



**HAL**  
open science

# The semi-classical spectrum and the Birkhoff normal form

Yves Colin de Verdière

► **To cite this version:**

Yves Colin de Verdière. The semi-classical spectrum and the Birkhoff normal form. 2009. hal-00361271

**HAL Id: hal-00361271**

**<https://hal.science/hal-00361271>**

Preprint submitted on 13 Feb 2009

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# The semi-classical spectrum and the Birkhoff normal form

Yves Colin de Verdière\*

February 13, 2009

## Introduction

The purposes of this note are

- To propose a direct and “elementary” proof of the main result of [3], namely that the semi-classical spectrum near a global minimum of the classical Hamiltonian determines the whole semi-classical Birkhoff normal form (denoted the BNF) in the non-resonant case. I believe however that the method used in [3] (trace formulas) are more general and can be applied to any non degenerate non resonant critical point provided that the corresponding critical value is “simple”.
- To present in the completely resonant case a similar problem which is NOT what is done in [3]: there, only the *non-resonant part* of the BNF is proved to be determined from the semi-classical spectrum!

## 1 A direct proof of the main result of [3]

### 1.1 The Theorem

Let us give a semi-classical Hamiltonian  $\hat{H}$  on  $\mathbb{R}^d$  (or even on a smooth connected manifold of dimension  $d$ ) which is the Weyl quantization of the symbol  $H \equiv H_0 + \hbar H_1 + \hbar^2 H_2 + \dots$ .

Let us assume that  $H_0$  has a global non degenerate non resonant minimum  $E_0$  at the point  $z_0$ : it means that after some affine symplectic change of variables

---

\*Institut Fourier, Unité mixte de recherche CNRS-UJF 5582, BP 74, 38402-Saint Martin d'Hères Cedex (France); yves.colin-de-verdiere@ujf-grenoble.fr

$H_0 = E_0 + \frac{1}{2} \sum_{j=1}^d \omega_j (x_j^2 + \xi_j^2) + \dots$  where the  $\omega_j$ 's are  $> 0$  and independent over the rationals. We can assume that  $0 < \omega_1 < \omega_2 < \dots < \omega_d$ . We will denote  $E_1 = H_1(z_0)$ .

We assume also that

$$\liminf_{(x,\xi) \rightarrow \infty} H(x,\xi) > E_0 .$$

Let us denote by  $\lambda_1(\hbar) < \lambda_2(\hbar) \leq \dots \leq \lambda_N(\hbar) \leq \dots$  the discrete spectrum of  $\hat{H}$ . This set can be finite for a fixed value of  $\hbar$ , but, if  $N$  is given,  $\lambda_N(\hbar)$  exists for  $\hbar$  small enough.

**Definition 1.1** *The semi-classical spectrum of  $\hat{H}$  is the set of all  $\lambda_N(\hbar)$  ( $N = 1, \dots$ ) modulo  $O(\hbar^\infty)$ . **NO uniformity with respect to  $N$  in the  $O(\hbar^\infty)$  is required.***

**Definition 1.2** *The semi-classical Birkhoff normal form is the following formal series expansion in  $\Omega = (\Omega_1, \dots, \Omega_d)$  and  $\hbar$ :*

$$\hat{B} \equiv E_0 + \hbar E_1 + \sum_{j=1}^d \omega_j \Omega_j + \sum_{l+|\alpha| \geq 2} c_{l,\alpha} \hbar^l \Omega^\alpha$$

with  $\Omega_j = \frac{1}{2} (-\hbar^2 \partial_j^2 + x_j^2)$ . The series  $\hat{B}$  is uniquely defined as being the Weyl quantization of some symbol  $B$  equivalent to the Taylor expansion at  $z_0$  of  $H$  by some automorphism of the semi-classical Weyl algebra (see [2]).

The main result is the

**Theorem 1.1 ([3])** *Assume as before that the  $\omega_j$ 's are linearly independent over the rationals. Then the semi-classical spectrum and the semi-classical Birkhoff normal form determine each other.*

The main difficulty is that the spectrum of  $\hat{B}$  is naturally labelled by  $d$ -uples  $\mathbf{k} \in \mathbb{Z}_+^d$  while the semi-classical spectrum is labelled by  $N \in \mathbb{N}$ . We will denote by  $\psi$  the bijection  $\psi : N \rightarrow \mathbf{k}$  of  $\mathbb{N}$  onto  $\mathbb{Z}_+^d := \{\mathbf{k} = (k_1, \dots, k_d) | \forall j, k_j \in \mathbb{Z}, k_j \geq 0\}$  given by ordering the numbers  $\langle \omega | \mathbf{k} \rangle$  in increasing order: they are pair-wise distincts because of the non-resonant assumption.

## 2 From the semi-classical Birkhoff normal form to the semi-classical spectrum

We have the following result

**Theorem 2.1** *The semi-classical spectrum is given by the following power series in  $\hbar$ :*

$$\lambda_N(\hbar) \equiv E_0 + \hbar \left( E_1 + \frac{1}{2} \langle \omega | \psi(N) + \frac{1}{2} \rangle \right) + \sum_{j=2}^{\infty} \hbar^j P_j(\psi(N)) \quad (1)$$

where the  $P_j$ 's are polynomials of degree  $j$  given by

$$P_j(Z) = \sum_{l+|\alpha|=j} c_{l,\alpha} \left( Z + \frac{1}{2} \right)^\alpha .$$

This result is an immediate consequence of results proved by B. Simon [5] and B. Helffer-J. Sjöstrand [4] concerning the first terms, and by J. Sjöstrand in [6] (Theorem 0.1) where he proved a much stronger result.

### 3 From the semi-classical spectrum to the $\omega_j$ 's

#### 3.1 Determining the $\omega_j$ 's

Because  $E_0 = \lim_{\hbar \rightarrow 0} \lambda_1(\hbar)$ , we can subtract  $E_0$  and assume  $E_0 = 0$ .

By looking at the limits, as  $\hbar \rightarrow 0$ ,  $\mu_N := \lim \lambda_N(\hbar)/\hbar$  ( $N$  fixed), we know the set of all  $E_1 + \sum_{j=1}^d \omega_j(k_j + \frac{1}{2})$ ,  $(k_1, \dots, k_d) \in \mathbb{Z}_+^d$ .

**Let us give 2 proofs that the  $\mu_N$ 's determine the  $\omega_j$ 's.**

1. **Using the partition function:** from the  $\mu_N$ 's, we know the meromorphic function

$$Z(z) := \sum e^{-z\mu_N} .$$

$$Z(z) := e^{-z(E_1 + \frac{1}{2} \sum_{j=1}^d \omega_j)} \sum_{\mathbf{k} \in \mathbb{Z}_+^d} e^{-z\langle \omega | \mathbf{k} \rangle} ,$$

We have

$$Z(z) = e^{-z(E_1 + \frac{1}{2} \sum_{j=1}^d \omega_j)} \prod_{j=1}^d (1 - e^{-z\omega_j})^{-1} ,$$

The poles of  $Z$  are  $\mathcal{P} := \cup_{j=1, \dots, d} \{ \frac{2\pi i \mathbb{Z}}{\omega_j} \}$ . The set of  $\omega_j$  is hence determined up to a permutation. We fix now  $\omega = (\omega_1, \dots, \omega_d)$  with  $\omega_1 < \omega_2 < \dots$ .

From the knowledge of the  $\omega_j$ 's, we get the bijection  $\psi$ .

2. **A more elementary proof:** subtract  $\mu_1 = E_1 + \frac{1}{2} \sum \omega_j$  from the whole sequence and denote  $\nu_N = \mu_N - \mu_1$ . Then  $\omega_1 = \nu_2$ . Then remove the multiples of  $\omega_1$ . The first remaining term is  $\omega_2$ . Remove all integer linear combinations of  $\omega_1$  and  $\omega_2$ , the first remaining term is  $\omega_3, \dots$

### 3.2 Determining the $c_{l,\alpha}$ 's

Let us first fix  $N$ : from Equation (1) and the knowledge of  $\lambda_N \bmod O(\hbar^\infty)$  we know the  $P_j(\psi(N))$ 's for all  $j$ 's.

Doing that for all  $N$ 's and using  $\psi$  determine the restriction of the  $P_j$ 's to  $\mathbb{Z}_+^d$  and hence the  $P_j$ 's.

## 4 A natural question in the resonant case

### 4.1 The context

For simplicity, we will consider the completely resonant case  $\omega_1 = \dots = \omega_d = 1$  and work with the Weyl symbols. Let us denote by  $\Sigma = \frac{1}{2} \sum (x_j^2 + \xi_j^2)$ .

The (Weyl symbol of the) QBNF is then of the form

$$B \equiv \Sigma + \hbar P_{0,1} + \sum_{n=2}^{\infty} \sum_{j+l=n} \hbar^j P_{2l,j}$$

where  $P_{2l,j}$  is an homogeneous polynomial of degree  $2l$  in  $(x, \xi)$ , Poisson commuting with  $\Sigma$ :  $\{\Sigma, P_{2l,j}\} = 0^1$ .

For example, the first non trivial terms are:

- for  $n = 2$ :  $P_{4,0} + \hbar P_{2,1} + \hbar^2 P_{0,2}$
- for  $n = 3$ :  $P_{6,0} + \hbar P_{4,1} + \hbar^2 P_{2,2} + \hbar^3 P_{0,3}$ .

The semi-classical spectrum splits into clusters  $C_N$  of  $N + 1$  eigenvalues in an interval of size  $O(\hbar^2)$  around each  $\hbar(N + \frac{1}{2}d + P_{0,1})$  with  $N = 0, 1, \dots$ .

The whole series  $B$  is however NOT unique, contrary to the non-resonant case, but defined up to automorphism of the semi-classical Weyl algebra commuting with  $\Sigma$ .

Let  $G$  be the group of such automorphisms (see [2]). The natural question is roughly:

**Is the QBNF determined modulo  $G$  from the semi-classical spectrum, i.e. from all the clusters?**

### 4.2 The group $G$

The linear part of  $G$  is the group  $M$  of all  $A$ 's in the symplectic group which commute with  $\hat{H}_2$ , i.e. the unitary group  $U(d)$ .

We have an exact sequence of groups:

$$0 \rightarrow K \rightarrow G \rightarrow M \rightarrow 0 .$$

---

<sup>1</sup> The Moyal bracket of any  $A$  with  $H_2$  reduces to the Poisson bracket

Let us describe  $K$  (the “pseudo-differential” part):

Let  $S = S_3 + \dots$  in the Weyl algebra (the formal power series in  $(\hbar, x, \xi)$  with the Moyal product and the usual grading degree( $\hbar^j x^\alpha \xi^\beta$ ) =  $2j + |\alpha| + |\beta|$ )

$$g_S(H) = e^{iS/\hbar} \star H \star e^{-iS/\hbar}$$

preserves  $\Sigma$  iff  $\{S_n, \Sigma\} = 0$ . This implies that  $n$  is even and  $S_n$  is a polynomial in  $z_j \overline{z_k}$  ( $z_j = x_j + i\xi_j$ ). Then  $K$  is the group of all  $g_S$ 's with  $\{S, \Sigma\} = 0$ .

## References

- [1] Laurent Charles & San Vũ Ngọc. Spectral asymptotics via the Birkhoff normal form. *ArXiv:math-SP/0605096*, *Duke Math. Journal* **143**:463–511 (2008).
- [2] Yves Colin de Verdière. An extension of the Duistermaat-Singer Theorem to the semi-classical Weyl algebra. *Preprint 2008*.
- [3] Victor Guillemin, Thierry Paul & Alejandro Uribe. “Bottom of the well” semi-classical wave trace invariants. *ArXiv:math-SP/0608617* and *Math. Res. Lett.* **14**:711–719 (2007).
- [4] Bernard Helffer & Johannes Sjöstrand. Puis multiples en semi-classique I. *Commun. PDE.* **9**:337–408 (1984).
- [5] Barry Simon. Semi-classical analysis of low lying eigenvalues I: Non degenerate minima. *Ann. IHP (phys. théo.)* **38**: 295–307 (1983).
- [6] Johannes Sjöstrand. Semi-excited states in nondegenerate potential wells. *Asymptotic Analysis* **6**:29–43 (1992).