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# On the properties of the solution path of the constrained and penalized L2-L0 problems 

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## 1 Domain of optimization

For $k \leqslant n$, we define the domain $\mathcal{D}_{k} \subset \mathbb{R}^{n}$ :

$$
\begin{equation*}
\mathcal{D}_{k}=\left\{\boldsymbol{x} \in \mathbb{R}^{n},\|\boldsymbol{x}\|_{0}=k\right\} . \tag{1}
\end{equation*}
$$

Theorem 1 For $k \geqslant 1, \mathcal{D}_{k}$ is not a closed set, and $\overline{\mathcal{D}_{k}}=\left\{\boldsymbol{x} \in \mathbb{R}^{n},\|\boldsymbol{x}\|_{0} \leqslant k\right\}$ (denoting by - the closure operator).

Proof $1 \quad$ - $\mathcal{D}_{k}$ is not a closed set: it is easy to find a sequence $\boldsymbol{x}_{j} \in \mathcal{D}_{k}(j \in \mathbb{N})$ whose limit is not in $\mathcal{D}_{k}$. For instance, $\boldsymbol{x}_{j}=(1 / j) \boldsymbol{e}$, where $\boldsymbol{e}$ is a given vector in $\mathcal{D}_{k}$. $\boldsymbol{x}_{j}$ tends towards $\mathbf{0} \notin \mathcal{D}_{k}$.

- $\overline{\mathcal{D}_{k}} \subseteq\left\{\boldsymbol{x},\|\boldsymbol{x}\|_{0} \leqslant k\right\}$. If $\boldsymbol{x} \in \overline{\mathcal{D}_{k}}$, then there exists a sequence $\boldsymbol{x}_{j} \in \mathcal{D}_{k}(j \in \mathbb{N})$ whose limit is equal to $\boldsymbol{x}$. Then,

$$
\forall \varepsilon>0, \exists J, j \geqslant J \Rightarrow \forall i,\left|\boldsymbol{x}(i)-\boldsymbol{x}_{j}(i)\right|<\varepsilon
$$

Applying this property with $\varepsilon=\min _{\boldsymbol{x}(i) \neq 0}|\boldsymbol{x}(i)|$, we deduce that there exists an iteration $J$, such that $\forall j \geqslant$ $J, \forall i, \boldsymbol{x}(i) \neq 0 \Rightarrow \boldsymbol{x}_{j}(i) \neq 0$. In other words, $\|\boldsymbol{x}\|_{0} \leqslant\left\|\boldsymbol{x}_{j}\right\|_{0}=k$.

- $\left\{\boldsymbol{x},\|\boldsymbol{x}\|_{0} \leqslant k\right\} \subseteq \overline{\mathcal{D}_{k}}$. Let us show that if $\boldsymbol{x}$ is such that $\|\boldsymbol{x}\|_{0} \leqslant k$, then there exists a sequence $\boldsymbol{x}_{j} \in \mathcal{D}_{k}$ whose limit is equal to $\boldsymbol{x}$. Given $\boldsymbol{x}$, we define $\boldsymbol{x}_{j}$ by setting $\boldsymbol{x}_{j}(i)=\boldsymbol{x}(i)$ if $i \in \mathcal{A}(\boldsymbol{x})$ (support of $\boldsymbol{x}$ ), and by replacing the $k-\|\boldsymbol{x}\|_{0}$ first zero valued entries of $\boldsymbol{x}$ by $1 / j$ in $\boldsymbol{x}_{j}$, and setting to 0 the remaining $n-k$ entries $\boldsymbol{x}_{j}(i)$. Obviously, $\boldsymbol{x}_{j} \in \mathcal{D}_{k}$ and this sequence tends towards $\boldsymbol{x}$.

The consequence of theorem 1 is that

$$
\underset{\boldsymbol{x} \in \mathcal{D}_{k}}{\arg \min }\left\{\mathcal{E}(\boldsymbol{x})=\|\boldsymbol{y}-\boldsymbol{A} \boldsymbol{x}\|^{2}\right\}
$$

is not always defined, although the minimal value $\min _{\boldsymbol{x} \in \mathcal{D}_{k}} \mathcal{E}(\boldsymbol{x})$ is defined. On the contrary, the set of minimizers

$$
\mathcal{X}_{c}(k)=\underset{\boldsymbol{x} \in \overline{\mathcal{D}_{k}}}{\arg \min } \mathcal{E}(\boldsymbol{x})=\underset{\|\boldsymbol{x}\|_{0} \leqslant k}{\arg \min } \mathcal{E}(\boldsymbol{x})
$$

is properly defined because $\overline{\mathcal{D}_{k}}$ is a closed set and $\mathcal{E}$ is quadratic and convex (to be completed).
Example 1 Let us consider the minimization of $\|\boldsymbol{x}\|^{2}$ over the domain $\mathcal{D}_{k}$. For $k>0$, there is no minimizer over $\mathcal{D}_{k}$, but the minimal cost $\min _{\boldsymbol{x} \in \mathcal{D}_{k}}\|\boldsymbol{x}\|^{2}$ is equal to 0 . The set of minimizers over $\overline{\mathcal{D}_{k}}$ is reduced to one vector: $\mathcal{X}_{c}(k)=\{\mathbf{0}\}$.

Example 2 The set $\mathcal{X}_{c}(k)$ is not always a singleton. Let us consider the minimization of the $2 D$ cost function $\mathcal{E}(\boldsymbol{x})=\boldsymbol{x}(1)^{2}$. It is easy to see that $\mathcal{X}_{c}(0)=\{\mathbf{0}\}, \mathcal{X}_{c}(1)=\left\{[0, \boldsymbol{x}(2)]^{T}, \boldsymbol{x}(2) \in \mathbb{R}\right\}$ and $\mathcal{X}_{c}(2)=\mathcal{X}_{c}(1)$.

Example 3 Let us consider the minimization of the $2 D$ cost function $\mathcal{E}(\boldsymbol{x})=(\boldsymbol{x}(1)-\alpha)^{2}$ for a given $\alpha \neq 0$. It is easy to see that $\mathcal{X}_{c}(0)=\{\mathbf{0}\}, \mathcal{X}_{c}(1)=\left\{[\alpha, 0]^{T}\right\}$ and $\mathcal{X}_{c}(2)=\left\{[\alpha, \boldsymbol{x}(2)]^{T}, \boldsymbol{x}(2) \in \mathbb{R}\right\}$.

Remark 1 Obviously, the sets $\overline{\mathcal{D}_{k}}$ have a nesting property $\left(\overline{\mathcal{D}_{k}} \subset \overline{\mathcal{D}_{k+1}}\right)$, therefore, for all $k$, we have

$$
\forall \boldsymbol{x}_{k} \in \mathcal{X}_{c}(k), \forall \boldsymbol{x}_{k+1} \in \mathcal{X}_{c}(k+1), \mathcal{E}\left(\boldsymbol{x}_{k+1}\right) \leqslant \mathcal{E}\left(\boldsymbol{x}_{k}\right)
$$

Theorem $2 \mathcal{X}_{c}(k+1) \cap \overline{\mathcal{D}_{k}} \subseteq \mathcal{X}_{c}(k)$.
Proof 2 Let us consider $\boldsymbol{x}_{k+1} \in \mathcal{X}_{c}(k+1) \cap \overline{\mathcal{D}_{k}}$. Since $\overline{\mathcal{D}_{k}} \subset \overline{\mathcal{D}_{k+1}}$ and $\boldsymbol{x}_{k+1}$ is a minimizer of $\mathcal{E}$ over $\overline{\mathcal{D}_{k+1}}$, we have $\forall \boldsymbol{x} \in \overline{\mathcal{D}_{k}}, \mathcal{E}\left(\boldsymbol{x}_{k+1}\right) \leqslant \mathcal{E}(\boldsymbol{x})$. As $\boldsymbol{x}_{k+1} \in \overline{\mathcal{D}_{k}}, \boldsymbol{x}_{k+1}$ is a minimizer of $\mathcal{E}$ over $\overline{\mathcal{D}_{k}}$.

## 2 Working assumptions and notion of constrained solution path

### 2.1 Unique representation property

We recall the definition of the unique representation property (URP), introduced in [1] for the underdetermined case (when $m \leqslant n$ ):

Definition 1 A matrix $\boldsymbol{A}$ of size $m \times n(m \leqslant n)$ satisfies the URP if and only if any selection of $m$ columns of $\boldsymbol{A}$ forms a family of linearly independent vectors.

Under the URP assumption, we can solve $\boldsymbol{y}=\boldsymbol{A} \boldsymbol{x}$ by assuming that $\boldsymbol{x} \in \overline{\mathcal{D}_{m}}$. The system is then is equivalent to $\boldsymbol{y}=\boldsymbol{B} \boldsymbol{z}$ where $\boldsymbol{B}$ is a matrix of size $m \times m$ extracted from $\boldsymbol{A}$, and $\boldsymbol{z}$ is the corresponding vector extracted from $\boldsymbol{x}$, of size $m \times 1$. According to the URP definition, $\boldsymbol{B}$ is always invertible, and we can find sparse solutions to $\boldsymbol{y}=\boldsymbol{A} \boldsymbol{x}$ with at most $m$ non-zero entries $\left(\boldsymbol{z}=\boldsymbol{B}^{-1} \boldsymbol{y}\right.$ and then $\boldsymbol{x}=\{\boldsymbol{z}, \mathbf{0}\}$ for all the possible extractions $\boldsymbol{B}$ from $\boldsymbol{A}$ ).

When $m>n$, we adopt the following definition:
Definition 2 A matrix $\boldsymbol{A}$ of size $m \times n(m>n)$ satisfies the $U R P$ if and only if it full rank.
When $m>n$, there is generally no solution to $\boldsymbol{y}=\boldsymbol{A} \boldsymbol{x}$ but the minimizer of $\mathcal{E}(\boldsymbol{x})$ over $\mathbb{R}^{n}$ is unique (although not necessarily sparse): $\mathcal{X}_{c}(n)=\left\{\left(\boldsymbol{A}^{T} \boldsymbol{A}\right)^{-1} \boldsymbol{A}^{T} \boldsymbol{y}\right\}$.

In the following, we will assume that $\boldsymbol{y} \neq \mathbf{0}$ and that $\boldsymbol{A}$ satisfies the URP.

### 2.2 Cardinality of the set $\mathcal{X}_{c}(k)$

Theorem 3 For $k \leqslant \min (m, n)$, the set $\mathcal{X}_{c}(k)$ is finite under the URP assumption.
Proof 3 Because of the URP assumption, any selection of $k \leqslant \min (m, n)$ columns of $\boldsymbol{A}$ yields a matrix $\boldsymbol{B}$ of size $m \times k$ whose rank is equal to $k$. Then, the energy reduces to $\mathcal{E}(\boldsymbol{x})=\mathcal{E}(\boldsymbol{z} ; \mathbf{0})=\|\boldsymbol{y}-\boldsymbol{B} \boldsymbol{z}\|^{2}$ (where $\boldsymbol{z} \in \mathbb{R}^{k}$ ), and there is only one minimizer of $\boldsymbol{z} \mapsto \mathcal{E}(\boldsymbol{z} ; \mathbf{0})$ over $\mathbb{R}^{k}$. Since the number of possible selections of $k$ columns of $\boldsymbol{A}$ is finite, the set $\mathcal{X}_{c}(k)$ is finite.

Remark 2 The minimal value of $\mathcal{E}(\boldsymbol{x})\left(\right.$ for $\left.\boldsymbol{x} \in \mathbb{R}^{n}\right)$ can be reached when minimizing $\mathcal{E}$ over $\overline{\mathcal{D}_{\min (m, n)}}$. Thus, when $m \leqslant n$, it is not necessary to compute $\mathcal{X}_{c}(k)$ for $k>m$. According to theorem 3, when $m>n$, all the sets $\mathcal{X}_{c}(k), k=0, \ldots, n$ are finite.

Theorem 4 When $m \leqslant n$ and $k$ is such that $m<k \leqslant n$, the set $\mathcal{X}_{c}(k)$ is of infinite cardinality.
Proof 4 Given a solution $\boldsymbol{x}_{m} \in \mathcal{X}_{c}(m)$, let $\mathcal{A}\left(\boldsymbol{x}_{m}\right)$ be the support of $\boldsymbol{x}_{m}$. We consider a support $\mathcal{B}$ of cardinality $k$ such that $\mathcal{A}\left(\boldsymbol{x}_{m}\right) \subset \mathcal{B} \subseteq\{1, \ldots, n\}$, and we extract from $\boldsymbol{A}$ the matrix $\boldsymbol{B}$ of size $m \times k$ formed of the columns $\boldsymbol{a}_{i}$ of $\boldsymbol{A}(i \in \mathcal{B})$. Then, let us add to $\boldsymbol{x}_{m}$ a vector $\boldsymbol{n}$ belonging to the null space of $\boldsymbol{B}$. Clearly, $\boldsymbol{x}_{m}+\boldsymbol{n} \in \mathcal{X}_{c}(k)$ because $\left\|\boldsymbol{x}_{m}+\boldsymbol{n}\right\|_{0} \leqslant k$ and $\mathcal{E}\left(\boldsymbol{x}_{m}+\boldsymbol{n}\right)=\mathcal{E}\left(\boldsymbol{x}_{m}\right)=\min _{\boldsymbol{x} \in \mathbb{R}^{n}} \mathcal{E}(\boldsymbol{x})$. Since the null space of $\boldsymbol{B}$ is of dimension $k-m>0, \mathcal{X}_{c}(k)$ is of infinite cardinality.

As a conclusion, the constrained solution path is defined in the following way, for any case ( $m \leqslant n$ or $m>n$ ).
Definition 3 The constrained solution path is the (finite) set

$$
\mathcal{X}_{c}=\bigcup_{k=0}^{\min (m, n)} \mathcal{X}_{c}(k)
$$

## 3 Properties of the penalized solution path

### 3.1 Penalized solution path

For a given $\lambda \geqslant 0$, we define the set of minimizers of $\mathcal{J}(\boldsymbol{x} ; \lambda)=\mathcal{E}(\boldsymbol{x})+\lambda\|\boldsymbol{x}\|_{0}$ :

$$
\mathcal{X}_{p}(\lambda)=\underset{\boldsymbol{x} \in \mathbb{R}^{n}}{\arg \min }\{\mathcal{J}(\boldsymbol{x} ; \lambda)\} .
$$

By extension, we define $\mathcal{X}_{p}(+\infty)=\{\mathbf{0}\}$.
Theorem 5 If $\lambda>0$, the set $\mathcal{X}_{p}(\lambda)$ is finite and $\mathcal{X}_{p}(\lambda) \subseteq \overline{\mathcal{D}_{\min (m, n)}}$.
Proof 5 Let $\boldsymbol{x}_{p}(\lambda)$ be a minimizer of $\mathcal{J}(\boldsymbol{x} ; \lambda)$, and let $\mathcal{A}(\lambda)$ denote the support of $\boldsymbol{x}_{p}(\lambda)$ (actually, the support depends on $\boldsymbol{x}_{p}(\lambda)$ but we use the simplified notation $\mathcal{A}(\lambda)$ instead of $\mathcal{A}\left(\boldsymbol{x}_{p}(\lambda)\right)$ ).

- First, we show that

$$
\boldsymbol{x}_{p}(\lambda) \in \underset{\mathcal{A}(\boldsymbol{x}) \subseteq \mathcal{A}(\lambda)}{\arg \min } \mathcal{E}(\boldsymbol{x}) .
$$

Since $\boldsymbol{x}_{p}(\lambda)$ is a minimizer of $\mathcal{J}(\boldsymbol{x} ; \lambda)$, the following equivalent inequalities hold for all $\boldsymbol{x}$ such that $\mathcal{A}(\boldsymbol{x}) \subseteq \mathcal{A}(\lambda)$ :

$$
\begin{aligned}
\mathcal{J}(\boldsymbol{x} ; \lambda) & \geqslant \mathcal{J}\left(\boldsymbol{x}_{p}(\lambda) ; \lambda\right) \\
\mathcal{E}(\boldsymbol{x})+\lambda\|\boldsymbol{x}\|_{0} & \geqslant \mathcal{E}\left(\boldsymbol{x}_{p}(\lambda)\right)+\lambda\|\mathcal{A}(\lambda)\|_{0} \\
\mathcal{E}(\boldsymbol{x})-\mathcal{E}\left(\boldsymbol{x}_{p}(\lambda)\right) & \geqslant \lambda\left(\|\mathcal{A}(\lambda)\|_{0}-\|\boldsymbol{x}\|_{0}\right) \geqslant 0
\end{aligned}
$$

$\left(\|\mathcal{A}(\lambda)\|_{0}\right.$ denotes the cardinality of $\left.\mathcal{A}(\lambda)\right)$. We finally deduce that $\boldsymbol{x}_{p}(\lambda)$ is a minimizer of $\mathcal{E}$ over the set $\{\boldsymbol{x}, \mathcal{A}(\boldsymbol{x}) \subseteq \mathcal{A}(\lambda)\}$.

- We study the following question: what is the cardinality of $\mathcal{A}(\lambda)$, and how many minimizers of $\mathcal{E}$ over the set $\{\boldsymbol{x}, \mathcal{A}(\boldsymbol{x}) \subseteq \mathcal{A}(\lambda)\}$ ?
- When $\|\mathcal{A}(\lambda)\|_{0} \leqslant \min (m, n)$, there is only one minimizer of $\mathcal{E}$ over the set $\{\boldsymbol{x}, \mathcal{A}(\boldsymbol{x}) \subseteq \mathcal{A}(\lambda)\}$ (URP assumption).
- The case where $\|\mathcal{A}(\lambda)\|_{0}>\min (m, n)$ never occurs. If it does, remark 2 shows that there exists $\boldsymbol{x} \in$ $\overline{\mathcal{D}_{\min (m, n)}}$ such that $\mathcal{E}(\boldsymbol{x})=\mathcal{E}\left(\boldsymbol{x}_{p}(\lambda)\right)$. Since $\|\boldsymbol{x}\|_{0} \leqslant \min (m, n)<\left\|\boldsymbol{x}_{p}(\lambda)\right\|_{0}=\|\mathcal{A}(\lambda)\|_{0}, \mathcal{J}(\boldsymbol{x} ; \lambda)<$ $\mathcal{J}\left(\boldsymbol{x}_{p}(\lambda) ; \lambda\right)$, which is in contradiction with $\boldsymbol{x}_{p}(\lambda) \in \mathcal{X}_{p}(\lambda)$.

In conclusion, there are at most $\sum_{k=0}^{\min (m, n)} C_{n}^{k}$ distinct values $\boldsymbol{x}_{p}(\lambda)$ (i.e., $\sum_{k=0}^{\min (m, n)} C_{n}^{k}$ sets which are candidate to be a set $\mathcal{A}(\lambda)$ and one optimal $\boldsymbol{x}$-value per set), which shows that $\mathcal{X}_{p}(\lambda)$ is a finite set. Additionally, we have seen that for each solution, $\left\|\boldsymbol{x}_{p}(\lambda)\right\|_{0}=\|\mathcal{A}(\lambda)\|_{0} \leqslant \min (m, n)$.

Definition 4 For a given active set $\mathcal{A}$ whose cardinality is lower than $\min (m, n)$, the corresponding least-square solution is unique (due to the URP assumption). We denote this solution by

$$
\begin{equation*}
\boldsymbol{x}_{\mathcal{A}} \triangleq \underset{\mathcal{A}(\boldsymbol{x}) \subseteq \mathcal{A}}{\arg \min } \mathcal{E}(\boldsymbol{x}) \tag{2}
\end{equation*}
$$

and the corresponding least-square cost by

$$
\begin{equation*}
\mathcal{E}_{\mathcal{A}} \triangleq \mathcal{E}\left(\boldsymbol{x}_{\mathcal{A}}\right)=\min _{\{\boldsymbol{x} \mid \mathcal{A}(\boldsymbol{x}) \subseteq \mathcal{A}\}} \mathcal{E}(\boldsymbol{x}) \tag{3}
\end{equation*}
$$

Finally, we define the corresponding value of $\mathcal{J}$ by

$$
\begin{equation*}
\mathcal{J}_{\mathcal{A}}(\lambda) \triangleq \mathcal{J}\left(\boldsymbol{x}_{\mathcal{A}} ; \lambda\right)=\mathcal{E}_{\mathcal{A}}+\lambda\left\|\boldsymbol{x}_{\mathcal{A}}\right\|_{0} \tag{4}
\end{equation*}
$$

Definition 5 The penalized solution path is defined as the reunion of sets

$$
\mathcal{X}_{p}=\bigcup_{\lambda>0} \mathcal{X}_{p}(\lambda)
$$

Note that imposing $\lambda>0$ (rather than $\lambda \geqslant 0$ ) guarantees that $\mathcal{X}_{p}(\lambda)$ is of finite cardinality for all $\lambda$. Moreover, it is easy to see (from proof 5) that the solution path is of finite cardinality, since all the sets $\mathcal{X}_{p}(\lambda)$ are included in a common set of cardinality $\sum_{k=0}^{\min (m, n)} C_{n}^{k}:\left\{\boldsymbol{x} \in \mathbb{R}^{n}, \exists \mathcal{A} \subseteq\{1, \ldots, n\},\|\mathcal{A}\|_{0} \leqslant \min (m, n)\right.$ and $\left.\boldsymbol{x}=\boldsymbol{x}_{\mathcal{A}}\right\}$.

### 3.2 Piecewise constant property

Theorem 6 The dependence of the set $\mathcal{X}_{p}(\lambda)$ w.r.t. $\lambda(\lambda>0)$ is piecewise constant, with a finite number of intervals. The minimal cost value $\mathcal{J}(\lambda) \triangleq \min _{\boldsymbol{x} \in \mathbb{R}^{n}} \mathcal{J}(\boldsymbol{x} ; \lambda)$ is a continuous and piecewise linear function of $\lambda$.

Definition 6 In the following, we will refer to the values of $\lambda_{i}^{\star}(i=1, \ldots, I)$ as the critical values. These values, together with $\lambda_{0}^{\star}=0$ and $\lambda_{I+1}^{\star}=+\infty$, define the piecewise constant domain $\mathcal{X}_{p}(\lambda)$ :

$$
\begin{equation*}
0=\lambda_{0}^{\star}<\lambda_{1}^{\star}<\ldots<\lambda_{I}^{\star}<\lambda_{I+1}^{\star}=+\infty \tag{5}
\end{equation*}
$$

These values are also the $\lambda$-values at which the derivative of $\mathcal{J}$ is changing. At $\lambda=\lambda_{i}^{\star}, \mathcal{J}$ is not differentiable.
Proof 6 When $\lambda$ is fixed, let $\boldsymbol{x}_{p}(\lambda)$ be included in the set of minimizers of $\mathcal{J}(\boldsymbol{x} ; \lambda)$, and let $\mathcal{A}(\lambda)$ denote the support of $\boldsymbol{x}_{p}(\lambda)$. We have shown in proof 5 , that $\|\mathcal{A}(\lambda)\|_{0} \leqslant \min (m, n)$ and that $\boldsymbol{x}_{p}(\lambda)=\boldsymbol{x}_{\mathcal{A}(\lambda)}$

Now, let us show that the set $\mathcal{X}_{p}(\lambda)$ is piecewise constant w.r.t. $\lambda$. This can be done with geometric arguments, by considering the affine curves $\lambda \mapsto \mathcal{J}_{\mathcal{A}}(\lambda)$ for all the possible supports $\mathcal{A}$ such that $\|\mathcal{A}\|_{0} \leqslant \min (m, n)$, and finding the minimal cost values

$$
\begin{equation*}
\forall \lambda, \mathcal{J}(\lambda)=\min _{\|\mathcal{A}\|_{0} \leqslant \min (m, n)} \mathcal{J}_{\mathcal{A}}(\lambda) \tag{6}
\end{equation*}
$$

It is easy to see that $\mathcal{J}$ is a continuous and piecewise linear function of $\lambda$. It is described by a finite set of values $\left\{\left(\lambda_{i}^{\star}, \mathcal{E}_{\mathcal{A}_{i}},\left\|\boldsymbol{x}_{\mathcal{A}_{i}}\right\|_{0}\right), i=0, \ldots, I\right\}$, where $\lambda_{0}^{\star}=0<\lambda_{1}^{\star}<\ldots<\lambda_{I}^{\star}<\lambda_{I+1}^{\star}=+\infty$. Each value $\lambda_{i}^{\star}(i=1, \ldots, I)$ corresponds to the intersection between a pair of affine curves, and the restriction of $\mathcal{J}$ to a given interval $\left[\lambda_{i}^{\star}, \lambda_{i+1}^{\star}\right]$ is linear:

$$
\begin{equation*}
\forall \lambda \in\left[\lambda_{i}^{\star}, \lambda_{i+1}^{\star}\right], \mathcal{J}(\lambda)=\mathcal{E}_{\mathcal{A}_{i}}+\lambda\left\|\boldsymbol{x}_{\mathcal{A}_{i}}\right\|_{0} \tag{7}
\end{equation*}
$$

In particular, for $i=0$, we have

$$
\begin{equation*}
\forall \lambda \in\left[0, \lambda_{1}\right], \mathcal{J}(\lambda)=\mathcal{E}_{\mathcal{A}_{0}}+\lambda\left\|\boldsymbol{x}_{\mathcal{A}_{0}}\right\|_{0} \tag{8}
\end{equation*}
$$

where $\mathcal{E}_{\mathcal{A}_{0}}=\min _{\boldsymbol{x} \in \mathbb{R}^{n}} \mathcal{E}(\boldsymbol{x})$ is the minimal least-square error, and $\left\|\boldsymbol{x}_{\mathcal{A}_{0}}\right\|_{0}$ is the minimal L0-norm of the minimizers of $\mathcal{E}$ over $\mathbb{R}^{n}$. For $i=I$, we have necessarily $\boldsymbol{x}_{\mathcal{A}_{I}}=\mathbf{0}$ and $\mathcal{E}_{\mathcal{A}_{I}}=\|\boldsymbol{y}\|^{2}$, thus

$$
\begin{equation*}
\forall \lambda \in\left[\lambda_{I}^{\star},+\infty\right), \mathcal{J}(\lambda)=\|\boldsymbol{y}\|^{2} \tag{9}
\end{equation*}
$$

For a given interval $\left[\lambda_{i}^{\star}, \lambda_{i+1}^{\star}\right]$, let us show that when $\lambda \in\left(\lambda_{i}^{\star}, \lambda_{i+1}^{\star}\right)$, $\mathcal{X}_{p}(\lambda)$ is a constant set. For some given $\lambda$-value $\in\left(\lambda_{i}^{\star}, \lambda_{i+1}^{\star}\right)$, we consider $\boldsymbol{x} \in \mathcal{X}_{p}(\lambda)$, then necessarily, the following equivalent equations hold:

$$
\begin{aligned}
\mathcal{J}(\lambda) & =\mathcal{J}(\boldsymbol{x} ; \lambda) \\
\mathcal{E}_{\mathcal{A}_{i}}+\lambda\left\|\boldsymbol{x}_{\mathcal{A}_{i}}\right\|_{0} & =\mathcal{E}(\boldsymbol{x})+\lambda\|\boldsymbol{x}\|_{0}
\end{aligned}
$$

Imagine that $\mathcal{E}_{\mathcal{A}_{i}} \neq \mathcal{E}(\boldsymbol{x})$, then, necessarily, the two functions $\mathcal{J}\left(\lambda^{\prime}\right)=\mathcal{E}_{\mathcal{A}_{i}}+\lambda^{\prime}\left\|\boldsymbol{x}_{\mathcal{A}_{i}}\right\|_{0}$ and $\mathcal{J}\left(\boldsymbol{x} ; \lambda^{\prime}\right)=\mathcal{E}(\boldsymbol{x})+$ $\lambda^{\prime}\|\boldsymbol{x}\|_{0}$ do not coincide for $\lambda^{\prime} \in\left[\lambda_{i}^{\star}, \lambda_{i+1}^{\star}\right] \backslash\{\lambda\}$. Moreover, $\mathcal{J}\left(\boldsymbol{x} ; \lambda^{\prime}\right)$ is strictly lower than $\mathcal{J}\left(\lambda^{\prime}\right)$ either for $\lambda^{\prime} \in\left[\lambda_{i}^{\star}, \lambda\right)$ or for $\lambda^{\prime} \in\left(\lambda, \lambda_{i+1}^{\star}\right]$. This is in contradiction with (7) and the definition of $\mathcal{J}\left(\lambda^{\prime}\right)$ in theorem 6 .

We have shown that $\mathcal{E}_{\mathcal{A}_{i}}=\mathcal{E}(\boldsymbol{x})$. Since $\mathcal{J}(\lambda)=\mathcal{J}(\boldsymbol{x} ; \lambda)$ and $\lambda>0$, we deduce that $\left\|\boldsymbol{x}_{\mathcal{A}_{i}}\right\|_{0}=\|\boldsymbol{x}\|_{0}$, and that $\forall \lambda^{\prime} \in\left[\lambda_{i}^{\star}, \lambda_{i+1}^{\star}\right], \mathcal{J}\left(\lambda^{\prime}\right)=\mathcal{J}\left(\boldsymbol{x} ; \lambda^{\prime}\right)$. Finally, if $\lambda \in\left(\lambda_{i}^{\star}, \lambda_{i+1}^{\star}\right)$ and $\boldsymbol{x} \in \mathcal{X}_{p}(\lambda)$, then $\boldsymbol{x} \in \mathcal{X}_{p}\left(\lambda^{\prime}\right)$ for all $\lambda^{\prime} \in\left[\lambda_{i}^{\star}, \lambda_{i+1}^{\star}\right] . \mathcal{X}_{p}(\lambda)$ is then a constant set when $\lambda \in\left(\lambda_{i}^{\star}, \lambda_{i+1}^{\star}\right)$, and $\mathcal{X}_{p}(\lambda) \subseteq \mathcal{X}_{p}\left(\lambda_{i}^{\star}\right) \cap \mathcal{X}_{p}\left(\lambda_{i+1}^{\star}\right)$.

Since the number of affine curves $\lambda \mapsto \mathcal{J}_{\mathcal{A}}(\lambda)$ is finite, the number of their intersections is finite, and the number of intervals $\left(\lambda_{i}^{\star}, \lambda_{i+1}^{\star}\right)$ is finite as well. This completes the proof.

Lemma 1 The function $\lambda \mapsto \mathcal{J}(\lambda)$ is increasing and concave.
Proof $7 \mathcal{J}$ is an increasing and concave function as the minimum of a finite set of increasing and concave functions.

Lemma $2 \forall i, \forall \lambda \in\left(\lambda_{i}^{\star}, \lambda_{i+1}^{\star}\right), \mathcal{X}_{p}(\lambda) \subseteq \mathcal{X}_{p}\left(\lambda_{i}^{\star}\right)$. For $\lambda=0, \mathcal{X}_{p}(0) \cap \mathcal{X}_{p} \neq \emptyset$, and if $m \geqslant n$, then $\mathcal{X}_{p}(0) \subset \mathcal{X}_{p}$.
Proof $8 \quad \forall i, \forall \lambda \in\left(\lambda_{i}^{\star}, \lambda_{i+1}^{\star}\right), \mathcal{X}_{p}(\lambda) \subseteq \mathcal{X}_{p}\left(\lambda_{i}^{\star}\right)$ : see proof 6 .

- The application of the result above with $i=0$ reads $\forall \lambda \in\left(0, \lambda_{1}^{\star}\right), \mathcal{X}_{p}(\lambda) \subseteq \mathcal{X}_{p}(0)$. Thus, we always have $\mathcal{X}_{p}(0) \cap \mathcal{X}_{p} \neq \emptyset$. For $m \geqslant n, \mathcal{X}_{p}(0)$ is formed of only one vector, thus $\forall \lambda \in\left(0, \lambda_{1}^{\star}\right), \mathcal{X}_{p}(\lambda)=\mathcal{X}_{p}(0)$, and $\mathcal{X}_{p}(0) \subseteq \mathcal{X}_{p}$. Since $\boldsymbol{y} \neq \mathbf{0}$ and $\boldsymbol{A}$ is full rank, the domain (5) is formed of at least two intervals $(I \geqslant 1)$, thus $\mathcal{X}_{p}(0) \subset \mathcal{X}_{p}$.

Theorem 7 For a given $\lambda$-value which is distinct from $\lambda_{0}^{\star}, \lambda_{1}^{\star}, \ldots, \lambda_{I}^{\star}$, all the elements of $\mathcal{X}_{p}(\lambda)$ are of same L0-norm, which is equal to the derivative of $\mathcal{J}(\lambda)$, and yield the same least-square cost.

Proof 9 This result has been shown in proof 6: if $\lambda \in\left(\lambda_{i}^{\star}, \lambda_{i+1}^{\star}\right)$ and $\boldsymbol{x} \in \mathcal{X}_{p}(\lambda)$, then $\|\boldsymbol{x}\|_{0}=\left\|\boldsymbol{x}_{\mathcal{A}_{i}}\right\|_{0}=\mathcal{J}^{\prime}(\lambda)$ and $\mathcal{E}(\boldsymbol{x})=\mathcal{E}_{\mathcal{A}_{i}}$.

Theorem 8 Let $\boldsymbol{x}_{p}(\lambda)$ be a sequence such that $\forall \lambda, \boldsymbol{x}_{p}(\lambda) \in \mathcal{X}_{p}(\lambda)$. Then, necessarily, $\left\|\boldsymbol{x}_{p}(\lambda)\right\|_{0}$ is a decreasing function of $\lambda$, and $\mathcal{E}\left(\boldsymbol{x}_{p}(\lambda)\right)$ is an increasing function of $\lambda$.

Proof 10 - Recall that for $i \in\{0, \ldots, I\}$, there exists a set $\mathcal{A}_{i}$ such that if $\lambda \in\left(\lambda_{i}^{\star}, \lambda_{i+1}^{\star}\right),\left\|\boldsymbol{x}_{p}(\lambda)\right\|_{0}=$ $\left\|\boldsymbol{x}_{\mathcal{A}_{i}}\right\|_{0}$ (see proof 6);

- The proof of the first result is a direct consequence of theorem 7: $\forall \lambda \notin\left\{\lambda_{0}^{\star}, \ldots, \lambda_{I}^{\star}\right\},\left\|\boldsymbol{x}_{p}(\lambda)\right\|_{0}=\mathcal{J}^{\prime}(\lambda)$, and of lemma 1: $\mathcal{J}$ is a concave function, thus its derivative (when it is defined) is a decreasing function of $\lambda$. At this point, we know that $\lambda \mapsto\left\|\boldsymbol{x}_{p}(\lambda)\right\|_{0}$ is piecewise constant on $\mathbb{R}_{+}$, and that its restriction to $\mathbb{R}_{+} \backslash\left\{\lambda_{0}^{\star}, \ldots, \lambda_{I}^{\star}\right\}$ is decreasing: $\forall i \in\{1, \ldots, I\},\left\|\boldsymbol{x}_{\mathcal{A}_{i-1}}\right\|_{0} \geqslant\left\|\boldsymbol{x}_{\mathcal{A}_{i}}\right\|_{0}$. The remaining part is to study the behavior of $\left\|\boldsymbol{x}_{p}(\lambda)\right\|_{0}$ at $\lambda=\lambda_{i}^{\star}, i=0, \ldots, I$.
For $i \in\{1, \ldots, I\}$, let us show that $\boldsymbol{x}_{p}\left(\lambda_{i}^{\star}\right)$ is such that $\left\|\boldsymbol{x}_{\mathcal{A}_{i-1}}\right\|_{0} \geqslant\left\|\boldsymbol{x}_{p}\left(\lambda_{i}^{\star}\right)\right\|_{0} \geqslant\left\|\boldsymbol{x}_{\mathcal{A}_{i}}\right\|_{0}$ :
$-\lambda \mapsto \mathcal{J}\left(\boldsymbol{x}_{p}\left(\lambda_{i}^{\star}\right) ; \lambda\right)$ and $\lambda \mapsto \mathcal{J}(\lambda)$ coincide at $\lambda=\lambda_{i}^{\star}$;
$-\mathcal{J}^{\prime}(\lambda)$ is equal to $\left\|\boldsymbol{x}_{\mathcal{A}_{i-1}}\right\|_{0}$ when $\lambda \in\left(\lambda_{i-1}^{\star}, \lambda_{i}^{\star}\right)$, and to $\left\|\boldsymbol{x}_{\mathcal{A}_{i}}\right\|_{0}$ when $\lambda \in\left(\lambda_{i}^{\star}, \lambda_{i+1}^{\star}\right)$.
- the derivative of $\lambda \mapsto \mathcal{J}\left(\boldsymbol{x}_{p}\left(\lambda_{i}^{\star}\right) ; \lambda\right)$ is equal to $\left\|\boldsymbol{x}_{p}\left(\lambda_{i}^{\star}\right)\right\|_{0}$.

Due to the definition of $\mathcal{J}$ in theorem 6, the affine function $\lambda \mapsto \mathcal{J}\left(\boldsymbol{x}_{p}\left(\lambda_{i}^{\star}\right) ; \lambda\right)$ is necessarily larger or equal to $\lambda \mapsto \mathcal{J}(\lambda)$ for $\lambda \in\left(\lambda_{i-1}^{\star}, \lambda_{i}^{\star}\right)$ and for $\lambda \in\left(\lambda_{i}^{\star}, \lambda_{i+1}^{\star}\right)$. This implies that $\left\|\boldsymbol{x}_{\mathcal{A}_{i-1}}\right\|_{0} \geqslant\left\|\boldsymbol{x}_{p}\left(\lambda_{i}^{\star}\right)\right\|_{0} \geqslant\left\|\boldsymbol{x}_{\mathcal{A}_{i}}\right\|_{0}$. A similar argument can be given to show that $\left\|\boldsymbol{x}_{p}\left(\lambda_{0}^{\star}\right)\right\|_{0} \geqslant\left\|\boldsymbol{x}_{\mathcal{A}_{0}}\right\|_{0}$.
Finally, we have shown that $\lambda \mapsto\left\|\boldsymbol{x}_{p}(\lambda)\right\|_{0}$ is decreasing on $\mathbb{R}_{+}$.

- Second result: for a given $i \in\{1, \ldots, I\}$, the continuity of $\mathcal{J}$ at $\lambda=\lambda_{i}^{\star}$ reads $\mathcal{E}_{\mathcal{A}_{i-1}}+\lambda_{i}^{\star}\left\|\boldsymbol{x}_{\mathcal{A}_{i-1}}\right\|_{0}=$ $\mathcal{E}_{\mathcal{A}_{i}}+\lambda_{i}^{\star}\left\|\boldsymbol{x}_{\mathcal{A}_{i}}\right\|_{0}$. Because $\left\|\boldsymbol{x}_{\mathcal{A}_{i-1}}\right\|_{0} \geqslant\left\|\boldsymbol{x}_{\mathcal{A}_{i}}\right\|_{0}, \mathcal{E}_{\mathcal{A}_{i-1}} \leqslant \mathcal{E}_{\mathcal{A}_{i}}$.
When $\lambda$ varies from 0 to $+\infty$ and $\lambda \notin\left\{\lambda_{1}^{\star}, \ldots, \lambda_{I}^{\star}\right\}, \mathcal{E}\left(\boldsymbol{x}_{p}(\lambda)\right)$ takes sequentially the values $\mathcal{E}_{\mathcal{A}_{i}}, i=0, \ldots, I$. Thus, the restriction of $\lambda \mapsto \mathcal{E}\left(\boldsymbol{x}_{p}(\lambda)\right)$ to $\mathbb{R}_{+} \backslash\left\{\lambda_{1}^{\star}, \ldots, \lambda_{I}^{\star}\right\}$ is increasing. With similar arguments than in the first result, we can show that for $i \in\{1, \ldots, I\}, \mathcal{E}_{\mathcal{A}_{i-1}} \leqslant \mathcal{E}\left(\boldsymbol{x}_{p}\left(\lambda_{i}^{\star}\right)\right) \leqslant \mathcal{E}_{\mathcal{A}_{i}}$. Finally, $\lambda \mapsto \mathcal{E}\left(\boldsymbol{x}_{p}(\lambda)\right)$ is increasing on $\mathbb{R}_{+}$.


Figure 1: Representation of the affine curves $\lambda \mapsto \mathcal{J}_{\mathcal{A}}(\lambda)=\mathcal{E}_{\mathcal{A}}+\lambda\left\|\boldsymbol{x}_{\mathcal{A}}\right\|_{0}$ for all the possible supports $\mathcal{A}$ such that $\|\mathcal{A}\|_{0} \leqslant \min (m, n)$. The piecewise linear function $\lambda \mapsto \mathcal{J}(\lambda)$ is defined according to (6) and is represented in bold lines. From this illustration, we can try to explain that both solution paths $\mathcal{X}_{c}$ and $\mathcal{X}_{p}$ are not equivalent. By following the bold curve representing $\lambda \mapsto \mathcal{J}(\lambda)$, we see that the penalized solution path corresponds to the active sets $\mathcal{A}_{0}, \mathcal{A}_{2}$ and $\mathcal{A}_{3}$ such that $\left\|\boldsymbol{x}_{\mathcal{A}}\right\|_{0}$ is equal to 0,2 and 3 , respectively. No active set such that $\left\|\boldsymbol{x}_{\mathcal{A}}\right\|_{0}=1$ is present in $\mathcal{X}_{p}$. $\mathcal{A}_{2}^{\prime}$ is an active set such that $\left\|\boldsymbol{x}_{\mathcal{A}_{2}^{\prime}}\right\|_{0}=2$ but $\mathcal{E}_{\mathcal{A}_{2}^{\prime}}>\mathcal{E}_{\mathcal{A}_{2}}$. $\mathcal{A}_{1}$ is the active set such that $\left\|\boldsymbol{x}_{\mathcal{A}_{1}}\right\|_{0}=1$ whose energy is the lowest among all the active set such that $\left\|\boldsymbol{x}_{\mathcal{A}}\right\|_{0} \leqslant 1$, however, $\forall \lambda, \mathcal{J}_{\mathcal{A}_{1}}(\lambda)>\mathcal{J}(\lambda)$. On the contrary, for all $\lambda \neq\left\{\lambda_{1}^{\star}, \ldots, \lambda_{I}^{\star}\right\}, \mathcal{X}_{p}(\lambda)=\mathcal{X}_{c}\left(k_{\lambda}\right)$, with $k_{\lambda}=\mathcal{J}^{\prime}(\lambda)=0,2$ or 3 .

### 3.3 Cardinality of $\mathcal{X}_{p}(\lambda)$

It is easy to see that:

- For all $i \in\{1, \ldots, I\}$, the cardinality of $\mathcal{X}_{p}\left(\lambda_{i}^{\star}\right)$ is larger than 2 , because at $\lambda=\lambda_{i}^{\star}$, at least two distinct affine curves $\lambda \mapsto \mathcal{J}_{\mathcal{A}}(\lambda)=\mathcal{E}_{\mathcal{A}}+\lambda\left\|\boldsymbol{x}_{\mathcal{A}}\right\|_{0}$ intersect.
- $\mathcal{X}_{p}(\lambda)$ is reduced to the unique vector $\mathbf{0}$ for the largest $\lambda$-values $\left(\lambda>\lambda_{I}^{\star} \Rightarrow \mathcal{X}_{p}(\lambda)=\{\mathbf{0}\}\right)$.
- For $\lambda=0$ (the least-square error $\mathcal{E}(\boldsymbol{x})$ is minimized with no penalty), $\mathcal{X}_{p}(0)$ is either reduced to the unique vector $\left(\boldsymbol{A}^{T} \boldsymbol{A}\right)^{-1} \boldsymbol{A}^{T} \boldsymbol{y}$ when $m \geqslant n$, or is of infinite cardinality otherwise.

We can conclude that at least for $m \geqslant n$, the cardinality of $\mathcal{X}_{p}(\lambda)$ is not monotonic w.r.t. $\lambda$.

### 3.4 Relationship between the constrained and the penalized solution paths

Generally, the solution paths $\mathcal{X}_{c}$ and $\mathcal{X}_{p}$ do not coincide. This is a consequence of the non-convexity of the L0-norm [2]. However, $\mathcal{X}_{p} \subseteq \mathcal{X}_{c}$ is always true (a well-known result in the literature of multi-objective optimization?).

In general, the proposition " $\forall k, \exists \lambda, \mathcal{X}_{c}(k) \subseteq \mathcal{X}_{p}(\lambda)$ " is false (see Fig. 1).
Theorem 9 If $\lambda \neq\left\{\lambda_{0}^{\star}, \ldots, \lambda_{I}^{\star}\right\}$, then there exists $k$ such that $\mathcal{X}_{p}(\lambda)=\mathcal{X}_{c}(k)$.

Proof 11 For a given $\lambda$-value, let $\boldsymbol{x} \in \mathcal{X}_{p}(\lambda)$, and let $\mathcal{A} \triangleq \mathcal{A}(\boldsymbol{x})$ denote the support of $\boldsymbol{x}$, and $k_{\boldsymbol{x}} \triangleq\|\boldsymbol{x}\|_{0}=\|\mathcal{A}\|_{0}$. According to theorem 5 and proof $5,\|\mathcal{A}\|_{0} \leqslant \min (m, n)$ and $\boldsymbol{x}=\boldsymbol{x}_{\mathcal{A}}$.

- Let us show that $\boldsymbol{x} \in \mathcal{X}_{c}\left(k_{\boldsymbol{x}}\right)$. If there exists a support $\mathcal{B}$ such that $\|\mathcal{B}\|_{0} \leqslant k_{\boldsymbol{x}}$ and $\mathcal{E}_{\mathcal{B}}<\mathcal{E}(\boldsymbol{x})=\mathcal{E}_{\mathcal{A}}$, then $\mathcal{J}\left(\boldsymbol{x}_{\mathcal{B}} ; \lambda\right)<\mathcal{J}\left(\boldsymbol{x}_{\mathcal{A}} ; \lambda\right)$. This is in contradiction with $\boldsymbol{x} \in \mathcal{X}_{p}(\lambda)$.

At this point, we have shown that

$$
\forall \lambda, \forall \boldsymbol{x} \in \mathcal{X}_{p}(\lambda), \exists k_{\boldsymbol{x}}, \boldsymbol{x} \in \mathcal{X}_{c}\left(k_{\boldsymbol{x}}\right),
$$

or equivalently,

$$
\mathcal{X}_{p} \subseteq \mathcal{X}_{c}
$$

- The following of the proof requires the assumption $\lambda \neq\left\{\lambda_{0}^{\star}, \ldots, \lambda_{I}^{\star}\right\}$. We have seen that if $\boldsymbol{x}$ and $\boldsymbol{y} \in \mathcal{X}_{p}(\lambda)$, then $\boldsymbol{x} \in \mathcal{X}_{c}\left(k_{\boldsymbol{x}}\right)$ and $\boldsymbol{y} \in \mathcal{X}_{c}\left(k_{\boldsymbol{y}}\right)$. According to theorem 7, all the elements of $\mathcal{X}_{p}(\lambda)$ are of same L0-norm. Therefore, $k_{\boldsymbol{y}}=k_{\boldsymbol{x}}$. At this point, we have shown that

$$
\forall \lambda \neq\left\{\lambda_{0}^{\star}, \ldots, \lambda_{I}^{\star}\right\}, \exists k_{\lambda}, \mathcal{X}_{p}(\lambda) \subset \mathcal{X}_{c}\left(k_{\lambda}\right)
$$

- Now, let us prove the reverse inclusion. Given $\lambda$, there exists at least one $\boldsymbol{x}$ such that $\boldsymbol{x} \in \mathcal{X}_{p}(\lambda), k_{\lambda}=\|\boldsymbol{x}\|_{0}$ and $\boldsymbol{x} \in \mathcal{X}_{c}\left(k_{\lambda}\right)$. For all $\boldsymbol{y} \in \mathcal{X}_{c}\left(k_{\lambda}\right)$, we have necessarily $\mathcal{E}(\boldsymbol{y})=\mathcal{E}(\boldsymbol{x})$ and $\|\boldsymbol{y}\|_{0} \leqslant\|\boldsymbol{x}\|_{0}$, thus $\mathcal{J}(\boldsymbol{y} ; \lambda) \leqslant \mathcal{J}(\boldsymbol{x} ; \lambda)$. Since $\boldsymbol{x} \in \mathcal{X}_{p}(\lambda)$, we deduce that $\mathcal{J}(\boldsymbol{y} ; \lambda)=\mathcal{J}(\boldsymbol{x} ; \lambda)$ and that $\boldsymbol{y} \in \mathcal{X}_{p}(\lambda)$. This completes the proof, since we have shown that

$$
\forall \lambda \neq\left\{\lambda_{0}^{\star}, \ldots, \lambda_{I}^{\star}\right\}, \exists k_{\boldsymbol{x}}, \mathcal{X}_{c}\left(k_{\lambda}\right) \subseteq \mathcal{X}_{p}(\lambda)
$$

### 3.5 Other results concerning $\mathcal{X}_{p}(\lambda)$

Lemma 3 If $\lambda_{i-1}^{\star}<\lambda<\lambda_{i}^{\star}<\lambda^{\prime}<\lambda_{i+1}^{\star}$, then $\mathcal{X}_{p}(\lambda) \cap \mathcal{X}_{p}\left(\lambda^{\prime}\right)=\emptyset$.
Proof 12 According to theorem 7, all the vectors of $\mathcal{X}_{p}(\lambda)$ (respectively of $\mathcal{X}_{p}\left(\lambda^{\prime}\right)$ ) are of same L0-norm, which is the derivative of $\mathcal{J}$ at $\lambda$ (resp. $\lambda^{\prime}$ ). Thus, if $\mathcal{X}_{p}(\lambda) \cap \mathcal{X}_{p}\left(\lambda^{\prime}\right) \neq \emptyset$, the derivative of $\mathcal{J}$ is constant on $\left(\lambda_{i-1}^{\star}, \lambda_{i+1}^{\star}\right) \backslash\left\{\lambda_{i}^{\star}\right\}$, which is in contradiction with the definition of $\lambda_{i}^{\star}$ (critical point, at which the derivative of $\mathcal{J}$ is changing).

Theorem 10 If $\lambda_{i-1}^{\star}<\lambda<\lambda_{i}^{\star}<\lambda^{\prime}<\lambda_{i+1}^{\star}$, then $\mathcal{X}_{p}(\lambda) \cup \mathcal{X}_{p}\left(\lambda^{\prime}\right) \subseteq \mathcal{X}_{p}\left(\lambda_{i}^{\star}\right)$, thus $\operatorname{Card}\left(\mathcal{X}_{p}(\lambda)\right)+\operatorname{Card}\left(\mathcal{X}_{p}\left(\lambda^{\prime}\right)\right) \leqslant$ $\operatorname{Card}\left(\mathcal{X}_{p}\left(\lambda_{i}^{\star}\right)\right)$ (Card denotes the cardinality). If $\lambda \in\left(\lambda_{i-1}^{\star}, \lambda_{i+1}^{\star}\right) \backslash\left\{\lambda_{i}^{\star}\right\}$, then $\operatorname{Card}\left(\mathcal{X}_{p}(\lambda)\right)<\operatorname{Card}\left(\mathcal{X}_{p}\left(\lambda_{i}^{\star}\right)\right)$.

Proof 13 - First result: according to lemma 2, if $\lambda \in\left(\lambda_{i}^{\star}, \lambda_{i+1}^{\star}\right)$, then $\mathcal{X}_{p}(\lambda) \subseteq \mathcal{X}_{p}\left(\lambda_{i}^{\star}\right)$. According to lemma 3, if $\lambda_{i-1}^{\star}<\lambda<\lambda_{i}^{\star}<\lambda^{\prime}<\lambda_{i+1}^{\star}$, then $\mathcal{X}_{p}(\lambda) \cap \mathcal{X}_{p}\left(\lambda^{\prime}\right)=\emptyset$. Thus, $\mathcal{X}_{p}(\lambda) \cup \mathcal{X}_{p}\left(\lambda^{\prime}\right) \subseteq \mathcal{X}_{p}\left(\lambda_{i}^{\star}\right)$ and $\operatorname{Card}\left(\mathcal{X}_{p}(\lambda)\right)+\operatorname{Card}\left(\mathcal{X}_{p}\left(\lambda^{\prime}\right)\right) \leqslant \operatorname{Card}\left(\mathcal{X}_{p}\left(\lambda_{i}^{\star}\right)\right)$.

- Second result: since neither $\mathcal{X}_{p}(\lambda)$ nor $\mathcal{X}_{p}\left(\lambda^{\prime}\right)$ is empty, their L0-norm is larger or equal to 1 , then, applying $\operatorname{Card}\left(\mathcal{X}_{p}(\lambda)\right)+\operatorname{Card}\left(\mathcal{X}_{p}\left(\lambda^{\prime}\right)\right) \leqslant \operatorname{Card}\left(\mathcal{X}_{p}\left(\lambda_{i}^{\star}\right)\right)$, we deduce that $\operatorname{Card}\left(\mathcal{X}_{p}(\lambda)\right)<\operatorname{Card}\left(\mathcal{X}_{p}\left(\lambda_{i}^{\star}\right)\right)$ and $\operatorname{Card}\left(\mathcal{X}_{p}\left(\lambda^{\prime}\right)\right)<\operatorname{Card}\left(\mathcal{X}_{p}\left(\lambda_{i}^{\star}\right)\right)$.


## 4 SBR and CSBR algorithms

### 4.1 SBR iterates and output

Let us consider the SBR algorithm for a given $\lambda$-value. An SBR iterate takes the form of:

- an active set $\mathcal{A}$ (for simplicity, we omit the dependence w.r.t. $\lambda$ );
- the corresponding least-square minimizer $\boldsymbol{x}_{\mathcal{A}}=\underset{\mathcal{A}(\boldsymbol{x}) \subseteq \mathcal{A}}{\arg \min } \mathcal{E}(\boldsymbol{x})$.
$\widehat{\boldsymbol{x}}(\lambda)=\boldsymbol{x}_{\mathcal{A}}$ is chosen as the estimator of a minimizer (there may be several) of $\mathcal{J}(\boldsymbol{x} ; \lambda)=\mathcal{E}(\boldsymbol{x})+\lambda\|\boldsymbol{x}\|_{0}$ over $\mathbb{R}^{n}$.

First, recall that for the SBR iterates (and in particular when SBR terminates), $\left\|\boldsymbol{x}_{\mathcal{A}}\right\|_{0}=\|\mathcal{A}\|_{0}$. This property can be guaranteed by including in the SBR loops a small procedure which removes from the active set $\mathcal{A}$ all the indices $i \in \mathcal{A}$ such that $\boldsymbol{x}_{\mathcal{A}}(i)=0$ (however, this case rarely happens in practice). The following remark follows from this property.

Remark 3 For a given $\lambda$-value, the cost of an $S B R$ iterate $\boldsymbol{x}_{\mathcal{A}}$ is $\mathcal{J}\left(\boldsymbol{x}_{\mathcal{A}} ; \lambda\right)=\mathcal{E}\left(\boldsymbol{x}_{\mathcal{A}}\right)+\lambda\|\mathcal{A}\|_{0}$.
Definition 7 Because the cost of the SBR iterates only depend on their support and for convenience, we use the simplified notations $\mathcal{E}_{\mathcal{A}} \triangleq \mathcal{E}\left(\boldsymbol{x}_{\mathcal{A}}\right)$ and $\mathcal{J}_{\mathcal{A}}(\lambda) \triangleq \mathcal{J}\left(\boldsymbol{x}_{\mathcal{A}} ; \lambda\right)=\mathcal{E}_{\mathcal{A}}+\lambda\|\mathcal{A}\|_{0}$.

Remark 4 SBR terminates after a finite number of iterations. Moreover, a set $\mathcal{A}$ cannot be explored twice while running $S B R$.

Proof $14 S B R$ is a descent algorithm and the number of sets $\mathcal{A}$ which are reachable is finite (i.e., the number of subsets of $\{1, \ldots, n\}$ ).

Remark 5 When SBR terminates, the estimate $\widehat{\boldsymbol{x}}(\lambda)=\boldsymbol{x}_{\mathcal{A}}$ is generally not included in $\mathcal{X}_{p}(\lambda)$ because $S B R$ is a sub-optimal algorithm.

Remark 6 At the $S B R$ output $\mathcal{A}$, $\mathcal{J}$ is "locally minimum w.r.t. $\mathcal{A}$ ": any replacement of $\mathcal{A}$ by $\mathcal{A} \bullet i$ (where $\bullet \triangleq \cup$ or $\backslash$ ) does not yield a decrease of the cost $\mathcal{J}_{\mathcal{A}}(\lambda)$ ). Formally, this property reads:

$$
\begin{equation*}
\forall i, \mathcal{E}_{\mathcal{A}}+\lambda\|\mathcal{A}\|_{0} \leqslant \mathcal{E}_{\mathcal{A} \bullet i}+\lambda\left\|\boldsymbol{x}_{\mathcal{A} \bullet i}\right\|_{0} \tag{10}
\end{equation*}
$$

or equivalently,

$$
\begin{equation*}
\forall i, \mathcal{E}_{\mathcal{A}}-\mathcal{E}_{\mathcal{A} \bullet i} \leqslant \lambda\left(\left\|\boldsymbol{x}_{\mathcal{A} \bullet i}\right\|_{0}-\|\mathcal{A}\|_{0}\right) \tag{11}
\end{equation*}
$$

Here, we do not consider the small procedure described above (update of $\mathcal{A} \bullet i$ by removing the indices corresponding to the zero valued entries of $\left.\boldsymbol{x}_{\mathcal{A} \bullet i}\right)$, therefore we use $\left\|\boldsymbol{x}_{\mathcal{A} \bullet}\right\|_{0}$, which may be lower than $\|\mathcal{A} \bullet i\|_{0}$.

### 4.2 Iterative computation of $\lambda$ in the CSBR algorithm

When $\lambda=\lambda_{q}>0\left(q\right.$-th call to SBR ), let $\mathcal{A}=\mathcal{A}_{q}$ be the support of the output of $\operatorname{SBR}\left(\lambda_{q}\right)$. Then, property (11) holds. For simplicity, we will omit, when possible, the dependence of $\mathcal{A}$ w.r.t. $q$. When $\lambda_{q}$ is replaced by another value $\lambda \leqslant \lambda_{q}$ and $\mathcal{A}=\mathcal{A}_{q}$ is kept fixed, for which $\lambda$-values does (11) remain valid?

When $\bullet=\backslash$, both terms on the left- and right-hand sides of the inequality are strictly negative $\left(\left\|\boldsymbol{x}_{\mathcal{A} \backslash i}\right\|_{0}<\right.$ $\|\mathcal{A}\|_{0}$ ), while when $\bullet=\cup$, both terms are positive since $\mathcal{E}_{\mathcal{A}}-\mathcal{E}_{\mathcal{A} \cup i} \geqslant 0$ and (11) holds for $\lambda=\lambda_{q}$. Therefore, (11) remains valid for $\lambda \neq \lambda_{q}$ if and only if

$$
\begin{equation*}
\max _{i \notin \mathcal{A} \text { and }\left\|\boldsymbol{x}_{\mathcal{A} \cup i}^{*}\right\|_{0}=\|\mathcal{A}\|_{0}+1}\left(\mathcal{E}_{\mathcal{A}}-\mathcal{E}_{\mathcal{A} \cup i}\right) \leqslant \lambda \leqslant \min _{i \in \mathcal{A}}\left[\frac{\mathcal{E}_{\mathcal{A}}-\mathcal{E}_{\mathcal{A} \backslash i}}{\left\|\boldsymbol{x}_{\mathcal{A} \backslash i}\right\|_{0}-\|\mathcal{A}\|_{0}}\right] \tag{12}
\end{equation*}
$$

The lower bound of (12) can be simplified to $\max _{i \notin \mathcal{A}}\left(\mathcal{E}_{\mathcal{A}}-\mathcal{E}_{\mathcal{A} \cup i}\right)$ because if $i \notin \mathcal{A}$ is such that $\left\|\boldsymbol{x}_{\mathcal{A} \cup i}^{\star}\right\|_{0}=\|\mathcal{A}\|_{0}$, then (11) implies that $\mathcal{E}_{\mathcal{A}}-\mathcal{E}_{\mathcal{A} \cup i}=0$. Thus, including these indices $i$ in the computation of the lower bound of (12) does not change its value, and (12) simplifies to

$$
\begin{equation*}
\max _{i \notin \mathcal{A}}\left(\mathcal{E}_{\mathcal{A}}-\mathcal{E}_{\mathcal{A} \cup i}\right) \leqslant \lambda \leqslant \min _{i \in \mathcal{A}}\left[\frac{\mathcal{E}_{\mathcal{A}}-\mathcal{E}_{\mathcal{A} \backslash i}}{\left\|\boldsymbol{x}_{\mathcal{A} \backslash i}\right\|_{0}-\|\mathcal{A}\|_{0}}\right] \tag{13}
\end{equation*}
$$

Given $\lambda_{q}$, the next $\lambda$-value $\lambda_{q+1}<\lambda_{q}$ is found by computing the lower bound of (13).
How to choose $\lambda_{q+1}$ ? Setting $\lambda_{q+1}$ to the lower bound of (13) is not judicious, since for this $\lambda$-value, $\mathcal{J}$ is still "locally minimum w.r.t. $\mathcal{A}$ " in the sense of (10). One possibility is to set $\lambda_{q+1}$ to the lower bound of (13)
minus some $\varepsilon>0$, without guarantee that this value is larger than the "next lower bound" of (13). Another possibility is to sort the values of

$$
\begin{equation*}
\tilde{\lambda}_{i} \triangleq \mathcal{E}_{\mathcal{A}}-\mathcal{E}_{\mathcal{A} \cup i} \tag{14}
\end{equation*}
$$

for all indices $i \notin \mathcal{A}$, and then to set $\lambda_{q+1}$ as the mean of the two largest values. This setting ensures that the inequality $\widetilde{\lambda_{i}} \leqslant \lambda_{q+1}$ does not hold for one value of $\widetilde{\lambda_{i}}$ only.

- If the number of indices $i \notin \mathcal{A}$ such that $\widetilde{\lambda}_{i}>0$ is equal to 1 , then we set $\lambda_{q+1}$ to half of the value of $\widetilde{\lambda_{i}}$.
- If all indices $i \notin \mathcal{A}$ are such that $\widetilde{\lambda}_{i}=0$, then we terminate CSBR.
- If $\mathcal{A}$ is the complete set $\{1, \ldots, n\}$, the lower bound of (13) is undefined, and we terminate CSBR.

Remark 7 For a given $i \notin \mathcal{A}$ for which $\left\|\boldsymbol{x}_{\mathcal{A} \cup i}\right\|_{0}=\|\mathcal{A}\|_{0}+1, \widetilde{\lambda}_{i}$ is the $\lambda$-value for which both affine curves $\lambda \mapsto \mathcal{J}_{\mathcal{A}}(\lambda)$ and $\lambda \mapsto \mathcal{J}_{A c \cup i}(\lambda)$ intersect. Similarly, for $i \in \mathcal{A}$, the value

$$
\begin{equation*}
\tilde{\lambda}_{i} \triangleq \frac{\mathcal{E}_{\mathcal{A}}-\mathcal{E}_{\mathcal{A} \backslash i}}{\left\|\boldsymbol{x}_{\mathcal{A} \backslash i}\right\|_{0}-\|\mathcal{A}\|_{0}} \tag{15}
\end{equation*}
$$

is the $\lambda$-value for which both affine curves $\lambda \mapsto \mathcal{J}_{\mathcal{A}}(\lambda)$ and $\lambda \mapsto \mathcal{J}_{\mathcal{A} \backslash i}(\lambda)$ intersect.
Proof 15 For $i \notin \mathcal{A}$ and for the $\lambda$-value $\tilde{\lambda}_{i}$, (11) is an equality, then $\mathcal{J}_{\mathcal{A}}\left(\lambda_{i}\right)=\mathcal{J}_{\mathcal{A}} \cup i\left(\lambda_{i}\right)$. Since $\left\|\boldsymbol{x}_{\mathcal{A} \cup i}\right\|_{0}$ is supposed to be different from $\|\mathcal{A}\|_{0}$, both affine curves $\lambda \mapsto \mathcal{J}_{\mathcal{A}}(\lambda)$ and $\lambda \mapsto \mathcal{J}_{\mathcal{A} \cup i}(\lambda)$ are not parallel (their derivatives are equal to $\|\mathcal{A}\|_{0}$ and $\left\|\boldsymbol{x}_{\mathcal{A} \cup i}\right\|_{0}=\|\mathcal{A}\|_{0}+1$ respectively), and they intersect at $\lambda=\lambda_{i}$. A similar proof holds in the case where $i \in \mathcal{A}$ and $\cup$ is replaced by $\backslash$.

Remark 8 A set $\mathcal{A}$ of cardinality larger than $\min (m, n)$ cannot be explored.
Proof 16 If a set $\mathcal{A}$ of cardinality larger than $\min (m, n)$ is explored, then CSBR has earlier explored at least one set $\mathcal{B}$ of cardinality $\min (m, n)$ (recall that the initial solution is $\mathcal{A}=\emptyset$ ). Due to the URP assumption, $\mathcal{E}_{\mathcal{B}}=\mathcal{E}_{\mathcal{A}}=\min _{\boldsymbol{x} \in \mathbb{R}^{n}} \mathcal{E}(\boldsymbol{x})$ is the optimal least-square cost. Therefore, $\|\mathcal{A}\|_{0}>\|\mathcal{B}\|_{0} \Rightarrow \forall \lambda>0, \mathcal{J}_{\mathcal{A}}(\lambda)>\mathcal{J}_{\mathcal{B}}(\lambda)$.

### 4.3 Termination of CSBR

Remark 9 When $C S B R$ terminates, $\mathcal{E}=\mathcal{J}_{\mathcal{A}}(0)$ is locally minimum w.r.t. $\mathcal{A}$, then $\forall i \notin \mathcal{A}, \mathcal{E}_{\mathcal{A} \cup i}=\mathcal{E}_{\mathcal{A}}$.
Remark 10 CSBR terminates after a finite number of SBR iterations.
Proof 17 - According to remark 4, for a given $\lambda$-value $\lambda_{q}, S B R\left(\lambda_{q}\right)$ terminates after a finite number of iterations.

- According to remark 7, each value of $\lambda_{q}$ is such that there exists $\underline{\mu_{q}}$ and $\overline{\mu_{q}}$ such that
$-0 \leqslant \underline{\mu_{q}}<\lambda_{q}<\overline{\mu_{q}} ;$
$-\lambda_{q}=\left(\underline{\mu_{q}}+\overline{\mu_{q}}\right) / 2$;
- $\underline{\mu_{q}}$ and $\overline{\mu_{q}}$ are critical values for which two affine curves $\lambda \mapsto \mathcal{J}_{\mathcal{A}}(\lambda)$ and $\lambda \mapsto \mathcal{J}_{\mathcal{A}}(\lambda \cup i)$ intersect.

From the recursive construction of the sequence $\left(\lambda_{q}, q \geqslant 0\right)$, it is clear that $\forall q, \overline{\mu_{q}}<\lambda_{q-1}$, thus $\forall q, \overline{\mu_{q}}<$ $\overline{\mu_{q-1}}$. Since each value of $\overline{\mu_{q}}$ can be associated to a given intersection between two affine curves $\lambda \mapsto \mathcal{J}_{\mathcal{A}}(\lambda)$ and $\lambda \mapsto \mathcal{J}_{\mathcal{B}}(\lambda)$ and the number of possible subsets $\mathcal{A}$ and $\mathcal{B}$ of $\{1, \ldots, n\}$ whose cardinality is lower than $\min (m, n)$ is finite, the number of possible values taken by $\overline{\mu_{q}}$ is also finite. Since the sequence $\left(\overline{\mu_{q}}, q \geqslant 0\right)$ satisfies $\forall q, \overline{\mu_{q}}<\overline{\mu_{q-1}}$, we conclude that the number of iterations $q$ at which $\operatorname{SBR}\left(\lambda_{q}\right)$ is run is finite.


Figure 2: A given set $\mathcal{A}$ may be explored twice during the CSBR procedure, at different $\lambda$-values. Here, each vertical line corresponds to one call to SBR, i.e., to a fixed $\lambda$-value while the plain lines are the affine curves $\lambda \mapsto \mathcal{J}_{\mathcal{A}}(\lambda)$ and $\lambda \mapsto \mathcal{J}_{\mathcal{A}^{\prime}}(\lambda)$.

Note that despite remark 10, we cannot claim that a given set $\mathcal{A}$ is never explored twice during the CSBR procedure. In remark 4 , we have seen that a given set $\mathcal{A}$ can never be explored twice while running SBR for a given $\lambda$-value. However, $\mathcal{A}$ may be explored several times while running CSBR, i.e., once while running SBR at some $\lambda$-value $\lambda_{q}$, and then another time while running SBR at another $\lambda$-value $\lambda_{q^{\prime}} \leqslant \lambda_{q}$. Please see Fig. 2 for a simple illustration.

Remark 11 When CSBR terminates, the solution $\boldsymbol{x}_{\mathcal{A}}$ is an unconstrained least-square estimate.
Proof 18 Let us define the residual $\boldsymbol{r}=\boldsymbol{y}-\boldsymbol{A} \boldsymbol{x}_{\mathcal{A}}$ and the unit vectors $\boldsymbol{e}_{i}$ in which all entries are equal to 0 except the $i$-th entry, equal to 1 .

Firstly, we prove that $\forall i \notin \mathcal{A}, \boldsymbol{a}_{i}^{T} \boldsymbol{r}=0$. According to remark 9 , $\mathcal{A}$ is such that $\forall i \notin \mathcal{A}, \mathcal{E}_{\mathcal{A} \cup i}=\mathcal{E}_{\mathcal{A}}$. Then, the following inequalities hold:

$$
\begin{array}{ll}
\forall i \notin \mathcal{A}, & \forall \varepsilon \in \mathbb{R}, \mathcal{E}\left(\boldsymbol{x}_{\mathcal{A}}+\varepsilon \boldsymbol{e}_{i}\right)-\mathcal{E}\left(\boldsymbol{x}_{\mathcal{A}}\right) \geqslant 0 \\
\forall i \notin \mathcal{A}, & \forall \varepsilon \in \mathbb{R},\left\|\boldsymbol{r}-\varepsilon \boldsymbol{a}_{i}\right\|^{2}-\|\boldsymbol{r}\|^{2} \geqslant 0 \\
\forall i \notin \mathcal{A}, & \forall \varepsilon \in \mathbb{R}, \varepsilon^{2}\left\|\boldsymbol{a}_{i}\right\|^{2}-2 \varepsilon \boldsymbol{a}_{i}^{T} \boldsymbol{r} \geqslant 0 \\
\forall i \notin \mathcal{A}, & \boldsymbol{a}_{i}^{T} \boldsymbol{r}=0,
\end{array}
$$

where $\boldsymbol{a}_{i}$ stands for the $i$-th column of $\boldsymbol{A}$.
Secondly, because $\boldsymbol{x}_{\mathcal{A}}$ is a solution to the constrained problem (2):

$$
\begin{array}{ll}
\forall i \in \mathcal{A}, & \forall \varepsilon \in \mathbb{R}, \mathcal{E}\left(\boldsymbol{x}_{\mathcal{A}}+\varepsilon \boldsymbol{e}_{i}\right)-\mathcal{E}\left(\boldsymbol{x}_{\mathcal{A}}\right) \geqslant 0 \\
\forall i \in \mathcal{A}, & \boldsymbol{a}_{i}^{T} \boldsymbol{r}=0
\end{array}
$$

Finally, we deduce that

$$
\begin{aligned}
\forall i \in\{1, \ldots, n\}, \boldsymbol{a}_{i}^{T} \boldsymbol{r} & =0 \\
\boldsymbol{A}^{T} \boldsymbol{r} & =\mathbf{0} \\
\boldsymbol{A}^{T} \boldsymbol{A} \boldsymbol{x}_{\mathcal{A}} & =\boldsymbol{A}^{T} \boldsymbol{y} \\
\nabla \mathcal{E}\left(\boldsymbol{x}_{\mathcal{A}}\right) & =\mathbf{0}
\end{aligned}
$$

We have shown that $\boldsymbol{x}_{\mathcal{A}}$ is an unconstrained least-square estimate.

## References

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