\chi(\omega) \int_{\partial D} \lambda \tilde{u}_\omega(y) \varrho(y) dy\right] + (1/2) \mathbb{M}\left[\chi(\omega) \int_{\bar{D}} (\mathbf{a} + \bar{\mathbf{H}})_{ij}(\tau_x \omega) \partial_{x_i} \tilde{u}_\omega(x) \partial_{x_j} \varrho(x) dx\right] \\ &= \lambda (P\mathbf{u}, P(\chi \odot \check{\varrho}))_2 + (1/2) \int_{\Omega^+} (\mathbf{a}^+ + \bar{\mathbf{H}}^+)_{ij} \partial_i \mathbf{u} \partial_j(\chi \odot \check{\varrho}) d\mu^+ \\ &= B_\lambda(\mathbf{u}, \chi \odot \check{\varrho}) = (\chi \odot \check{\varrho}, P^* \varphi) = \mathbb{M}\left[\chi(\omega) \int_{\partial D} \varphi(\tau_y \omega) \varrho(y) dy\right] \end{aligned}$$

so that  $\mu$  a.s.  $\lambda \tilde{u}_\omega(y) - \gamma_i(\tau_y \omega) \partial_{x_i} \tilde{u}_\omega(y) = \varphi(\tau_y \omega)$ ,  $y \in \partial D$ .  $\square$

**Proof of Lemma A.1.** Consider  $\varrho \in C_c^\infty(\bar{D})$  and  $\chi \in \mathcal{C}$ . We use the invariance of  $\mu$  under translations:

$$\begin{aligned} \mathbb{M}\left[\chi(\omega) \int_{\bar{D}} \tilde{u}_\omega \varrho(x) dx\right] &= \mathbb{M}\left[\int_{(x_1, y) \in \bar{D}} \chi(\omega) \mathbf{u}(x_1, \tau_{(0, y)} \omega) \varrho(x_1, y) dx_1 dy\right] \\ &= \mathbb{M}\left[\int_{(x_1, y) \in \bar{D}} \chi(\tau_{-(0, y)} \omega) \mathbf{u}(x_1, \omega) \varrho(x_1, y) dx_1 dy\right]. \end{aligned}$$

After making the change of variables  $u = -y$ , it is readily seen that the latter quantity exactly matches  $\int_{\Omega^+} \mathbf{u}(\chi \odot \tilde{\varrho}) d\mu^+$ . We let the reader follow the same strategy to prove 2).  $\square$

**Proof of Proposition 4.10.** We adapt the Stampacchia truncation method. More precisely, we introduce a function  $H : \mathbb{R} \rightarrow \mathbb{R}$  of class  $C^1(\mathbb{R})$  such that

$$i) \forall s \in \mathbb{R}, |H'(s)| \leq C, \quad ii) \forall s > 0, H'(s) > 0, \quad iii) \forall s \leq 0, H'(s) = 0.$$

We define  $K = |\varphi|_\infty / \lambda$  and  $\mathbf{u}_\lambda = G_\lambda P^* \varphi$ . We let the reader check that  $H(\mathbf{u}_\lambda - K) \in W^1$ . Then we plug  $\mathbf{g} = H(\mathbf{u}_\lambda - K)$  into (35) and we obtain:

$$\lambda(P\mathbf{u}_\lambda, PH(\mathbf{u}_\lambda - K))_2 + \frac{1}{2} \int_{\Omega^+} \mathbf{a}_{ij}^+ \partial_i \mathbf{u}_\lambda \partial_j \mathbf{u}_\lambda H'(\mathbf{u}_\lambda - K) d\mu^+ = (PH(\mathbf{u}_\lambda - K), \varphi)_2.$$

Hence

$$\lambda(P\mathbf{u}_\lambda - K, H(P\mathbf{u}_\lambda - K))_2 + \frac{1}{2} \int_{\Omega^+} \mathbf{a}_{ij}^+ \partial_i \mathbf{u}_\lambda \partial_j \mathbf{u}_\lambda H'(\mathbf{u}_\lambda - K) d\mu^+ = (H(P\mathbf{u}_\lambda - K), \varphi - \lambda K)_2.$$

Observe that the right-hand side is negative since  $\varphi - \lambda K \leq 0$  and  $H(s) \geq 0$  for any  $s \geq 0$ . Furthermore, the left-hand side is positive since  $H'(s) \geq 0$  and  $sH(s) \geq 0$  for  $s \in \mathbb{R}$ . We deduce that both terms of the left-hand side reduce to 0. This implies  $P\mathbf{u}_\lambda \leq K$  and (from (4))  $|\partial \mathbf{u}_\lambda|^2 H'(\mathbf{u}_\lambda - K) = 0$   $\mu^+$  a.s., in particular  $PH(\mathbf{u}_\lambda - K) = 0$  and  $\frac{1}{2} \int_{\Omega^+} |\partial \mathbf{u}_\lambda H'(\mathbf{u}_\lambda - K)|^2 d\mu^+ = 0$ . In other words,  $N(H(\mathbf{u}_\lambda - K)) = 0$ . So  $H(\mathbf{u}_\lambda - K) = 0$  and this means  $\mathbf{u}_\lambda \leq K$ .  $\square$

**Proof of Lemma 4.1.** The proof is quite similar to that of Proposition 4.10. So we let the reader check the details.  $\square$

## B Proofs of Section 5

**Lemma B.1.** For any functions  $f \in C_c^\infty(D)$  and  $g, h \in C_b^\infty(\bar{D})$ , there exists a unique classical solution  $w_\varepsilon \in C^\infty([0, T]; \bar{D}) \cap C_b^{1,2}$  to the problem

(62)

$$\partial_t w_\varepsilon = \mathcal{L}_V^\varepsilon w_\varepsilon + g w_\varepsilon + h \text{ on } [0, T] \times D, \quad \gamma_i(\tau_{\cdot/\varepsilon} \omega) \partial_{x_i} w_\varepsilon = 0 \text{ on } [0, T] \times \partial D, \quad \text{and } w_\varepsilon(0, \cdot) = f.$$

**Proof.** First of all, we remind the reader that all the coefficients involved in the operator  $\mathcal{L}_V^\varepsilon$  (see the definition of  $\mathcal{L}_V^\varepsilon$  in (6)) belong to  $C_b^\infty(\bar{D})$ . From [14, Th V.7.4], we can find a unique generalized solution  $w'_\varepsilon$  in  $C_b^{1,2}$  to the equation

$$\partial_t w'_\varepsilon = \mathcal{L}_V^\varepsilon w'_\varepsilon + g w'_\varepsilon + \mathcal{L}_V^\varepsilon f + g f + h, \quad w'_\varepsilon(0, \cdot) = 0 \text{ on } D, \quad \gamma(\tau_{\cdot/\varepsilon} \omega) \partial_{x_i} w'_\varepsilon = 0 \text{ on } [0, T] \times \partial D.$$

From [14, IV.§10], we can prove that  $w'_\varepsilon$  is smooth up to the boundary. Then the function

$$w_\varepsilon(t, x) = w'_\varepsilon(t, x) + f(x) \in C^\infty([0, T] \times \bar{D}) \cap C_b^{1,2}$$

is a classical solution to the problem (62).  $\square$

**Lemma B.2.** *The solution  $w_\varepsilon$  given by Lemma B.1 admits the following probabilistic representation:*

$$\forall (t, x) \in [0, T] \times \bar{D}, \quad w_\varepsilon(t, x) = \mathbb{E}_x^{\varepsilon*} \left[ f(X_t^\varepsilon) \exp \left( \int_0^t g(X_r^\varepsilon) dr \right) \right].$$

**Proof.** Applying the Itô formula to the function  $(r, x, y) \mapsto w_\varepsilon(t - r, x) \exp(y)$  and to the triple of processes  $(r, X_r^\varepsilon, \int_0^r g(X_u^\varepsilon) du)$  yields (see for instance [10, Ch. II, Th. 5.1]):

$$\begin{aligned} d \left( w_\varepsilon(t - r, X_r^\varepsilon) \exp \left( \int_0^r g(X_u^\varepsilon) du \right) \right) &= \exp \left( \int_0^r g(X_u^\varepsilon) du \right) \left( (-\partial_t w_\varepsilon + \mathcal{L}_V^\varepsilon w_\varepsilon)(t - r, X_r^\varepsilon) \right) dr \\ &\quad + \exp \left( \int_0^r g(X_u^\varepsilon) du \right) \partial_{x_i} w_\varepsilon(t - r, X_r^\varepsilon) \sigma_{ij}(\tau_{X_r^\varepsilon/\varepsilon} \omega) dB_r^{*j} \\ &\quad + w_\varepsilon(t - r, X_r^\varepsilon) g(X_r^\varepsilon) \exp \left( \int_0^r g(X_u^\varepsilon) du \right) dr \\ &\quad + \exp \left( \int_0^r g(X_u^\varepsilon) du \right) \gamma_i(\tau_{X_r^\varepsilon/\varepsilon} \omega) \partial_{x_i} w_\varepsilon(t - r, X_r^\varepsilon) dK_r^\varepsilon, \end{aligned}$$

that is, by using (62),

$$\begin{aligned} f(X_t^\varepsilon) \exp \left( \int_0^t g(X_u^\varepsilon) du \right) &= w_\varepsilon(t, x) - \int_0^t h(X_r^\varepsilon) \exp \left( \int_0^r g(X_u^\varepsilon) du \right) dr \\ &\quad + \int_0^t \exp \left( \int_0^r g(X_u^\varepsilon) du \right) \partial_{x_i} w_\varepsilon(t - r, X_r^\varepsilon) \sigma_{ij}(\tau_{X_r^\varepsilon/\varepsilon} \omega) dB_r^{*j}. \end{aligned}$$

Since  $w_\varepsilon \in C_b^{1,2}$ , the stochastic integral is a martingale and its expectation reduces to 0. So it just remains to take the expectation in the above expression to prove the Lemma.  $\square$

## C J-topology

We summarized below the main properties of the Jakubowski topology (J-topology) on the space  $D([0, T]; \mathbb{R})$  and refer the reader to [11] for further details and proofs. We denote by  $\mathbb{V}$  the set of functions  $v : [0, T] \rightarrow \mathbb{R}$  with bounded variations. The J-topology is a sequential topology defined by

**Definition C.1.** *A sequence  $(x_n)_n$  in  $D([0, T]; \mathbb{R})$  converges to  $x_0 \in D([0, T]; \mathbb{R})$  if for every  $\varepsilon > 0$ , one can find elements  $(v_{n,\varepsilon})_{n \in \mathbb{N}} \subset \mathbb{V}$  such that*

$$1) \forall n \in \mathbb{N}, \sup_{[0, T]} |x_n - v_{n,\varepsilon}| \leq \varepsilon,$$

$$2) \text{ for each continuous function } f : [0, T] \rightarrow \mathbb{R}, \int_0^T f(r) dv_{n,\varepsilon}(r) \rightarrow \int_0^T f(r) dv_{0,\varepsilon}(r) \text{ as } n \rightarrow +\infty.$$

Gathering [11, Th. 3.8] and [11, Th. 3.10], one can state:

**Theorem C.2.** *Let  $(V_\alpha)_\alpha \subset D([0, T]; \mathbb{R})$  be a family of nondecreasing stochastic processes. Suppose that the family  $(V_\alpha(T))_\alpha$  is tight. Then the family  $(V_\alpha)_\alpha$  is tight for the J-topology. Moreover, there exists a sequence  $(V_n)_n \subset (V_\alpha)_\alpha$ , a nondecreasing right-continuous process  $V_0$  and a countable subset  $C \subset [0, T[$  such that for all finite sequence  $(t_1, \dots, t_p) \subset [0, T] \setminus C$ , the family  $(V_n(t_1), \dots, V_n(t_p))_n$  converges in law towards  $(V_0(t_1), \dots, V_0(t_p))_n$ .*

Equip the set  $\mathbb{V}_c^+([0, T]; \mathbb{R})$  of continuous nondecreasing functions on  $[0, T]$  with the J-topology and  $C([0, T]; \mathbb{R})$  with the sup-norm topology. We claim:

**Lemma C.3.** *Let  $(V_n)_n$  be a sequence in  $\mathbb{V}_c^+$  converging towards  $V_0 \in \mathbb{V}_c^+$ . Then  $(V_n)_n$  converges towards  $V_0$  for the sup-norm topology.*

**Proof** This results from Corollary 2.9 in [11] and the Dini theorem. □

**Lemma C.4.** *The following mapping is continuous*

$$(x, v) \in C([0, T]; \mathbb{R}) \times \mathbb{V}_c^+([0, T]; \mathbb{R}) \mapsto \int_0^\cdot x_r dv(r) \in C([0, T]; \mathbb{R}).$$

**Proof** This results from Lemma C.3 and the continuity of the mapping

$$(x, v) \in C([0, T]; \mathbb{R}) \times \mathbb{V}_c^+([0, T]; \mathbb{R}) \mapsto \int_0^\cdot x_r dv(r) \in C([0, T]; \mathbb{R}),$$

where both  $C([0, T]; \mathbb{R})$  and  $\mathbb{V}_c^+([0, T]; \mathbb{R})$  are equipped with the sup-norm topology. The reader may find a proof of the continuity of the above mapping in the proof of Lemma 3.3 in [18] (remark that the S-topology on  $C([0, T]; \mathbb{R})$  coincides with the sup-norm topology). □

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