Comparing Markov chains: Combining aggregation and precedence relations applied to sets of states

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Abstract

Numerical methods for solving Markov chains are in general inefficient if the state space of the chain is very large (or infinite) and lacking a simple repeating structure. One alternative to solving such chains is to construct models that are simple to analyze and that provide bounds for a reward function of interest. We present a new bounding method for Markov chains inspired by Markov reward theory; our method constructs bounds by redirecting selected sets of transitions, facilitating an intuitive interpretation of the modifications on the original system. We show that our method is compatible with strong aggregation of Markov chains; thus we can obtain bounds for the initial chain by analyzing a much smaller chain. We illustrate our method on a problem of order fill rates for an inventory system of service tools.

1 Introduction

In Markov chain modeling, one often faces the problem of combinatorial state space explosion: modeling a system completely requires an unmanageable - combinatorial - number of states. Many high-level formalisms, such as queueing networks or stochastic Petri nets, have been developed to simplify the specification and storage of the Markov chain. However, these models only rarely have closed-form solutions, and numerical methods are inefficient when the size of the state space becomes very large or for models with infinite state space that do not exhibit special repeating structure that admits a matrix analytic approach [16]. Typically, the latter approach is quite limited if the state space is infinite in more than one dimension. An alternative approach to cope with state space explosion is to design new models that (i) provide bounds for a specific measure

of interest (for instance the probability of a failure in a complex system); and (ii) are simpler to analyze than the original system.

Establishing (i): the bounding relationship between the original and new (bounding) systems may be based on different arguments. Potentially the most general way of obtaining bounds is by stochastic comparison, which gives bounds for a whole family of reward functions (for instance increasing or convex functions). Furthermore, stochastic comparison provides bounds for both the steady-state and transient behavior of the studied model. Many results have been obtained using strong stochastic order (i.e. generated by increasing functions) and coupling arguments [11]. Recently, an algorithmic approach has been proposed in [8] to construct stochastic bounds, based on stochastic monotonicity; this stochastic monotonicity provides simple algebraic sufficient conditions for stochastic comparison of Markov chains. Ben Mamoun et al. showed in [3] that an algorithmic approach is also possible using increasing convex ordering that allows one to compare variability. The clear advantage of stochastic comparison is its generality: it provides bounds for a whole family of rewards, both for the steady-state and transient behavior of the studied system. Its drawback is that in its generality, it may not apply to the system of interest. For more details on the theoretical aspects of stochastic comparison we refer the reader to [14, 18].

For this reason more specialized methods than stochastic comparison have also been developed, which apply only to one specific function, and only in the steady-state. Muntz et al. [15] proposed an algebraic approach to derive bounds of steady-state rewards without computing the steady-state distribution of the chain, founded on results of Courtois and Semal on eigenvectors of non-negative matrices [5, 6]. This approach was specially designed for reliability analysis of highly reliable systems, and requires special constraints on the structure of the underlying Markov chain. This approach was further improved and generalized by various authors [12, 17, 4, 13], but the primary assumption for its applicability is still that there is a very small portion of the state space that has a very high probability of occurrence, while the other states are almost never visited.

Similarly, Van Dijk and van der Wal [21, 22] proposed a different method for comparing two chains in terms of a particular reward function, often referred to as the Markov reward approach. This method allows the comparison of mean cumulated and stationary rewards for two given chains. A simplified version of the Markov reward approach, called the precedence relation method, was proposed by van Houtum et al. in [24]. The origin of the precedence relation method dates back to Adan et al. [1], and it has been successfully applied to various problems [23, 20, 10]. The advantage of this method is its straightforward description of the modifications of the initial model. The precedence relation method consists of two steps. Precedence relations are first established on the states of the system, based on the reward function (or familly of functions) one wants to study. Then an upper (resp. lower) bound for the initial model can be obtained simply by redirecting the transitions to greater (resp. smaller) states with respect to the precedence relations established in the first step. A significant drawback of the precedence relation method is that it can be applied only to derive bounding models obtained by replacing one transition by another with the same probability: the method does not allow the modification of the probability of a transition, nor the replacement of one transition by more than one new transition. Such a modification is typically needed, for example, if one wants to keep the mean behavior of a part of the system (for instance arrivals to a queue), but change its variability. (One small example of such a system is given in Section 3.)

We propose a generalization of precedence relations to *sets of states*. This significantly increases the applicability of the precedence relation method, by allowing the replacement of one set of transitions by another set. The modification of the probability of a transition can also be seen as replacement of one transition by two new transitions, one of which is a loop.

We now discuss point (ii): how to derive models that are simpler to solve. In the context of

stochastic comparison, different types of bounding models have been used: models having closed form solutions, models that are easier to analyze using numerical methods, and aggregation [8, 7]. To our knowledge, the precedence relation method has been combined only to the first two simplifications. We show here that it is also compatible with aggregation. Thus we prove the validity of applying the precedence relation method on sets of states, in concert with the simplifying technique of aggregation.

To illustrate our new technique, we use as an example the *service tool problem*. This problem, introduced by Vliegen and Van Houtum in [25], models a single-location multi-item inventory system in which customers demand different *sets* of service tools, needed for a specific maintenance action, from a stock point. Items that are available in stock are delivered immediately; items that are not in stock are delivered via an emergency shipment, and lost for the location under consideration. (This is called partial order service [19].) All non-emergency items are replenished (returned from the customer) together after a stochastic replenishment time. The service level in this problem is defined as the *aggregate order fill rate*, the percentage of orders for which *all* requested items can be delivered from stock immediately. Vliegen and Van Houtum [25] developed an efficient and accurate approximate evaluation model for this problem that combines two different evaluation models. In their numerical study, one of their models (M_1) always led to an overestimation of the order fill rates compared to the original model, while the other (M_2) always led to an underestimation. Using the generalization of the precedence relation method in combination with aggregation, we prove that these two models indeed provide analytical bounds for the original model.

This paper is organized as follows. In Section 2 we give an overview of precedence relation method proposed in [24]. In Section 3 we show the limits of this method and we propose and prove the validity of our generalization. Section 4 is devoted to aggregation and its connections with our method. In Section 5 we illustrate our technique on the service tool problem, proving that the approximations proposed in [25] do provide a lower and an upper bound for the original model. These bounding models have a state space that is highly reduced compared to the original system: its dimension is equal to the number of different types of tools (I), while the original model has dimension 2^I. Finally, Appendix A contains a supermodularity characterization on a discrete lattice, that is used in the proof of supermodularity for order fill rates for the bounding models (M_1 and M_2), given in Appendix B.

2 Precedence relations

Let $\{X_n\}_{n\geq 0}$ be an irreducible, aperiodic and positive recurrent discrete time Markov chain (DTMC) on a countable state space $\mathcal{S}$. We will denote by P the corresponding transition matrix and by π the stationary distribution. For a given reward (or cost) function $r : \mathcal{S} \to \mathbb{R}$, the mean stationary reward is given by:

$$a = \sum_{x\in\mathcal{S}} r(x)\pi(x).$$

Directly computing the stationary distribution π is often difficult if, for instance, the state space is infinite in many dimensions or finite, but prohibitively large. The main idea of the precedence relation method proposed in [24] is to obtain upper or lower bounds for a without explicitly computing π. By redirecting selected transitions of the original model, the graph of the chain is modified to obtain a new chain that is significantly easier to analyze. For example, one might essentially truncate the chain by blocking the outgoing transitions from a subset of states. Note that this might produce a modified chain that is not irreducible. We will assume in this case that the modified chain has only one recurrent class, which is positive recurrent. Then we can restrict our attention

to this recurrent class $\widetilde{\mathcal{S}} \subset \mathcal{S}$, and thus the stationary distribution $\widetilde{\pi}$ of the modified chain is still well defined.

Some special care needs to be taken in order to ensure that the reward function of the new chain provides bounds on the reward function of the original chain. We denote by $v_t(x)$ (resp. by $\widetilde{v}_t(x)$) the expected cumulated reward over the first t periods for the original (resp. modified) chain when starting in state $x \in \mathcal{S}$:

$$v_t(x) = r(x) + \sum_{y \in \mathcal{S}} P[x,y] v_{t-1}(y),\ t \geq 1, \tag{1}$$

where $v_0(x) := 0,\ \forall x \in \mathcal{S}$. If we can show that

$$v_t(x) \leq \widetilde{v}_t(x), \forall x,\ \forall t \geq 0, \tag{2}$$

then we have also the comparison of mean stationary rewards:

$$a = \lim_{t \to \infty} \frac{v_t(x)}{t} \leq \lim_{t \to \infty} \frac{\widetilde{v}_t(x)}{t} = \widetilde{a}.$$

In order to establish (2), a precedence relation $\preceq$ is defined on state space $\mathcal{S}$ as follows:

$$x \preceq y \quad \text{if} \quad v_t(x) \leq v_t(y),\ \forall t \geq 0.$$

When we are talking about rewards, we will often say that a state x is *less attractive* than y if $x \preceq y$.

The following theorem states that redirecting transitions to less (more) attractive states results in an lower (upper) bound for mean stationary reward.

Theorem 1. *[24, Theorem 4.1] Let $\{X_n\}$ be a DTMC and let $\{Y_n\}$ be a chain obtained from $\{X_n\}$ by replacing some transitions (x,y) with transitions (x,y') such that $y \preceq y'$. Then:*

$$v_t(x) \leq \widetilde{v}_t(x), \forall x,\ \forall t \geq 0.$$

If both chains have steady state distributions, then $a \leq \widetilde{a}$.

The above theorem allows one to easily construct bounding models by redirecting possibly only a few transitions. In [24] this approach is illustrated on the example of a system with the Join the Shortest Queue routing. In the following section we illustrate some of the limits of the precedence relation approach before proposing its generalization.

3 Precedence relations on sets of states

The precedence relation method allows one to redirect transitions: the destination of the transition is modified, while its probability remains the same. The following simple example shows that we cannot use the precedence relation method to compare models with the same average arrival rate, but different variabilities.

Example 1 (Single queue with batch arrivals)**.** We consider a single queue with two types of jobs:

- Class 1 jobs arrive individually following a Poisson process with rate λ_1.
- Class 2 jobs arrive by batch of size 2, following a Poisson process with rate λ_2.

We assume a single exponential server with rate μ, and let x denote the number of jobs in the system. Then the following events can occur in the system:

event	rate	transition	condition
type 1 arrival	λ_1	$x+1$	-
type 2 arrival	λ_2	$x+2$	-
service	μ	$x-1$	$x>0$

Without loss of generality, we assume that $\lambda_1+\lambda_2+\mu=1$. Thus we can consider λ_1, λ_2 and μ as the probabilities of the events in the corresponding discrete time (uniformized) chain. Suppose that we are interested in the mean number of jobs. The appropriate reward function is thus $r(x)=x$, $\forall x$. The corresponding t-period rewards satisfy:

$$v_{t+1}(x)=r(x)+\lambda_1 v_t(x+1)+\lambda_2 v_t(x+2)+\mu v_t(x-1)1_{\{x>0\}}+\mu v_t(x)1_{\{x=0\}},\ x\geq 0,\ t\geq 0.$$

Denote respectively by A_1, A_2 and S the t-period rewards in new states after an arrival of type 1, an arrival of type 2 and a service in state x:

- $A_1(x,t)=v_t(x+1)$,
- $A_2(x,t)=v_t(x+2)$,
- $S(x,t)=v_t(x-1)1_{\{x>0\}}+v_t(x)1_{\{x=0\}}$.

Then:

$$v_{t+1}(x)=r(x)+\lambda_1 A_1(x,t)+\lambda_2 A_2(x,t)+\mu S(x,t).$$

Now, suppose some class 1 jobs become class 2 jobs, keeping the total arrival rate constant. This means that these jobs arrive less often (only half of the previous rate), but they arrive in batches of size 2 (Figure 1). Then:

$$\lambda_1^{'}=\lambda_1-\epsilon \text{ and } \lambda_2^{'}=\lambda_2+\frac{\epsilon}{2},$$

where $0<\epsilon\leq\lambda_1$. The total arrival rate is the same in both models, but the arrival process of the second system is more variable.

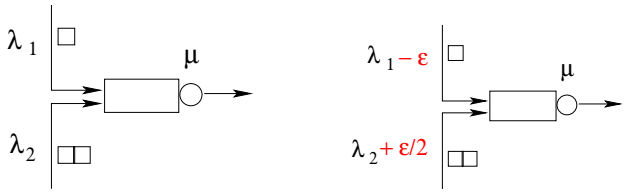

Figure 1: Batch arrivals.

Different transitions for both models are shown in Figure 2. Note that a part of the transition rate that corresponds to the arrivals of type 1 is replaced by a new transition that corresponds to the arrivals of type 2, but the rate is divided by two. This can be also seen as replacing one transition with rate ϵ by two transitions, each with rate $\epsilon/2$; we can consider a "fake" transition (x,x) with rate $\epsilon/2$ in the continuous time model, that is transformed into a strictly positive diagonal term in the discrete time model, after uniformization. Thus we cannot directly apply Theorem 1, since it allows neither replacing only a part of a transition, nor replacing one transition with two new ones.

In the following we propose a more general method, that allows us to replace a transition or more generally a set of transitions by another set of transitions having the same aggregate rate. Also, only a part of some transitions might be redirected.

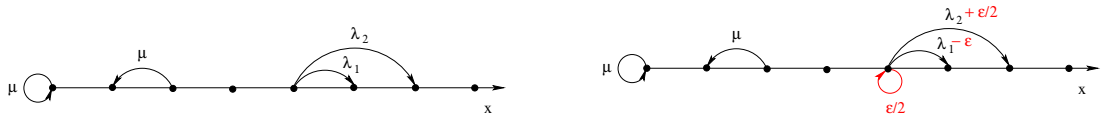

Figure 2: Batch arrivals: redirecting transitions.

3.1 Generalization of precedence relations

To aid intuition, we will introduce the main ideas by considering a single state $x \in \mathcal{S}$. (The general result will be proved later, in Theorem 3.) Assume we want to replace (redirect) the outgoing transitions from x to a subset A by transitions to another subset B. For instance, in Figure 3 we want to replace transitions to $A = \{a_1, a_2\}$ (blue transitions on the left) by transitions to $B = \{b_1, b_2, b_3\}$ (red transitions on the right). We might also have some transitions from x to states that are not in A and that we do not want to redirect (transitions to states u and v in Figure 3).

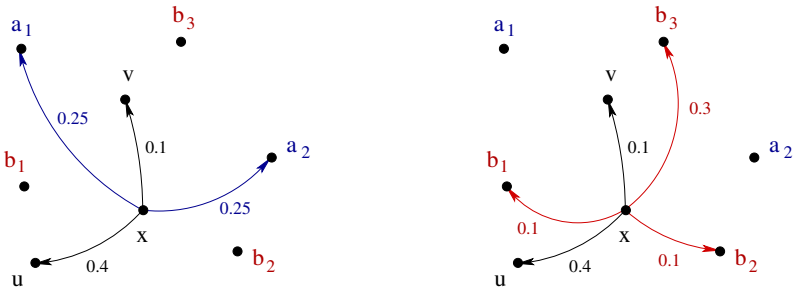

Figure 3: Redirecting the sets of transitions.

Furthermore, we might want to redirect transitions only partially: in Figure 4 only the half of probability of transitions to A is replaced by transitions to B. Thus in order to describe the

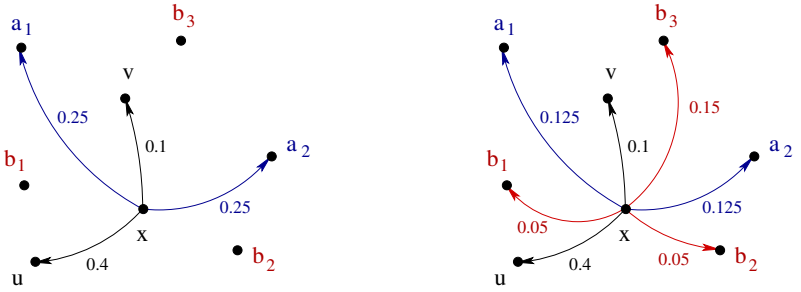

Figure 4: Partial redirection of transitions.

redirection of a set of transitions we will need to provide:

- the set A (resp. B) and the probabilities of transitions to each state in A (resp. B);
- the weight factor Δ (the amount of each transition to be redirected; the same scalar Δ is applied to all transitions to states in A).

Information on sets A and B and the corresponding transition probabilities will be given by two vectors $\alpha = (\alpha(z))_{z \in \mathcal{S}}$ and $\beta = (\beta(z))_{z \in \mathcal{S}}$. Since the information on the amount to be redirected will be given by a weight factor, we only need the relative probabilities of transitions to the respective

sets A and B. Therefore it is convenient to renormalize the vectors α and β. The modifications in Figures 3 and 4 can now be described by vectors $\alpha = 0.5\delta_{a_1} + 0.5\delta_{a_2}$ and $\beta = 0.2\delta_{b_1} + 0.2\delta_{b_2} + 0.6\delta_{b_3}$, where δ_a denotes the Dirac measure in a:

$$\delta_a(z) = \begin{cases} 1, & z = a, \\ 0, & z \neq a \end{cases}, z \in \mathcal{S}.$$

The weight factor Δ is equal to 0.5 in Figure 3 and to 0.25 in Figure 4.

We now formally generalize the previous example. Let α and β be two stochastic vectors: $\alpha(z) \geq 0,\ \beta(z) \geq 0,\ \forall z \in \mathcal{S}$ and $||\alpha||_1 = ||\beta||_1 = 1$ (where $||\alpha||_1 := \sum_{z\in\mathcal{S}} \alpha(z)$ is the usual 1-norm). Let $\{X_n\}$ be an irreducible, aperiodic and positive recurrent DTMC with transition probability matrix P and t-period reward functions v_t, satisfying the following relation:

$$\sum_{z\in\mathcal{S}} \alpha(z)v_t(z) \leq \sum_{z\in\mathcal{S}} \beta(z)v_t(z),\ t \geq 0. \tag{3}$$

Let A and B denote the supports of vectors α and β respectively:

$$A = \text{supp}(\alpha) = \{z \in \mathcal{S}\ :\ \alpha(z) > 0\}, \quad B = \text{supp}(\beta) = \{z \in \mathcal{S}\ :\ \beta(z) > 0\}.$$

If (3) holds, we will say that the set of states A is less attractive than the set B with respect to probability vectors α and β, and we will denote this:

$$A \preceq_{\alpha,\beta} B.$$

We will show that if $A \preceq_{\alpha,\beta} B$, replacing the outgoing transitions to A (with probabilities α) by the outgoing transitions to B (with probabilities β) leads to an upper bound for t-period rewards (and thus also for the mean stationary reward, when it exists). Before giving this result in Theorem 3, note that relation (3) is indeed a generalization of precedence relations of states:

Remark 1. Suppose $x \preceq y$, for some $x, y \in \mathcal{S}$. Set $\alpha = \delta_x$ and $\beta = \delta_y$. Then (3) becomes:

$$v_t(x) \leq v_t(y),\ t \geq 0,$$

which is equivalent to $x \preceq y$ by definition.

To see that (3) indeed is more general than the method in [24], let $\alpha = \delta_x$ and $\beta = \frac{1}{2}\delta_y + \frac{1}{2}\delta_z$, $x, y, z \in \mathcal{S}$. Then (3) becomes:

$$v_t(x) \leq \frac{1}{2}v_t(y) + \frac{1}{2}v_t(z),\ t \geq 0.$$

We can write this $\{x\} \preceq_{\delta_x, \frac{1}{2}\delta_y + \frac{1}{2}\delta_z} \{y, z\}$. By taking $y = x + 1$ and $z = x - 1$ this is exactly the relation we need to prove in Example 1. The proof of this relation for Example 1 will be given in Section 3.2.

Replacing the outgoing transitions that correspond to the set A and probabilities α by the transitions that correspond to the set B and probabilities β (called (α, β)-redirection in the following), can be also represented in matrix form. The matrix $T_{\alpha,\beta}(x)$ defined as:

$$T_{\alpha,\beta}(x)[w, z] = \begin{cases} \beta(z) - \alpha(z), & w = x, \\ 0, & w \neq x, \end{cases}$$

describes the (α, β)-redirection of the outgoing transitions from state $x \in \mathcal{S}$. The transition matrix of the modified chain after (α, β)-redirection of the outgoing transitions from state $x \in \mathcal{S}$ is then given by:

$$\widetilde{P} = P + \Delta_{\alpha,\beta}(x) T_{\alpha,\beta}(x),$$

with the weight factor $\Delta_{\alpha,\beta}(x)$, $0 \leq \Delta_{\alpha,\beta}(x) \leq 1$. (Note that if $\Delta_{\alpha,\beta}(x) = 0$, we do not modify the chain.) In order for $\widetilde{P}$ to be a stochastic matrix, the weight factor $\Delta_{\alpha,\beta}(x)$ must satisfy:

$$0 \leq P[x, y] + \Delta_{\alpha,\beta}(x)(\beta(y) - \alpha(y)) \leq 1, \ y \in \mathcal{S}, \tag{4}$$

which can be also written as:

$$\Delta_{\alpha,\beta}(x) \leq \min \left\{ \min_{y \,:\, \alpha(y) > \beta(y)} \left\{ \frac{P[x, y]}{\alpha(y) - \beta(y)} \right\}, \min_{y \,:\, \alpha(y) < \beta(y)} \left\{ \frac{1 - P[x, y]}{\beta(y) - \alpha(y)} \right\} \right\}.$$

Without loss of generality, we can assume that the supports of α and β are disjoint: $A \cap B = \emptyset$. Indeed, if there exists $y \in \mathcal{S}$ such that $\alpha(y) > 0$ and $\beta(y) > 0$, then we can define new vectors $\alpha' = \frac{1}{1-c}(\alpha - c e_y)$ and $\beta' = \frac{1}{1-c}(\beta - c e_y)$, where $c = \min\{\alpha(y), \ \beta(y)\}$. Relation (3) is then equivalent to:

$$\sum_{z \in \mathcal{S}} \alpha'(z) v_t(z) \leq \sum_{z \in \mathcal{S}} \beta'(z) v_t(z), \ t \geq 0.$$

Assuming A and B are disjoint, relation (4) has an intuitive interpretation given as Proposition 2: one can only redirect the existing transitions.

Proposition 2. *For vectors α and β with supports $A \cap B = \emptyset$ condition (4) is equivalent to:*

$$\alpha(y) \Delta_{\alpha,\,\beta}(x) \leq P[x, y], \ y \in \mathcal{S}. \tag{5}$$

Proof. Relation (5) follows trivially from (4) as $\alpha(y) > 0$ and $A \cap B = \emptyset$ imply $\beta(y) = 0$. In order to see that (5) implies (4), we will consider the following cases for an arbitrary $y \in \mathcal{S}$:

- $\alpha(y) > 0$. Then $\beta(y) = 0$ so relation (4) becomes: $0 \leq P[x, y] - \Delta_{\alpha,\beta}(x)\alpha(y) \leq 1$. As we assumed that $0 \leq \Delta_{\alpha,\beta}(x) \leq 1$, and $0 \leq \alpha(y) \leq 1$, the right inequality is trivial and the left one is simply relation (5).

- $\beta(y) > 0$. Then $\alpha(y) = 0$ so relation (4) becomes: $0 \leq P[x, y] + \Delta_{\alpha,\beta}(x)\beta(y) \leq 1$. The left inequality is trivial. For the right one we have, using the fact that P is a stochastic matrix and $\beta(y) \leq 1$:

$$\begin{aligned} P[x, y] + \Delta_{\alpha,\beta}(x)\beta(y) &\leq 1 - \sum_{z \neq y} P[x, z] + \Delta_{\alpha,\beta}(x) \\ &\leq 1 - \sum_{z \neq y} \Delta_{\alpha,\beta}(x)\alpha(z) + \Delta_{\alpha,\beta}(x) \\ &= 1 + \Delta_{\alpha,\beta}(x) \left(1 - \sum_{z \neq y} \alpha(z) \right) \\ &= 1 + \Delta_{\alpha,\beta}(x)\alpha(y) = 1, \end{aligned}$$

  where the second inequality follows from (5).

- Finally the case $\alpha(y) = \beta(y) = 0$ is trivial. □

Until now we have considered only one state x and only one relation (α, β). Typically, we will redirect outgoing transitions for a subset of the state space. For any state $x \in \mathcal{S}$, we might use more than one relation: we will denote by $\mathcal{R}_x$ the set of all relations that will be used for a state $x \in \mathcal{S}$. (If the outgoing transitions for a state $x \in \mathcal{S}$ do not change, we will set $\mathcal{R}_x := \emptyset$.) Note that the same relation (α, β) could be applied to different states x and x' (i.e. $(\alpha, \beta) \in \mathcal{R}_x \cap \mathcal{R}_{x'}$). In that case the corresponding weight factors $\Delta_{(\alpha,\beta)}(x)$ and $\Delta_{(\alpha,\beta)}(x')$ need not to be equal. Also, there could be different relations (α, β) and (α', β') that have the same supports A and B; we might even have that $A \preceq_{(\alpha,\beta)} B$, but $B \preceq_{(\alpha',\beta')} A$ (if $\mathrm{supp}(\alpha) = \mathrm{supp}(\beta') = A$ and $\mathrm{supp}(\beta) = \mathrm{supp}(\alpha') = B$). Thus our method can be made arbitrarily general. The following theorem states that under similar conditions as (3) and (4) (for all states and for a family of precedence relations), the t-period rewards $\widetilde{v}_t$ of the modified chain satisfy:

$$v_t(x) \leq \widetilde{v}_t(x),\ x \in \mathcal{S}, t \geq 0.$$

Theorem 3. *Let $\{X_n\}$ be an irreducible, aperiodic and positive recurrent DTMC with transition probability matrix P and a reward r that is bounded from below:*

$$\exists m \in \mathbb{R},\ r(x) \geq m,\ \forall x \in \mathcal{S}. \tag{6}$$

Denote by $v_t,\ t \geq 0$, the corresponding t-period rewards. Let $\mathcal{R}$ be a set of couples of stochastic vectors such that for all pairs $(\alpha, \beta) \in \mathcal{R}$:

$$\sum_{y \in \mathcal{S}} \alpha(y) v_t(y) \leq \sum_{y \in \mathcal{S}} \beta(y) v_t(y),\ t \geq 0. \tag{7}$$

Let $\mathcal{R}_x \subset \mathcal{R},\ x \in \mathcal{S}$ (the precedence relations that will be applied to a state $x \in \mathcal{S}$). Let $\{Y_n\}$ be a DTMC with transition probability matrix $\widetilde{P}$ given by:

$$\widetilde{P} = P + \sum_{x \in \mathcal{S}} \sum_{(\alpha,\beta) \in \mathcal{R}_x} \Delta_{\alpha,\beta}(x) T_{\alpha,\beta}(x),$$

where the factors $\Delta_{\alpha,\beta}(x),\ x \in \mathcal{S},\ (\alpha, \beta) \in \mathcal{R}_x$, satisfy:

$$0 \leq P[x, y] + \sum_{(\alpha,\beta) \in \mathcal{R}_x} \Delta_{\alpha,\beta}(x)(\beta(y) - \alpha(y)) \leq 1,\ x, y \in \mathcal{S} \tag{8}$$

(i.e. $0 \leq \widetilde{P}[x, y] \leq 1,\ x, y \in \mathcal{S}$).

Then the t-period rewards $\widetilde{v}_t$ of the modified chain satisfy:

$$v_t(x) \leq \widetilde{v}_t(x),\ x \in \mathcal{S},\ t \geq 0. \tag{9}$$

Symmetrically, if (7) is replaced by:

$$\sum_{y \in \mathcal{S}} \alpha(y) v_t(y) \geq \sum_{y \in \mathcal{S}} \beta(y) v_t(y),\ t \geq 0, \tag{10}$$

for all $(\alpha, \beta) \in \mathcal{R}$, then the t-period rewards $\widetilde{v}_t$ of the modified chain satisfy:

$$v_t(x) \geq \widetilde{v}_t(x),\ x \in \mathcal{S},\ t \geq 0. \tag{11}$$

Proof. We will prove (9) by induction on t. For $t = 0$ we have $v_0(x) = \widetilde{v}_0(x) := 0,\ x \in \mathcal{S}$, so (9) is trivially satisfied. Suppose (9) is satisfied for $t \geq 0$. Then for $t+1$ we have:

$$\begin{aligned}
\widetilde{v}_{t+1}(x) &= r(x) + \sum_{y \in \mathcal{S}} \widetilde{P}[x,y]\widetilde{v}_t(y) \\
&\geq r(x) + \sum_{y \in \mathcal{S}} \widetilde{P}[x,y] v_t(y) \\
&\geq r(x) + \sum_{y \in \mathcal{S}} \left(P[x,y] + \sum_{(\alpha,\beta) \in \mathcal{R}_x} \Delta_{\alpha,\beta}(x) T_{\alpha,\beta}(x)[x,y] \right) v_t(y).
\end{aligned}$$

Relation (8) implies that for all $y \in \mathcal{S}$ the series absolutely converges (in $\mathbb{R}^+ \cup \{+\infty\}$) if v_t is bounded from below. (Note that if r is bounded from below then v_t is positive or bounded from below for each t.) Thus, simplifying and interchanging summation:

$$\begin{aligned}
\widetilde{v}_{t+1}(x) &\geq v_{t+1}(x) + \sum_{(\alpha,\beta) \in \mathcal{R}_x} \Delta_{\alpha,\beta}(x) \sum_{y \in \mathcal{S}} \left(\beta(y) - \alpha(y)\right) v_t(y) \\
&\geq v_{t+1}(x).
\end{aligned}$$

□

Corollary 4. *Under the same conditions as in Theorem 3, the mean stationary reward (when it exists) of the modified chain $\{Y_n\}$ is an upper bound (a lower bound in case of (10)) for the mean stationary reward of the original chain $\{X_n\}$.*

3.2 Proving the relations

Thus the steps in order to prove a bound are to first identify the set $\mathcal{R}$, and then to prove the corresponding relations for the t-period rewards. We will illustrate this steps on our simple example of a queue with batch arrivals, discussed in Example 1.

Example 2. Consider again the two models from Example 1. We will show that:

$$v_t(x) \leq \widetilde{v}_t(x),\ x \geq 0,\ t \geq 0, \tag{12}$$

where v_t denotes the t-period rewards for the original chain and $\widetilde{v}_t$ for the modified chain for reward function $r(x) = x,\ \forall x$. For each $x \geq 0$, we want to replace a part of the transition that goes to $x+1$ by two new transitions that go to x and $x+2$. We will define vectors α_x and β_x as follows:

$$\alpha_x(y) = \delta_{x+1}(y), \quad \beta_x(y) = \frac{1}{2}\left(\delta_x(y) + \delta_{x+2}(y)\right),\ y \in \mathcal{S}.$$

Let $\mathcal{R}_x = \{(\alpha_x, \beta_x)\},\ x \in \mathcal{S}$. Then $\mathcal{R} = \cup_{x \in \mathcal{S}} \mathcal{R}_x = \{(\alpha_x, \beta_x)\ :\ x \in \mathcal{S}\}$. Furthermore, let:

$$\Delta(x) := \Delta_{\alpha_x,\beta_x}(x) = \epsilon,\ x \in \mathcal{S},$$

with $0 < \epsilon \leq \lambda_1$. Let P be the transition probability matrix of the original discrete time model. The transition matrix of the modified chain is then given by:

$$\widetilde{P} = P + \sum_{x \in \mathcal{S}} \sum_{(\alpha,\beta) \in \mathcal{R}_x} \Delta_{\alpha,\beta}(x) T_{\alpha,\beta}(x) = P + \sum_{x \in \mathcal{S}} \epsilon T_{\alpha_x,\beta_x}(x).$$

Relation (7) for $(\alpha_x, \beta_x),\ x \geq 0$, is equivalent to convexity of functions $v_t,\ t \geq 0$:

$$v_t(x+1) \leq \frac{1}{2}v_t(x+2) + \frac{1}{2}v_t(x), x \geq 0,\ t \geq 0.$$

Thus, if we prove that $v_t,\ t \geq 0$ are convex, then Theorem 3 implies (12), as our reward function r is positive, and (8) holds from the definition of ϵ.

In the proof of convexity of $v_t,\ t \geq 0$, we will also use an additional property. We will show by induction on t that for each $t \geq 0$, the function v_t is:

1. Non-decreasing: $v_t(x) \leq v_t(x+1), x \geq 0$.
2. Convex: $2v_t(x+1) \leq v_t(x+2) + v_t(x), x \geq 0$.

Assume this holds for a given $t \geq 0$ (for $t = 0$, $v_0 := 0$ is obviously non-decreasing and convex). Then for $t+1$ we have:

$$v_{t+1}(x) = r(x) + \lambda_1 v_t(x+1) + \lambda_2 v_t(x+2) + \mu v_t(x-1)1_{\{x>0\}} + \mu v_t(x)1_{\{x=0\}},\ x \geq 0.$$

We consider separately one period rewards, arrivals of each type, and service.

- *One period rewards.* $v_1 = r$ is obviously non-decreasing and convex.
- *Type* 1 *arrivals.*
  - Non-decreasing. For $x \geq 0$, $A_1(x+1,t) - A_1(x,t) = v_t(x+2) - v_t(x+1) \geq 0$, since v_t is non-decreasing.
  - Convex. For $x \geq 0$, $A_1(x+2,t)+A_1(x,t)-2A_1(x+1,t) = v_t(x+3)+v_t(x+1)-2v_t(x+2) \geq 0$, since v_t is convex.
- *Type* 2 *arrivals.*
  - Non-decreasing. For $x \geq 0$, $A_2(x+1,t) - A_2(x,t) = v_t(x+3) - v_t(x+2) \geq 0$, since v_t is non-decreasing.
  - Convex. For $x \geq 0$, $A_2(x+2,t)+A_2(x,t)-2A_2(x+1,t) = v_t(x+4)+v_t(x+2)-2v_t(x+3) \geq 0$, since v_t is convex.
- *Service.*
  - Non-decreasing. For $x \geq 0$, $S(x+1,t)-S(x,t) = v_t(x)-v_t(x-1)1_{\{x>0\}}-v_t(x)1_{\{x=0\}} = (v_t(x)-v_t(x-1))1_{\{x>0\}} \geq 0$, since v_t is non-decreasing.
  - Convex. For $x \geq 0$, $S(x+2,t)+S(x,t)-2S(x+1,t) = v_t(x+1)+v_t(x-1)1_{\{x>0\}}+v_t(x)1_{\{x=0\}}-2v_t(x) = 1_{\{x>0\}}(v_t(x+1)+v_t(x-1)-2v_t(x))+1_{\{x=0\}}(v_t(x+1)-v_t(x)) \geq 0$, since v_t is non-decreasing and convex.

  Thus v_{t+1} is non-decreasing and convex.

Applying Theorem 3 we have $v_t(x) \leq \widetilde{v}_t(x),\ x \geq 0,\ t \geq 0$, that is the number of jobs in the system increases if we have more variable arrivals.

Remark 2. The goal of this example is only to illustrate the generalization of precedence relation method. Note that the above result can be also obtained by using stochastic recurrences and icx-order (see for instance [2]).

A primary use of precedence relations is to enable bounds to be established by analyzing simpler (smaller) systems. As mentioned in the introduction, one common simplification of a chain is to reduce its state space using aggregated bounds. We examine this technique next.

4 Aggregation

In this and the following sections we assume that the state space of the chain is finite, as we will use results in [9] on the aggregation of finite Markov chains. Let $\mathcal{C} = \{C_k\}_{k\in K}$ be a partition of the state space S into macro-states:

$$\cup_{k\in K} C_k = \mathcal{S}, \quad C_i \cap C_j = \emptyset,\ \forall i \neq j.$$

Definition 1. [9] A Markov chain $X = \{X_n\}_{n\geq 0}$ is strongly aggregable (or lumpable) with respect to partition $\mathcal{C}$ if the process obtained by merging the states that belong to the same set into one state is still a Markov chain, for all initial distributions of X_0.

There are necessary and sufficient conditions for a chain to be aggregable:

Theorem 5 (Matrix characterization [9, Theorem 6.3.2])**.** *A DTMC $X = \{X_n\}_{n\geq 0}$ with probability transition matrix P is strongly aggregable with respect to $\mathcal{C}$ if and only if :*

$$\forall i \in K, \forall j \in K, \sum_{y\in C_j} P[x,y] \text{ is constant for all } x \in C_i. \tag{13}$$

Then we can define a new (aggregated) chain $Y = \{Y_n\}_{n\geq 0}$ with transition matrix Q. For all $i, j \in K$:

$$Q[i,j] = \sum_{y\in C_j} P[x,y],\ x \in C_i.$$

There are many results on aggregation of Markov chains, however they primarily consider the steady state distribution. Surprisingly, we were not able to find the following simple property, so we provide it here with a proof.

Proposition 6. *Let $X = \{X_n\}_{n\geq 0}$ be a Markov chain satisfying (13) and $Y = \{Y_n\}_{n\geq 0}$ the aggregated chain. Let $r : \mathcal{S} \to \mathbb{R}$ be a reward function that is constant within each macro-state, i.e. there exist $r_k \in \mathbb{R},\ k \in K$ such that for all $k \in K$:*

$$r(x) = r_k,\ \forall x \in C_k.$$

Denote by v_t and w_t the t-period rewards for chains X and Y. Then for all $k \in K$:

$$v_t(x) = w_t(k),\ x \in C_k,\ t \geq 0. \tag{14}$$

Proof. We will show (14) by induction on t.

Suppose that (14) is satisfied for $t \geq 0$ (for $t = 0$ this is trivially satisfied). Then for $t+1$ and $k \in K$:

$$v_{t+1}(x) = r(x) + \sum_{y\in\mathcal{S}} P[x,y]v_t(y) = r_k + \sum_{j\in K}\sum_{y\in C_j} P[x,y]v_t(y).$$

By the induction hypothesis $v_t(y) = w_t(j),\ j \in K,\ y \in C_j$, and from Theorem 5, for $x \in C_k$, $\sum_{y\in C_j} P[x,y] = Q[k,j],\ j \in K$. Thus:

$$v_{t+1}(x) = r_k + \sum_{j\in K} Q[k,j]w_t(j) = w_{t+1}(k).$$

□

By taking the limit, the above result also gives us the equality of mean stationary rewards.

Corollary 7. *Let X and Y be two Markov chains satisfying the assumptions of Proposition 6. If both chains are irreducible and aperiodic, then the mean stationary rewards $a = \lim_{t\to\infty} \frac{v_t(x)}{t}, x \in \mathcal{S}$ and $\widetilde{a} = lim_{t\to\infty}\frac{w_t(k)}{t},\ k \in K$ satisfy: $a = \widetilde{a}$.*

5 An inventory system of service tools

In this section we illustrate our analytical technique - applying precedence relations on sets of states - on the example of an inventory system of service tools, introduced in [25]. We consider a multi-item inventory system; we denote by I the number of different item types. For each $i \in \{1, \ldots, I\}$, S_i denotes the maximal stock level for item type i: a total of S_i items of type i are always either in stock or being used for a maintenance action at one of the customers. Demands occur for set of items. Let A be any subset of $\{1, \ldots, I\}$. We assume that demands of set A follow a Poisson process with rate λ_A. When a demand occurs, items that are available in stock are delivered immediately, any others are delivered via an emergency shipment with a considerably higher cost. Items that are delivered together from stock return together after an exponential amount of time with rate μ: we have joint returns to stock. The service level in this problem is defined as the aggregate order fill rate, the percentage of orders for which *all* requested items can be delivered from stock immediately.

As the items that are delivered together return together, we need to keep track of the sets of items delivered together. Consider a very simple case of two different item types. Then the state of the system is given by a vector $(n_{\{1\}}, n_{\{2\}}, n_{\{1,2\}})$ where $n_{\{1\}}$ (resp. $n_{\{2\}}$) is the number of items of type 1 (resp. 2) at the customer that were delivered individually, and $n_{\{1,2\}}$ is the number of sets $\{1, 2\}$ at the customer that were delivered together and that will return together. Given, for example stock levels $S_1 = 1$, $S_2 = 2$, all possible states of the system are: $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, $(1, 1, 0)$, $(0, 0, 1)$, $(0, 2, 0)$, $(1, 2, 0)$ and $(0, 1, 1)$. Note that if set $\{1, 2\}$ is demanded, and item type 2 is out of stock, this becomes a demand for item type 1 (and similarly if item 1 is out of stock). The Markov chain for this case is given in Figure 5.

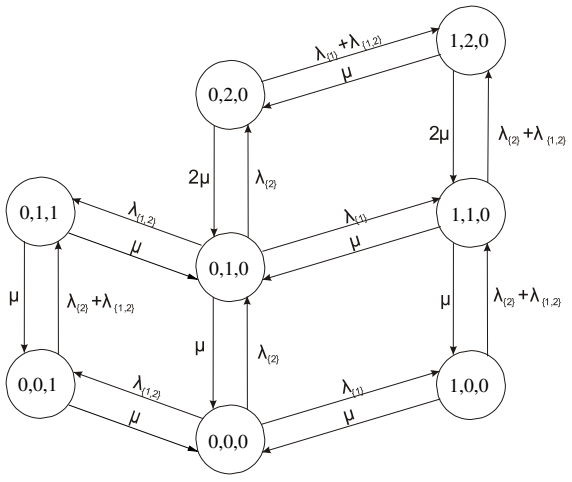

Figure 5: Markov chain for the original model for $I = 2$, $S_1 = 1$ and $S_2 = 2$.

For the general I-item case, the state of the system is given by a vector $n = (n_A)_{\emptyset \neq A \subset \{1,\ldots,I\}}$ where $n_A \geq 0$ is the number of sets A at the customer that were delivered together. For each

$i \in \{1, \ldots, I\}$, we denote by $\xi_i(n)$ the total number of items of type i at the customer:

$$\xi_i(n) = \sum_{A \subset \{1,\ldots,I\},\ i \in A} n_A. \tag{15}$$

The state space of the model is then:

$$\mathcal{S} = \left\{ n = (n_A)_{\emptyset \neq A \subset \{1,\ldots,I\}} \ : \ \xi_i(n) \leq S_i, \ \forall i \in \{1, \ldots, I\} \right\}.$$

For each $A \subset \{1, \ldots, I\}$, $A \neq \emptyset$, we will denote by e_A the state $n = (n_B)_{\emptyset \neq B \subset \{1,\ldots,I\}} \in \mathcal{S}$ such that $n_A = 1$ and $n_B = 0, \ B \neq A$.

We will consider the uniformized chain: without loss of generality, throughout the paper we assume that:

$$\sum_{\emptyset \neq A \subset \{1,\ldots,I\}} \lambda_A + \left(\sum_{i=1}^{I} S_i \right) \mu = 1.$$

Then for a state $n \in \mathcal{S}$ we have the following transitions:

- *Demands.* For all the subsets $A \subset \{1, \ldots, I\}$, $A \neq \emptyset$:
  - Probability: λ_A.
  - Destination: $d_A(n)$. For all $i \notin A$ the amount of items of type i at the customer stays the same. For $i \in A$, we can deliver an item of type i only if $\xi_i(n) < S_i$. Thus:

$$d_A(n) = n + e_{\{i \in A \,:\, \xi_i(n) < S_i\}}.$$

- *Returns.* For all the subsets $A \subset \{1, \ldots, I\}$ such that $n_A > 0$:
  - Probability: μn_A.
  - Destination: $r_A(n) = n - e_A$.
- *Uniformization.*
  - Probability: $\mu(\sum_{i=1}^{I} S_i - \sum_{\emptyset \neq A \subset \{1,\ldots,I\}} n_A)$.
  - Destination: n.

We will refer to this (original) model as M_0. Though its state space is finite, its dimension is equal to $2^I - 1$, thus the Markov chain becomes numerically intractable even for small values of I and S_i, $i \in \{1, \ldots, I\}$. For example, for $I = 5$ and $S_i = 5$, $\forall i$, the cardinality of the state space is $|\mathcal{S}| = 210\ 832\ 854$.

5.1 Models M_1 and M_2

The complexity of the original model in [25] comes from the need to track which items were delivered together. We will consider two extreme, simplifying cases: model M_1 assumes that all the items return individually, while model M_2 assumes returns of sets of maximal cardinality. Both of these cases remove the need to track which items were delivered together. Thus the state of the system for models M_1 and M_2 is fully described by the number of items at the customer for each item type:

$$x = (x_1, \ldots, x_I).$$

We have the following state space:

$$\mathcal{X} = \{x \ : \ 0 \leq x_i \leq S_i, \ i = 1, \ldots, I\},$$

and we will denote by e_i the state $x \in \mathcal{X}$ such that $x_j = 0, \ j \neq i$, and $x_i = 1$. The cardinality of $\mathcal{X}$, $|\mathcal{X}| = \prod_{i=1}^{I}(S_i + 1)$, is considerably lower than for the original model. For example, for $I = 5$ and $S_i = 5$, $\forall i$, we have $|\mathcal{X}| = 7776$ (compared to $|\mathcal{S}| = 210\ 832\ 854$).

Note that models M_1 and M_2 can be obtained from the original model in two steps:

1. By redirecting transitions that correspond to returns. We will denote by M_1' the model obtained from the original by replacing all joint returns by individual returns (see Figure 6, on the left). For example, in the original model in state $(0, 1, 1)$ we have one joint return of set $\{1, 2\}$ and a uniformization loop; these are replaced by two new transitions: one to state $(0, 2, 0)$ (corresponding to an individual return of item 1) and one to state $(1, 1, 0)$ (an individual return of item 2). Similarly, we denote by M_2' the model in which the returns are defined as follows : we greedily partition the set of items at the customers into disjoint sets; we have a return with probability μ for each of these sets (see Figure 6, on the right). For instance, in state $(1, 2, 0)$ we have one item of type 1 and two items of type 2 at the customer. The greedy partition gives the sets $\{1, 2\}$ and $\{2\}$. Thus in state $(1, 2, 0)$, we will have a return of set $\{1, 2\}$ and a return of set $\{2\}$, each with probability μ.

   Note that in this case the destination of new transitions is not uniquely specified: for example consider $I = 3$ and state $n = (n_{\{1\}}, n_{\{2\}}, n_{\{3\}}, n_{\{1,2\}}, n_{\{1,3\}}, n_{\{2,3\}}, n_{\{1,2,3\}}) = (0, 0, 0, 1, 1, 0, 0)$. Then in model M_0 we have a return of set $\{1, 2\}$ that goes to state $(0, 0, 0, 0, 1, 0, 0)$, and a return of set $\{2, 3\}$ that goes to state $(0, 0, 0, 1, 0, 0, 0)$. In M_2' we will have one return of the set of maximal cardinality $\{1, 2, 3\}$, and one return of set $\{1\}$ (left over after considering the return of $\{1, 2, 3\}$). The return of set $\{1, 2, 3\}$ goes to state $(1, 0, 0, 0, 0, 0, 0)$, but the return of the set $\{1\}$ can go to $(0, 1, 0, 0, 1, 0, 0)$, $(0, 0, 1, 1, 0, 0, 0)$, or $(1, 1, 1, 0, 0, 0, 0)$: we can choose any state m such that $\xi(m) = (1, 1, 1)$, see (15). To simplify the notation in Section 5.2, where the formal description of the transformation of the chain and the proof of the bounds will be given, we will assume that in model M_2' all the returns go to states with only individual items at the customer. In Figure 6 (on the right), the transition from state $(0, 1, 1)$ to $(0, 0, 1)$ is thus replaced by a transition to state $(1, 1, 0)$.

2. Noticing that the obtained models M_1' and M_2' are lumpable with respect to the partition of the state space induced by function $\xi = (\xi_i)_{i \in \{1,\ldots,I\}}$, see (15). The model M_1 is the lumped version of M_1' and M_2 is the lumped version of M_2'. We now need not track the history of joint demands, only the total number of items of each type at the customer.

We describe now in detail the transitions in models M_1 and M_2. Note that transitions corresponding to demands are the same in both models. Markov chains for case $I = 2$, $S_1 = 1$ and $S_2 = 2$ are given in Figure 7.

Model M_1. For a state $x \in \mathcal{X}$ we have the following transitions:

- *Demands.* For all the subsets $A \subset \{1, \ldots, I\}$, $A \neq \emptyset$:
  - Probability: λ_A.

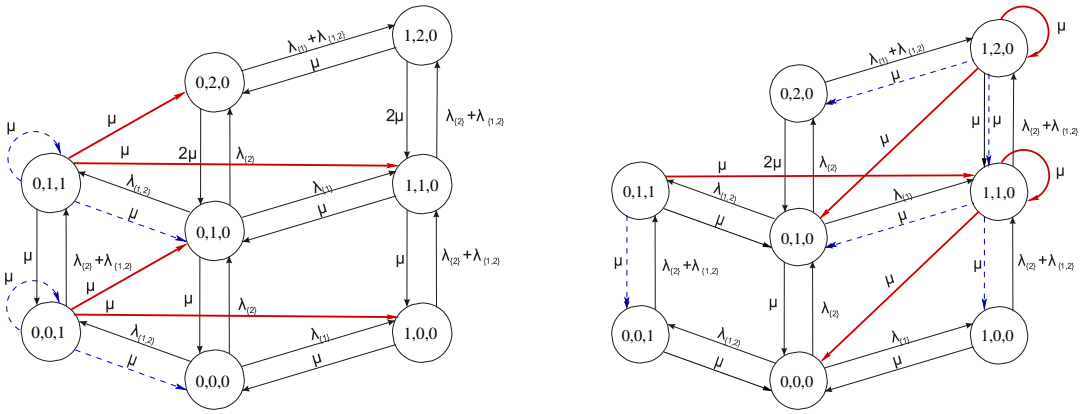

Figure 6: Markov chains M_1' (left) and M_2' (right) for $I = 2$, $S_1 = 1$ and $S_2 = 2$. The blue (dashed) transitions represent the transitions of the original model that have been replaced by red (bold) transitions. To simplify the figures, the (uniformization) loops are not shown when they are not a part of redirected transitions. See Section 5.2 for more formal description of the corresponding transformation of the chain.

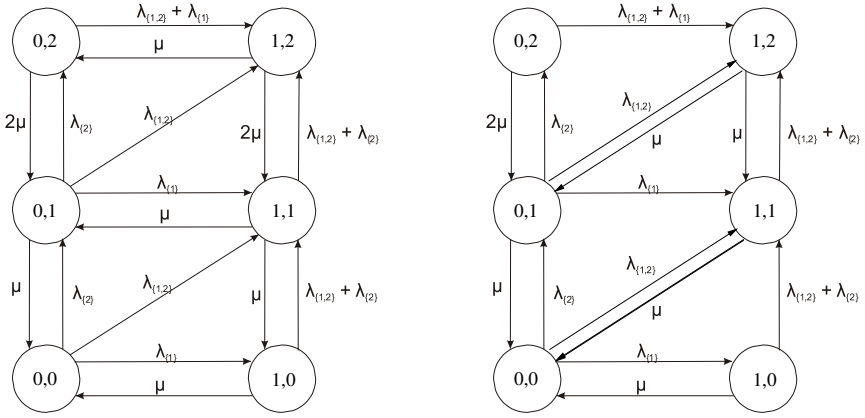

Figure 7: Markov chains for models M_1 (left) and M_2 (right) for $I = 2$, $S_1 = 1$ and $S_2 = 2$.

  - Destination: $d'_A(x)$. For all $i \notin A$ the amount of items of type i at the customer stays the same: $(d'_A(x))_i = x_i,\ i \notin A$. For $i \in A$, we can deliver an item of type i only if $x_i < S_i$: $(d'_A(x))_i = \min\{x_i + 1,\ S_i\},\ i \in A$. We can write the both cases together as:

$$d'_A(x) = x + \sum_{i \in A} 1_{\{x_i < S_i\}} e_i.$$

- *Returns.* We have only individual returns. Thus for each item type i, $1 \le i \le I$ we have the following transition:
  - Probability: μx_i.
  - Destination: $r'_i(x) = x - 1_{\{x_i > 0\}} e_i$.
- *Uniformization.*
  - Probability: $\mu \sum_{i=1}^{I} (S_i - x_i)$.
  - Destination: x.

Model M_2. For a state $x \in \mathcal{X}$ we have the following transitions:

- *Demands.* Same as in model M_1. For all the subsets $A \subset \{1, \ldots, I\}$, $A \neq \emptyset$:
  - Probability: λ_A.
  - Destination: $d''_A(x) = d'_A(x) = x + \sum_{i \in A} 1_{\{x_i < S_i\}} e_i$.
- *Returns.* We consider joint returns : we greedily partition the set of items at the customer into disjoint sets; each set corresponds to a return with probability μ. For example, if $I = 3$, $S_1 = S_2 = S_3 = 5$, then in state $x = (1, 5, 3)$ we have the following joint returns:
  - return of set $\{1, 2, 3\}$ with probability μ (remaining items at the customer: $(0, 4, 2)$),
  - return of set $\{2, 3\}$ with probability 2μ (remaining items at the customer: $(0, 2, 0)$),
  - return of item 2 with probability 2μ.

  Generally, for all the subsets $B \subset \{1, \ldots, I\}$, $B \neq \emptyset$:
  - Probability: $\mu[\min_{k \in B} x_k - \max_{k \notin B} x_k]^+$ (with $\max \emptyset := 0$).
  - Destination: $r''_B(x) = x - \sum_{k \in B} 1_{\{x_k > 0\}} e_k$.
- *Uniformization.*
  - Probability: $\mu(\sum_{k=1}^{I} S_k - \max_{k=1 \ldots I} x_k)$.
  - Destination: x.

In the following, we will show that model M_1 gives a lower bound and model M_2 an upper bound for the aggregate order fill rate of the original model.

5.2 Proof of the bounds

5.2.1 Model M_1

Let us first consider model M_1. It is obtained from the original model by replacing the returns of set of items by individual returns. In order to describe this transformation formally, for each $n \in \mathcal{S}$ and each $A \subset \{1, \ldots, I\}$ such that $|A| > 1$ and $n_A > 0$ we will define probability vectors $\alpha_{n,A}$ and $\beta_{n,A}$ as follows:

$$\alpha_{n,A} = \frac{1}{|A|} \left(\delta_{n - e_A} + (|A| - 1)\delta_n \right), \quad \beta_{n,A} = \frac{1}{|A|} \sum_{i \in A} \delta_{n - e_A + e_{A \setminus \{i\}}},$$

and the weight factor $\Delta_{n,A}$:

$$\Delta_{n,A} = \mu n_A |A|.$$

Let P be the transition matrix of the original chain. The transition matrix P'_1 of the chain M'_1 is then given by:

$$P'_1 = P + \sum_{n \in \mathcal{S}} \sum_{A \subset \{1, \ldots, I\} \,:\, n_A > 0} \mu n_A |A| T_{\alpha_{n,A}, \beta_{n,A}}(n).$$

For example, if $I = 2$ and $S_1 = 1$, $S_2 = 2$ (as in Figure 6, on the left), then for state $n = (0, 0, 1)$ and $A = \{1, 2\}$:

$$\begin{aligned} \alpha_{(0,0,1),\{1,2\}} &= \frac{1}{2} \left(\delta_{(0,0,1) - e_{\{1,2\}}} + \delta_{(0,0,1)} \right) = \frac{1}{2} \left(\delta_{(0,0,0)} + \delta_{(0,0,1)} \right), \\ \beta_{(0,0,1),\{1,2\}} &= \frac{1}{2} \left(\delta_{(0,0,1) - e_{\{1,2\}} + e_{\{2\}}} + \delta_{(0,0,1) - e_{\{1,2\}} + e_{\{1\}}} \right) = \frac{1}{2} \left(\delta_{(0,1,0)} + \delta_{(1,0,0)} \right), \end{aligned}$$

and $\Delta_{(0,0,1),\{1,2\}} = 2\mu$. Vectors $\alpha_{(0,0,1),\{1,2\}}$ and $\beta_{(0,0,1),\{1,2\}}$ formally describe redirection of outgoing transitions from state $(0,0,1)$ (see Figure 6, on the left): old (blue) transitions to states $(0,0,0)$ (joint return of set $\{1,2\}$) and $(0,0,1)$ (uniformization loop) are replaced by new (red) transitions to states $(0,1,0)$ (individual return of item type 1) and $(1,0,0)$ (individual return of item type 2). The corresponding weight factor $\Delta_{(0,0,1),\{1,2\}} = 2\mu$ states that the redirected amount of the transitions should be equal to $\alpha_{(0,0,1),\{1,2\}}\Delta_{(0,0,1),\{1,2\}} = P_{(0,0,1),(0,0,0)}$. Intuitively, we redirect entirely the transitions corresponding to joint returns. Note that after applying the above modification we still have a loop at state $(0,0,1)$, but with a modified probability: the new probability of the loop transition is now equal to μ, compared to 2μ in the original chain.

Let $r : \mathcal{S} \to \mathbb{R}$ be any reward function and let $v_t,\ t \geq 0$, denote the t-period rewards for the original model:

$$\begin{aligned} v_{t+1}(n) &= r(n) + \sum_{\emptyset \neq A \subset \{1,\ldots,I\}} \lambda_A v_t(d_A(n)) + \sum_{B \subset \{1,\ldots,I\}\,:\, n_B > 0} \mu n_B v_t(r_B(n)) \\ &\quad + \mu \left(\sum_{k=1}^{I} S_k - \sum_{\emptyset \neq A \subset \{1,\ldots,I\}} n_A \right) v_t(n),\ \forall n \in \mathcal{S}, \end{aligned}$$

where $v_0(n) := 0,\ n \in \mathcal{S}$. Similarly, denote by v'_t the t-period rewards for model M'_1. If we show that for all $n \in \mathcal{S}$ and for all $A \subset \{1,\ldots,I\}$ such that $n_A > 0$, functions v_t satisfy:

$$\sum_{k \in \mathcal{S}} \alpha_{n,A}(k) v_t(k) \geq \sum_{k \in \mathcal{S}} \beta_{n,A}(k) v_t(k),\ t \geq 0, \tag{16}$$

then by Theorem 3 it follows that:

$$v_t(n) \geq v'_t(n),\ n \in \mathcal{S},\ t \geq 0. \tag{17}$$

For $n \in \mathcal{S}$ and $A \subset \{1,\ldots,I\}$ such that $n_A > 0$, relation (16) is equivalent to:

$$v_t(n - e_A) + (|A| - 1) v_t(n) \geq \sum_{i \in A} v_t(n - e_A + e_{A\setminus\{i\}}),\ t \geq 0. \tag{18}$$

Due to the complex structure of the state space $\mathcal{S}$, relation (18) is difficult to check (and might even not hold). However, (18) is only a sufficient condition for (17). The "dual" sufficient condition for (17) it to show that for all $n \in \mathcal{S}$ and for all $A \subset \{1,\ldots,I\}$, $A \neq \emptyset$, functions v'_t satisfy:

$$v'_t(n - e_A) + (|A| - 1) v'_t(n) \geq \sum_{i \in A} v'_t(n - e_A + e_{A\setminus\{i\}}),\ t \geq 0. \tag{19}$$

Intuitively, instead of starting with model M_0 as the original model, we can start with model M'_1. Then the transformation of model M'_1 to model M_0 can be described using probability vectors $\alpha'_{n,A} = \beta_{n,A}$ and $\beta'_{n,A} = \alpha_{n,A}$, and the weight factor $\Delta'_{n,A} = \Delta_{n,A}$, for each $n \in \mathcal{S}$ and each $A \subset \{1,\ldots,I\}$ such that $|A| > 1$ and $n_A > 0$. Transition matrices P (model M_0) and P'_1 (model M'_1) clearly satisfy:

$$P = P'_1 + \sum_{n \in \mathcal{S}} \sum_{A \subset \{1,\ldots,I\}\,:\, n_A > 0} \mu n_A |A| T_{\alpha'_{n,A}, \beta'_{n,A}}(n).$$

Furthermore, relation (19) is clearly equivalent to:

$$\sum_{k \in \mathcal{S}} \alpha'_{n,A}(k) v'_t(k) \leq \sum_{k \in \mathcal{S}} \beta'_{n,A}(k) v'_t(k),\ t \geq 0. \tag{20}$$

So proving (19), and using Theorem 3, we will show that model M_0 gives an upper bound for model M_1', which is equivalent to showing that M_1' gives a lower bound for model M_0.

The advantage of (19) is the lumpability of model M_1'. Let r be a reward that is constant within every macro-state $\mathcal{C}_x = \xi^{-1}(x),\ x \in \mathcal{X}$, and denote this common value by $\widetilde{r}(x),\ x \in \mathcal{X}$:

$$r(n) = \widetilde{r}(x),\ \forall n \in C_x.$$

Let v_t and v_t' be as before, the t-period rewards for original model and model M_1', and let w_t be the t-period reward for model M_1:

$$w_{t+1}(x) = \widetilde{r}(x) + \sum_{\emptyset \neq A \subset \{1,\ldots,I\}} \lambda_A w_t(d_A'(x)) + \sum_{k=1}^{I} \mu x_k w_t(r_k'(x)) + \mu \left(\sum_{k=1}^{I} (S_k - x_k) \right) w_t(x), \quad (21)$$

for all $x \in \mathcal{X},\ t \geq 0$, where $w_0(x) := 0,\ x \in \mathcal{X}$. Now by Proposition 6, for all $x \in \mathcal{X}$:

$$v_t'(n) = w_t(x),\ \forall n \in \mathcal{C}_x. \quad (22)$$

This property allows us to consider the relations for model M_1, instead of M_1': relations (19) and (22) imply that to show (17), it is sufficient to show that for all $x \in \mathcal{S}$ and for all $A \subset \{1, \ldots, I\}$, $A \neq \emptyset$, functions w_t satisfy:

$$w_t(x - \sum_{i \in A} e_i) + (|A| - 1) w_t(x) \geq \sum_{i \in A} w_t(x - e_i),\ t \geq 0. \quad (23)$$

Relation (23) is equivalent to the supermodularity of functions $w_t,\ t \geq 0$ (see Proposition 13 in Appendix A).

Proposition 8. *Let $r : \mathcal{X} \to \mathbb{R}$ be any supermodular reward function and let w_t denote the corresponding t-period reward for model M_1. Then w_t is supermodular for all $t \geq 0$.*

The proof is given in Appendix B.

Theorem 9. *Let $r : \mathcal{S} \to \mathbb{R}$ be a reward function that is constant within every macro-state $\mathcal{C}_x = \xi^{-1}(x),\ x \in \mathcal{X}$, and define by:*

$$\widetilde{r}(x) = r(n),\ n \in \mathcal{C}_x.$$

Let $v_t : \mathcal{S} \to \mathbb{R}$ and $w_t : \mathcal{X} \to \mathbb{R}$, $t \geq 0$ be t-period rewards respectively for the original model and model M_1. If the reward function $\widetilde{r}$ is supermodular, then:

$$v_t(n) \geq w_t(x),\ n \in \mathcal{C}_x,\ t \geq 0,$$

and the mean stationary rewards satisfy:

$$a = \lim_{t \to \infty} \frac{v_t(n)}{t} \geq \lim_{t \to \infty} \frac{w_t(x)}{t} = \widetilde{a}.$$

Proof. Let $r : \mathcal{S} \to \mathbb{R}$ be a reward function that satisfies the assumptions of the theorem. Then Proposition 8 and Proposition 13 (in Appendix A) imply that the t-period rewards w_t, $t \geq 0$, for model M_1 satisfy (23). By (22), this is equivalent to (19) and to (20). The result now follows from Theorem 3 and its Corollary 4. □

5.2.2 Model M_2

The proof for model M_2 is similar to, yet more technical than, the proof for model M_1, as we need to compare individual and joint returns with the returns of maximal cardinality. Let $\widetilde{w}_t$ denote the t-period reward for model M_2:

$$\begin{aligned}\widetilde{w}_{t+1}(x) &= \widetilde{r}(x) + \sum_{\emptyset \neq A \subset \{1,\ldots,I\}} \lambda_A \widetilde{w}_t(d''_A(x)) + \sum_{\emptyset \neq B \subset \{1,\ldots,I\}} \mu[\min_{k\in B} x_k - \max_{k \notin B} x_k]^+ \widetilde{w}_t(r''_B(x)) \\ &\quad + \mu\left(\sum_{k=1}^{I} S_k - \max_{k=1\ldots I} x_k\right)\widetilde{w}_t(x),\ x \in \mathcal{X},\ t \geq 0, \end{aligned} \tag{24}$$

where $\widetilde{w}_0(x) := 0,\ x \in \mathcal{X}$.

Proposition 10. *Let $r : \mathcal{X} \to \mathbb{R}$ be any supermodular reward function and let $\widetilde{w}_t$ denote the corresponding t-period reward for model M_2. Then $\widetilde{w}_t$ is supermodular for all $t \geq 0$.*

The proof is given in Appendix B. By following similar steps as for model M_1, we obtain the following result for model M_2:

Theorem 11. *Let $r : \mathcal{S} \to \mathbb{R}$ be a reward function that satisfies the same conditions as in Theorem 9 and denote by $\widetilde{r}$ the corresponding reward function on $\mathcal{X}$: $\widetilde{r}(x) = r(n),\ n \in \mathcal{C}_x$. Let $v_t : \mathcal{S} \to \mathbb{R}$ and $\widetilde{w}_t : \mathcal{X} \to \mathbb{R},\ t \geq 0$ be t-period rewards respectively for the original model and model M_2. If the reward function $\widetilde{r}$ is supermodular, then:*

$$v_t(n) \leq \widetilde{w}_t(x),\ n \in \mathcal{C}_x,\ t \geq 0,$$

and the mean stationary rewards satisfy:

$$a = \lim_{t\to\infty} \frac{v_t(n)}{t} \leq \lim_{t\to\infty} \frac{\widetilde{w}_t(x)}{t} = \widetilde{a}.$$

Proof. We consider model M'_2 with a state space $\mathcal{S}$, defined as follows (see Figure 6, on the right):

- *Demands.* Demands are defined in the same way as the original model: for every $n \in X$ and every set $A \subset \{1,\ldots,I\},\ A \neq \emptyset$, there is a transition to state:

  $$n + e_{\{i\in A\,:\,\xi_i(n)<S_i\}},$$

  with probability λ_A.

- *Returns.* For each equivalence class of states $\mathcal{C}_x, x \in \mathcal{X}$ define as a representative state $n^x \in \mathcal{C}_x$ that has only individual items at the customer. Formally, for sets of items A such that $|A| > 1$, $n^x_A = 0$ and for sets $A = \{i\},\ i \in \{1,\ldots,I\},\ n^x_{\{i\}} = x_i$:

  $$n^x = \sum_{i=1}^{I} x_i e_{\{i\}}.$$

  Returns in model M'_2 from any state $n \in \mathcal{S}$ go only to the representative states $\{n^x,\ x \in \mathcal{X}\}$. Let $x \in \mathcal{X}$. For every state $n \in \mathcal{C}_x$ and every set $B \subset \{1,\ldots,I\},\ B \neq \emptyset$, there is a transition to state:

  $$n^{x - \sum_{k\in B} 1_{\{x_k>0\}} e_k},$$

  with probability $\mu[\min_{k\in B} x_k - \max_{k\notin B} x_k]^+$ (with $\max\emptyset := 0$).

- *Uniformization.* For $x \in \mathcal{X}$ and a state $n \in \mathcal{C}_x$, the probability of the uniformization loop is equal to: $\mu(\sum_{k=1}^{I} S_k - \max_{k=1\ldots I} x_k)$.

Then model M_2' is obviously aggregable and its aggregated model is exactly model M_2.

As before, we will start from model M_2' and show that model M_0 gives a lower bound for the aggregate order fill rate of model M_2'. Model M_0 can be obtained from M_2' by modifying the transitions that correspond to returns and uniformization. Denote by $\widetilde{v}_t'$ t-period rewards for model M_2'. We will show that for any $x \in \mathcal{S}$ and any $n \in C_x$:

$$\sum_{\emptyset \neq B \subset \{1,\ldots,I\}} [\min_{k\in B} x_k - \max_{k \notin B} x_k]^+ \widetilde{v}_t'(n^{x-\sum_{k\in B} 1_{\{x_k>0\}} e_k}) + \left(\sum_{\emptyset \neq A \subset \{1,\ldots,I\}} n_A - \max_{k=1}^{I} x_k \right) \widetilde{v}_t'(n) \geq$$
$$\sum_{\emptyset \neq A \subset \{1,\ldots,I\}} n_A \widetilde{v}_t'(n - 1_{\{n_A>0\}} e_A),\ t \geq 0. \tag{25}$$

The first term on the left-hand side of (25) corresponds to returns in M_2', and the second term is the difference in uniformization terms between model M_2' and M_0. The right-hand side of (25) corresponds to returns in M_0.

Relation (25) corresponds to the following probability vectors α_n, β_n:

$$\begin{aligned}
\alpha_n &= \frac{1}{\sum_{\emptyset \neq A \subset \{1,\ldots,I\}} n_A} \left(\sum_{\emptyset \neq B \subset \{1,\ldots,I\}} [\min_{k\in B} x_k - \max_{k \notin B} x_k]^+ \delta_{n^{x-\sum_{k\in B} 1_{\{x_k>0\}} e_k}} \right. \\
&\quad \left. + \left(\sum_{\emptyset \neq A \subset \{1,\ldots,I\}} n_A - \max_{k=1}^{I} x_k \right) \delta_n \right),\ x \in \mathcal{X},\ n \in C_x, \\
\beta_n &= \frac{1}{\sum_{\emptyset \neq A \subset \{1,\ldots,I\}} n_A} \sum_{\emptyset \neq A \subset \{1,\ldots,I\}} n_A \delta_{n - 1_{\{n_A>0\}} e_A},\ x \in \mathcal{X},\ n \in C_x,
\end{aligned}$$

and weight factors Δ_n:

$$\Delta_n = \mu \sum_{A \subset \{1,\ldots,I\}} n_A,\ x \in \mathcal{X},\ n \in C_x.$$

If we denote the transition matrix of model M_2' by P_2', then the transition matrix P of model M_0 can be obtained as:

$$P = P_2' + \sum_{x \in \mathcal{X}} \sum_{n \in C_x} \Delta_n T_{\alpha_n, \beta_n}(n).$$

Therefore, if we show that (25) holds, then Theorem 3 implies that model M_0 gives a lower bound for model M_2', which is equivalent to show that M_2' gives an upper bound for model M_0.

By Proposition 6, for any $x \in \mathcal{S}$ and any $n \in C_x$ relation (25) is equivalent to:

$$\sum_{\emptyset \neq B \subset \{1,\ldots,I\}} [\min_{k\in B} x_k - \max_{k \notin B} x_k]^+ \widetilde{w}_t(x - \sum_{k\in B} 1_{\{x_k>0\}} e_k) + \left(\sum_{\emptyset \neq A \subset \{1,\ldots,I\}} n_A - \max_{k=1}^{I} x_k \right) \widetilde{w}_t(x) \geq$$
$$\sum_{\emptyset \neq A \subset \{1,\ldots,I\}} n_A \widetilde{w}_t(x - \sum_{i\in A} e_i),\ t \geq 0. \tag{26}$$

We will show next that the above relation follows from supermodularity of $\widetilde{w}_t,\ t \geq 0$ (Proposition 10), which will end the proof.

As $\widetilde{w}_t, t \geq 0$, is supermodular, Proposition 13, relation (35), implies that for all $K \in \mathbb{N}$, $A_1, \ldots, A_K \subset \{1, \ldots, n\}$ and for all $x \in S$ such that $x - \sum_{k=1}^{K} e_{A_k} \in S$:

$$\sum_{j=1}^{K} \widetilde{w}_t \left(x - e_{\cup_{1 \leq i_1 < \ldots < i_j \leq K} (\cap_{l=1}^{j} A_{i_l})} \right) \geq \sum_{k=1}^{K} \widetilde{w}_t (x - e_{A_k}),\ t \geq 0, \tag{27}$$

where $e_A := \sum_{i \in A} e_i$. For $x \in \mathcal{S}$ and any $n \in C_x$, let $K = \sum_{\emptyset \neq A \subset \{1,\ldots,I\}} n_A$ and choose sets $A_1, \ldots, A_K$ in (27) such that for each subset $A \subset \{1, \ldots, I\}$, $A \neq \emptyset$, there are n_A subsets among $A_1, \ldots, A_K$ that are equal to A. For example, if $I = 2$, $x = (3, 5)$ and $n = (n_{\{1\}}, n_{\{2\}}, n_{\{1,2\}}) = (1, 3, 2)$, we set $K = 6$ and $A_1 = \{1\}$, $A_2 = A_3 = A_4 = \{2\}$, and $A_5 = A_6 = \{1, 2\}$.

For this collection of sets $A_1, \ldots, A_K$, the right-hand side in (27) is obviously equal to the right-hand side in (26), thus:

$$\sum_{j=1}^{K} \widetilde{w}_t \left(x - e_{\cup_{1 \leq i_1 < \ldots < i_j \leq K} (\cap_{l=1}^{j} A_{i_l})} \right) \geq \sum_{\emptyset \neq A \subset \{1,\ldots,I\}} n_A \widetilde{w}_t (x - e_A),\ t \geq 0.$$

We will show that the left-hand side in (27) is also equal to the left-hand side in (26). Denote by $H_j = \cup_{1 \leq i_1 < \ldots < i_j \leq K} (\cap_{l=1}^{j} A_{i_l})$, $1 \leq j \leq K$, the set of items that appear in at least j sets among $A_1, \ldots, A_K$. Clearly, $H_1 \supset H_2 \supset \ldots \supset H_K$. Consider now an arbitrary subset $B \subset \{1, \ldots, I\}$, and denote by $h(B)$ the number of sets H_j, $1 \leq j \leq K$, that are equal to B: $h(B) = \sum_{j=1}^{K} 1_{\{H_j = B\}}$. Then:

$$\sum_{j=1}^{K} \widetilde{w}_t \left(x - e_{\cup_{1 \leq i_1 < \ldots < i_j \leq K} (\cap_{l=1}^{j} A_{i_l})} \right) = \sum_{B \subset \{1,\ldots,I\}} h(B) \widetilde{w}_t (x - e_B),\ t \geq 0,$$

with $e_\emptyset := (0, \ldots, 0)$. We will show that:

$$h(B) = [\min_{k \in B} x_k - \max_{k \notin B} x_k]^+,\ B \neq \emptyset, \tag{28}$$

and

$$h(\emptyset) = \sum_{\emptyset \neq A \subset \{1,\ldots,I\}} n_A - \max_{k=1}^{I} x_k. \tag{29}$$

Relation (29) follows from the fact that there are in total $K = \sum_{\emptyset \neq A \subset \{1,\ldots,I\}} n_A$ sets H_j, $1 \leq j \leq K$, and the sets H_j, $1 \leq j \leq \max_{k=1}^{I} x_k$ are non empty (they contain at least the items $i \in \{1, \ldots, I\}$ such that $x_i = \max_{k=1}^{I} x_k$).

For $B \neq \emptyset$, denote by $\mathcal{G}(B) = \{C \subset \{1, \ldots, I\} \ :\ B \subset C\}$ the family of all subsets of items that contain set B. Then clearly $\sum_{C \in \mathcal{G}(B)} h(C) = \min_{k \in B} x_k$. We have two cases:

- If $B = \cup_{k=1}^{K} A_k$, then (28) follows from the fact that $\sum_{C \in \mathcal{G}(B)} h(C) = h(B) = \min_{k \in B} x_k$, and $\max_{k \notin B} x_k = 0$.
- If $B \neq \cup_{k=1}^{K} A_k$. For each $i \notin B$, $\sum_{C \in \mathcal{G}(B \cup \{i\})} h(C) = \min_{k \in B \cup \{i\}} x_k$, and

$$\sum_{C \in \mathcal{G}(B)} h(C) = h(B) + \max_{i \notin B} \left\{ \sum_{C \in \mathcal{G}(B \cup \{i\})} h(C) \right\}.$$

Thus:

$$\begin{aligned} h(B) &= \min_{k \in B} x_k - \max_{i \notin B} \left\{ \min_{k \in B \cup \{i\}} x_k \right\} = \min_{k \in B} x_k - \min \left\{ \min_{k \in B} x_k, \max_{i \notin B} x_i \right\} \\ &= [\min_{k \in B} x_k - \max_{i \notin B} x_i]^+, \end{aligned}$$

so (28) holds.

Therefore (26) holds, and so does (25). In other words, model M_2' is an upper bound for the original model, and so is the aggregated model M_2 by Proposition 6. □

It remains for us to show that the aggregate order fill rate is indeed a function satisfying the conditions of Theorems 9 and 11. The aggregated order fill rate is a linear combination of order fill rates for individual demand streams. Denote by $OFR_A : \mathcal{S} \to \{0, 1\}$ order fill rate for stream of set of items $A \subset \{1, \ldots, I\}$, $A \neq \emptyset$:

$$OFR_A(n) = \prod_{i \in A} 1_{\{\xi_i(n) < S_i\}},\ n \in \mathcal{S}.$$

Reward function OFR_A is clearly constant within every macro-state and we will denote also by $OFR_A : \mathcal{X} \to \{0, 1\}$ its aggregated version:

$$OFR_A(x) = \prod_{i \in A} 1_{\{x_i < S_i\}},\ x \in \mathcal{X}. \tag{30}$$

Lemma 12. *Reward function OFR_A is supermodular for all $A \subset \{1, \ldots, I\}$, $A \neq \emptyset$.*

Proof. Let $A \subset \{1, \ldots, I\}$, $A \neq \emptyset$, and $i, j \in \{1, \ldots, I\}$, $i \neq j$. We need to show that for all $x \in \mathcal{X}$ such that $x - e_i - e_j \in \mathcal{X}$ (see Proposition 13):

$$OFR_A(x - e_i - e_j) + OFR_A(x) \geq OFR_A(x - e_i) + OFR_A(x - e_j). \tag{31}$$

Suppose first that $i, j \in A$. We have 4 different cases:

- There is $k \in A \backslash \{i, j\}$ such that $x_k = S_k$. Then $OFR_A(x - e_i - e_j) = OFR_A(x) = OFR_A(x - e_i) = OFR_A(x - e_j) = 0$, and relation (31) clearly holds.
- For all $k \in A \backslash \{i, j\}$, $x_k < S_k$, $x_i = S_i$ and $x_j = S_j$. Then $OFR_A(x - e_i - e_j) = 1$ and all the other terms are equal to 0, thus relation (31) holds.
- For all $k \in A \backslash \{i, j\}$, $x_k < S_k$, $x_i = S_i$ and $x_j < S_j$ ($x_i < S_i$ and $x_j = S_j$ is symmetrical). Then $OFR_A(x - e_i - e_j) = OFR_A(x - e_i) = 1$ and the other two terms are equal to 0, so the both sides of relation (31) are equal to 1.
- Finally, if $x_k < S_k$, $\forall k \in A$, then the both sides of (31) are equal to 2.

If $i \notin A$, then $OFR_A(x - e_i - e_j) = OFR_A(x - e_j)$ and $OFR_A(x) = OFR_A(x - e_i)$ so (31) clearly holds. □

Thus the aggregated order fill rate is supermodular as an linear combination of supermodular functions.

6 Conclusions

We have established a new method to compare Markov chains: a generalization of the precedence relation method to sets of states, which we have shown is compatible with aggregation. The precedence relation method on sets of states, combined with aggregation, is then used to prove the bounds for the service tools problem, conjectured in [25].

The core advantage of precedence relations is still preserved: the modifications of the original model are easy to understand and allow intuitive interpretation. On the other hand, establishing precedence relations for *sets* of states allows one to construct bounding chains by replacing one set of transitions with another set. As illustrated on the example of a queue with batch arrivals, this can be used, for example, to compare systems with arrivals that have the same mean but different variability.

One can expect that our technique could be applied to derive bounds by replacing a part of the system with a simplified version having the same mean behavior. Note that this is not typically possible using some classical methods for Markov chain comparison, for example strong stochastic ordering or the classical precedence relation method. Along this line, the generalization of precedence relations to sets of states can be, to some extent, compared with the generalization of strong stochastic order to other integral orders, such as convex or increasing convex order. One possible future research direction is to compare the method presented here with comparison techniques based on stochastic monotonicity and different integral stochastic orders. One could expect, for selected families of functions and under certain conditions, that the two methods would be equivalent. If true, such an equivalence could allow on one hand the definition of a model-driven family of functions for which the precedence relations hold, and on the other hand enable the use of arguments of integral stochastic orders that allow both steady-state and transient comparison of Markov chains.

The second major contribution of the paper is showing that our new method is compatible with strong aggregation. This property allows the construction of bounding chains with a state space of significantly reduced cardinality. This much smaller chain is then used to derive bounds on the reward function. We have also shown that the precedence relations can be established both on the original or the aggregated bounding chain; in some cases the latter may be much easier. For example, in the service tools problem, we have shown that the precedence relations we need to compare the two chains are equivalent to the supermodularity property of cumulated rewards for the aggregated chain.

Finally, we studied here the service tools model in which all the returns have equal rates, which is a natural assumption in our application. It may be interesting to study a generalization of this model to allow different return rates. Note that the bounding models M_1 and M_2 strongly rely on the assumption of equal return rates. These models may still be used as bounds if the difference in rates are not too high. Otherwise, they will give very loose bounds and it is reasonable to expect that more accurate bounding models might need to be found.

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A Supermodularity and its characterization

Definition 2. Let $(S, \preceq)$ be a lattice and f a real function on S. Then f is said to be supermodular if

$$f(x \wedge y) + f(x \vee y) \geq f(x) + f(y), \ \forall x, y \in S. \tag{32}$$

In Proposition 13 we will give a characterization of supermodularity for the case of a finite-dimensional lattice. Without loss of generality, we will assume the set $(S, \preceq)$ to be a subset of $(\mathbb{N}^n, \leq)$. In that case, the $\wedge$ (meet) and $\vee$ (join) operators are defined componentwise:

$$\begin{aligned}(x_1, x_2, \ldots, x_n) \wedge (y_1, y_2, \ldots, y_n) &= (x_1 \wedge y_1, x_2 \wedge y_2, \ldots, x_n \wedge y_n),\\ (x_1, x_2, \ldots, x_n) \vee (y_1, y_2, \ldots, y_n) &= (x_1 \vee y_1, x_2 \vee y_2, \ldots, x_n \vee y_n),\end{aligned}$$

where $x_i \wedge y_i = \min\{x_i, y_i\}$ and $x_i \vee y_i = \max\{x_i, y_i\}$, for all i.

Before stating the proposition, we introduce some additional notation that will be used. We recall that e_i, $1 \leq i \leq n$, denotes the vector with all the coordinates equal to 0, except the coordinate i that is equal to 1. Similarly, we will denote by e_A the vector with all the coordinates equal to 0, except the coordinates that belong to the set A: $e_A := \sum_{i \in A} e_i$, $A \subset \{1, \ldots, n\}$, $A \neq \emptyset$, and $e_\emptyset := (0, \ldots, 0)$. Finally, let $A_1, \ldots, A_K$ be any collection of sets such that $K \in \mathbb{N}$, $A_1, \ldots, A_K \subset \{1, \ldots, n\}$, and $A_i \neq \emptyset$, $1 \leq i \leq K$. Then for each j, $1 \leq j \leq K$, the set $H_j := \cup_{1 \leq i_1 < \ldots < i_j \leq K} (\cap_{l=1}^{j} A_{i_l})$ is the set of the elements in $\{1, \ldots, n\}$ that appear in at least j sets among the sets $A_1, \ldots, A_K$. For example, if $n = 5$, $K = 3$, $A_1 = \{1, 2, 4\}$, $A_2 = \{2, 3, 4\}$, and $A_3 = \{3, 5\}$, then:

$$\begin{aligned}H_1 &= A_1 \cup A_2 \cup A_3 = \{1, 2, 3, 4, 5\},\\ H_2 &= (A_1 \cap A_2) \cup (A_1 \cap A_3) \cup (A_2 \cap A_3) = \{2, 3, 4\},\\ H_3 &= (A_1 \cap A_2 \cap A_3) = \emptyset.\end{aligned}$$

Proposition 13. *Let $(S, \preceq)$ be a subspace of $(\mathbb{N}^n, \leq)$ and $f : S \to \mathbb{R}$. The following statements are equivalent:*

1. *f is supermodular.*

2. *For all* $i, j \in \{1, \ldots, n\}$, $i \neq j$, *and for all* $x \in S$ *such that* $x - e_i - e_j \in S$*:*

$$f(x - e_i - e_j) + f(x) \geq f(x - e_i) + f(x - e_j). \tag{33}$$

3. *For all* $A \subset \{1, \ldots, n\}$, $A \neq \emptyset$, *and for all* $x \in S$ *such that* $x - \sum_{i \in A} e_i \in S$*:*

$$f(x - \sum_{i \in A} e_i) + (|A| - 1) f(x) \geq \sum_{i \in A} f(x - e_i). \tag{34}$$

4. *For all* $K \in \mathbb{N}$, $A_1, \ldots, A_K \subset \{1, \ldots, n\}$, $A_i \neq \emptyset$, $1 \leq i \leq K$, *and for all* $x \in S$ *such that* $x - \sum_{k=1}^{K} e_{A_k} \in S$*:*

$$\sum_{j=1}^{K} f\left(x - e_{\cup_{1 \leq i_1 < \ldots < i_j \leq K} (\cap_{l=1}^{j} A_{i_l})}\right) \geq \sum_{k=1}^{K} f(x - e_{A_k}). \tag{35}$$

Proof. • 1) $\Rightarrow$ 4). We will show this implication by induction on K. For $K = 1$ relation (35) is trivially satisfied: $f(x - e_{A_1}) \geq f(x - e_{A_1})$. In order to better understand relation (35), we will write it explicitly also for $K = 2$:

$$f(x - e_{A_1 \cup A_2}) + f(x - e_{A_1 \cap A_2}) \geq f(x - e_{A_1}) + f(x - e_{A_2}).$$

This follows trivially from supermodularity of f, as $(x - e_{A_1}) \wedge (x - e_{A_2}) = x - e_{A_1 \cup A_2}$ and $(x - e_{A_1}) \vee (x - e_{A_2}) = x - e_{A_1 \cap A_2}$.

Assume now that relation (35) holds for some $K \geq 2$. Then for $K+1$ by induction hypothesis we have:

$$\sum_{k=1}^{K+1} f(x - e_{A_k}) \leq \sum_{j=1}^{K} f\left(x - e_{\cup_{1 \leq i_1 < \ldots < i_j \leq K} (\cap_{l=1}^{j} A_{i_l})}\right) + f(x - e_{A_{K+1}}).$$

Since:

(i) f is supermodular,

(ii) $\left(x - e_{\cup_{1 \leq i_1 \leq K} A_{i_1}}\right) \wedge \left(x - e_{A_{K+1}}\right) = x - e_{\cup_{1 \leq i_1 \leq K+1} A_{i_1}}$, and

(iii) $\left(x - e_{\cup_{1 \leq i_1 \leq K} A_{i_1}}\right) \vee \left(x - e_{A_{K+1}}\right) = x - e_{(\cup_{1 \leq i_1 \leq K} A_{i_1}) \cap A_{K+1}} = x - e_{\cup_{1 \leq i_1 \leq K} (A_{i_1} \cap A_{K+1})}$,

we obtain:

$$\begin{aligned} \sum_{k=1}^{K+1} f(x - e_{A_k}) \quad \leq \quad & \sum_{j=2}^{K} f\left(x - e_{\cup_{1 \leq i_1 < \ldots < i_j \leq K} (\cap_{l=1}^{j} A_{i_l})}\right) + f(x - e_{\cup_{1 \leq i_1 \leq K+1} A_{i_1}}) \\ & + f\left(x - e_{\cup_{1 \leq i_1 \leq K} (A_{i_1} \cap A_{K+1})}\right). \end{aligned} \tag{36}$$

Assume now that for some $1 \leq m < K$ we have shown (for $m = 1$ this is equivalent to (36)):

$$\begin{aligned} \sum_{k=1}^{K+1} f(x - e_{A_k}) \quad \leq \quad & \sum_{j=m+1}^{K} f\left(x - e_{\cup_{1 \leq i_1 < \ldots < i_j \leq K} (\cap_{l=1}^{j} A_{i_l})}\right) + \sum_{j=1}^{m} f\left(x - e_{\cup_{1 \leq i_1 < \ldots < i_j \leq K+1} (\cap_{l=1}^{j} A_{i_l})}\right) \\ & + f\left(x - e_{\cup_{1 \leq i_1 < \ldots < i_m \leq K} ((\cap_{l=1}^{m} A_{i_l}) \cap A_{K+1})}\right). \end{aligned} \tag{37}$$

As

$$\begin{aligned} & \left(\cup_{1\le i_1<\ldots<i_{m+1}\le K}(\cap_{l=1}^{m+1}A_{i_l})\right)\cup\left(\cup_{1\le i_1<\ldots<i_m\le K}((\cap_{l=1}^{m}A_{i_l})\cap A_{K+1})\right) \\ = \;& \cup_{1\le i_1<\ldots<i_{m+1}\le K+1}(\cap_{l=1}^{m+1}A_{i_l}) \end{aligned}$$

and

$$\begin{aligned} & \left(\cup_{1\le i_1<\ldots<i_{m+1}\le K}(\cap_{l=1}^{m+1}A_{i_l})\right)\cap\left(\cup_{1\le i_1<\ldots<i_m\le K}((\cap_{l=1}^{m}A_{i_l})\cap A_{K+1})\right) \\ = \;& \left(\cup_{1\le i_1<\ldots<i_{m+1}\le K}(\cap_{l=1}^{m+1}A_{i_l})\right)\cap\left(\cup_{1\le i_1<\ldots<i_m\le K}(\cap_{l=1}^{m}A_{i_l})\right)\cap A_{K+1} \\ = \;& \left(\cup_{1\le i_1<\ldots<i_{m+1}\le K}(\cap_{l=1}^{m+1}A_{i_l})\right)\cap A_{K+1}=\cup_{1\le i_1<\ldots<l_{m+1}\le K}((\cap_{l=1}^{m+1}A_{i_l})\cap A_{K+1}), \end{aligned}$$

we have:

$$\begin{aligned} & \left(x-e_{\cup_{1\le i_1<\ldots<i_{m+1}\le K}(\cap_{l=1}^{m+1}A_{i_l})}\right)\wedge\left(x-e_{\cup_{1\le i_1<\ldots<i_m\le K}((\cap_{l=1}^{m}A_{i_1})\cap A_{K+1})}\right) \\ = \;& x-e_{\cup_{1\le i_1<\ldots<i_{m+1}\le K+1}(\cap_{l=1}^{m+1}A_{i_l})} \end{aligned}$$

and

$$\begin{aligned} & \left(x-e_{\cup_{1\le i_1<\ldots<i_{m+1}\le K}(\cap_{l=1}^{m+1}A_{i_l})}\right)\vee\left(x-e_{\cup_{1\le i_1<\ldots<i_m\le K}((\cap_{l=1}^{m}A_{i_1})\cap A_{K+1})}\right) \\ = \;& x-e_{\cup_{1\le i_1<\ldots<l_{m+1}\le K}((\cap_{l=1}^{m+1}A_{i_l})\cap A_{K+1})}. \end{aligned}$$

Supermodularity of f and (37) thus imply:

$$\begin{aligned} \sum_{k=1}^{K+1}f(x-e_{A_k}) \;\le\; & \sum_{j=m+2}^{K}f\left(x-e_{\cup_{1\le i_1<\ldots<i_j\le K}(\cap_{l=1}^{j}A_{i_l})}\right)+\sum_{j=1}^{m+1}f\left(x-e_{\cup_{1\le i_1<\ldots<i_j\le K+1}(\cap_{l=1}^{j}A_{i_l})}\right) \\ & +f\left(x-e_{\cup_{1\le i_1<\ldots<l_{m+1}\le K}((\cap_{l=1}^{m+1}A_{i_l})\cap A_{K+1})}\right), \end{aligned}$$

so (37) is valid for any $m\le K$. Finally, for $m=K$ (37) gives:

$$\begin{aligned} \sum_{k=1}^{K+1}f(x-e_{A_k}) \;&\le\; \sum_{j=1}^{K}f\left(x-e_{\cup_{1\le i_1<\ldots<i_j\le K+1}(\cap_{l=1}^{j}A_{i_l})}\right)+f\left(x-e_{(\cap_{k=1}^{K}A_k)\cap A_{K+1}}\right) \\ &=\; \sum_{j=1}^{K+1}f\left(x-e_{\cup_{1\le i_1<\ldots<i_j\le K+1}(\cap_{l=1}^{j}A_{i_l})}\right), \end{aligned}$$

which is exactly what we needed to show.

- 4) $\Rightarrow$ 3). Consider an arbitrary but fixed subset $A\subset\{1,\ldots,n\}$, $A\neq\emptyset$, and a state $x\in S$ such that $x-\sum_{i\in A}e_i\in S$. Let $K=|A|$, and denote by $i_1,\ldots,i_K$ the elements of A. Define $A_k=\{i_k\}$, $k=1,\ldots,K$. Sets A_k, $k=1,\ldots,K$ are disjoint so for $j>1$ we have: $\cap_{l=1}^{j}A_{i_l}=\emptyset$, for all $i_1,\ldots,i_j\in A$ such that $i_1\neq\ldots\neq i_j$, and for $j=1$: $\cup_{i_1=1}^{K}A_{i_1}=A$. Thus (35) becomes:

$$f\left(x-e_A\right)+\sum_{j=2}^{K}f(x)\ge\sum_{k=1}^{K}f(x-e_{i_k}),$$

which is precisely (34).

- 3) $\Rightarrow$ 2) follows directly by taking $A=\{i,j\}$.

- 2) $\Rightarrow$ 1). Consider arbitrary two states $x, y \in S$. Then y can be written as:

$$y = x + \sum_{i \in A} \alpha_i e_i - \sum_{i \in B} \beta_i e_i,$$

where $A, B \subset \{1, \ldots, n\}$ and $A \cap B = \emptyset$. Then:

$$x \wedge y = x - \sum_{i \in B} \beta_i e_i, \text{ and } x \vee y = x + \sum_{i \in A} \alpha_i e_i.$$

If $B = \emptyset$, then $x \leq y$, which implies $x \wedge y = x$ and $x \vee y = y$. Therefore relation (32) is trivially satisfied. The case $A = \emptyset$ is similar. We consider now the non-trivial case where both $A \neq \emptyset$ and $B \neq \emptyset$. We will first show the following relation for arbitrary $i \in B$ and $j \in A$:

$$f(x - \beta_i e_i) + f(x + e_j) \geq f(x) + f(x + e_j - \beta_i e_i). \tag{38}$$

Indeed, the above relation can be obtained by adding the following relations (relation (33) for states $x + e_j - m e_i$, $0 \leq m \leq (\beta_i - 1)$):

$$\begin{aligned}
f(x - e_i) + f(x + e_j) &\geq f(x) + f(x + e_j - e_i) \\
f(x - 2e_i) + f(x + e_j - e_i) &\geq f(x - e_i) + f(x + e_j - 2e_i) \\
&\vdots \\
f(x - \beta_i e_i) + f(x + e_j - (\beta_i - 1)e_i) &\geq f(x - (\beta_i - 1)e_i) + f(x + e_j - \beta_i e_i)
\end{aligned}$$

Denote the elements of B by $B = \{b_1, b_2, \ldots, b_{|B|}\}$. Then adding the following relations (obtained by applying relation (38) for $k = 1, \ldots, |B|$ to state $x - \sum_{l=1}^{k-1} \beta_{b_l} e_{b_l}$, with $i = b_k$ and $j \in A$):

$$\begin{aligned}
f(x - \beta_{b_1} e_{b_1}) + f(x + e_j) &\geq f(x) + f(x + e_j - \beta_{b_1} e_{b_1}) \\
f(x - \beta_{b_1} e_{b_1} - \beta_{b_2} e_{b_2}) + f(x + e_j - \beta_{b_1} e_{b_1}) &\geq f(x - \beta_{b_1} e_{b_1}) + f(x + e_j - \beta_{b_1} e_{b_1} - \beta_{b_2} e_{b_2}) \\
&\vdots \\
f(x - \sum_{l=1}^{|B|} \beta_{b_l} e_{b_l}) + f(x + e_j - \sum_{l=1}^{|B|-1} \beta_{b_l} e_{b_l}) &\geq f(x - \sum_{l=1}^{|B|-1} \beta_{b_l} e_{b_l}) + f(x + e_j - \sum_{l=1}^{|B|} \beta_{b_l} e_{b_l})
\end{aligned}$$

gives:

$$f(x - \sum_{l=1}^{|B|} \beta_{b_l} e_{b_l}) + f(x + e_j) \geq f(x) + f(x + e_j - \sum_{l=1}^{|B|} \beta_{b_l} e_{b_l}),\ j \in A. \tag{39}$$

By adding the following equations (obtained by applying relation (39) for $k = 0, \ldots, \alpha_j - 1$ to state $x + k e_j$, $j \in A$):

$$\begin{aligned}
f(x - \sum_{l=1}^{|B|} \beta_{b_l} e_{b_l}) + f(x + e_j) &\geq f(x) + f(x + e_j - \sum_{l=1}^{|B|} \beta_{b_l} e_{b_l}) \\
f(x + e_j - \sum_{l=1}^{|B|} \beta_{b_l} e_{b_l}) + f(x + 2e_j) &\geq f(x + e_j) + f(x + 2e_j - \sum_{l=1}^{|B|} \beta_{b_l} e_{b_l}) \\
&\vdots \\
f(x + (\alpha_j - 1)e_j - \sum_{l=1}^{|B|} \beta_{b_l} e_{b_l}) + f(x + \alpha_j e_j) &\geq f(x + (\alpha_j - 1)e_j) + f(x + \alpha_j e_j - \sum_{l=1}^{|B|} \beta_{b_l} e_{b_l})
\end{aligned}$$

we obtain:

$$f(x-\sum_{l=1}^{|B|}\beta_{b_l}e_{b_l})+f(x+\alpha_j e_j)\geq f(x)+f(x+\alpha_j e_j-\sum_{l=1}^{|B|}\beta_{b_l}e_{b_l}),\ j\in A. \tag{40}$$

Denote the elements of A by $A=\{a_1,a_2,\ldots,a_{|A|}\}$. Then adding the following relations (obtained by applying relation (40) for $k=1,\ldots,|A|$ to state $x+\sum_{l=1}^{k-1}\alpha_{a_l}e_{a_l}$, with $j=a_k$):

$$\begin{aligned}
f(x-\sum_{l=1}^{|B|}\beta_{b_l}e_{b_l})+f(x+\alpha_{a_1}e_{a_1}) &\geq f(x)+f(x+\alpha_{a_1}e_{a_1}-\sum_{l=1}^{|B|}\beta_{b_l}e_{b_l})\\
f(x+\alpha_{a_1}e_{a_1}-\sum_{l=1}^{|B|}\beta_{b_l}e_{b_l})+f(x+\sum_{l=1}^{2}\alpha_{a_l}e_{a_l}) &\geq f(x+\alpha_{a_1}e_{a_1})\\
&\quad +f(x+\sum_{l=1}^{2}\alpha_{a_l}e_{a_l}-\sum_{l=1}^{|B|}\beta_{b_l}e_{b_l})\\
&\vdots\\
f(x+\sum_{l=1}^{|A|-1}\alpha_{a_l}e_{a_l}-\sum_{l=1}^{|B|}\beta_{b_l}e_{b_l})+f(x+\sum_{l=1}^{|A|}\alpha_{a_l}e_{a_l}) &\geq f(x+\sum_{l=1}^{|A|-1}\alpha_{a_l}e_{a_l})\\
&\quad +f(x+\sum_{l=1}^{|A|}\alpha_{a_l}e_{a_l}-\sum_{l=1}^{|B|}\beta_{b_l}e_{b_l})
\end{aligned}$$

gives:

$$f(x-\sum_{l=1}^{|B|}\beta_{b_l}e_{b_l})+f(x+\sum_{l=1}^{|A|}\alpha_{a_l}e_{a_l})\geq f(x)+f(x+\sum_{l=1}^{|A|}\alpha_{a_l}e_{a_l}-\sum_{l=1}^{|B|}\beta_{b_l}e_{b_l}),$$

what we needed to show.

□

B Supermodularity proof for models M_1 and M_2

Proof of Proposition 8. After Proposition 13, proving supermodularity of w_t is equivalent to showing that for all $i,j\in\{1,\ldots,I\}$, $i\neq j$, and for all $x\in\mathcal{X}$ such that $x-e_i-e_j\in\mathcal{X}$:

$$w_t(x-e_i-e_j)+w_t(x)\geq w_t(x-e_i)+w_t(x-e_j),\ \forall t\geq 0. \tag{41}$$

Recall that (relation (21)):

$$w_{t+1}(x)=\widetilde{r}(x)+\sum_{\emptyset\neq A\subset\{1,\ldots,I\}}\lambda_A w_t(d'_A(x))+\sum_{k=1}^{I}\mu x_k w_t(r'_k(x))+\mu\left(\sum_{k=1}^{I}(S_k-x_k)\right)w_t(x), \tag{42}$$

for all $x\in\mathcal{X},\ t\geq 0$, where $w_0(x):=0,\ x\in\mathcal{X}$.

We will show relation (41) by induction on t. Suppose that relation (41) holds for a given $t\geq 0$ (the case $t=0$ is trivial). We will show that then it also holds for $t+1$. Let $i,j\in\{1,\ldots,I\}$, $i\neq j$, be arbitrary and fixed. We need to show that for all $x\in\mathcal{X}$ such that $x-e_i-e_j\in\mathcal{X}$:

$$w_{t+1}(x-e_i-e_j)+w_{t+1}(x)\geq w_{t+1}(x-e_i)+w_{t+1}(x-e_j). \tag{43}$$

To simplify the discussion, we will consider demands separately.

- *Demands.* Consider a demand of an arbitrary and fixed subset $A \subset \{1, \ldots, I\}$, $A \neq \emptyset$. We will show that for all $x \in \mathcal{X}$ such that $x - e_i - e_j \in \mathcal{X}$:

$$w_t(d'_A(x - e_i - e_j)) + w_t(d'_A(x)) \geq w_t(d'_A(x - e_i)) + w_t(d'_A(x - e_j)). \tag{44}$$

Denote by $C \subset A$ the subset of item types that are out of stock in state x:

$$C = \{j \in A \ : \ x_j = S_j\}.$$

We have 3 different cases:

1. $i \notin C$ and $j \notin C$. Then (44) becomes:

$$w_t(x - e_i - e_j + \sum_{k \in A \setminus C} e_k) + w_t(x + \sum_{k \in A \setminus C} e_k) \geq w_t(x - e_i + \sum_{k \in A \setminus C} e_k) + w_t(x - e_j + \sum_{k \in A \setminus C} e_k),$$

which holds, by induction hypothesis, from relation (41) for state $x + \sum_{k \in A \setminus C} e_k$.

2. $i \in C$ and $j \notin C$ (case $i \notin C$ and $j \in C$ is symmetrical). Then (44) becomes:

$$w_t(x - e_j + \sum_{k \in A \setminus C} e_k) + w_t(x + \sum_{k \in A \setminus C} e_k) \geq w_t(x + \sum_{k \in A \setminus C} e_k) + w_t(x - e_j + \sum_{k \in A \setminus C} e_k),$$

which is trivialy satisfied.

3. Finally, if $i \in C$ and $j \in C$, then (44) becomes:

$$w_t(x + \sum_{k \in A \setminus C} e_k) + w_t(x + \sum_{k \in A \setminus C} e_k) \geq w_t(x + \sum_{k \in A \setminus C} e_k) + w_t(x + \sum_{k \in A \setminus C} e_k),$$

which is also trivialy satisfied.

- *Returns and uniformization.* Denote by $R_k(x),\ 1 \leq k \leq I$, the terms corresponding to returns of item type k in state x, and by $U(x)$ the uniformization term in state x (without the scalar factor μ):

$$R_k(x) = x_k w_t(r'_k(x)),\ 1 \leq k \leq I, \quad U(x) = \sum_{k=1}^{I} (S_k - x_k) w_t(x).$$

We will show that the function $\sum_{k=1}^{I} R_k + U$ is supermodular, i.e. that for all $x \in \mathcal{X}$ such that $x - e_i - e_j \in \mathcal{X}$:

$$\begin{aligned} & \sum_{k=1}^{I} R_k(x - e_i - e_j) + U(x - e_i - e_j) + \sum_{k=1}^{I} R_k(x) + U(x) \\ \geq \ & \sum_{k=1}^{I} R_k(x - e_i) + U(x - e_i) + \sum_{k=1}^{I} R_k(x - e_j) + U(x - e_j). \end{aligned} \tag{45}$$

For uniformization terms we have:

$$\begin{aligned}
& U(x-e_i-e_j)+U(x)-U(x-e_i)-U(x-e_j) \\
= & \sum_{k=1}^{I}(S_k-x_k+1_{\{k\in\{i,j\}\}})w_t(x-e_i-e_j)+\sum_{k=1}^{I}(S_k-x_k)w_t(x) \\
& -\sum_{k=1}^{I}(S_k-x_k+1_{\{k=i\}})w_t(x-e_i)-\sum_{k=1}^{I}(S_k-x_k+1_{\{k=j\}})w_t(x-e_j) \\
= & \left(\sum_{k=1}^{I}(S_k-x_k)\right)(w_t(x-e_i-e_j)+w_t(x)-w_t(x-e_i)-w_t(x-e_j)) \\
& +2w_t(x-e_i-e_j)-w_t(x-e_i)-w_t(x-e_j).
\end{aligned}$$

Relation (41) gives $w_t(x-e_i-e_j)+w_t(x)-w_t(x-e_i)-w_t(x-e_j)\geq 0$, thus:

$$U(x-e_i-e_j)+U(x)-U(x-e_i)-U(x-e_j)\geq 2w_t(x-e_i-e_j)-w_t(x-e_i)-w_t(x-e_j). \quad (46)$$

For returns of type k we have two cases:

– For $k\notin\{i,j\}$:

$$\begin{aligned}
& R_k(x-e_i-e_j)+R_k(x)-R_k(x-e_i)-R_k(x-e_j) \\
= & x_k w_t(x-e_i-e_j-1_{\{x_k>0\}}e_k)+x_k w_t(x-1_{\{x_k>0\}}e_k) \\
& -x_k w_t(x-e_i-1_{\{x_k>0\}}e_k)-x_k w_t(x-e_j-1_{\{x_k>0\}}e_k) \\
= & x_k\big(w_t(x-e_i-e_j-1_{\{x_k>0\}}e_k)+w_t(x-1_{\{x_k>0\}}e_k) \\
& -w_t(x-e_i-1_{\{x_k>0\}}e_k)-w_t(x-e_j-1_{\{x_k>0\}}e_k)\big).
\end{aligned}$$

Relation (41) for state $x-1_{\{x_k>0\}}e_k$ implies $w_t(x-e_i-e_j-1_{\{x_k>0\}}e_k)+w_t(x-1_{\{x_k>0\}}e_k)-w_t(x-e_i-1_{\{x_k>0\}}e_k)-w_t(x-e_j-1_{\{x_k>0\}}e_k)\geq 0$, thus:

$$R_k(x-e_i-e_j)+R_k(x)-R_k(x-e_i)-R_k(x-e_j)\geq 0,\ k\notin\{i,j\}. \quad (47)$$

– For $k=i$ (case $k=j$ is symmetrical):

$$\begin{aligned}
& R_i(x-e_i-e_j)+R_i(x)-R_i(x-e_i)-R_i(x-e_j) \\
= & (x_i-1)w_t(x-e_i-e_j-1_{\{x_i>1\}}e_i)+x_i w_t(x-e_i) \\
& -(x_i-1)w_t(x-e_i-1_{\{x_i>1\}}e_i)-x_i w_t(x-e_i-e_j).
\end{aligned}$$

For $x_i=1$, the above equation becomes:

$$R_i(x-e_i-e_j)+R_i(x)-R_i(x-e_i)-R_i(x-e_j)=w_t(x-e_i)-w_t(x-e_i-e_j),$$

and for $x_i>1$ we have:

$$\begin{aligned}
& R_i(x-e_i-e_j)+R_i(x)-R_i(x-e_i)-R_i(x-e_j) \\
= & (x_i-1)\big(w_t(x-2e_i-e_j)+w_t(x-e_i)-w_t(x-2e_i)-w_t(x-e_i-e_j)\big) \\
& +w_t(x-e_i)-w_t(x-e_i-e_j).
\end{aligned}$$

Relation (41) for state $x - e_i$ implies:

$$w_t(x - 2e_i - e_j) + w_t(x - e_i) - w_t(x - 2e_i) - w_t(x - e_i - e_j) \geq 0,$$

thus for $k = i$:

$$R_i(x - e_i - e_j) + R_i(x) - R_i(x - e_i) - R_i(x - e_j) \geq w_t(x - e_i) - w_t(x - e_i - e_j). \quad (48)$$

By symmetry, for $k = j$:

$$R_j(x - e_i - e_j) + R_j(x) - R_j(x - e_i) - R_j(x - e_j) \geq w_t(x - e_j) - w_t(x - e_i - e_j). \quad (49)$$

Now from (46), (47), (48) and (49) it follows that:

$$\begin{aligned} & \sum_{k=1}^{I} R_k(x - e_i - e_j) + U(x - e_i - e_j) + \sum_{k=1}^{I} R_k(x) + U(x) \\ & - \sum_{k=1}^{I} R_k(x - e_i) - U(x - e_i) - \sum_{k=1}^{I} R_k(x - e_j) - U(x - e_j) \\ \geq \; & 2w_t(x - e_i - e_j) - w_t(x - e_i) - w_t(x - e_j) \\ & + w_t(x - e_i) - w_t(x - e_i - e_j) + w_t(x - e_j) - w_t(x - e_i - e_j) = 0. \end{aligned}$$

Thus relation (45) holds.

Relation (43) follows follows now easily from relations (42), (44), (45) and supermodularity of reward function $\widetilde{r}$. □

Proof of Proposition 10. The proof is similar to the proof of Proposition 8. We will show, by induction on t, that for all $i, j \in \{1, \ldots, I\}$, $i \neq j$, and for all $x \in \mathcal{X}$ such that $x - e_i - e_j \in \mathcal{X}$:

$$\widetilde{w}_t(x - e_i - e_j) + \widetilde{w}_t(x) \geq \widetilde{w}_t(x - e_i) + \widetilde{w}_t(x - e_j), \; \forall t \geq 0. \quad (50)$$

Recall that (relation (24)):

$$\begin{aligned} \widetilde{w}_{t+1}(x) \; = \; & \widetilde{r}(x) + \sum_{\emptyset \neq A \subset \{1,\ldots,I\}} \lambda_A \widetilde{w}_t(d''_A(x)) + \sum_{\emptyset \neq B \subset \{1,\ldots,I\}} \mu[\min_{k \in B} x_k - \max_{k \notin B} x_k]^+ \widetilde{w}_t(r''_B(x)) \\ & + \mu \left(\sum_{k=1}^{I} S_k - \max_{k=1\ldots I} x_k \right) \widetilde{w}_t(x), \; x \in \mathcal{X}, \; t \geq 0, \end{aligned} \quad (51)$$

where $\widetilde{w}_0(x) := 0, \; x \in \mathcal{X}$. Suppose that relation (50) holds for a given $t \geq 0$ (the case $t = 0$ is trivial). We will show that then it also holds for $t + 1$. Let $i, j \in \{1, \ldots, I\}$, $i \neq j$, be arbitrary and fixed. We need to show that for all $x \in \mathcal{X}$ such that $x - e_i - e_j \in \mathcal{X}$:

$$\widetilde{w}_{t+1}(x - e_i - e_j) + \widetilde{w}_{t+1}(x) \geq \widetilde{w}_{t+1}(x - e_i) + \widetilde{w}_{t+1}(x - e_j). \quad (52)$$

- *Demands.* Demands in model M_2 have the same rate and destination as in model M_1, therefore the same arguments as in proof of Proposition 8 can be used to show the equivalent of relation (44) for the M_2 model: for all $A \subset \{1, \ldots, I\}$, $A \neq \emptyset$, and for all $x \in \mathcal{X}$ such that $x - e_i - e_j \in \mathcal{X}$:

$$\widetilde{w}_t(d''_A(x - e_i - e_j)) + \widetilde{w}_t(d''_A(x)) \geq \widetilde{w}_t(d''_A(x - e_i)) + \widetilde{w}_t(d''_A(x - e_j)). \quad (53)$$

- *Returns and uniformization.* Similar to the proof of Proposition 8, we denote by $\widetilde{R}_B(x)$, $B \subset \{1,\ldots,I\}$, $B \neq \emptyset$, the term corresponding to joint returns of set B in state x, and by $\widetilde{U}(x)$ the uniformization term in state x (without the scalar factor μ):

$$\widetilde{R}_B(x) = [\min_{k\in B} x_k - \max_{k\notin B} x_k]^+ \widetilde{w}_t(r''_B(x)),\ B \neq \emptyset, \quad \widetilde{U}(x) = \left(\sum_{k=1}^{I} S_k - \max_{k=1\ldots I} x_k\right) \widetilde{w}_t(x).$$

We will show that the function $\sum_{k=1}^{I} \widetilde{R}_k + \widetilde{U}$ is supermodular, i.e. that for all $x \in \mathcal{X}$ such that $x - e_i - e_j \in \mathcal{X}$:

$$\begin{aligned} & \sum_{\emptyset\neq B\subset\{1,\ldots,I\}} \widetilde{R}_B(x-e_i-e_j) + \widetilde{U}(x-e_i-e_j) + \sum_{\emptyset\neq B\subset\{1,\ldots,I\}} \widetilde{R}_B(x) + \widetilde{U}(x) \\ \geq & \sum_{\emptyset\neq B\subset\{1,\ldots,I\}} \widetilde{R}_B(x-e_i) + \widetilde{U}(x-e_i) + \sum_{\emptyset\neq B\subset\{1,\ldots,I\}} \widetilde{R}_B(x-e_j) + \widetilde{U}(x-e_j). \quad (54) \end{aligned}$$

For uniformization terms we have:

$$\begin{aligned} & \widetilde{U}(x-e_i-e_j) + \widetilde{U}(x) - \widetilde{U}(x-e_i) - \widetilde{U}(x-e_j) \\ = & \left(\sum_{k=1}^{I} S_k - \max_{k=1\ldots I}\{x_k - 1_{k\in\{i,j\}}\}\right) \widetilde{w}_t(x-e_i-e_j) + \left(\sum_{k=1}^{I} S_k - \max_{k=1\ldots I} x_k\right) \widetilde{w}_t(x) \\ & - \sum_{r\in\{i,j\}} \left(\sum_{k=1}^{I} S_k - \max_{k=1\ldots I}\{x_k - 1_{k=r}\}\right) \widetilde{w}_t(x-e_r). \quad (55) \end{aligned}$$

Denote by $M(x) = \max_{k=1\ldots I} x_k$. Let $C = \{k \mid x_k = M(x)\}$. We have 3 different cases:

- $C \not\subset \{i,j\}$. Then (55) becomes:

$$\begin{aligned} & \widetilde{U}(x-e_i-e_j) + \widetilde{U}(x) - \widetilde{U}(x-e_i) - \widetilde{U}(x-e_j) \\ = & \left(\sum_{k=1}^{I} S_k - M(x)\right) \left(\widetilde{w}_t(x-e_i-e_j) + \widetilde{w}_t(x) - \widetilde{w}_t(x-e_i) - \widetilde{w}_t(x-e_j)\right) \geq 0, \end{aligned}$$

  by induction hypothesis (relation (50)).

- $C = \{i,j\}$. Then (55) becomes:

$$\begin{aligned} & \widetilde{U}(x-e_i-e_j) + \widetilde{U}(x) - \widetilde{U}(x-e_i) - \widetilde{U}(x-e_j) \\ = & \left(\sum_{k=1}^{I} S_k - M(x)\right) (\widetilde{w}_t(x-e_i-e_j) + \widetilde{w}_t(x) - \widetilde{w}_t(x-e_i) - \widetilde{w}_t(x-e_j)) \\ & + \widetilde{w}_t(x-e_i-e_j) \geq \widetilde{w}_t(x-e_i-e_j). \end{aligned}$$

- $C = \{i\}$ (case $C = \{j\}$ is symmetrical). Then (55) becomes:

$$\begin{aligned} & \widetilde{U}(x-e_i-e_j) + \widetilde{U}(x) - \widetilde{U}(x-e_i) - \widetilde{U}(x-e_j) \\ = & \left(\sum_{k=1}^{I} S_k - M(x)\right) (\widetilde{w}_t(x-e_i-e_j) + \widetilde{w}_t(x) - \widetilde{w}_t(x-e_i) - \widetilde{w}_t(x-e_j)) \\ & + \widetilde{w}_t(x-e_i-e_j) - \widetilde{w}_t(x-e_i) \geq \widetilde{w}_t(x-e_i-e_j) - \widetilde{w}_t(x-e_i). \end{aligned}$$

All three cases can be now written together as:

$$\begin{aligned} & \widetilde{U}(x-e_i-e_j)+\widetilde{U}(x)-\widetilde{U}(x-e_i)-\widetilde{U}(x-e_j) \\ \geq\ & 1_{C\subset\{i,j\}}\widetilde{w}_t(x-e_i-e_j)-1_{C=\{i\}}\widetilde{w}_t(x-e_i)-1_{C=\{j\}}\widetilde{w}_t(x-e_j). \end{aligned} \tag{56}$$

Let us consider now the returns. Let $B\subset\{1,\ldots,I\}$, $B\neq\emptyset$, be arbitrary and fixed. Then for returns of set B we have:

$$\begin{aligned} & \widetilde{R}_B(x-e_i-e_j)+\widetilde{R}_B(x)-\widetilde{R}_B(x-e_i)-\widetilde{R}_B(x-e_j) \\ =\ & [\min_{k\in B}(x_k-1_{k\in\{i,j\}})-\max_{k\notin B}(x_k-1_{k\in\{i,j\}})]^+\widetilde{w}_t(x-e_i-e_j-\sum_{k\in B}1_{\{x_k>1_{\{k\in\{i,j\}\}}\}}e_k) \\ & +[\min_{k\in B}x_k-\max_{k\notin B}x_k]^+\widetilde{w}_t(x-\sum_{k\in B}1_{\{x_k>0\}}e_k) \\ & -\sum_{r\in\{i,j\}}[\min_{k\in B}(x_k-1_{k=r})-\max_{k\notin B}(x_k-1_{k=r})]^+\widetilde{w}_t(x-e_r-\sum_{k\in B}1_{\{x_k>1_{\{k=r\}}\}}e_k). \end{aligned}$$

Note that:

$$x-e_i-e_j-\sum_{k\in B}1_{\{x_k>1_{\{k\in\{i,j\}\}}\}}e_k=x-\sum_{k\in B}1_{\{x_k>0\}}e_k-\sum_{r\in\{i,j\}}1_{\{x_r>1_{\{r\in B\}}\}}e_r$$

and

$$x-e_r-\sum_{k\in B}1_{\{x_k>1_{\{k=r\}}\}}e_k=x-\sum_{k\in B}1_{\{x_k>0\}}e_k-1_{\{x_r>1_{\{r\in B\}}\}}e_r,\ r\in\{i,j\}.$$

Let

$$x'=x-\sum_{k\in B}1_{\{x_k>0\}}e_k.$$

Then the above relation becomes:

$$\begin{aligned} & \widetilde{R}_B(x-e_i-e_j)+\widetilde{R}_B(x)-\widetilde{R}_B(x-e_i)-\widetilde{R}_B(x-e_j) \\ =\ & [\min_{k\in B}(x_k-1_{k\in\{i,j\}})-\max_{k\notin B}(x_k-1_{k\in\{i,j\}})]^+\widetilde{w}_t(x'-\sum_{r\in\{i,j\}}1_{\{x_r>1_{\{r\in B\}}\}}e_r) \\ & +[\min_{k\in B}x_k-\max_{k\notin B}x_k]^+\widetilde{w}_t(x') \\ & -\sum_{r\in\{i,j\}}[\min_{k\in B}(x_k-1_{k=r})-\max_{k\notin B}(x_k-1_{k=r})]^+\widetilde{w}_t(x'-1_{\{x_r>1_{\{r\in B\}}\}}e_r). \end{aligned}$$

For an arbitrary subset $D\subset\{1,\ldots,I\}$ we define $l(D)$ (resp. $u(D)$) to be the minimal (resp. maximal) value of components of state x that belong to the set D:

$$l(D)=\min_{k\in D}x_k\ \text{ and }\ u(D)=\max_{k\in D}x_k.$$

We also denote by $L(D)$ (resp. $U(D)$) the components for which x reaches the minimal (resp. maximal) value:

$$L(D)=\{k\in D\ :\ x_k=l(D)\}\ \text{ and }\ U(D)=\{k\in D\ :\ x_k=u(D)\}.$$

Let $\overline{B} = \{k : 1 \leq k \leq I,\ k \notin B\}$ denote the complement of set B. Then:

$$[\min_{k\in B} x_k - \max_{k\notin B} x_k]^+ = [l(B) - u(\overline{B})]^+, \tag{57}$$

$$\begin{aligned} & [\min_{k\in B}(x_k - 1_{k\in\{i,j\}}) - \max_{k\notin B}(x_k - 1_{k\in\{i,j\}})]^+ \\ = & \begin{cases} [l(B) - 1 - u(\overline{B})]^+, & L(B)\cap\{i,j\} \neq \emptyset,\ U(\overline{B}) \not\subset \{i,j\} \\ [l(B) - u(\overline{B}) + 1]^+, & L(B)\cap\{i,j\} = \emptyset,\ U(\overline{B}) \subset \{i,j\} \\ [l(B) - u(\overline{B})]^+, & \text{otherwise} \end{cases} \end{aligned} \tag{58}$$

For $r \in \{i,j\}$:

$$[\min_{k\in B}(x_k - 1_{k=r}) - \max_{k\notin B}(x_k - 1_{k=r})]^+ = \begin{cases} [l(B) - 1 - u(\overline{B})]^+, & r \in L(B) \\ [l(B) - u(\overline{B}) + 1]^+, & U(\overline{B}) = \{r\} \\ [l(B) - u(\overline{B})]^+, & \text{otherwise} \end{cases} \tag{59}$$

Note first that if $l(B) < u(\overline{B})$, then (57), (58) and (59) become all equal to 0 and therefore:

$$\widetilde{R}_B(x - e_i - e_j) + \widetilde{R}_B(x) - \widetilde{R}_B(x - e_i) - \widetilde{R}_B(x - e_j) = 0.$$

If $l(B) = u(\overline{B})$, then we have:

$$[\min_{k\in B} x_k - \max_{k\notin B} x_k]^+ = [l(B) - u(\overline{B})]^+ = 0,$$

$$[\min_{k\in B}(x_k - 1_{k\in\{i,j\}}) - \max_{k\notin B}(x_k - 1_{k\in\{i,j\}})]^+ = \begin{cases} 1, & L(B)\cap\{i,j\} = \emptyset,\ U(\overline{B}) \subset \{i,j\} \\ 0, & \text{otherwise} \end{cases}$$

and for $r \in \{i,j\}$,

$$[\min_{k\in B}(x_k - 1_{k=r}) - \max_{k\notin B}(x_k - 1_{k=r})]^+ = \begin{cases} 1, & U(\overline{B}) = \{r\} \\ 0, & \text{otherwise} \end{cases}$$

Therefore, if $l(B) = u(\overline{B})$:

$$\begin{aligned} & \widetilde{R}_B(x - e_i - e_j) + \widetilde{R}_B(x) - \widetilde{R}_B(x - e_i) - \widetilde{R}_B(x - e_j) \\ = & 1_{\{L(B)\cap\{i,j\}=\emptyset,\, U(\overline{B})\subset\{i,j\}\}} \widetilde{w}_t(x' - \sum_{r\in\{i,j\}} 1_{\{x_r > 1_{\{r\in B\}}\}} e_r) \\ & - \sum_{r\in\{i,j\}} 1_{\{U(\overline{B})=\{r\}\}} \widetilde{w}_t(x' - 1_{\{x_r > 1_{\{r\in B\}}\}} e_r). \end{aligned}$$

The second term is non-zero only if $U(\overline{B}) = \{r\}$ for $r \in \{i,j\}$, that is:

$$\begin{aligned} & \widetilde{R}_B(x - e_i - e_j) + \widetilde{R}_B(x) - \widetilde{R}_B(x - e_i) - \widetilde{R}_B(x - e_j) \\ = & 1_{\{L(B)\cap\{i,j\}=\emptyset,\, U(\overline{B})\subset\{i,j\}\}} \widetilde{w}_t(x' - \sum_{r\in\{i,j\}} 1_{\{x_r > 1_{\{r\in B\}}\}} e_r) \\ & -1_{\{U(\overline{B})=\{r\}\subset\{i,j\}\}} \widetilde{w}_t(x' - e_r), \end{aligned} \tag{60}$$

where

$$x' = x - \sum_{k \in B} 1_{\{x_k > 0\}} e_k.$$

Finally, we consider the case $l(B) > u(\overline{B})$. We have:

$$\begin{aligned}
& \widetilde{R}_B(x - e_i - e_j) + \widetilde{R}_B(x) - \widetilde{R}_B(x - e_i) - \widetilde{R}_B(x - e_j) \\
= \; & (l(B) - u(\overline{B}) - 1_{\{L(B) \cap \{i,j\} \neq \emptyset,\, U(\overline{B}) \not\subset \{i,j\}\}} \\
& + 1_{\{L(B) \cap \{i,j\} = \emptyset,\, U(\overline{B}) \subset \{i,j\}\}}) \widetilde{w}_t(x' - \sum_{r \in \{i,j\}} 1_{\{x_r > 1_{\{r \in B\}}\}} e_r) \\
& + (l(B) - u(\overline{B})) \widetilde{w}_t(x') \\
& - \sum_{r \in \{i,j\}} (l(B) - u(\overline{B}) - 1_{\{r \in L(B)\}} + 1_{\{U(\overline{B}) = \{r\}\}}) \widetilde{w}_t(x' - 1_{\{x_r > 1_{\{r \in B\}}\}} e_r). \qquad (61)
\end{aligned}$$

We will first show the following relation:

$$\widetilde{w}_t(x' - \sum_{r \in \{i,j\}} 1_{\{x_r > 1_{\{r \in B\}}\}} e_r) + \widetilde{w}_t(x') - \sum_{r \in \{i,j\}} \widetilde{w}_t(x' - 1_{\{x_r > 1_{\{r \in B\}}\}} e_r) \geq 0. \qquad (62)$$

- If $r \in B$ and $x_r = 1$, for $r \in \{i, j\}$, then relation (62) becomes:

  $$\widetilde{w}_t(x') + \widetilde{w}_t(x') - 2\widetilde{w}_t(x') \geq 0,$$

  which is trivially satisfied.
- If ($i \in B$ and $x_i = 1$) and ($j \notin B$ or $x_j > 1$), then relation (62) becomes:

  $$\widetilde{w}_t(x' - e_j) + \widetilde{w}_t(x') - \widetilde{w}_t(x') - \widetilde{w}_t(x' - e_j) \geq 0,$$

  which is also trivially satisfied. The case ($i \notin B$ or $x_i > 1$) and ($j \in B$ and $x_j = 1$) is symmetrical.
- If $r \notin B$ or $x_r > 1$, for $r \in \{i, j\}$, then relation (62) becomes:

  $$\widetilde{w}_t(x' - e_i - e_j) + \widetilde{w}_t(x') - \widetilde{w}_t(x' - e_i) - \widetilde{w}_t(x' - e_j) \geq 0,$$

  which is satisfied by the induction hypothesis for state x'.

Therefore, relation (62) holds.

Let us now go back to relation (61). We have the following cases:

- $L(B) \cap \{i, j\} = \emptyset$. Then:

$$\begin{aligned}
& \widetilde{R}_B(x - e_i - e_j) + \widetilde{R}_B(x) - \widetilde{R}_B(x - e_i) - \widetilde{R}_B(x - e_j) \\
= \; & (l(B) - u(\overline{B}) + 1_{\{U(\overline{B}) \subset \{i,j\}\}}) \widetilde{w}_t(x' - \sum_{r \in \{i,j\}} 1_{\{x_r > 1_{\{r \in B\}}\}} e_r) + (l(B) - u(\overline{B})) \widetilde{w}_t(x') \\
& - \sum_{r \in \{i,j\}} (l(B) - u(\overline{B}) + 1_{\{U(\overline{B}) = \{r\}\}}) \widetilde{w}_t(x' - 1_{\{x_r > 1_{\{r \in B\}}\}} e_r) \\
= \; & (l(B) - u(\overline{B})) \Bigg(\widetilde{w}_t(x' - \sum_{r \in \{i,j\}} 1_{\{x_r > 1_{\{r \in B\}}\}} e_r) + \widetilde{w}_t(x') \\
& - \sum_{r \in \{i,j\}} \widetilde{w}_t(x' - 1_{\{x_r > 1_{\{r \in B\}}\}} e_r) \Bigg) + 1_{\{U(\overline{B}) \subset \{i,j\}\}} \widetilde{w}_t(x' - \sum_{r \in \{i,j\}} 1_{\{x_r > 1_{\{r \in B\}}\}} e_r) \\
& - 1_{\{U(\overline{B}) = \{r\} \subset \{i,j\}\}} \widetilde{w}_t(x' - e_r).
\end{aligned}$$

Then from relation (62) it follows that:

$$\begin{aligned}
&\widetilde{R}_B(x-e_i-e_j)+\widetilde{R}_B(x)-\widetilde{R}_B(x-e_i)-\widetilde{R}_B(x-e_j)\\
\geq\ & 1_{\{U(\overline{B})\subset\{i,j\}\}}\widetilde{w}_t(x'-\sum_{r\in\{i,j\}}1_{\{x_r>1_{\{r\in B\}}\}}e_r)-1_{\{U(\overline{B})=\{r\}\subset\{i,j\}\}}\widetilde{w}_t(x'-e_r).
\end{aligned}$$

– $L(B)\cap\{i,j\}\neq\emptyset$. Then:

$$\begin{aligned}
&\widetilde{R}_B(x-e_i-e_j)+\widetilde{R}_B(x)-\widetilde{R}_B(x-e_i)-\widetilde{R}_B(x-e_j)\\
=\ & (l(B)-u(\overline{B})-1_{\{U(\overline{B})\not\subset\{i,j\}\}})\widetilde{w}_t(x'-\sum_{r\in\{i,j\}}1_{\{x_r>1_{\{r\in B\}}\}}e_r)\\
&+(l(B)-u(\overline{B}))\widetilde{w}_t(x')\\
&-\sum_{r\in\{i,j\}}(l(B)-u(\overline{B})-1_{\{r\in L(B)\}}+1_{\{U(\overline{B})=\{r\}\}})\widetilde{w}_t(x'-1_{\{x_r>1_{\{r\in B\}}\}}e_r)\\
=\ & (l(B)-u(\overline{B})-1)\Bigg(\widetilde{w}_t(x'-\sum_{r\in\{i,j\}}1_{\{x_r>1_{\{r\in B\}}\}}e_r)+\widetilde{w}_t(x')\\
&-\sum_{r\in\{i,j\}}\widetilde{w}_t(x'-1_{\{x_r>1_{\{r\in B\}}\}}e_r)\Bigg)\\
&+1_{\{U(\overline{B})\subset\{i,j\}\}}\widetilde{w}_t(x'-\sum_{r\in\{i,j\}}1_{\{x_r>1_{\{r\in B\}}\}}e_r)+\widetilde{w}_t(x')\\
&-\sum_{r\in\{i,j\}}(1-1_{\{r\in L(B)\}}+1_{\{U(\overline{B})=\{r\}\}})\widetilde{w}_t(x'-1_{\{x_r>1_{\{r\in B\}}\}}e_r).
\end{aligned}$$

Then from relation (62) it follows that:

$$\begin{aligned}
&\widetilde{R}_B(x-e_i-e_j)+\widetilde{R}_B(x)-\widetilde{R}_B(x-e_i)-\widetilde{R}_B(x-e_j)\\
\geq\ & 1_{\{U(\overline{B})\subset\{i,j\}\}}\widetilde{w}_t(x'-\sum_{r\in\{i,j\}}1_{\{x_r>1_{\{r\in B\}}\}}e_r)+\widetilde{w}_t(x')\\
&-\sum_{r\in\{i,j\}}(1-1_{\{r\in L(B)\}}+1_{\{U(\overline{B})=\{r\}\}})\widetilde{w}_t(x'-1_{\{x_r>1_{\{r\in B\}}\}}e_r).
\end{aligned}$$

Therefore, if we put the both cases together we obtain:

$$\begin{aligned}
&\widetilde{R}_B(x-e_i-e_j)+\widetilde{R}_B(x)-\widetilde{R}_B(x-e_i)-\widetilde{R}_B(x-e_j)\\
\geq\ & 1_{\{U(\overline{B})\subset\{i,j\}\}}\widetilde{w}_t(x'-\sum_{r\in\{i,j\}}1_{\{x_r>1_{\{r\in B\}}\}}e_r)+1_{\{L(B)\cap\{i,j\}\neq\emptyset\}}\widetilde{w}_t(x')\\
&-1_{\{L(B)\cap\{i,j\}\neq\emptyset\}}\sum_{r\in\{i,j\}\setminus L(B)}\widetilde{w}_t(x'-1_{\{x_r>1_{\{r\in B\}}\}}e_r)\\
&-1_{\{U(\overline{B})=\{r\}\subset\{i,j\}\}}\widetilde{w}_t(x'-e_r),
\end{aligned}\tag{63}$$

where

$$x'=x-\sum_{k\in B}1_{\{x_k>0\}}e_k.$$

In order to show relation (54), we need to consider together uniformization terms (relation (56)) and returns (relations (60) and (63)). Without loss of generality we can assume that:

$$x_1 \leq x_2 \leq \ldots \leq x_I.$$

Consider the partition $\{G_1, \ldots, G_n\}$ of set $\{1, \ldots, I\}$ into sets of components having equal values:

1. For all $1 \leq k \leq n$, $x_i = x_j$, $\forall i, j \in G_k$.
2. For $1 \leq k < l \leq n$, $x_i < x_j$, $\forall i \in G_k$, $\forall j \in G_l$.

Furthermore, for all $1 \leq k \leq n$, we will denote by g_k the value of components in G_k:

$$x_i = g_k, \ \forall i \in G_k.$$

For a given set B we will denote by $s(B)$ the index of the component set in B for which x has the minimal value:

$$s(B) = \min\{k \ : \ G_k \cap B \neq \emptyset\}.$$

Then for a fixed value of $s(B) = s$ the non-trivial sets B for returns are given by:

$$l(B) \geq u(\overline{B}) \ \Leftrightarrow \ B = \cup_{k>s} G_k \cup F,$$

where

$$\emptyset \neq F = L(B) \subset G_s.$$

In other words, B must contain all elements of sets G_k with index k larger than s and some subset of G_s.

Now returns and uniformization term can be written as:

$$\begin{aligned}
\widetilde{RU}(x) &= \sum_{\emptyset \neq B \subset \{1,\ldots,I\}} \widetilde{R}_B(x - e_i - e_j) + \widetilde{U}(x - e_i - e_j) + \sum_{\emptyset \neq B \subset \{1,\ldots,I\}} \widetilde{R}_B(x) + \widetilde{U}(x) \\
&\quad - \sum_{r \in \{i,j\}} \left(\sum_{\emptyset \neq B \subset \{1,\ldots,I\}} \widetilde{R}_B(x - e_r) + \widetilde{U}(x - e_r) \right) \\
&= \sum_{s=1}^{n} \sum_{\emptyset \neq F \subset G_s} \left(\widetilde{R}_{\cup_{k>s} G_k \cup F}(x - e_i - e_j) + \widetilde{R}_{\cup_{k>s} G_k \cup F}(x) - \sum_{r \in \{i,j\}} \widetilde{R}_{\cup_{k>s} G_k \cup F}(x - e_r) \right) \\
&\quad + \widetilde{U}(x - e_i - e_j) + \widetilde{U}(x) - \widetilde{U}(x - e_i) - \widetilde{U}(x - e_j).
\end{aligned}$$

We have two different types of sets B:

1. Sets for which $G_{s(B)} \not\subset B$ (i.e. $F \neq G_{s(B)}$). Then

   $$L(B) = F \subsetneq G_{s(B)}, \quad U(\overline{B}) = G_{s(B)} \backslash F,$$

   and

   $$l(B) = u(\overline{B}) = g_{s(B)}.$$

   For returns of this type we use relation (60).

2. Sets for which $G_{s(B)} \subset B$ (i.e. $F = G_{s(B)}$). Then

$$L(B) = F = G_{s(B)}, \quad U(\overline{B}) = G_{s(B)-1} \ (G_0 := \emptyset),$$

and

$$l(B) = g_{s(B)}, \quad u(\overline{B}) = g_{s(B)-1} \ (g_0 := 0).$$

For returns of this type we use relation (63).

Finally, for uniformization term we use relation (56). Note that we have here $I \in C = G_n$. Then from (60) for the first three lines, (63) for the next four lines, and (56) for the last line, it follows:

$$\begin{aligned}
\widetilde{RU}(x) \ \geq \ & \sum_{s=1}^{n} \Bigg(\sum_{\emptyset \neq F \subsetneq G_s} \Big(1_{\{F \cap \{i,j\} = \emptyset, \, G_s \setminus F \subset \{i,j\}\}} \widetilde{w}_t\big(x - \sum_{m \in \cup_{k>s} G_k \cup F} 1_{\{x_m > 0\}} e_m \\
& - \sum_{r \in \{i,j\}} 1_{\{x_r > 1_{\{r \in \cup_{k>s} G_k \cup F\}}\}} e_r \big) \\
& - \sum_{r \in \{i,j\}} 1_{\{G_s \setminus F = \{r\}\}} \widetilde{w}_t(x - \sum_{m \in \cup_{k>s} G_k \cup F} 1_{\{x_m > 0\}} e_m - e_r) \Big) \\
& + 1_{\{\emptyset \neq G_{s-1} \subset \{i,j\}\}} \widetilde{w}_t(x - \sum_{m \in \cup_{k \geq s} G_k} 1_{\{x_m > 0\}} e_m - \sum_{r \in \{i,j\}} 1_{\{x_r > 1_{\{r \in \cup_{k \geq s} G_k\}}\}} e_r) \\
& + 1_{\{G_s \cap \{i,j\} \neq \emptyset\}} \widetilde{w}_t(x - \sum_{m \in \cup_{k \geq s} G_k} 1_{\{x_m > 0\}} e_m) \\
& - 1_{\{G_s \cap \{i,j\} \neq \emptyset\}} \sum_{r \in \{i,j\} \setminus G_s} \widetilde{w}_t(x - \sum_{m \in \cup_{k \geq s} G_k} 1_{\{x_m > 0\}} e_m - 1_{\{x_r > 1_{\{r \in \cup_{k \geq s} G_k\}}\}} e_r) \\
& - \sum_{r \in \{i,j\}} 1_{\{G_{s-1} = \{r\}\}} \widetilde{w}_t(x - \sum_{m \in \cup_{k \geq s} G_k} 1_{\{x_m > 0\}} e_m - e_r) \Bigg) \\
& + 1_{\{G_n \subset \{i,j\}\}} \widetilde{w}_t(x - e_i - e_j) - 1_{\{G_n = \{I\} \subset \{i,j\}\}} \widetilde{w}_t(x - e_I).
\end{aligned}$$

Now we can rewrite the terms that contain G_{s-1} as follows:

$$\begin{aligned}
& \sum_{s=1}^{n} 1_{\{\emptyset \neq G_{s-1} \subset \{i,j\}\}} \widetilde{w}_t(x - \sum_{m \in \cup_{k \geq s} G_k} 1_{\{x_m > 0\}} e_m - \sum_{r \in \{i,j\}} 1_{\{x_r > 1_{\{r \in \cup_{k \geq s} G_k\}}\}} e_r) \\
= \ & \sum_{s=1}^{n-1} 1_{\{G_s \subset \{i,j\}\}} \widetilde{w}_t(x - \sum_{m \in \cup_{k \geq s+1} G_k} 1_{\{x_m > 0\}} e_m - \sum_{r \in \{i,j\}} 1_{\{x_r > 1_{\{r \in \cup_{k \geq s+1} G_k\}}\}} e_r) \\
= \ & \sum_{s=1}^{n-1} 1_{\{G_s \subset \{i,j\}\}} \widetilde{w}_t(x - \sum_{m \in \cup_{k \geq s} G_k} 1_{\{x_m > 0\}} e_m - \sum_{r \in \{i,j\} \setminus G_s} 1_{\{x_r > 1_{\{r \in \cup_{k \geq s+1} G_k\}}\}} e_r)
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{s=1}^{n}\sum_{r\in\{i,j\}} 1_{\{G_{s-1}=\{r\}\}}\widetilde{w}_t(x-\sum_{m\in\cup_{k\geq s}G_k}1_{\{x_m>0\}}e_m-e_r)\\
=\;&\sum_{s=1}^{n-1}\sum_{r\in\{i,j\}} 1_{\{G_s=\{r\}\}}\widetilde{w}_t(x-\sum_{m\in\cup_{k\geq s+1}G_k}1_{\{x_m>0\}}e_m-e_r)\\
=\;&\sum_{s=1}^{n-1}\sum_{r\in\{i,j\}} 1_{\{G_s=\{r\}\}}\widetilde{w}_t(x-\sum_{m\in\cup_{k\geq s}G_k}1_{\{x_m>0\}}e_m).
\end{aligned}$$

Note that for $s=n$:

$$1_{\{G_n\subset\{i,j\}\}}\widetilde{w}_t(x-e_i-e_j)=1_{\{G_n\subset\{i,j\}\}}\widetilde{w}_t(x-\sum_{m\in\cup_{k\geq s}G_k}1_{\{x_m>0\}}e_m-\sum_{r\in\{i,j\}\backslash G_n}1_{\{x_r>0\}}e_r)$$

and

$$1_{\{G_n=\{I\}\subset\{i,j\}\}}\widetilde{w}_t(x-e_I)=\sum_{r\in\{i,j\}}1_{\{G_n=\{r\}\}}\widetilde{w}_t(x-\sum_{m\in\cup_{k\geq n}G_k}1_{\{x_m>0\}}e_m)$$

Therefore:

$$\begin{aligned}
\widetilde{RU}(x)\;\geq\;&\sum_{s=1}^{n}\Bigg(\sum_{\emptyset\neq F\subsetneq G_s}\Big(1_{\{F\cap\{i,j\}=\emptyset,\,G_s\backslash F\subset\{i,j\}\}}\widetilde{w}_t\big(x-\sum_{m\in\cup_{k>s}G_k\cup F}1_{\{x_m>0\}}e_m \qquad (64)\\
&-\sum_{r\in\{i,j\}}1_{\{x_r>1_{\{r\in\cup_{k>s}G_k\cup F\}}\}}e_r\big)\\
&-\sum_{r\in\{i,j\}}1_{\{G_s\backslash F=\{r\}\}}\widetilde{w}_t(x-\sum_{m\in\cup_{k>s}G_k\cup F}1_{\{x_m>0\}}e_m-e_r)\Big) \qquad (65)\\
&+1_{\{G_s\subset\{i,j\}\}}\widetilde{w}_t(x-\sum_{m\in\cup_{k\geq s}G_k}1_{\{x_m>0\}}e_m-\sum_{r\in\{i,j\}\backslash G_s}1_{\{x_r>1_{\{r\in\cup_{k\geq s+1}G_k\}}\}}e_r) \qquad (66)\\
&+1_{\{G_s\cap\{i,j\}\neq\emptyset\}}\widetilde{w}_t(x-\sum_{m\in\cup_{k\geq s}G_k}1_{\{x_m>0\}}e_m)\\
&-1_{\{G_s\cap\{i,j\}\neq\emptyset\}}\sum_{r\in\{i,j\}\backslash G_s}\widetilde{w}_t(x-\sum_{m\in\cup_{k\geq s}G_k}1_{\{x_m>0\}}e_m-1_{\{x_r>1_{\{r\in\cup_{k\geq s}G_k\}}\}}e_r)\\
&-\sum_{r\in\{i,j\}}1_{\{G_s=\{r\}\}}\widetilde{w}_t(x-\sum_{m\in\cup_{k\geq s}G_k}1_{\{x_m>0\}}e_m)\Bigg). \qquad (67)
\end{aligned}$$

Consider now terms (64) and (66) in the above relation. First, if $F\cap\{i,j\}=\emptyset$ and $G_s\backslash F\subset\{i,j\}$, then:

$$\begin{aligned}
&x-\sum_{m\in\cup_{k>s}G_k\cup F}1_{\{x_m>0\}}e_m-\sum_{r\in\{i,j\}}1_{\{x_r>1_{\{r\in\cup_{k>s}G_k\cup F\}}\}}e_r\\
=\;&x-\sum_{m\in\cup_{k\geq s}G_k}1_{\{x_m>0\}}e_m-\sum_{r\in\{i,j\}\backslash G_s}1_{\{x_r>1_{\{r\in\cup_{k\geq s}G_k\}}\}}e_r.
\end{aligned}$$

Now (66) can be seen as a special case of (64) for $F = \emptyset$. Similarly, for $G_s \backslash F = \{r\}$,

$$x - \sum_{m \in \cup_{k>s} G_k \cup F} 1_{\{x_m>0\}} e_m - e_r = x - \sum_{m \in \cup_{k \geq s} G_k \cup F} 1_{\{x_m>0\}} e_m.$$

Now (67) is a special case of (65) for $F = \emptyset$. Therefore,

$$\begin{aligned}
\widetilde{RU}(x) \;\geq\; & \sum_{s=1}^{n} \Bigg(\sum_{F \subsetneqq G_s} \Big(1_{\{F \cap \{i,j\} = \emptyset,\, G_s \backslash F \subset \{i,j\}\}} \widetilde{w}_t \big(x - \sum_{m \in \cup_{k \geq s} G_k} 1_{\{x_m>0\}} e_m \\
& - \sum_{r \in \{i,j\} \backslash G_s} 1_{\{x_r > 1_{\{r \in \cup_{k \geq s} G_k\}}\}} e_r \big) \\
& - \sum_{r \in \{i,j\}} 1_{\{G_s \backslash F = \{r\}\}} \widetilde{w}_t (x - \sum_{m \in \cup_{k \geq s} G_k} 1_{\{x_m>0\}} e_m) \Big) \\
& + 1_{\{G_s \cap \{i,j\} \neq \emptyset\}} \widetilde{w}_t (x - \sum_{m \in \cup_{k \geq s} G_k} 1_{\{x_m>0\}} e_m) \\
& - 1_{\{G_s \cap \{i,j\} \neq \emptyset\}} \sum_{r \in \{i,j\} \backslash G_s} \widetilde{w}_t (x - \sum_{m \in \cup_{k \geq s} G_k} 1_{\{x_m>0\}} e_m - 1_{\{x_r > 1_{\{r \in \cup_{k \geq s} G_k\}}\}} e_r) \Bigg).
\end{aligned}$$

Note now that for $F \subsetneqq G_s$,

$$1_{\{F \cap \{i,j\} = \emptyset,\, G_s \backslash F \subset \{i,j\}\}} = 1_{\{G_s \cap \{i,j\} \neq \emptyset,\, F = G_s \backslash \{i,j\}\}}$$

and

$$1_{\{G_s \backslash F = \{r\}\}} = 1_{\{r \in G_s,\, F = G_s \backslash \{r\}\}},\ r \in \{i,j\}.$$

Therefore, in the above summation over all $F \subsetneqq G_s$, the first term is non-zero only for $F = G_s \backslash \{i,j\}$ and the second one is non-zero only for $F = G_s \backslash \{r\}$, which gives:

$$\begin{aligned}
\widetilde{RU}(x) \;\geq\; & \sum_{s=1}^{n} \Bigg(1_{\{G_s \cap \{i,j\} \neq \emptyset\}} \widetilde{w}_t \big(x - \sum_{m \in \cup_{k \geq s} G_k} 1_{\{x_m>0\}} e_m - \sum_{r \in \{i,j\} \backslash G_s} 1_{\{x_r > 1_{\{r \in \cup_{k \geq s} G_k\}}\}} e_r \big) \\
& - \sum_{r \in \{i,j\}} 1_{\{r \in G_s\}} \widetilde{w}_t (x - \sum_{m \in \cup_{k \geq s} G_k} 1_{\{x_m>0\}} e_m) \\
& + 1_{\{G_s \cap \{i,j\} \neq \emptyset\}} \widetilde{w}_t (x - \sum_{m \in \cup_{k \geq s} G_k} 1_{\{x_m>0\}} e_m) \\
& - 1_{\{G_s \cap \{i,j\} \neq \emptyset\}} \sum_{r \in \{i,j\} \backslash G_s} \widetilde{w}_t (x - \sum_{m \in \cup_{k \geq s} G_k} 1_{\{x_m>0\}} e_m - 1_{\{x_r > 1_{\{r \in \cup_{k \geq s} G_k\}}\}} e_r) \Bigg).
\end{aligned}$$

In order to show that $\widetilde{RU}(x) \geq 0$, will show that for each $s \in \{1, \ldots, n\}$:

$$\begin{aligned}
& 1_{\{G_s \cap \{i,j\} \neq \emptyset\}} \Bigg(\widetilde{w}_t \big(x - \sum_{m \in \cup_{k \geq s} G_k} 1_{\{x_m>0\}} e_m - \sum_{r \in \{i,j\} \backslash G_s} 1_{\{x_r > 1_{\{r \in \cup_{k \geq s} G_k\}}\}} e_r \big) \\
& - \sum_{r \in \{i,j\}} 1_{\{r \in G_s\}} \widetilde{w}_t (x - \sum_{m \in \cup_{k \geq s} G_k} 1_{\{x_m>0\}} e_m) + \widetilde{w}_t (x - \sum_{m \in \cup_{k \geq s} G_k} 1_{\{x_m>0\}} e_m) \\
& - \sum_{r \in \{i,j\} \backslash G_s} \widetilde{w}_t (x - \sum_{m \in \cup_{k \geq s} G_k} 1_{\{x_m>0\}} e_m - 1_{\{x_r > 1_{\{r \in \cup_{k \geq s} G_k\}}\}} e_r) \Bigg) \geq 0. \qquad (68)
\end{aligned}$$

For a fixed $s \in \{1, \ldots, n\}$, we have the following cases:

1. $G_s \cap \{i, j\} = \emptyset$. Then relation (68) trivially holds.
2. $i \in G_s$ and $j \notin G_s$ (the case $j \in G_s$ and $i \notin G_s$ is symmetrical). Then (68) becomes:

$$\begin{aligned}
&\widetilde{w}_t\big(x - \sum_{m \in \cup_{k \geq s} G_k} 1_{\{x_m > 0\}} e_m - 1_{\{x_j > 1_{\{j \in \cup_{k \geq s} G_k\}}\}} e_j\big) \\
&-\widetilde{w}_t(x - \sum_{m \in \cup_{k \geq s} G_k} 1_{\{x_m > 0\}} e_m) + \widetilde{w}_t(x - \sum_{m \in \cup_{k \geq s} G_k} 1_{\{x_m > 0\}} e_m) \\
&-\widetilde{w}_t(x - \sum_{m \in \cup_{k \geq s} G_k} 1_{\{x_m > 0\}} e_m - 1_{\{x_j > 1_{\{j \in \cup_{k \geq s} G_k\}}\}} e_j) \ \geq \ 0,
\end{aligned}$$

   which clearly holds.

3. $\{i, j\} \subset G_s$. Then (68) becomes:

$$\begin{aligned}
&\widetilde{w}_t(x - \sum_{m \in \cup_{k \geq s} G_k} 1_{\{x_m > 0\}} e_m) - 2\widetilde{w}_t(x - \sum_{m \in \cup_{k \geq s} G_k} 1_{\{x_m > 0\}} e_m) \\
&+\widetilde{w}_t(x - \sum_{m \in \cup_{k \geq s} G_k} 1_{\{x_m > 0\}} e_m) + 0 \ \geq \ 0,
\end{aligned}$$

   which also holds.

Therefore, $\widetilde{RU}(x) \geq 0$, so we proved relation (54).

Now relation (52) follows directly from (51), (53), (54) and supermodularity of reward function $\widetilde{r}$. □