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ON THE SPECTRUM OF α -RIGID MAPS

E. H. EL ABDALAOU

ABSTRACT. It is shown that there exists an α -rigid transformation with α less or equal to $\frac{1}{2}$ whose spectrum has Lebesgue component. This answers the question raised by Klemes and Reinhold in [30]. We exhibit also a large class of α -rigid transformations with singular spectrum.

1. INTRODUCTION

In his 1980's paper [27], A. Katok show that the interval exchange maps are not mixing. The proof yields that the interval exchange maps are α -rigid. Later, Veech in [41] investigated the spectral properties of interval exchange maps and showed that almost every minimal interval exchange maps had singular simple spectrum. Before, Oseledets in [21] proved that for an interval exchange transformation the maximal spectral multiplicity is bounded above by $r - 1$ where r is the number of intervals exchanged. Moreover he constructed the first example of a transformation with continuous spectrum and finite multiplicity greater than 1. Since the example is an exchange of 30 intervals, the maximal spectral multiplicity m satisfies $2 \leq m \leq 29$. Robinson [35] constructed ergodic interval exchange transformations with arbitrary finite maximal spectral multiplicity. It follows from [28] that all the examples constructed by Oseledets and Robinson have singular spectrum. However Katok's result remains at the present time, the only universal result about the spectrum of interval exchanges. Following Veech one may asked if there is any other "universal" spectral property satisfied by interval exchange maps. More precisely as in [1] one may asked the following

Question 1. *Does any interval exchange map have singular spectral type?*

The answer to the above question is affirmative in the case of three interval exchange maps. In fact, using the result in [17] one may show that every ergodic interval exchange transformation on three intervals has singular spectrum.

On the other hand, Klemes and Reinhold in [30] obtained that for any $\alpha \in]0, 1[$, there exists an α -rigid rank one transformation with singular spectrum and asked

Question 2. *Does any α -rigid transformation have singular spectral type?*

We point out that the positive answer to Klemes-Reinhold question yields a positive answer to the question 1. Unfortunately, in this paper we shall proved that the Mathew-Nadkarni transformation is $\frac{1}{2}$ -rigid. We recall that Mathew-Nadkarni introduced this transformation in 1984 in [19] to answer the question raised by

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Helson and Parry [18] of whether there exists an ergodic measure preserving transformation which has a Lebesgue component in its spectrum with finite non-zero multiplicity. Helson and Parry also mentioned the problem, attributed to Banach, whether there exists an ergodic measure preserving transformation on a finite measure space whose spectrum is simple Lebesgue. In [36], Rokhlin mentioned the problem of finding an ergodic measure preserving transformation on a finite measure space whose spectrum is of Lebesgue type with finite multiplicity. Another contribution to the Banach-Rokhlin question is due to M. Queffélec [32] who proved that the spectrum of the Rudin-Shapiro substitution has Lebesgue component of multiplicity two. It turns out that the Rudin-Shapiro substitution is $\frac{1}{2}$ -rigid. We will present other example of dynamical systems arising from substitutions with Lebesgue component of multiplicity two whose rigidity constant is less than $\frac{1}{2}$.

It is an easy exercise to prove that α -rigid transformations are not mixing. Since α -rigidity does not imply that the spectrum is singular as we shall see, the absence of mixing does not imply that the spectrum is singular. Let us mention that any aperiodic measure-preserving transformation can be realized as an exchange of an infinite number of intervals [3].

Our work on the question of Klemes and Reinhold includes a survey of the results of Dekking-Keane [8] and Queffélec [33]. F. M. Dekking and M. Keane showed that the dynamical systems arising from substitutions are not mixing. The ingredients of the proof established that the dynamical systems arising from substitutions are α -rigid. The procedure to check the constant α will be presented in the last section without proof. Queffélec showed that the substitution which gives rise to the Rudin-Shapiro sequence has Lebesgue component of multiplicity two together with a discrete component. (Kamae [25] had earlier shown that the correlation measure of the Rudin-Shapiro sequence is Lebesgue).

In this paper we shall exhibit also a large class of α -rigid transformations with singular spectrum. More precisely we shall prove that the transformation satisfying the restricted Beurling condition is singular. We say that the transformation T satisfy the **restricted Beurling condition** if the following holds

$$\left\{ \sum_{i \in \mathbb{Z}} a_i U_T^i : a_i > 0 \text{ for some } i \text{ and } \sum_{n \geq 0} \frac{\log(\sum_{k \leq -n} a_k^2)}{n^2} = -\infty \right\} \\ \cap \left(\overline{\{U_T^n, n \in \mathbb{Z}\}}^W \setminus (\{U_T^n, n \in \mathbb{Z}\}) \right) \neq \emptyset.$$

Where U_T is the operator define by $U_T f(x) = f(T^{-1}x)$ and $\overline{\{U_T^n, n \in \mathbb{Z}\}}^W$ is the weak closure power of T .

We mention that the α -rigid rank one transformations constructed by Klemes & Reinhold is contained in this class. Thus, our result can be considered as a generalization of the Klemes-Reinhold result. There is another motivation of our work. In 1979 D. Rudolph [37] introduced the notion of minimal self-joinings as the foundation for a machinery that yields a wide variety of counterexamples in ergodic theory. But the property of minimal self-joinings is meager with respect to the

weak topology on the group of the all automorphisms. Later, in 1983 A. del Junco and M. Lemańczyk in [9] show that these constructions, as well as many others in [37], can be based on a much weaker property that is in fact generic (residual in the weak topology). This property has as a special case of the property of κ -mixing (see Katok [26], Stepin [40]) which implies the mutual singularity of the convolution powers of the maximal spectral type of an automorphism T . We remark also that the κ -mixing transformation is $(1 - \kappa)$ -rigid.

Question 3. *Are the convolution powers of the maximal spectral type of any non-mixing transformation with minimal self-joinings property pairwise singular?*

Recently, some progress was obtained by Prikhod'ko and Ryzhikov in [24]. The authors show that the well-known Chacon transformation [7] possesses this property. It is well known that the Chacon transformation has the minimal self-joining property [11]. This result can be extended easily to the case of staircase transformations with bounded cutting parameter. These latter examples include as a special case the Klemes-Reinhold examples of α -rigid rank one transformations.

We recall now some basic fact from the spectral theory. A nice account can be found in the appendix of [22].

1.1. Spectral measures. Given $T : X \mapsto X$ a measure preserving invertible transformation and denoting as above by $U_T f$ the operator $U_T f(x) = f(T^{-1}x)$, recall that to any $f \in L^2(X)$ there corresponds a positive measure σ_f on S^1 , the unit circle, defined by $\hat{\sigma}_f(n) = \langle U_T^n f, f \rangle$.

Definition . The maximal spectral type of T is the equivalence class of Borel measures σ on \mathbb{T} (under the equivalence relation $\mu_1 \sim \mu_2$ if and only if $\mu_1 \ll \mu_2$ and $\mu_2 \ll \mu_1$), such that $\sigma_f \ll \sigma$ for all $f \in L^2(X)$ and if ν is another measure for which $\sigma_f \ll \nu$ for all $f \in L^2(X)$ then $\sigma \ll \nu$.

By the canonical decomposition of $L^2(X)$ into decreasing cycles with respect to the operator U_T , there exists a Borel measure $\sigma = \sigma_f$ for some $f \in L^2(X)$, such that σ is in the equivalence class defining the maximal spectral type of T . By abuse of notation, we will call this measure the maximal spectral type measure. It can be replaced by any other measure in its equivalence class. The reduced maximal type $\sigma_T^{(0)}$ is the maximal spectral type of U_T on $L_0^2(X) \stackrel{\text{def}}{=} \{f \in L^2(X) : \int f d\mu = 0\}$. the spectrum of T is said to be discrete (resp. continuous, resp. singular, resp. absolutely continuous, resp. Lebesgue) if $\sigma_T^{(0)}$ is discrete (resp. continuous, resp. singular, resp. absolutely continuous with respect to the Lebesgue measure or equivalent to the Lebesgue measure). We write

$$Z(h) \stackrel{\text{def}}{=} \overline{\text{span}\{U_T^n h, n \in \mathbb{Z}\}}.$$

T is said to have simple spectrum, if there exist $h \in L^2(X)$ such that

$$Z(h) = L^2(X).$$

Two dynamical systems (X, \mathcal{A}, μ, T) and (Y, \mathcal{B}, ν, S) are spectrally disjoint if $\sigma_T^{(0)}$ and $\sigma_S^{(0)}$ are mutually singular.

1.2. α -rigid transformations.

Definition . A measure-preserving transformation T on the probability space (X, \mathcal{B}, μ) is said to be α -rigid where $\alpha \in]0, 1]$ if there exists a sequence of integers $\{n_k\}_{k \in \mathbb{N}}$ such that

$$\lim_{k \rightarrow \infty} \mu(T^{n_k} A \cap A) \geq \alpha \mu(A), \quad \forall A \in \mathcal{B}.$$

The α -rigidity has been formulated in 1987 by N. Friedman in [14]. Besides this, in 1969 J. Baxter in [5] proved

Theorem 4 (Baxter). *The spectrum of any α -rigid transformation is singular if $\alpha > \frac{1}{2}$.*

Proof. (due to Ryzhikov.) We shall prove more, namely we show that the α -rigid transformation T , $\alpha > \frac{1}{2}$, is spectrally disjoint from any mixing transformation S . By assumption, there exist a sequence of integers and a Markov operator P such that $(U_T^{n_i})$ converges weakly to $\alpha I + (1 - \alpha)P$. Let J be any operator such that

$$U_T J = J U_S,$$

then, for any $f \in L_0^2(X)$ we have

$$\alpha Jf + (1 - \alpha)PJf = 0.$$

Since the norm of P is equal to 1, the operator $\alpha I + (1 - \alpha)P$ is invertible; hence, Jf must be 0, thus $J \equiv 0$. We deduce that T is spectrally disjoint from S and the proof of the theorem is complete. \square

Remark 1. The proof above gives more, namely if the weak closure of the powers of the operator U_T contains any invertible operator, then T is spectrally disjoint from any mixing maps. In fact, Assume that there exist a invertible operator V in the weak closure of the power of T which means that there exists a sequence of interger (n_i) such that, for any $f \in L_0^2(X)$ we have

$$\lim_{i \rightarrow +\infty} (U_T^{n_i} f, f) = (Vf, f).$$

Let S any mixing maps. By the Lebesgue decomposition of $\sigma_T^{(0)}$ with respect to the maximal spectral type of S we have

$$\sigma_T^{(0)} = \sigma_s + \sigma_a.$$

It is well known that there exists a function $f \in L_0^2(X)$ such that $\sigma_f = \sigma_a$. But for any function $g \in L_0^2(X)$ we have

$$\lim_{j \rightarrow +\infty} \lim_{i \rightarrow +\infty} (U_T^{n_i - n_j} g, g) = (Vg, Vg).$$

We deduce from the Riemann-Lebesgue theorem that

$$(Vf, Vf) = 0,$$

therefore $f = 0$ and the proof is complete.

1.3. Joinings. Let T be any ergodic automorphism of a Lebesgue space (X, \mathcal{B}, μ) . The centralizer of T , $C(T)$, is the semi-group of all endomorphisms $S : (X, \mathcal{B}, \mu) \rightarrow (X, \mathcal{B}, \mu)$ such that $ST = TS$. Denote by $J(T, \dots, T)$ the space of all n -joinings of T :

$$\begin{aligned} \lambda \in J(T, \dots, T) \text{ if} \\ \lambda \text{ is a } T^{\otimes n} \text{ - invariant probability measure on} \\ \mathcal{B}^{\otimes n}, \text{ such that } \lambda_{\mathcal{B}} = \mu. \\ \text{Where } T^{\otimes n} \stackrel{\text{def}}{=} T \times T \cdots \times T. \end{aligned}$$

A standard example of ergodic 2-joinings comes from the centralizer of T . More precisely, if $S \in C(T)$ then the measure given by

$$\Delta_S(A \times B) = \mu(A \cap S^{-1}B)$$

is a 2-joining. For $S = T^n$, $n \in \mathbb{Z}$, we put

$$\begin{aligned} \Delta_n &\stackrel{\text{def}}{=} \Delta_{T^n} \quad \text{and} \\ \Delta &\stackrel{\text{def}}{=} \Delta_0. \end{aligned}$$

Following [10], [42] we call T is 2-*fold* simple if each ergodic 2-joining of T either is on the graph of some $S \in C(T)$ or is the product measure $\mu \times \mu$. It is an easy observation that 2-fold simplicity implies that $C(T)$ is a group (consider $\mu_S(A \times B) = \mu(S^{-1}A \cap B)$).

Now, for $n > 1$ and any $S_i \in C(T)$, $i = 1, \dots, n$, the measure (n -joining) μ_{S_1, \dots, S_n} given by

$$\mu_{S_1, \dots, S_n}(A_1 \times \cdots \times A_n) = \mu(S_1^{-1}A_1 \cap \cdots \cap S_n^{-1}A_n).$$

is said to be *off-diagonal*.

T is said to be *simple* if $C(T)$ is a group and if for every $n \geq 2$ and every ergodic n -joining λ the set $\{1, \dots, n\}$ can be split into subsets s_1, \dots, s_k such that $\lambda|_{\mathcal{B}^{s_i}}$ is a off-diagonal and λ is the product of these off-diagonals [D-R]. If T is 2-fold simple and $C(T)$ is trivial i.e. $C(T) = \{T^i : i \in \mathbb{Z}\}$, then T is said to have the property of minimal self-joining of order 2 ($T \in MSJ(2)$). T is said to have a minimal self-joinings of any order if T is simple and $C(T)$ is trivial. The class of $MSJ(2)$ transformations is contains in the class of transformations with finite joining rank. The joining rank of any ergodic transformation T , written $\text{jrk}(T)$, is defined as the minimum of $r \in \mathbb{N}$ for which each ergodic r -fold self-joining ν has some pair $i < j$ such that the two-dimensional marginal $\nu_{\mathcal{B}^{(i)} \times \mathcal{B}^{(j)}}$ is trivial (i.e. $\nu_{\mathcal{B}^{(i)} \times \mathcal{B}^{(j)}} \in \{\mu \times \mu\} \cup \{\Delta_n\}_{n \in \mathbb{Z}}$). The standard examples of transformations with finite joining rank great then 2 is given by the powers of any weak mixing transformation T in $MSJ(2)$. In addition, a non-mixing transformation with the minimal self-joining property is α -rigid. Indeed, we have the following

Lemma 1 ([38]). *Let T be a non-mixing map in $MSJ(2)$. Then T is α -rigid for some $\alpha \in]0, 1[$.*

Proof. Since the set $J(T, T)$ is compact and T is not mixing, there is a sequence $n_k \rightarrow \infty$ such that the sequence of diagonal measures Δ_{n_k} converges to some measure $\lambda \in J(T, T)$, where $\lambda \neq \mu \times \mu$. Since $T \in MSJ(2)$, by the ergodic

decomposition, we have

$$\lambda = \beta\mu \times \mu + \sum_{i \in \mathbb{Z}} a_i \Delta_i, \quad (1.1)$$

$$\text{where } a_i \geq 0 \text{ and } \sum_{i \in \mathbb{Z}} a_i = 1 - \beta.$$

Hence, for some i we have

$$\lambda \geq a_i \Delta_i, \quad a_i > 0.$$

Thus, for the sequence $m_k = n_k - i$ we have

$$\lim_{k \rightarrow \infty} \mu(T^{m_k} A \cap A) \geq \alpha \mu(A)$$

where $\alpha \stackrel{\text{def}}{=} a_i$. The proof is complete. \square

Remark 1. It is well known and it is easy to see that for any joining λ between two dynamical systems $(X, \mathcal{A}, \mathcal{T}, \mu)$ and $(Y, \mathcal{B}; \mathcal{S}, \nu)$ there exists a Markov operator, $J : L_0^2(X, \mu) \longrightarrow L_0^2(Y, \nu)$ such that

- (1) $J \circ U_T = U_S \circ J$,
- (2) $J\chi_X = \chi_Y$ and $J^*\chi_Y = \chi_X$,
- (3) $f > 0$ and $g > 0$ implies $Jf > 0$ and $J^*g > 0$,
- (4) for any $f \in L_0^2(X, \mu)$ and $g \in L_0^2(Y, \nu)$ we have

$$\int_{X \times Y} f(x)g(y)d\lambda(x, y) = \int_Y Jf(y)g(y)d\nu(y).$$

(Here J^* is the adjoint of J defined by $(Jf, g) = (f, J^*g)$ for $f \in L_0^2(X, \mu)$ and $g \in L_0^2(Y, \nu)$. Moreover, the correspondence is one-to-one. The systems $(X, \mathcal{A}, \mathcal{T}, \mu)$ and $(Y, \mathcal{B}; \mathcal{S}, \nu)$ are disjoint (in the sens of Furstenberg) if J is trivial which means $Jf = \int f d\mu$, for any $f \in L_0^2(X, \mu)$. With this in mind, Ryzhikov [39] and [12] shows that the α -rigid maps with $\alpha > 0$ is disjoint from all mixing maps. Indeed, Let λ any joining between a α -rigid maps T and mixing maps S and let J be a associated Markov operator. By the α -rigidity of T it is easy to see that there exists a sequence of integers $(n_k)_{k \in \mathbb{N}}$ and a Markov operator P such that $(U_T^{n_k})$ converge weakly to $\alpha Id + (1 - \alpha)P$. It follows that $J \circ U_T^{n_k}$ converge weakly to $\alpha J + (1 - \alpha)J \circ P$. But

$$J \circ U_T^{n_k} = U_S^{n_k} J,$$

and since S is mixing, we have

$$(U_S^{n_k} Jf, g) = 0,$$

for each $f \in L_0^2(X, \mu)$ and $g \in L_0^2(Y, \nu)$. We deduce that

$$\alpha(Jf, g) + (1 - \alpha)(JPf, g) = 0$$

for each $f \in L_0^2(X, \mu)$ and $g \in L_0^2(Y, \nu)$. Hence,

$$Jf = 0, \quad \forall f \in L_0^2(X, \mu).$$

In the following we shall prove the connection between the old problem of Banach on the existence of dynamical system with simple Lebesgue spectrum and the MSJ property. For that, we recall the following conjecture.

Conjecture 5. ([22], pp. 50) *If a reduced maximal spectral type of some transformation with simple spectrum is absolutely continuous then it is Lebesgue.*

Theorem 6. *If T is a transformation with finite joining rank and simple Lebesgue spectrum then the conjecture 5 implies that T is in the class MSJ.*

Proof. By the King-Thouvenot theorem [29], T is the e -extension ($e \in \mathbb{N}^*$) of the power (say p) of some transformation S with MSJ property. But S^p is a factor of T . It follows that S^p and S have simple Lebesgue spectrum and since the multiplicity of S^p is p this implies that $p = 1$. We get that T is an e -extension of S and the classical spectral decomposition of the group extension yields that $e = 1$. \square

2. BEURLING CONDITION.

In this section we shall prove the following

Theorem 7. *Assume that the map T satisfy the following Beurling condition*

$$\left\{ \sum_{i \in \mathbb{Z}} a_i U_T^i : a_i \neq 0 \text{ for some } i \text{ and } \sum_{n \geq 0} \frac{\log(\sum_{k \leq -n} a_k^2)}{n^2} = -\infty \right\} \\ \cap \left(\overline{\{U_T^n, n \in \mathbb{Z}\}}^W \setminus (\{U_T^n, n \in \mathbb{Z}\}) \right) \neq \emptyset.$$

Then T is singular.

We shall need the following lemma. The proof of it follows word by word the proof given in [34, pp. 66] in the case of Lebesgue measure.

Lemma 2 (Translation lemma). *Suppose μ is a probability measure on \mathbb{T} and $(n_k)_{k \in \mathbb{N}}$ is a sequence of distinct integers. Define μ_k by*

$$\mu_k = e^{in_k \theta} d\mu(\theta), \quad k = 1, 2, 3, \dots$$

If $(\mu_k)_{k \in \mathbb{Z}}$ converges in the weak topology to σ , then σ is singular with respect to any Rajchman measure ν (a measure ν on the circle is called a Rajchman measure if $\lim_{n \rightarrow \infty} \widehat{\nu}(n) = 0$). In fact, if $\mu = \mu_s + \mu_a$ is the Lebesgue decomposition of μ with respect to the Rajchman measure ν on \mathbb{T} , then*

$$|\sigma|(E) \leq \mu_s(E), \text{ for every Borel set } E \text{ in } \mathbb{T}.$$

Theorem 8 (Beurling's Theorem [4]). *Let f be a function on torus \mathbb{T} define by*

$$f(\theta) = \sum_{j \in \mathbb{Z}} a_j e^{ij\theta}.$$

and assume that the following condition holds

$$\sum_{n \geq 0} \frac{\log(\sum_{k \leq -n} a_k^2)}{n^2} = -\infty,$$

Then, $f = 0$ on the set of positive measure implies that $f = 0$ a.e. on \mathbb{T} .

Proof. Let $\sigma \stackrel{\text{def}}{=} \sigma_h$ be the reduced maximal spectral type of T . Then from the Beurling condition it follows that there exists a sequence of integers $(n_k)_{k \in \mathbb{N}}$ such that

$$e^{in_k \theta} d\sigma \xrightarrow{w^*} \sum_{j \in \mathbb{Z}} a_j e^{ij\theta} d\sigma. \quad (2.2)$$

Let $\sigma = \sigma_s + \sigma_a$ be the Lebesgue decomposition of σ with respect to the Lebesgue measure m on \mathbb{T} . Put

$$f(\theta) \stackrel{\text{def}}{=} \sum_{j \in \mathbb{Z}} a_j e^{ij\theta}.$$

Then, by translation lemma, we have

$$\sigma_a \{x : f(x) = 0\} = \sigma_a(\mathbb{T}). \quad (2.3)$$

Now observe that Beurling's theorem (Theorem 8) says that

$$f \neq 0 \Rightarrow m\{x : f(x) = 0\} = 0.$$

Hence $\sigma_a = 0$ and this completes the proof of theorem. \square

As immediate consequence of our result we obtain the following

Corollary 1. *If the map T satisfy the following restricted Beurling condition*

$$\left\{ \sum_{i \in \mathbb{Z}} a_i U_T^i : a_i > 0 \text{ for some } i \text{ and } \sum_{n \geq 0} \frac{\log(\sum_{k \leq -n} a_k^2)}{n^2} = -\infty \right\} \\ \cap \left(\overline{\{U_T^n, n \in \mathbb{Z}\}}^w \setminus (\{U_T^n, n \in \mathbb{Z}\}) \right) \neq \emptyset.$$

Then T is α -rigid with singular spectrum.

Remark 2. Note that we have actually proved that T is spectrally disjoint from any mixing maps S provided that

$$\sigma_S^{(0)} \{x : f(x) = 0\} = 0, \quad (2.4)$$

and observe that (2.4) holds if f is a analytic function on the circle.

2.1. Applications. Using our result we shall give, in this section, a simple proof of some well known results on the singularity of the spectrum of some special case of rank one maps. More precisely, we shall give a simple proof of the singularity of Chacon maps and the staircase maps with bounded cutting parameter. We recall that for any $\varepsilon > 0$ one may construct a staircase maps with bounded cutting parameter say p such that the α -rigid constant is smaller that ε . In fact the α -rigid constant is $\frac{1}{p}$.

Let us recall the definition of rank one maps and by the way the staircase maps. Using the cutting and stacking method described in [15], [16], one can defines inductively a family of measure preserving transformations, called rank one transformations, as follows

Let B_0 be the unit interval equipped with the Lebesgue measure. At stage one we divide B_0 into p_0 equal parts, add spacers and form a stack of height h_1 in the usual fashion. At the k^{th} stage we divide the stack obtained at the $(k-1)^{\text{th}}$ stage

into p_{k-1} equal columns, add spacers and obtain a new stack of height h_k . If during the k^{th} stage of our construction the number of spacers put above the j^{th} column of the $(k-1)^{\text{th}}$ stack is $a_j^{(k-1)}$, $0 \leq a_j^{(k-1)} \leq \infty$, $1 \leq j \leq p_k$, then we have

$$h_k = p_{k-1}h_{k-1} + \sum_{j=1}^{p_{k-1}} a_j^{(k-1)}.$$

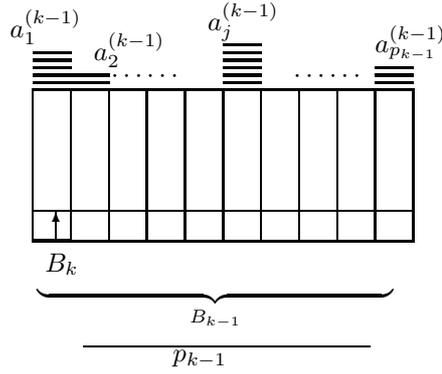


Figure 1 : k^{th} -tower.

Proceeding in this way we get a rank one transformation T on a certain measure space (X, \mathcal{B}, ν) which may be finite or σ -finite depending on the number of spacers added.

The construction of any rank one transformation thus needs two parameters, $(p_k)_{k=0}^{\infty}$ (parameter of cutting and stacking), and $((a_j^{(k)})_{j=1}^{p_k})_{k=0}^{\infty}$ (parameter of spacers). Put

$$T \stackrel{\text{def}}{=} T_{(p_k, (a_j^{(k)})_{j=1}^{p_k})_{k=0}^{\infty}}$$

In the case of staircase maps the parameter of spacers is given by

$$a_j^{(k)} = j - 1, \text{ for } j = 1 \text{ to } p_k - 1 \text{ and } a_{p_k}^{(k)} = 0.$$

The classical Chacon map [7] correspond to the case $p_k = 3$, for every $k \in \mathbb{N}^*$. It easy to see that Chacon map is $\frac{1}{3}$ -rigid. More generally it is easy to prove that the staircase with bound cutting parameter (say p) is $\frac{1}{p}$ -rigid (in fact, for any measurable set A we have $\liminf \mu(T^{h_n} A \cap A) \geq \frac{1}{p} \mu(A)$). Using the theorem 7. and the remark 2. we have the following

Corollary 2. *The staircase maps with bounded cutting parameter are spectrally disjoint from any mixing maps.*

Proof. By the definition of staircase maps it is easy to prove that the sequence T^{h_n} converge weakly to $\sum_{j=0}^{r-1} \frac{1}{r-1} T^j$ where T^0 is a identity map. \square

One may apply our result also to the historically Chacon map [6]. The historically Chacon map is given by

$$p_k = 2, \forall k \in \mathbb{N}^* \text{ and } a_1 = 1, a_2 = 0.$$

From the construction one may check that the sequence $(T^{h_k})_{k \in \mathbb{N}^*}$ converge weakly to $\sum_{k=0}^{+\infty} \frac{1}{2^k} T^k$.

Now, it is easy to get from our result the following

Corollary 3. *The historically Chacon map is spectrally disjoint from any mixing maps.*

Remark 2. Using the same methods and the continuous version of Beurling theorem we can extend the theorem 7 to the case of the flows and by the results of Frączek-Lemańczyk [12], [13], we can give examples for which the conditions of the theorem are satisfied.

3. MATHEW-NADKARNI TRANSFORMATION

The Mathew-Nadkarni transformation is a \mathbb{Z}_2 -extension of the odometer or von Neumann-Kakutani adding machine τ^{-1} , $\mathbb{Z}_2 \stackrel{\text{def}}{=} \{0, 1\}$ endowed with its Haar measure $h \stackrel{\text{def}}{=} \frac{1}{2}\delta_0 + \frac{1}{2}\delta_1$. Explicitly, define $T_\phi : [0, 1) \times \mathbb{Z}_2 \mapsto [0, 1) \times \mathbb{Z}_2$ by $T_\phi(x, g) = (Tx, \phi(x) + g)$, where T is the adding machine.

T is defined by mapping the interval $[1 - \frac{1}{2^n}, 1 - \frac{1}{2^{n+1}})$ linearly onto the interval $[\frac{1}{2^{n+1}}, \frac{1}{2^n})$, $n = 0, 1, 2, \dots$. T is a rank one transformation which is also easily described using cutting and stacking. The cocycle ϕ is defined (inductively over all "levels" of the tower associated to T except the last one) to be 0 on the intervals $[1 - \frac{1}{2^n}, 1 - \frac{1}{2^n} + \frac{1}{2^{n+2}})$, and 1 on the intervals $[1 - \frac{1}{2^n} + \frac{1}{2^{n+2}}, 1 - \frac{1}{2^{n+1}})$, $n = 0, 1, 2, \dots$.

Theorem 9 (Mathew-Nadkarni [19]). *The Mathew-Nadkarni transformation has a spectrum consisting of a Lebesgue component with multiplicity 2, together with discrete component.*

Theorem 10. *The Mathew-Nadkarni transformation is $\frac{1}{2}$ -rigid.*

Proof. Recall that the operator $U_{T_\phi} : L^2([0, 1) \times \mathbb{Z}_2, \mu \otimes h) \rightarrow L^2([0, 1) \times \mathbb{Z}_2, \mu \otimes h)$ has a direct sum decomposition

$$L^2([0, 1) \times \mathbb{Z}_2, \mu \otimes h) = L_0 \oplus L_1.$$

where $L_0 = \{f \otimes 1 : f \in L^2([0, 1))\}$ and $L_1 = \{f \otimes \chi : f \in L^2([0, 1)) \text{ and } \chi(g) = (-1)^g, \text{ for } g \in \mathbb{Z}_2\}$.

Let A be a Borel set and $\varepsilon \in \mathbb{Z}_2$ then

$$\mathbb{I}_{A \times \{\varepsilon\}} = f_1 + \chi f_2.$$

where $f_1 = \frac{1}{2}\mathbb{I}_A$, $f_2 = \chi(\varepsilon)f_1$. Observe that the discrete part σ_d of the spectral measure $\sigma_{\mathbb{I}_{A \times \{\varepsilon\}}}$ verify that there exists a sequence of integers (n_k) such that

$$\lim_{k \rightarrow \infty} \hat{\sigma}_{A \times \{\varepsilon\}}^\wedge(n_k) = \lim_{k \rightarrow \infty} \hat{\sigma}_d^\wedge(n_k) = (f_1, f_1) = \frac{1}{2}m \otimes h(A \times \{\varepsilon\}).$$

¹see [23], [15] or [20].

Hence by a standard argument, for any Borel set B in the σ -algebra of $[0, 1) \times \mathbb{Z}_2$ we have

$$\liminf_{k \rightarrow \infty} \hat{\sigma}_B(n_k) \geq \frac{1}{2} m \otimes h(B).$$

and this completes the proof of the theorem. \square

Remark 3. The Mathew-Nadkarni transformation is not weakly mixing but using the Ageev's construction [2] one can produce an α -rigid transformations with continuous spectrum and Lebesgue component of even multiplicity. Let us remark also that Mathew-Nadkarni's construction contains continuum pairwise non-isomorphic dynamical system [31]. It follows that one can produce a continuum of α -rigid transformations with 2-fold Lebesgue spectrum. Moreover, M. Lemańczyk pointed out to us that one can generalize the above result as follows :

If T is rigid and the cocycle $\phi : X \rightarrow \mathbb{Z}_2$ gives Lebesgue spectrum for T_ϕ (such ϕ exists over each rigid maps by Helson-Parry [18]), then T_ϕ is $\frac{1}{2}$ -rigid.

4. SUBSTITUTION EXAMPLES

A vast literature is devoted to substitutions, whose Bible is [33]. They appear as symbolic systems defined on a finite alphabet $A = \{0, 1, \dots, k-1\}$; a substitution ξ is a mapping from A to the set A^* of all finite words of A . It extends naturally into a morphism of A^* by concatenation. We restrict ourselves to the case when $\xi(0)$ begins with 0 and the length of $\xi^n(0)$ tends to infinity with n . The infinite sequence u beginning with $\xi^n(0)$ for all n is then called a fixed point of ξ and the symbolic system associated to u is called the dynamical system associated to ξ .

When ξ is primitive, i.e. there exists n such that α appears in $\xi^n(\beta)$ for all $\alpha, \beta \in A$, the system is uniquely ergodic, and we can consider the measure-preserving system associated to ξ , to which we refer in short, "the substitution ξ ". The composition matrix M of the substitution ξ is the matrix whose entries are $l_{ij} = O_i(\xi(j))$, where $i, j \in A$ and $O_i(\xi(j))$ is the number of i 's occurring in $\xi(j)$. If ξ is primitive, it follows from the Perron-Frobenius theorem that M admits a strictly positive simple eigenvalue θ , such that $\theta > |\lambda|$, for any other eigenvalue λ and there exists a strictly positive eigenvector corresponding to θ . It is any easy exercise (see [33]) to see that, for any $a \in A$, the sequence of k -dimensional vector

$$\left(\frac{O_0(\xi^n(a))}{\theta^n}, \dots, \frac{O_k(\xi^n(a))}{\theta^n} \right),$$

converges to a strictly positive eigenvector $v(a)$ corresponding to θ .

One classical result on primitive substitution we need to recall here is the result of M. Keane and M. Dekking [8]. They proved that the dynamical system arising from a primitive substitution is not mixing. Their proof, with the above notations, contains the following.

Theorem 11. *The dynamical system arising from a primitive substitution is $r\rho$ -rigid. Where, r is the maximum of the measure of the cylinders set $[aa]$, $a \in A$, and ρ is the ℓ_1 -norm of the strictly positive eigenvector $v(a_r)$ corresponding to the Perron-Frobenius eigenvalue θ of ξ , a_r is a letter for which the measure of $[a_r a_r]$ is r .*

It is an easy exercise to compute the constant r . In fact, it is sufficient to compute the normalized positive eigenvector corresponding to the dominant eigenvalue of the composition matrix M_2 of the substitution ξ_2 . The substitution ξ_2 is defined on the alphabet $A_2 = \{(ab), a, b \in A\}$ in the following way :

$$\text{if } \xi(ab) = \xi(a)\xi(b) = y_0y_1y_2y_3$$

we set

$$\xi_2(ab) = (y_0y_1)(y_1y_2).$$

One of the classical examples of substitutions is the Rudin-Shapiro substitution. The Rudin-Shapiro substitution is defined on the alphabet $A = \{0, 1, 2, 3\}$ in the following way

$$\begin{aligned} \xi(0) &= 02, & \xi(1) &= 32, \\ \xi(2) &= 01, & \xi(3) &= 31. \end{aligned}$$

M. Queffélec in [32] shows that the continuous part of the Rudin-Shapiro dynamical system is Lebesgue with multiplicity 2 and it is easy to prove that the Rudin-Shapiro dynamical system is $\frac{1}{2}$ -rigid. Another example of a substitution with Lebesgue spectrum is given by the following substitution ξ on the alphabet $A = \{0, 1, 2\}$

$$\begin{aligned} \xi(0) &= 001 \\ \xi(1) &= 122 \\ \xi(2) &= 210. \end{aligned}$$

ξ has Lebesgue spectrum with multiplicity 2 in the orthocomplement of eigenfunctions [33, p. 221], we point out that one may use the standard computer program to compute approximatively the constant of α -rigidity of ξ which is approximatively equal to $0.3104979673 \times 10^{-7}$.

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