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Self-similar solutions of the p -Laplace heat equation: the case $p > 2$.

Marie Françoise Bidaut-Véron*

February 12, 2009

Abstract

We study the self-similar solutions of the equation

$$u_t - \operatorname{div}(|\nabla u|^{p-2} \nabla u) = 0,$$

in \mathbb{R}^N , when $p > 2$. We make a complete study of the existence and possible uniqueness of solutions of the form

$$u(x, t) = (\pm t)^{-\alpha/\beta} w((\pm t)^{-1/\beta} |x|)$$

of any sign, regular or singular at $x = 0$. Among them we find solutions with an expanding compact support or a shrinking hole (for $t > 0$), or a spreading compact support or a focussing hole (for $t < 0$). When $t < 0$, we show the existence of positive solutions oscillating around the particular solution $U(x, t) = C_{N,p}(|x|^p / (-t))^{1/(p-2)}$.

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1 Introduction and main results

Here we consider the self-similar solutions of the degenerate heat equation involving the p -Laplace operator

$$u_t - \operatorname{div}(|\nabla u|^{p-2} \nabla u) = 0. \quad (\mathbf{E}_u)$$

in \mathbb{R}^N , with $p > 2$. This study is the continuation of the work started in [4], relative to the case $p < 2$. It can be read independently. We set

$$\gamma = \frac{p}{p-2}, \quad \eta = \frac{N-p}{p-1}, \quad (1.1)$$

thus $\gamma > 1$, $\eta < N$,

$$\frac{N+\gamma}{p-1} = \eta + \gamma = \frac{N-\eta}{p-2}. \quad (1.2)$$

If u is a solution, then for any $\alpha, \beta \in \mathbb{R}$, $u_\lambda(x, t) = \lambda^\alpha u(\lambda x, \lambda^\beta t)$ is a solution of (\mathbf{E}_u) if and only if

$$\beta = \alpha(p-2) + p = (p-2)(\alpha + \gamma); \quad (1.3)$$

notice that $\beta > 0 \iff \alpha > -\gamma$. Given $\alpha \in \mathbb{R}$ such that $\alpha \neq -\gamma$, we search self-similar solutions, radially symmetric in x , of the form:

$$u = u(x, t) = (\varepsilon\beta t)^{-\alpha/\beta} w(r), \quad r = (\varepsilon\beta t)^{-1/\beta} |x|, \quad (1.4)$$

where $\varepsilon = \pm 1$. By translation, for any real T , we obtain solutions defined for any $t > T$ when $\varepsilon\beta > 0$, or $t < T$ when $\varepsilon\beta < 0$. We are lead to the equation

$$\left(|w'|^{p-2} w'\right)' + \frac{N-1}{r} |w'|^{p-2} w' + \varepsilon(rw' + \alpha w) = 0 \quad \text{in } (0, \infty). \quad (\mathbf{E}_w)$$

Our purpose is to give a complete description of all the solutions, with constant or changing sign. Equation (\mathbf{E}_w) is very interesting, because it is singular at any zero of w' , since $p > 2$, implying a nonuniqueness phenomena.

For example, concerning the constant sign solutions near the origin, it can happen that

$$\lim_{r \rightarrow 0} w = a \neq 0, \quad \lim_{r \rightarrow 0} w' = 0,$$

we will say that w is *regular*, or

$$\lim_{r \rightarrow 0} w = \lim_{r \rightarrow 0} w' = 0,$$

we say that w is *flat*. Or different kinds of singularities may occur, either at the level of w :

$$\lim_{r \rightarrow 0} w = \infty,$$

or at the level of the gradient:

$$\begin{aligned} \lim_{r \rightarrow 0} w = a \in \mathbb{R}, \quad \lim_{r \rightarrow 0} w' = \pm\infty, & \quad \text{when } p > N > 1, \\ \lim_{r \rightarrow 0} w = a \in \mathbb{R}, \quad \lim_{r \rightarrow 0} w' = b \neq 0 & \quad \text{when } p > N = 1. \end{aligned}$$

We first show that any local solution w of (\mathbf{E}_w) can be defined on $(0, \infty)$, thus any solution u of equation (\mathbf{E}_u) associated to w by (1.4) is defined on $\mathbb{R}^N \setminus \{0\} \times (0, \pm\infty)$. Then we prove the existence of regular solutions, flat ones, and of all singular solutions mentioned above.

Moreover, for $\varepsilon = 1$, there exist solutions w with a compact support $(0, \bar{r})$; then $u \equiv 0$ on the set

$$D = \left\{ (x, t) : x \in \mathbb{R}^N, \quad \beta t > 0, \quad |x| > (\beta t)^{1/\beta} \bar{r} \right\}.$$

For $\varepsilon = -1$, there exist solutions with a hole: $w(r) = 0 \iff r \in (0, \bar{r})$. Then $u \equiv 0$ on the set

$$H = \left\{ (x, t) : x \in \mathbb{R}^N, \quad \beta t < 0, \quad |x| < (-\beta t)^{1/\beta} \bar{r} \right\}.$$

The free boundary is of parabolic type for $\beta > 0$, of hyperbolic type for $\beta < 0$. This leads to four types of solutions, and we prove their existence:

- If $t > 0$, with $\varepsilon = 1, \beta > 0$, we say that u has an *expanding support*; the support increases from $\{0\}$ as t increases from 0.
- If $t > 0$, with $\varepsilon = -1, \beta < 0$, we say that u has a *shrinking hole*: the hole decreases from infinity as t increases from 0;
- If $t < 0$, with $\varepsilon = 1, \beta < 0$, we say that u has a *spreading support*: the support increases to be infinite as t increases to 0.
- If $t < 0$, with $\varepsilon = -1, \beta > 0$, we say that u has a *focussing hole*: the hole disappears as t increases to 0.

Up to our knowledge, some of them seem completely new, as for example the solutions with a shrinking hole or a spreading support. In particular we find again and improve some results of [8] concerning the existence of focussing type solutions.

Finally for $t < 0$ we also show the existence of positive solutions turning around the fundamental solution U given at (1.8) with a kind of periodicity, and also the existence of changing sign solutions doubly oscillating in $|x|$ near 0 and infinity.

As in [4] we reduce the problem to dynamical systems.

When $\varepsilon = -1$, a critical negative value of α is involved:

$$\alpha^* = -\gamma + \frac{\gamma(N + \gamma)}{(p-1)(N + 2\gamma)}. \quad (1.5)$$

1.1 Explicit solutions

Obviously if w is a solution of (\mathbf{E}_w) , $-w$ is also a solution. Some particular solutions are well-known.

The solution U . For any α such that $\varepsilon(\alpha + \gamma) < 0$, that means $\varepsilon\beta < 0$, there exist flat solutions of (\mathbf{E}_w) , given by

$$w(r) = \pm \ell r^\gamma, \quad (1.6)$$

where

$$\ell = \left(\frac{|\alpha + \gamma|}{\gamma^{p-1}(\gamma + N)} \right)^{1/(p-2)} > 0. \quad (1.7)$$

They correspond to a unique solution of (\mathbf{E}_u) called U , defined for $t < 0$, such that $U(0, t) = 0$, flat, blowing up at $t = 0$ for fixed $x \neq 0$:

$$U(x, t) = C \left(\frac{|x|^p}{-t} \right)^{1/(p-2)}, \quad C = ((p-2)\gamma^{p-1}(\gamma + N))^{1/(2-p)}. \quad (1.8)$$

The case $\alpha = N$. Then $\beta = \beta_N = N(p-2) + p > 0$, and the equation has a first integral

$$w + \varepsilon r^{-1} |w'|^{p-2} w' = Cr^{-N}. \quad (1.9)$$

All the solutions corresponding to $C = 0$ are given by

$$\begin{aligned} w = w_{K,\varepsilon}(r) &= \pm \left(K - \varepsilon \gamma^{-1} r^{p'} \right)_+^{(p-1)/(p-2)}, \quad K \in \mathbb{R}, \\ u = \pm u_{K,\varepsilon}(x, t) &= \pm (\varepsilon \beta_N t)^{-N/\beta_N} \left(K - \varepsilon \gamma^{-1} (\varepsilon \beta_N t)^{-p'/\beta_N} |x|^{p'} \right)_+^{(p-1)/(p-2)}. \end{aligned} \quad (1.10)$$

For $\varepsilon = 1$, $K > 0$, they are defined for $t > 0$, called *Barenblatt solutions*, regular with a compact support. Given $c > 0$, the function $u_{K,1}$, defined on $\mathbb{R}^N \times (0, \infty)$, is the unique solution of equation (\mathbf{E}_u) with initial data $u(0) = c\delta_0$, where δ_0 is the Dirac mass at 0, and K being linked by $\int_{\mathbb{R}^N} u_{K,1}(x, t) dt = c$. The $u_{K,1}$ are the only nonnegative solutions defined on $\mathbb{R}^N \times (0, \infty)$, such that $u(x, 0) = 0$ for any $x \neq 0$. For $\varepsilon = -1$, the $u_{K,-1}$ are defined for $t < 0$; for $K > 0$, w does not

vanish on $(0, \infty)$; for $K < 0$, w is flat with a hole near 0. For $K = 0$, we find again the function w given at (1.6).

The case $\alpha = \eta \neq 0$. We exhibit a family of solutions of (\mathbf{E}_w) :

$$w(r) = Cr^{-\eta}, \quad u(t, x) = C|x|^{-\eta}, \quad C \neq 0. \quad (1.11)$$

The solutions u , independent of t , are p -harmonic in \mathbb{R}^N ; they are fundamental solutions when $p < N$. When $p > N$, w satisfies $\lim_{r \rightarrow 0} w = 0$, and $\lim_{r \rightarrow 0} w' = \infty$ for $N > 1$, $\lim_{r \rightarrow 0} w' = b$ for $N = 1$.

The case $\alpha = -p'$. Equation (\mathbf{E}_w) admits regular solutions of the form

$$w(r) = \pm K \left(N(Kp')^{p-2} + \varepsilon r^{p'} \right), \quad u(x, t) = \pm K \left(N(Kp')^{p-2}t + |x|^{p'} \right), \quad K > 0. \quad (1.12)$$

Here $\beta > 0$; in the two cases $\varepsilon = 1, t > 0$ and $\varepsilon = -1, t < 0$, u is defined for any $t \in \mathbb{R}$ and of the form $\psi(t) + \Phi(|x|)$ with Φ nonconstant, and $u(\cdot, t)$ has a constant sign for $t > 0$ and changing sign for $t < 0$.

The case $\alpha = 0$. Equation (\mathbf{E}_w) can be explicitly solved: either $w' \equiv 0$, thus $w \equiv a \in \mathbb{R}$, u is a constant solution of (\mathbf{E}_u) , or there exists $K \in \mathbb{R}$ such that

$$|w'| = r^{-(\eta+1)} \left(K - \frac{\varepsilon}{\gamma + N} r^{N-\eta} \right)_+^{1/(p-2)}; \quad (1.13)$$

and w follows by integration, up to a constant, and then $u(x, t) = w(|x|/(\varepsilon pt)^{1/p})$. If $\varepsilon = 1$, then $t > 0$, $K > 0$ and w' has a compact support; up to a constant, u has a compact support. If $\varepsilon = -1$, then $t < 0$; for $K > 0$, w is strictly monotone; for $K < 0$, w is flat, constant near 0; for $K = 0$, we find again (1.6). For $\varepsilon = \pm 1, K > 0$, observe that $\lim_{r \rightarrow 0} w = \pm\infty$ if $p \leq N$; and $\lim_{r \rightarrow 0} w = a \in \mathbb{R}$, $\lim_{r \rightarrow 0} w' = \pm\infty$ if $p > N > 1$; and $\lim_{r \rightarrow 0} w = a \in \mathbb{R}$, $\lim_{r \rightarrow 0} w' = K$ if $p > N = 1$. In particular we find solutions such that $w = cr^{|\eta|}(1 + o(1))$ near 0, with $c > 0$.

(v) Case $N = 1$ and $\alpha = -(p-1)/(p-2) < 0$. Here $\beta = 1$, and we find the solutions

$$w(r) = \pm \left(Kr + \varepsilon |\alpha|^{p-1} |K|^p \right)_+^{(p-1)/(p-2)}, \quad u(x, t) = \pm \left(K|x| + |\alpha|^{p-1} |K|^p t \right)_+^{(p-1)/(p-2)}, \quad (1.14)$$

If $\varepsilon = 1, t > 0$, then w has a singularity at the level of the gradient, and either $K > 0, w > 0$, or $K < 0$ and w has a compact support. If $\varepsilon = -1, t < 0$ then $K > 0, w$ has a hole.

1.2 Main results

In the next sections we provide an exhaustive study of equation (\mathbf{E}_w) . Here we give the main results relative to the function u . Let us show how to return from w to u . Suppose that the behaviour of w is given by

$$\lim_{r \rightarrow 0} r^\lambda w(r) = c \neq 0, \quad \lim_{r \rightarrow \infty} r^\mu w(r) = c' \neq 0, \quad \text{where } \lambda, \mu \in \mathbb{R}.$$

(i) Then for fixed $t \neq 0$, the function u has a behaviour in $|x|^{-\lambda}$ near $x = 0$, and a behaviour in $|x|^{-\mu}$ for large $|x|$.

If $\lambda = 0$, then u is defined on $\mathbb{R}^N \times (0, \pm\infty)$. Either w is regular, then $u(\cdot, t) \in C^1(\mathbb{R}^N \times (0, \infty))$; we will say that u is **regular**; nevertheless the regular solutions u presents a singularity at time $t = 0$ if and only if $\alpha < -\gamma$ or $\alpha > 0$. Or a singularity can appear for u at the level of the gradient.

If $\lambda < 0$, thus u is defined on $\mathbb{R}^N \times (0, \pm\infty)$ and $u(0, t) = 0$; either w is flat, we also say that u is **flat**, or a singularity appears at the level of the gradient.

If $0 < \lambda < N$, then $u(\cdot, t) \in L^1_{loc}(\mathbb{R}^N)$ for $t \neq 0$, we say that $x = 0$ is a **weak singularity**. We will show that there exist no stronger singularity.

If $\lambda < N < \mu$; then $u(\cdot, t) \in L^1(\mathbb{R}^N)$.

(ii) For fixed $x \neq 0$, the behaviour of u near $t = 0$, depends on the sign of β :

$$\begin{aligned} \lim_{t \rightarrow 0} |x|^\mu |t|^{(\alpha-\mu)/\beta} u(x, t) &= C \neq 0 \quad \text{if } \alpha > -\gamma, \\ \lim_{t \rightarrow 0} |x|^\lambda |t|^{(\alpha-\lambda)/\beta} u(x, t) &= C \neq 0 \quad \text{if } \alpha < -\gamma. \end{aligned}$$

If $\mu < 0, \alpha > -\gamma$ or $\lambda < 0, \alpha < -\gamma$, then $\lim_{t \rightarrow 0} u(x, t) = 0$.

1.2.1 Solutions defined for $t > 0$

Here we look for solutions u of (\mathbf{E}_u) of the form (1.4) defined on $\mathbb{R}^N \setminus \{0\} \times (0, \infty)$. That means $\varepsilon\beta > 0$ or equivalently $\varepsilon = 1, -\gamma < \alpha$ (see Section 6) or $\varepsilon = -1, \alpha < -\gamma$ see (Section 7). We begin by the case $\varepsilon = 1$, treated at Theorem 6.1.

Theorem 1.1 *Assume $\varepsilon = 1$, and $-\gamma < \alpha$.*

(1) *Let $\alpha < N$.*

All regular solutions on $\mathbb{R}^N \setminus \{0\} \times (0, \infty)$ have a strict constant sign, in $|x|^{-\alpha}$ near ∞ for fixed t , with initial data $L|x|^{-\alpha}$ ($L \neq 0$) in \mathbb{R}^N ; thus $u(\cdot, t) \notin L^1(\mathbb{R}^N)$, and u is unbounded when $\alpha < 0$.

There exist nonnegative solutions such that near $x = 0$,

$$\left. \begin{aligned} \text{for } p < N, \quad & u \text{ has a weak singularity in } |x|^{-\eta}, \\ \text{for } p = N, \quad & u \text{ has a weak singularity in } \ln|x|, \\ \text{for } p > N, \quad & u \in C^0(\mathbb{R}^N \times (0, \infty)), \quad u(0, t) = a > 0, \text{ with a singular gradient,} \end{aligned} \right\} \quad (1.15)$$

*and u has an **expanding compact support** for any $t > 0$, with initial data $L|x|^{-\alpha}$ in $\mathbb{R}^N \setminus \{0\}$.*

There exist positive solutions with the same behaviour as $x \rightarrow 0$, in $|x|^{-\alpha}$ near ∞ for fixed t ; and also solutions such that u has one zero for fixed $t \neq 0$, and the same behaviour.

If $p > N$, there exist positive solutions satisfying (1.15), and also positive solutions such that

$$u \in C^0(\mathbb{R}^N \times (0, \infty)), \quad u(0, t) = 0, \quad \text{in } |x|^{|\eta|} \text{ near } 0, \quad \text{with a singular gradient,} \quad (1.16)$$

in $|x|^{-\alpha}$ near ∞ for fixed t , with and initial data $L|x|^{-\alpha}$ in $\mathbb{R}^N \setminus \{0\}$.

(2) Let $\alpha = N$.

All **regular (Barenblatt)** solutions are nonnegative, have a **compact support** for any $t > 0$. If $p \leq N$, all the other solutions have one zero for fixed t , satisfy (1.15) or (1.16) and have the same behaviour at ∞ .

(3) Let $N < \alpha$.

All regular solutions u have a finite number $m \geq 1$ of simple zeros for fixed t , and $u(\cdot, t) \in L^1(\mathbb{R}^N)$. Either they are in $|x|^{-\alpha}$ near ∞ for fixed t , then there exist solutions with m zeros, compact support, satisfying (1.15); or they have a compact support. All the solutions have m or $m + 1$ zeros. There exist solutions satisfying (1.15) with $m + 1$ zeros, and in $|x|^{-\alpha}$ near ∞ . If $p > N$, there exist solutions satisfying (1.15) with m zeros; there exist also solutions with m zeros, $u(0, t) = 0$, and a singular gradient, in $|x|^{-\alpha}$ near ∞ .

Next we come to the case $\varepsilon = -1$, which is the subject of Theorem 7.1.

Theorem 1.2 Assume $\varepsilon = -1$ and $\alpha < -\gamma$.

All the solutions u on $\mathbb{R}^N \setminus \{0\} \times (0, \infty)$, in particular the regular ones, are **oscillating around** 0 for fixed $t > 0$ and large $|x|$, and $r^{-\gamma}u$ is asymptotically periodic in $\ln r$. Moreover there exist

solutions such that $r^{-\gamma}u$ is **periodic** in $\ln r$, in particular $C_1 t^{-|\alpha/\beta|} \leq |u| \leq C_2 t^{-|\alpha/\beta|}$ for some $C_1, C_2 > 0$;

solutions $u \in C^1(\mathbb{R}^N \times [0, \infty))$, $u(x, 0) \equiv 0$, with a **shrinking hole**;

flat solutions $u \in C^1(\mathbb{R}^N \times [0, \infty))$, in $|x|^{|\alpha|}$ near 0, with initial data $L|x|^{|\alpha|}$ ($L \neq 0$);

solutions satisfying (1.15) near $x = 0$, and if $p > N$, solutions satisfying (1.16) near 0.

1.2.2 Solutions defined for $t < 0$

We look for solutions u of (\mathbf{E}_u) of the form (1.4) defined on $\mathbb{R}^N \setminus \{0\} \times (-\infty, 0)$. That means $\varepsilon\beta < 0$ or equivalently $\varepsilon = 1$, $\alpha < -\gamma$ (see Section 8, Theorem 8.1) or $\varepsilon = -1$, $\alpha > -\gamma$ (see Section 9). In the case $\varepsilon = 1$, we get the following:

Theorem 1.3 Assume $\varepsilon = 1$, and $\alpha < -\gamma$.

The function $U(x, t) = C \left(\frac{|x|^p}{-t} \right)^{1/(p-2)}$ is a positive **flat** solution on $\mathbb{R}^N \setminus \{0\} \times (-\infty, 0)$.

All regular solutions have a constant sign, are unbounded in $|x|^\gamma$ near ∞ for fixed t , and blow up at $t = 0$ like $(-t)^{-|\alpha|/|\beta|}$ for fixed $x \neq 0$.

There exist **flat positive** solutions $u \in C^1(\mathbb{R}^N \times (-\infty, 0])$, in $|x|^\gamma$ near ∞ for fixed t , with **final data** $L|x|^{|\alpha|}$ ($L > 0$).

There exist **nonnegative** solutions satisfying (1.15) near 0, with a **spreading compact support**, blowing up near $t = 0$ (like $|t|^{-(\eta+|\alpha|)/|\beta|}$ for $p < N$, or $|t|^{-|\alpha|/|\beta|} \ln |t|$ for $p = N$, or $(-t)^{-|\alpha|/|\beta|}$ for $p > N$).

There exist positive solutions with the same behaviour near 0, in $|x|^\gamma$ near ∞ , blowing up as above at $t = 0$, and solutions with one zero for fixed t , and the same behaviour. If $p > N$, there exist positive solutions satisfying (1.15) (resp. (1.16)) near 0, in $|x|^\gamma$ near ∞ for fixed t , blowing up at $t = 0$ like $|t|^{-|\alpha|/|\beta|}$ (resp. $|t|^{(|\eta|-|\alpha|)/|\beta|}$) for fixed x .

Up to a symmetry, all the solutions are described.

The most interesting case is $\varepsilon = -1, -\gamma < \alpha$. For simplicity we will assume that $p < N$. The case $p \geq N$ is much more delicate, and the complete results can be read in terms of w at Theorems 9.4, 9.6, 9.9, 9.10, 9.11 and 9.12. We discuss according to the position of α with respect to $-p'$ and α^* defined at (1.5). Notice that $\alpha^* < -p'$.

Theorem 1.4 Assume $\varepsilon = -1$, and $-p' \leq \alpha \neq 0$. The function U is still a flat solution on $\mathbb{R}^N \setminus \{0\} \times (-\infty, 0)$.

(1) Let $0 < \alpha$.

All regular solutions have a strict constant sign, in $|x|^\gamma$ near ∞ for fixed t , blowing up at $t = 0$ like $(-t)^{-1/(p-2)}$ for fixed $x \neq 0$.

There exist nonnegative solutions with a **focussing hole**: $u(x, t) \equiv 0$ for $|x| \leq C|t|^{1/\beta}$, $t > 0$, in $|x|^\gamma$ near ∞ for fixed t , blowing up at $t = 0$ like $(-t)^{-1/(p-2)}$ for fixed $x \neq 0$.

There exist positive solutions u with a (**weak**) **singularity** in $|x|^{-\eta}$ at $x = 0$, in $|x|^{-\alpha}$ near ∞ for fixed t , with $u(\cdot, t) \in L^1(\mathbb{R}^N)$ if $\alpha > N$, with final data $L|x|^{-\alpha}$ ($L > 0$) in $\mathbb{R}^N \setminus \{0\}$.

There exist positive solutions u in $|x|^{-\eta}$ at $x = 0$, in $|x|^\gamma$ near ∞ for fixed t , blowing up at $t = 0$ like $(-t)^{-1/(p-2)}$ for fixed $x \neq 0$; solutions with one zero and the same behaviour.

(2) Let $-p' < \alpha < 0$.

All regular solutions have **one zero** for fixed t , and the same behaviour. There exist solutions with one zero, in $|x|^{-\eta}$ at $x = 0$, in $|x|^{|\alpha|}$ near ∞ for fixed t , with final data $L|x|^{-\alpha}$ ($L > 0$) in $\mathbb{R}^N \setminus \{0\}$. There exist solutions with one zero, u in $|x|^{-\eta}$ at $x = 0$, in $|x|^\gamma$ near ∞ for fixed t , blowing up at $t = 0$ like $(-t)^{-1/(p-2)}$ for fixed $x \neq 0$; solutions with two zeros and the same behaviour.

3) Let $\alpha = -p'$.

All regular solutions have **one zero** and are in $|x|^{|\alpha|}$ near ∞ for fixed t , and with **final data** $L|x|^{|\alpha|}$ ($L > 0$). The other solutions have one or two zeros, are in $|x|^{-\eta}$ at $x = 0$, in $|x|^\gamma$ near ∞ for fixed t .

In any case, up to a symmetry, all the solutions are described.

Theorem 1.5 Assume $\varepsilon = -1, -\gamma < \alpha < -p'$. Then U is still a flat solution on $\mathbb{R}^N \setminus \{0\} \times (-\infty, 0)$.

(1) Let $\alpha \leq \alpha^*$.

Then there exist **positive flat solutions**, in $|x|^\gamma$ near 0, in $|x|^{|\alpha|}$ near ∞ for fixed t , with **final data** $L|x|^{-\alpha}$ ($L > 0$) in \mathbb{R}^N .

All the other solutions, among them the **regular ones**, have an **infinity of zeros**: $u(t, \cdot)$ is oscillating around 0 for large $|x|$. There exist solutions with a focussing hole, and solutions with a singularity in $|x|^{-\eta}$ at $x = 0$. There exist solutions **oscillating also for small** $|x|$, such that $r^{-\gamma}w$ is periodic in $\ln r$.

(2) There exist a **critical unique value** $\alpha_c \in (\max(\alpha^*, -p')$ such that for $\alpha = \alpha_c$, there exists nonnegative solutions with a **focussing hole** near 0, in $|x|^{|\alpha|}$ near ∞ for fixed t , with **final data** $L|x|^{-\alpha}$ ($L > 0$) in \mathbb{R}^N . And $\alpha_c > -(p-1)/(p-2)$.

There exist positive flat solutions, such that $|x|^{-\gamma}u$ is bounded on \mathbb{R}^N for fixed t , blowing up at $t = 0$ like $(-t)^{-1/(p-2)}$ for fixed $x \neq 0$. The regular solutions are oscillating around 0 as above. There exist solutions **oscillating around 0**, such that $r^{-\gamma}w$ is **periodic** in $\ln r$. There are solutions with a weak singularity in $|x|^{-\eta}$ at $x = 0$, and oscillating around 0 for large $|x|$.

(3) Let $\alpha^* < \alpha < \alpha_c$.

The regular solutions are as above. There exist solutions of the same types as above. Moreover there exist **positive** solutions, such that $r^{-\gamma}w$ is **periodic** in $\ln r$, thus there exist $C_1, C_2 > 0$ such that

$$C_1 \left(\frac{|x|^p}{|t|} \right)^{1/(p-2)} \leq u \leq C_2 \left(\frac{|x|^p}{|t|} \right)^{1/(p-2)}$$

There exist **positive** solutions, such that $r^{-\gamma}w$ is asymptotically periodic in $\ln r$ near 0 and in $|x|^\gamma$ near ∞ for fixed t ; and also, solutions with a hole, and oscillating around 0 for large $|x|$. There exist solutions positive near 0, oscillating near ∞ , and $r^{-\gamma}w$ is **doubly asymptotically periodic** in $\ln r$.

4) Let $\alpha_c < \alpha < -p'$.

There exist nonnegative solutions with a focussing hole near 0, in $|x|^\gamma$ near ∞ for fixed t , blowing up at $t = 0$ like $(-t)^{-1/(p-2)}$ for fixed $x \neq 0$. Either the regular solutions have an **infinity** of zeros for fixed t , then the same is true for all the other solutions. Or they have a **finite** number $m \geq 2$ of zeros, and can be in $|x|^\gamma$ or $|x|^{|\alpha|}$ near ∞ (in that case they have a final data $L|x|^{|\alpha|}$); all the other solutions have m or $m+1$ zeros.

In the case $\alpha = \alpha_c$, we find again the existence and uniqueness of the focussing solutions introduced in [8].

2 Different formulations of the problem

In all the sequel we assume

$$\alpha \neq 0,$$

recalling that the solutions w are given explicitly by (1.13) when $\alpha = 0$. Defining

$$J_N(r) = r^N \left(w + \varepsilon r^{-1} |w'|^{p-2} w' \right), \quad J_\alpha(r) = r^{\alpha-N} J_N(r), \quad (2.1)$$

equation (\mathbf{E}_w) can be written in an equivalent way under the forms

$$J'_N(r) = r^{N-1} (N - \alpha) w, \quad J'_\alpha(r) = -\varepsilon (N - \alpha) r^{\alpha-2} |w'|^{p-2} w'. \quad (2.2)$$

If $\alpha = N$, then J_N is constant, so we find again (1.9).

We mainly use logarithmic substitutions; given $d \in \mathbb{R}$, setting

$$w(r) = r^{-d} y_d(\tau), \quad Y_d = -r^{(d+1)(p-1)} |w'|^{p-2} w', \quad \tau = \ln r, \quad (2.3)$$

we obtain the equivalent system:

$$\left. \begin{aligned} y'_d &= dy_d - |Y_d|^{(2-p)/(p-1)} Y_d, \\ Y'_d &= (p-1)(d-\eta)Y_d + \varepsilon e^{(p+(p-2)d)\tau} (\alpha y_d - |Y_d|^{(2-p)/(p-1)} Y_d). \end{aligned} \right\} \quad (2.4)$$

At any point τ where $w'(\tau) \neq 0$, the functions y_d, Y_d satisfy the equations

$$y''_d + (\eta - 2d)y'_d - d(\eta - d)y_d + \frac{\varepsilon}{p-1} e^{((p-2)d+p)\tau} |dy_d - y'_d|^{2-p} (y'_d + (\alpha - d)y_d) = 0, \quad (2.5)$$

$$\begin{aligned} Y''_d + (p-1)(\eta - 2d - p')Y'_d + \varepsilon e^{((p-2)d+p)\tau} |Y_d|^{(2-p)/(p-1)} (Y'_d/(p-1) + (\alpha - d)Y_d) \\ - (p-1)^2(\eta - d)(p' + d)Y_d = 0, \end{aligned} \quad (2.6)$$

The main case is $d = -\gamma$: setting $y = y_{-\gamma}$,

$$w(r) = r^\gamma y(\tau), \quad Y = -r^{(-\gamma+1)(p-1)} |w'|^{p-2} w', \quad \tau = \ln r, \quad (2.7)$$

we are lead to the *autonomous* system

$$\left. \begin{aligned} y' &= -\gamma y - |Y|^{(2-p)/(p-1)} Y, \\ Y' &= -(\gamma + N)Y + \varepsilon(\alpha y - |Y|^{(2-p)/(p-1)} Y). \end{aligned} \right\} \quad (\mathbf{S})$$

Its study is fundamental: its phase portrait allows to study all the *signed* solutions of equation (\mathbf{E}_w) . Equation (2.5) takes the form

$$(p-1)y'' + (N+\gamma p)y' + \gamma(\gamma+N)y + \varepsilon |\gamma y + y'|^{2-p} (y' + (\alpha + \gamma)y) = 0, \quad (\mathbf{E}_y)$$

Notice that $J_N(r) = r^{N+\gamma}(y(\tau) - \varepsilon Y(\tau))$.

Remark 2.1 *Since (\mathbf{S}) is autonomous, for any solution w of (\mathbf{E}_w) of the problem, all the functions $w_\xi(r) = \xi^{-\gamma}w(\xi r), \xi > 0$, are also solutions.*

Notation 2.2 *In the sequel we set $\varepsilon\infty := +\infty$ if $\varepsilon = 1$, $\varepsilon\infty := -\infty$ if $\varepsilon = -1$.*

2.1 The phase plane of system (\mathbf{S})

In the phase plane (y, Y) we denote the four quadrants by

$$\mathcal{Q}_1 = (0, \infty) \times (0, \infty), \quad \mathcal{Q}_2 = (-\infty, 0) \times (0, \infty), \quad \mathcal{Q}_3 = -\mathcal{Q}_1, \quad \mathcal{Q}_4 = -\mathcal{Q}_2.$$

Remark 2.3 *The vector field at any point $(0, \xi), \xi > 0$ satisfies $y' = -\xi^{1/(p-1)} < 0$, thus points to \mathcal{Q}_2 ; moreover $Y' < 0$ if $\varepsilon = 1$. The field at any point $(\varphi, 0), \varphi > 0$ satisfies $Y' = \varepsilon\alpha\varphi$, thus points to \mathcal{Q}_1 if $\varepsilon\alpha > 0$ and to \mathcal{Q}_4 if $\varepsilon\alpha < 0$; moreover $y' = -\gamma\varphi < 0$.*

If $\varepsilon(\gamma + \alpha) \geq 0$, system (\mathbf{S}) has a unique stationary point $(0, 0)$. If $\varepsilon(\gamma + \alpha) < 0$, it admits three stationary points:

$$(0, 0), \quad M_\ell = (\ell, -(\gamma\ell)^{p-1}) \in \mathcal{Q}_4, \quad M'_\ell = -M_\ell \in \mathcal{Q}_2, \quad (2.8)$$

where ℓ is defined at (1.7). The point $(0, 0)$ is singular because $p > 2$; its study concern in particular the solutions w with a *double zero*. When $\varepsilon(\gamma + \alpha) < 0$, the point M_ℓ is associated to the solution $w \equiv \ell r^\gamma$ of equation (\mathbf{E}_w) given at (1.1).

Linearization around M_ℓ . Near the point M_ℓ , setting

$$y = \ell + \bar{y}, \quad Y = -(\gamma\ell)^{p-1} + \bar{Y}, \quad (2.9)$$

system (\mathbf{S}) is equivalent in \mathcal{Q}_4 to

$$\bar{y}' = -\gamma\bar{y} - \varepsilon\nu(\alpha)\bar{Y} + \Psi(\bar{Y}), \quad \bar{Y}' = \varepsilon\alpha\bar{y} - (\gamma + N + \nu(\alpha))\bar{Y} + \varepsilon\Psi(\bar{Y}), \quad (2.10)$$

where

$$\nu(\alpha) = -\frac{\gamma(N + \gamma)}{(p-1)(\gamma + \alpha)}, \quad \text{and } \Psi(\vartheta) = ((\gamma\ell)^{p-1} - \vartheta)^{1/(p-1)} - \gamma\ell + \frac{(\gamma\ell)^{2-p}}{p-1}\vartheta, \quad \vartheta < (\gamma\ell)^{p-1}, \quad (2.11)$$

thus $\varepsilon\nu(\alpha) > 0$. The linearized problem is given by

$$\bar{y}' = -\gamma\bar{y} - \varepsilon\nu(\alpha)\bar{Y}, \quad \bar{Y}' = \varepsilon\alpha\bar{y} - (\gamma + N + \nu(\alpha))\bar{Y}.$$

Its eigenvalues $\lambda_1 \leq \lambda_2$ are the solutions of equation

$$\lambda^2 + (2\gamma + N + \nu(\alpha))\lambda + p'(N + \gamma) = 0 \quad (2.12)$$

The discriminant Δ of the equation (2.12) is given by

$$\Delta = (2\gamma + N + \nu(\alpha))^2 - 4p'(N + \gamma) = (N + \nu(\alpha))^2 - 4\nu(\alpha)\alpha. \quad (2.13)$$

For $\varepsilon = 1$, M_ℓ is a *sink*, and a node point, since $\nu(\alpha) > 0$, and $\alpha < 0$, thus $\Delta > 0$. For $\varepsilon = -1$, we have $\nu(\alpha) < 0$; the nature of M_ℓ depends on the critical value α^* defined at (1.5); indeed

$$\alpha = \alpha^* \iff \lambda_1 + \lambda_2 = 0.$$

Then M_ℓ is a *sink* when $\alpha > \alpha^*$ and a *source* when $\alpha < \alpha^*$. Moreover α^* corresponds to a spiral point, and M_ℓ is a node point when $\Delta \geq 0$, that means $\alpha \leq \alpha_1$, or $\gamma > N/2 + \sqrt{p'(N + \gamma)}$ and $\alpha_2 \leq \alpha$, where

$$\alpha_1 = -\gamma + \frac{\gamma(N + \gamma)}{(p - 1)(2\gamma + N + 2(p'(N + \gamma))^{1/2})}, \quad \alpha_2 = -\gamma + \frac{\gamma(N + \gamma)}{(p - 1)(2\gamma + N - 2(p'(N + \gamma))^{1/2})}. \quad (2.14)$$

When $\Delta > 0$, and $\lambda_1 < \lambda_2$, one can choose a basis of eigenvectors

$$e_1 = (-\varepsilon\nu(\alpha), \lambda_1 + \gamma) \quad \text{and} \quad e_2 = (\varepsilon\nu(\alpha), -\gamma - \lambda_2). \quad (2.15)$$

Remark 2.4 *One verifies that $\alpha^* < -1$; and $\alpha^* < -(p - 1)/(p - 2)$ if and only if $p > N$. Also $\alpha_2 \leq 0$, and $\alpha_2 = 0 \iff N = p/(p - 2)^2$; and $\alpha_2 > -p' \iff \gamma^2 - 7\gamma - 8N < 0$, which is not always true.*

As in [4, Theorem 2.16] we prove that the Hopf bifurcation point is not degenerate, which implies the existence of small cycles near α^* .

Proposition 2.5 *Let $\varepsilon = -1$, and $\alpha = \alpha^* > -\gamma$. Then M_ℓ is a weak source. If $\alpha > \alpha^*$ and $\alpha - \alpha^*$ is small enough, there exists a unique limit cycle in \mathcal{Q}_4 , attracting at $-\infty$.*

2.2 Other systems for positive solutions

When w has a constant sign, we define two functions associated to (y, Y) :

$$\zeta(\tau) = \frac{|Y|^{(2-p)/(p-1)} Y}{y}(\tau) = -\frac{rw'(r)}{w(r)}, \quad \sigma(\tau) = \frac{Y}{y}(\tau) = -\frac{|w'(r)|^{p-2} w'(r)}{rw(r)}. \quad (2.16)$$

Thus ζ describes the behaviour of w'/w and σ is the slope in the phase plane (y, Y) . They satisfy the system

$$\left. \begin{aligned} \zeta' &= \zeta(\zeta - \eta) + \varepsilon |\zeta y|^{2-p} (\alpha - \zeta)/(p-1) = \zeta(\zeta - \eta + \varepsilon(\alpha - \zeta)/(p-1)\sigma), \\ \sigma' &= \varepsilon(\alpha - N) + \left(|\sigma y|^{(2-p)/(p-1)} \sigma - N \right) (\sigma - \varepsilon) = \varepsilon(\alpha - \zeta) + (\zeta - N)\sigma. \end{aligned} \right\} \quad (\mathbf{Q})$$

In particular, System **(Q)** provides a short proof of the local existence and uniqueness of the *regular* solutions: they correspond to its stationary point $(0, \varepsilon\alpha/N)$, see Section 3.1.

Moreover, if w and w' have a strict constant sign, that means in any quadrant \mathcal{Q}_i , we can define

$$\psi = \frac{1}{\sigma} = \frac{y}{Y} \quad (2.17)$$

We obtain a new system relative to (ζ, ψ) :

$$\left. \begin{aligned} \zeta' &= \zeta(\zeta - \eta + \varepsilon(\alpha - \zeta)\psi/(p-1)), \\ \psi' &= \psi(N - \zeta + \varepsilon(\zeta - \alpha)\psi). \end{aligned} \right\} \quad (\mathbf{P})$$

We are reduced to a polynomial system, thus with no singularity. System **(P)** gives the existence of singular solutions when $p > N$, corresponding to its stationary point $(\eta, 0)$, see Section 5.

We will also consider another system in any \mathcal{Q}_i : setting

$$\zeta = -1/g, \quad \sigma = -s, \quad d\tau = gsd\nu = |Y|^{(p-2)/(p-1)} d\nu, \quad (2.18)$$

we find

$$\left. \begin{aligned} dg/d\nu &= g(s(1 + \eta g) + \varepsilon(1 + \alpha g)/(p-1)), \\ ds/d\nu &= -s(\varepsilon(1 + \alpha g) + (1 + Ng)s). \end{aligned} \right\} \quad (\mathbf{R})$$

System **(R)** allows to get the existence of solutions w with a hole or a compact support, and other solutions, corresponding to its stationary points $(0, -\varepsilon)$ and $(-1/\alpha, 0)$; it provides a complete study of the singular point $(0, 0)$ of system **(S)**, see Sections 3.3, 5; and of the focussing solutions, see Section 9.

Remark 2.6 *The particular solutions can be found again in the different phase planes, where their trajectories are lines:*

For $\alpha = N$, the solutions (1.10) correspond to $Y \equiv \varepsilon y$, that means $\sigma \equiv \varepsilon$.

For $\alpha = \eta \neq 0$ the solutions (1.11) correspond to $\zeta \equiv \eta$.

For $\alpha = -p'$, the solutions (1.12) are given by $\zeta + \varepsilon N \sigma \equiv \alpha$.

For $N = 1$, $\alpha = -(p-2)/(p-1)$, the solutions (1.14) satisfy $\alpha g + \varepsilon s \equiv -1$.

3 Global existence

3.1 Local existence and uniqueness

Proposition 3.1 *Let $r_1 > 0$ and $a, b \in \mathbb{R}$. If $(a, b) \neq (0, 0)$, there exists a unique solution w of equation (\mathbf{E}_w) in a neighborhood \mathcal{V} of r_1 , such that w and $|w'|^{p-2} w' \in C^1(\mathcal{V})$ and $w(r_1) = a$, $w'(r_1) = b$. It extends on a maximal interval I where $(w(r), w'(r)) \neq (0, 0)$.*

Proof. If $b \neq 0$, the Cauchy theorem directly applies to system (\mathbf{S}). If $b = 0$ the system is a priori singular on the line $\{Y = 0\}$ since $p > 2$. In fact it is only singular at $(0, 0)$. Indeed near any point $(\xi, 0)$ with $\xi \neq 0$, one can take Y as a variable, and

$$\frac{dy}{dY} = F(Y, y), \quad F(Y, y) := \frac{\gamma y + |Y|^{(2-p)/(p-1)} Y}{(\gamma + N)Y + \varepsilon(|Y|^{(2-p)/(p-1)} Y - \alpha y)},$$

where F is continuous in Y and C^1 in y , hence local existence and uniqueness hold. ■

Notation 3.2 *For any point $P_0 = (y_0, Y_0) \in \mathbb{R}^2 \setminus \{(0, 0)\}$, the unique trajectory in the phase plane (y, Y) of system (\mathbf{S}) going through P_0 is denoted by $\mathcal{T}_{[P_0]}$. By symmetry, $\mathcal{T}_{[-P_0]} = -\mathcal{T}_{[P_0]}$.*

Next we show the existence of regular solutions. Our proof is short, based on phase plane portrait, and not on a fixed point method, rather delicate because $p > 2$, see [3].

Theorem 3.3 *For any $a \in \mathbb{R}$, $a \neq 0$, there exists a unique solution $w = w(\cdot, a)$ of equation (\mathbf{E}_w) in an interval $[0, r_0)$, such that w and $|w'|^{p-2} w' \in C^1([0, r_0))$ and*

$$w(0) = a, \quad w'(0) = 0; \tag{3.1}$$

and then $\lim_{r \rightarrow 0} |w'|^{p-2} w' / rw = -\varepsilon \alpha / N$. In other words in the phase plane (y, Y) there exists a unique trajectory \mathcal{T}_r such that $\lim_{\tau \rightarrow -\infty} y = \infty$, and $\lim_{\tau \rightarrow -\infty} Y / y = \varepsilon \alpha / N$.

Proof. We have assumed $\alpha \neq 0$ (when $\alpha = 0$, $w \equiv a$ from (1.13)). If such a solution w exists, then from (2.1) and (2.2), $J'_N(r) = r^{N-1}(N - \alpha)a(1 + o(1))$ near 0. Thus $J_N(r) = r^{N-1}(1 - \alpha/N)a(1 + o(1))$, hence $\lim_{r \rightarrow 0} |w'|^{p-2} w' / rw = -\varepsilon \alpha / N$; in other words, $\lim_{\tau \rightarrow -\infty} \sigma = \varepsilon \alpha / N$. And

$\lim_{\tau \rightarrow -\infty} y = \infty$, thus $\lim_{\tau \rightarrow -\infty} \zeta = 0$, and $\varepsilon\alpha\zeta > 0$ near $-\infty$. Reciprocally consider system **(Q)**. The point $(0, \varepsilon\alpha/N)$ is stationary. Setting $\sigma = \varepsilon\alpha/N + \bar{\sigma}$, the linearized system near this point is given by

$$\zeta' = p'\zeta, \quad \bar{\sigma}' = \varepsilon\zeta(\alpha - N)/N - N\bar{\sigma}.$$

One finds is a saddle point, with eigenvalues $-N$ and p' . Then there exists a unique trajectory \mathcal{T}'_r in the phase-plane (ζ, σ) starting at $-\infty$ from $(0, \varepsilon\alpha/N)$ with the slope $\varepsilon(\alpha - N)/N(N + p') \neq 0$ and $\varepsilon\alpha\zeta > 0$. It corresponds to a unique trajectory \mathcal{T}_r in the phase plane (y, Y) , and $\lim_{\tau \rightarrow -\infty} y = \infty$, since $y = |\sigma| |\zeta|^{1-p})^{1/(p-2)}$. For any solution (ζ, σ) describing \mathcal{T}'_r , the function $w(r) = r^\gamma (|\sigma| |\zeta|^{1-p}(\tau))^{1/(p-2)}$ satisfies $\lim_{r \rightarrow 0} |w'|^{p-2} w'/rw = -\varepsilon\alpha/N$. As a consequence, $w^{(p-2)/(p-1)}$ has a finite nonzero limit, and $\lim_{r \rightarrow 0} w' = 0$; thus w is regular. Local existence and uniqueness follows for any $a \neq 0$, by Remark 2.1. \blacksquare

Definition 3.4 *The trajectory \mathcal{T}_r in the plane (y, Y) and its opposite $-\mathcal{T}_r$ will be called regular trajectories. We shall say that y is regular. Observe that \mathcal{T}_r starts in \mathcal{Q}_1 if $\varepsilon\alpha > 0$, and in \mathcal{Q}_4 if $\varepsilon\alpha < 0$.*

Remark 3.5 *From Theorem 3.3 and Remark 2.1, all regular solutions are obtained from one one of them: $w(r, a) = aw(a^{-1/\gamma}r, 1)$. Thus they have the same behaviour near ∞ .*

3.2 Sign properties

Next we give informations on the zeros of w or w' , by using the monotonicity properties of the functions y_d, Y_d , in particular y, Y , and ζ and σ . At any extremal point τ , they satisfy respectively

$$y_d''(\tau) = y_d(\tau) \left(d(\eta - d) + \frac{\varepsilon(d - \alpha)}{p - 1} e^{((p-2)d+p)\tau} |dy_d(\tau)|^{2-p} \right), \quad (3.2)$$

$$Y_d''(\tau) = Y_d(\tau) \left((p - 1)^2(\eta - d)(p' + d) + \varepsilon(d - \alpha) e^{((p-2)d+p)\tau} |Y_d(\tau)|^{(2-p)/(p-1)} \right), \quad (3.3)$$

$$(p - 1)y''(\tau) = \gamma^{2-p}y(\tau) \left(-\gamma^{p-1}(N + \gamma) - \varepsilon(\gamma + \alpha) |y(\tau)|^{2-p} \right) = -|Y(\tau)|^{(2-p)/(p-1)} Y'(\tau), \quad (3.4)$$

$$Y''(\tau) = Y(\tau) \left(-\gamma(N + \gamma) - \varepsilon(\gamma + \alpha) |Y(\tau)|^{(2-p)/(p-1)} \right) = \varepsilon\alpha y'(\tau), \quad (3.5)$$

$$(p - 1)\zeta''(\tau) = -\varepsilon(p - 2)((\alpha - \zeta) |\zeta|^{2-p} |y|^{-p} yy')(\tau) = \varepsilon(p - 2)((\alpha - \zeta)(\gamma + \zeta) |\zeta y|^{2-p})(\tau), \quad (3.6)$$

$$(p - 1)\sigma''(\tau) = -(p - 2)((\sigma - \varepsilon) |\sigma|^{(2-p)/(p-1)} Y |y|^{(4-3p)/(p-1)} y')(\tau) = \zeta'(\tau)(\sigma(\tau) - \varepsilon). \quad (3.7)$$

Proposition 3.6 *Let $w \not\equiv 0$ be any solution of (\mathbf{E}_w) on an interval I .*

(i) *If $\varepsilon = 1$ and $\alpha \leq N$, then w has at most one simple zero; if $\alpha < N$ and w is regular, it has no zero. If $\alpha = N$ it has no simple zero and a compact support. If $\alpha > N$ and w is regular, it has at least one simple zero.*

(ii) If $\varepsilon = -1$ and $\alpha \geq \min(0, \eta)$, then w has at most one simple zero. If $w \not\equiv 0$ has a double zero, then it has no simple zero. If $\alpha > 0$ and w is regular, it has no zero.

(iii) If $\varepsilon = -1$ and $-p' \leq \alpha < \min(0, \eta)$, then w' has at most one simple zero, consequently w has at most two simple zeros, and at most one if w is regular. If $\alpha < -p'$, the regular solutions have at least two zeros.

Proof. (i) Let $\varepsilon = 1$. Consider two consecutive simple zeros $\rho_0 < \rho_1$ of w , with $w > 0$ on (ρ_0, ρ_1) ; hence $w'(\rho_1) < 0 < w'(\rho_0)$. If $\alpha \leq N$, we find from (2.1),

$$J_N(\rho_1) - J_N(\rho_0) = -\rho_1^{N-1} |w'(\rho_1)|^{p-2} - \rho_0^{N-1} w'(\rho_0)^{p-1} = (N - \alpha) \int_{\rho_0}^{\rho_1} s^{N-1} w ds,$$

which is contradictory; thus w has at most one simple zero. The contradiction holds as soon as ρ_0 is simple, even if ρ_1 is not. If w is regular with $w(0) > 0$, and ρ_1 is a first zero, and $\alpha < N$,

$$J_N(\rho_1) = -\rho_1^{N-1} |w'(\rho_1)|^{p-1} = (N - \alpha) \int_0^{\rho_1} s^{N-1} w ds > 0,$$

which is still impossible. If $\alpha = N$, the (Barenblatt) solutions are given by (1.10). Next suppose $\alpha > N$ and w regular. If $w > 0$, then $J_N < 0$, thus $w^{-1/(p-1)} w' + r^{1/(p-1)} < 0$. Then the function $r \mapsto r^{p'} + \gamma w^{(p-2)/(p-1)}$ is non increasing and we reach a contradiction for large r . Thus w has a first zero ρ_1 , and $J_N(\rho_1) < 0$, thus $w'(\rho_1) \neq 0$.

(ii) Let $\varepsilon = -1$ and $\alpha \geq \min(\eta, 0)$. Here we use the substitution (2.3) from some $d \neq 0$. If y_d has a maximal point, where it is positive, and is not constant, then (3.2) holds. Taking $d \in (0, \min(\alpha, \eta))$ if $\eta > 0$, $d = \eta$ if $\eta \leq 0$, we reach a contradiction. Hence y_d has at most a simple zero, and no simple zero if it has a double one. Suppose w regular and $\alpha > 0$. Then $w' > 0$ near 0, from Theorem 3.3. As long as w stays positive, any extremal point r is a strict minimum, from (\mathbf{E}_w) , thus in fact w' stays positive.

(iii) Let $\varepsilon = -1$ and $-p' \leq \alpha < \min(0, \eta)$. Suppose that w' has two consecutive zeros $\rho_0 < \rho_1$, and one of them is simple, and use again (2.3) with $d = \alpha$. Then the function Y_α has an extremal point τ , where it is positive and is not constant; from (3.3),

$$Y_\alpha''(\tau) = (p-1)^2(\eta - \alpha)(p' + \alpha)Y_\alpha(\tau), \quad (3.8)$$

thus $Y_\alpha''(\tau) \geq 0$, which is contradictory. Next consider the regular solutions. They satisfy $Y_\alpha(\tau) = e^{(\alpha(p-1)+p)\tau} (|\alpha| a/N)(1 + o(1))$ near $-\infty$, from Theorem 3.3 and (2.3), thus $\lim_{\tau \rightarrow -\infty} Y_\alpha = 0$. As above Y_α cannot have any extremal point, then Y_α is positive and increasing. In turn $w' < 0$ from (2.3), hence w has at most one zero. ■

Proposition 3.7 *Let $w \not\equiv 0$ be any solution of (\mathbf{E}_w) on an interval I . If $\varepsilon = 1$, then w has a finite number of isolated zeros. If $\varepsilon = -1$, it has a finite number of isolated zeros in any interval $[m, M] \cap I$ with $0 < m < M < \infty$.*

Proof. Let Z be the set of isolated zeros on I . If w has two consecutive isolated zeros $\rho_1 < \rho_2$, and $\tau \in (e^{\rho_1}, e^{\rho_2})$ is a maximal point of $|y_d|$, from (3.2), it follows that

$$\varepsilon e^{((p-2)d+p)\tau} |dy_d(\tau)|^{2-p} (d - \alpha) \leq (p - 1)d(d - \eta). \quad (3.9)$$

That means with $\rho = e^\tau \in (\rho_1, \rho_2)$,

$$\varepsilon \rho^p |w(\rho)|^{2-p} (d - \alpha) \leq (p - 1)d^{p-1}(d - \eta). \quad (3.10)$$

First suppose $\varepsilon = 1$ and fix $d > \alpha$. Consider the energy function

$$E(r) = \frac{1}{p'} |w'|^p + \frac{\alpha}{2} w^2.$$

It is nonincreasing since $E'(r) = -(N - 1)r^{-1} |w'|^p - r w'^2$, thus bounded on $I \cap [\rho_1, \infty)$. Then w is bounded, ρ_2 is bounded, Z is a bounded set. If Z is infinite, there exists a sequence of zeros (r_n) converging to some point $\bar{r} \in [0, \infty)$, and a sequence (τ_n) of maximal points of $|y_d|$ converging to $\bar{r} = \ln \bar{r}$. If $\bar{r} > 0$, then $w(\bar{r}) = w'(\bar{r}) = 0$; we get a contradiction by taking $\rho = \rho_n = e^{\tau_n}$ in (3.10), because the left-hand side tends to ∞ . If $\bar{r} = 0$, fixing now $d < \eta$, there exists a sequence (τ_n) of maximal points of $|y_d|$ converging to $-\infty$. Then $w(\rho_n) = O(\rho_n^{p/(p-2)})$, and $w'(\rho_n) = -d\rho_n^{-1}w(\rho_n) = O(\rho_n^{2/(p-2)})$, thus $E(\rho_n) = o(1)$. Since E is monotone, it implies $\lim_{r \rightarrow 0} E(r) = 0$, hence $E \equiv 0$, and $w \equiv 0$, which is contradictory. Next suppose $\varepsilon = -1$ and fix $d < \alpha$. If $Z \cap [m, M]$ is infinite, we construct a sequence converging vers some $\bar{r} > 0$ and reach a contradiction as above. \blacksquare

Proposition 3.8 *Let y be any non constant solution of (\mathbf{E}_y) , on a maximal interval I where $(y, Y) \neq (0, 0)$, and s be an extremity of I .*

- (i) *If y has a constant sign near s , then the same is true for Y .*
- (ii) *If $y > 0$ is strictly monotone near s , then Y, ζ, σ are monotone near s .*
- (iii) *If $y > 0$ is not strictly monotone near s , then $s = \pm\infty$, $\varepsilon(\gamma + \alpha) < 0$ and y oscillates around ℓ .*
- (iv) *If y is oscillating around 0 near s , then $\varepsilon = -1$, $s = \pm\infty$, $\alpha < -p'$; if $\alpha > -\gamma$, then $|y| > \ell$ at the extremal points.*

Proof. (i) The function w has at most one extremal point on I : at such a point, it satisfies $(|w'|^{p-2} w')' = -\varepsilon \alpha w$ with $\alpha \neq 0$. From (2.7), Y has a constant sign near s .

(ii) Suppose y strictly monotone near s . At any extremal point τ of Y , we find $Y''(\tau) = \varepsilon \alpha y'(\tau)$ from (3.5). Then $y'(\tau) \neq 0$, $Y''(\tau)$ has a constant sign. Thus τ is unique, and Y is strictly monotone near s . Next consider ζ . If there exists τ_0 such that $\zeta(\tau_0) = \alpha$, then $\zeta'(\tau_0) = \alpha(\alpha - \eta)$, from system (\mathbf{Q}) . If $\alpha \neq \eta$, then τ_0 is unique, thus $\alpha - \zeta$ has a constant sign near s . Then $\zeta''(\tau)$ has a constant

sign at any extremal point τ of ζ , from (3.6), thus ζ is strictly monotone near s . If $\alpha = \eta$, then $\zeta \equiv \alpha$. At last consider σ . If there exists τ_0 such that $\sigma(\tau_0) = \varepsilon$, then $\sigma'(\tau_0) = \varepsilon(\alpha - N)$ from System **(Q)**. If $\alpha \neq N$, then τ_0 is unique, and $\sigma - \varepsilon$ has a constant sign near s . Thus $\sigma''(\tau)$ has a constant sign at any extremal point τ of σ , from (3.7) and assertion (i). If $\alpha = N$, then $\sigma \equiv \varepsilon$.

(iii) Let y be positive and not strictly monotone near s . There exists a sequence (τ_n) strictly monotone, converging to $\pm\infty$, such that $y'(\tau_n) = 0$, $y''(\tau_{2n}) > 0 > y''(\tau_{2n+1})$. Since $y(\tau_n) = \gamma^{-1} |Y|^{(2-p)/(p-1)} Y(\tau_n)$, we deduce $Y < 0$ near s , from (i). From (3.5),

$$-\varepsilon(\gamma + \alpha)y(\tau_{2n+1})^{2-p} \leq \gamma^{p-1}(N + \gamma) \leq -\varepsilon(\gamma + \alpha)y(\tau_{2n})^{2-p}, \quad (3.11)$$

thus $\varepsilon(\gamma + \alpha) < 0$ and $y(\tau_{2n}) < \ell < y(\tau_{2n+1})$, and $Y(\tau_{2n+1}) < -(\gamma\ell)^{p-1} < Y(\tau_{2n})$. If s is finite, then $y(s) = y'(s) = 0$, which is impossible; thus $s = \pm\infty$.

(iv) If y is changing sign, then $\varepsilon = -1$ and $\alpha < -p'$, from Propositions 3.6 and 3.7. At any extremal point τ ,

$$(\alpha + \gamma)|y(\tau)|^{2-p} \leq \gamma^{p-1}(N + \gamma)$$

from (3.4); if $\alpha > -\gamma$ it means $|y| > \ell$. ■

3.3 Double zeros and global existence

Theorem 3.9 *For any $\bar{r} > 0$, there exists a unique solution w of (E_w) defined in a interval $[\bar{r}, \bar{r} \pm h)$ such that*

$$w > 0 \quad \text{on } (\bar{r}, \bar{r} \pm h) \quad \text{and} \quad w(\bar{r}) = w'(\bar{r}) = 0.$$

Moreover $\varepsilon h < 0$ and

$$\lim_{r \rightarrow \bar{r}} |(\bar{r} - r)|^{(p-1)/(2-p)} \bar{r}^{1/(2-p)} w(r) = \pm((p-2)/(p-1))^{(p-1)/(p-2)}. \quad (3.12)$$

In other words in the phase plane (y, Y) there exists a unique trajectory T_ε converging to $(0, 0)$ at $\varepsilon\infty$. It has the slope ε and converges in finite time; it depends locally continuously of α .

Proof. Suppose that a solution $w \not\equiv 0$ exists on $[\bar{r}, \bar{r} \pm h)$ with $w(\bar{r}) = w'(\bar{r}) = 0$. From Propositions 3.7 and 3.8, up to a symmetry, $y > 0$, $|Y| > 0$ near $\bar{r} = \ln \bar{r}$, and $\lim_{\tau \rightarrow \ln \bar{r}} y = 0$, and σ, ζ are monotone near $\ln r$. Let μ and λ be their limits. If $|\mu| = \infty$, then $|\lambda| = \infty$, because $\zeta = |Y|^{(2-p)/(p-1)} \sigma$, $|\zeta|^{p-2} \zeta = \sigma y^{2-p}$; then $f = 1/\zeta$ tends to 0; but

$$f' = -1 + \eta f + \varepsilon \frac{1 - \alpha f}{(p-1)\sigma}, \quad (3.13)$$

thus f' tends to -1 , which is impossible. Thus μ is finite. If λ is finite, then $\mu = 0$, thus $\lambda = \alpha$, from system **(Q)**, $\ln w$ is integrable at \bar{r} , which is not true. Then $\lambda = \varepsilon\infty$, hence

$$\mu = \lim_{\tau \rightarrow \ln \bar{r}} \sigma = \varepsilon,$$

from system **(Q)**. Then $\varepsilon Y > 0$ near $\bar{\tau}$, then $\varepsilon w' < 0$ near $\bar{\tau}$, thus $\varepsilon h < 0$. Consider system **(R)**: as τ tends to $\bar{\tau}$, ν tends to $\pm\infty$, and (g, s) converges to the stationary point $(0, -\varepsilon)$.

Reciprocally, setting $s = -\varepsilon/\beta + h$, the linearized system of system **(R)** at this point is given by

$$\frac{dg}{d\nu} = -\varepsilon \frac{p-2}{p-1} g, \quad \frac{dh}{d\nu} = (\alpha - N)g + \varepsilon h.$$

The eigenvalues are $-\varepsilon(p-2)/(p-1)$ and ε , thus we find a saddle point. There are two trajectories converging to $(0, -\varepsilon)$. The first one satisfies $g \equiv 0$, it does not correspond to a solution of the initial problem. Then there exists a unique trajectory converging to $(0, -\varepsilon)$, as ν tends to $\varepsilon\infty$, with $g > 0$ near $\varepsilon\infty$. It is associated to the eigenvalue $-\varepsilon(p-2)/(p-1)$ and the eigenvector $((2p-3)/(p-1), \varepsilon(N-\alpha))$. It satisfies $dg/d\nu = -\varepsilon((p-2)/(p-1))g(1+o(1))$, thus $dg/d\tau = ((p-2)/(p-1))(1+o(1))$. Then τ has a finite limit $\bar{\tau}$, and τ increases to $\bar{\tau}$ if $\varepsilon = 1$ and decreases to $\bar{\tau}$ if $\varepsilon = -1$. In turn $|Y|^{(p-2)/(p-1)} = gs$ tends to 0, and s tends to ε , thus (y, Y) tends to $(0, 0)$ as τ tends to $\bar{\tau}$. Then w and w' converges to 0 at $\bar{r} = e^{\bar{\tau}}$. And $w'w^{-1/(p-1)} + (\varepsilon + o(1))r^{1/(p-1)} = 0$, which implies (3.12).

Corollary 3.10 *Let $r_1 > 0$, and $a, b \in \mathbb{R}$ and w be any local solution such that $w(r_1) = a$, $w'(r_1) = b$.*

- (i) *If $(a, b) = (0, 0)$, then w has a unique extension by 0 on (r_1, ∞) if $\varepsilon = 1$, on $(0, r_1)$ if $\varepsilon = -1$.*
- (ii) *If $(a, b) \neq (0, 0)$, w has a unique extension to $(0, \infty)$.*

■

Proof. (i) Assume $a = b = 0$, the function $w \equiv 0$ is a solution. Let w be any local solution near r_1 , defined in an interval $(r_1 - h_1, r_1 + h_1)$ with $w(r_1) = w'(r_1) = 0$. Suppose that there exists $h_2 \in (0, h_1)$ such that $w(r_1 + \varepsilon h_1) \neq 0$. Let $\bar{h} = \inf \{h \in (0, h_1) : w(r_1 + \varepsilon h) \neq 0\}$, and $\bar{r} = r_1 + \varepsilon \bar{h}$, thus $w(\bar{r}) = w'(\bar{r}) = 0$, and for example $w > 0$ on some interval $(\bar{r}, \bar{r} + \varepsilon k)$ with $k > 0$. This contradicts theorem 3.9. Thus $w \equiv 0$ on $(r_1, r_1 + \varepsilon h_1)$.

(ii) From Theorems 3.9 and 3.3, w has no double zero for $\varepsilon(r - r_1) < 0$, and has a unique extension to a maximal interval with no double zero. From (i) it has a unique extension to $(0, \infty)$. In particular any local regular solution is defined on $[0, \infty)$. ■

4 Asymptotic behaviour

Next the function y is supposed to be monotone, thus w has a constant sign near 0 or ∞ , we can assume that $w > 0$.

Proposition 4.1 *Let y be any solution of (E_y) strictly monotone and positive near $s = \pm\infty$.*

(1) Then (ζ, σ) has a limit (λ, μ) near s , given by is some of the values

$$\begin{aligned} A_\gamma &= \left(-\gamma, \varepsilon \frac{\alpha + \gamma}{N + \gamma} \right), \quad A_r = (0, \varepsilon \alpha / N), \quad A_\alpha = (\alpha, 0), \\ L_\eta &= \eta(1, \infty) \text{ (if } p \neq N), \quad L_+ = (0, \infty) \text{ (if } p \geq N), \quad L_- = (0, -\infty) \text{ (if } p > N). \end{aligned} \quad (4.1)$$

(2) More precisely,

(i) Either $\varepsilon(\gamma + \alpha) < 0$ and (y, Y) converges to $\pm M_\ell$. Then $(\lambda, \mu) = A_\gamma$ and $(\varepsilon = 1, s = \infty)$ or $(\varepsilon = -1, s = -\infty)$ for $\alpha \leq \alpha^*$, $s = \infty$ for $\alpha > \alpha^*$.

(ii) Or (y, Y) converges to $(0, 0)$. Then $(s = \infty \text{ and } -\gamma < \alpha)$ or $(s = -\infty \text{ and } \alpha < -\gamma)$, or $(s = \varepsilon \infty \text{ and } \alpha = -\gamma)$ and $(\lambda, \mu) = A_\alpha$.

(iii) Or $\lim_{\tau \rightarrow s} y = \infty$. Then $s = -\infty$. If $p < N$, then $(\lambda, \mu) = A_r$ or L_η . If $p = N$, then $(\lambda, \mu) = A_r$ or L_+ . If $p > N$, then $(\lambda, \mu) = A_r, L_\eta, L_+$ or L_- .

Proof. (1) The functions Y, σ, ζ are also monotone, and by definition $\zeta \sigma > 0$. Thus ζ has a limit $\lambda \in [-\infty, \infty]$ and σ has a limit $\mu \in [-\infty, \infty]$, and $\lambda \mu \geq 0$.

(i) λ is finite. Indeed if $\lambda = \pm\infty$, then $f = 1/\zeta$ tends to 0. From (3.13), either $\mu = \pm\infty$, then f' tends to -1 , which is impossible; or μ is finite, thus $\mu = \varepsilon$ from system **(Q)**, then f' tends to $(2-p)/(p-1)$, which is still contradictory.

(ii) Either μ is finite, thus (λ, μ) is a stationary point of system **(Q)**, equal to A_γ, A_r or A_α .

(iii) Or $\mu = \pm\infty$ and $(\lambda, 0)$ is a stationary point of system **(P)**.

- If $p \neq N$, either $\lambda = \eta \neq 0$ and $(\lambda, \mu) = L_\eta$; or $\lambda = 0$ and $(\lambda, \mu) = L_+$ or L_- . In the last case (ζ, ψ) converges to $(0, 0)$, and $\zeta'/\psi' = -(\eta\zeta/N\psi)(1 + o(1))$, thus $\eta < 0$, that means $p > N$.

- If $p = N$, then again (ζ, ψ) converges to $(0, 0)$, thus $\mu = \pm\infty$, and $\psi' = N\psi(1 + o(1))$, and necessarily $s = -\infty$. We make the substitution (2.4) with $d = 0$. Then $y_0(\tau) = w(r)$, and y_0 satisfies

$$y_0' = -|Y_0|^{(2-p)/(p-1)} Y_0 = -\zeta y_0 = o(y_0), \quad Y_0' = \varepsilon e^{p\tau} y_0 (\alpha - \zeta) = \varepsilon e^{p\tau} y_0 \alpha (1 + o(1)).$$

Thus for any $v > 0$, we get $y_0 = O(e^{-v\tau})$ and $1/y_0 = O(e^{v\tau})$. Then Y_0' is integrable, and Y_0 has a finite limit $|k|^{p-2} k$. Suppose that $k = 0$. Then $Y_0 = O(e^{(p-v)\tau})$, and y_0 has a finite limit $a \geq 0$. If $a \neq 0$, then $Y_0' = \varepsilon \alpha a e^{p\tau} (1 + o(1))$; in turn $Y_0 = p^{-1} \varepsilon \alpha a e^{p\tau} (1 + o(1))$, and $\psi = e^{p\tau} y_0 / Y_0$ does not tend to 0. If $a = 0$, then $y_0 = O(e^{p'\tau})$, which contradicts the estimate of $1/y_0$. Thus $k > 0$ and

$$y_0 = -k\tau(1 + o(1)), \quad Y_0 = k^{p-1}(1 + o(1)); \quad (4.2)$$

hence $(\lambda, \mu) = L_+$.

(2) Since y is monotone, we encounter one of the three following cases:

(i) (y, Y) converges to $\pm M_\ell$. Then $(\lambda, \mu) = A_\gamma$ and M_ℓ is a source (or a weak source) for $\alpha \leq \alpha^*$, a sink for $\alpha > \alpha^*$.

(ii) y tends to 0. Since λ is finite, (y, Y) converges to $(0, 0)$. And $|\sigma| = |\zeta|^{p-1} y^{p-2}$ tends to 0, thus $(\lambda, \mu) = A_\alpha$. If $-\gamma < \alpha$, seeing that $y' = -y(\gamma + \zeta) < 0$ we find $s = \infty$. If $\alpha < -\gamma$, then $s = -\infty$. If $\alpha = -\gamma < 0$, then $\varepsilon(\gamma + \zeta) > 0$, from the first equation of **(Q)**, thus $\varepsilon y' < 0$, hence $s = \varepsilon\infty$.

(iii) y tends to ∞ . Either $\lambda \neq 0$, thus $|\sigma| = |\zeta|^{p-1} y^{p-2}$ tends to ∞ , and $\lambda = \eta$ from system **(Q)**, thus $p \neq N$, $(\lambda, \mu) = L_\eta$. Or $\lambda = 0$ and μ is finite, thus $\mu = \varepsilon\alpha/N$, $(\lambda, \mu) = A_r$. Or $(\lambda, \mu) = L_0$; then either $p = N$, $L_0 = L_\eta$, or $p > N$. In any case, $y' = -y(\gamma + \zeta) < 0$, from (1.2), hence $s = -\infty$.

■

Next we apply these results to the functions w :

Proposition 4.2 *We keep the assumptions of Proposition 4.1. Let w be the solution of **(E_w)** associated to y by (2.7).*

(i) If $(\lambda, \mu) = A_\gamma$ (near 0 or ∞), then

$$\lim r^{-\gamma} w = \ell. \quad (4.3)$$

(ii) If $(\lambda, \mu) = A_\alpha$ (near 0 or ∞), then

$$\lim r^\alpha w = L > 0 \quad \text{if } \alpha \neq -\gamma, \quad (4.4)$$

$$\lim r^{-\gamma} (\ln r)^{1/(p-2)} w = ((p-2)\gamma^{p-1}(N+\gamma))^{-1/(p-2)} \quad \text{if } \alpha = -\gamma. \quad (4.5)$$

(iii) If $p < N$ and $(\lambda, \mu) = L_\eta$, then

$$\lim_{r \rightarrow 0} r^\eta w = c > 0. \quad (4.6)$$

(iv) If $p > N$ and $(\lambda, \mu) = L_\eta$, then

$$\lim_{r \rightarrow 0} r^{-|\eta|} w = c > 0. \quad (4.7)$$

(v) If $p = N$ and $(\lambda, \mu) = L_+$, then

$$\lim_{r \rightarrow 0} |\ln r|^{-1} w = k > 0, \quad \lim_{r \rightarrow 0} r w' = -k \quad \text{if } p = N. \quad (4.8)$$

(vi) If $p > N$ and $(\lambda, \mu) = L_+$, or L_- , then

$$\lim_{r \rightarrow 0} w = a > 0, \quad \lim_{r \rightarrow 0} (-r^{(N-1)/(p-1)} w') = c > 0, \quad (4.9)$$

or

$$\lim_{r \rightarrow 0} w = a > 0, \quad \lim_{r \rightarrow 0} (-r^{(N-1)/(p-1)} w') = c < 0. \quad (4.10)$$

Proof. (i) This follows directly from (2.7).

(ii) From (2.16), $rw'(r) = -\alpha w(r)(1 + o(1))$. We are lead to three cases.

- Either $-\gamma < \alpha$, and $s = \infty$. For any $v > 0$, we find $w = O(r^{-\alpha+v})$ and $1/w = O(r^{\alpha+v})$ near ∞ and $w' = O(r^{-\alpha-1+v})$. Then $J'_\alpha(r) = O(r^{\alpha(2-p)-p-1+v})$, hence J'_α is integrable, J_α has a limit L . And $\lim r^\alpha w = L$, seeing that $J_\alpha(r) = r^\alpha w(1 + o(1))$. If $L = 0$, then $r^\alpha w = O(r^{\alpha(2-p)-p+v})$, which contradicts the estimate of $1/w = O(r^{\alpha+v})$ for v small enough. Thus $L > 0$.

- Or $\alpha < -\gamma$ and $s = -\infty$. For any $v > 0$, we find $w = O(r^{-\alpha-v})$ and $1/w = O(r^{\alpha+v})$ near 0 and $w' = O(r^{-\alpha-1-v})$. Then $J'_\alpha(r) = O(r^{\alpha(2-p)-p-1-v})$, and J'_α is still integrable, J_α has a limit L , and $\lim r^\alpha w = L$. If $L = 0$, then $r^\alpha w = O(r^{\alpha(2-p)-p-v})$, which contradicts the estimate of $1/w$. Thus again $L > 0$.

- Or $\alpha = -\gamma$ and $s = \varepsilon\infty$. Then $Y = -\gamma^{p-1}y^{p-1}(1 + o(1))$, and $\mu = 0$, thus $y - \varepsilon Y = y(1 + o(1))$. From System (S),

$$(y - \varepsilon Y)' = \varepsilon(N + \gamma)Y = -\varepsilon(N + \gamma)\gamma^{p-1}(y - \varepsilon Y)^{p-1}(1 + o(1)).$$

Then $y = (N + \gamma)\gamma^{p-1}(p-2)|\tau|^{-1/(p-2)}(1 + o(1))$, which is equivalent to (4.5).

(iii) From (2.16), we get $rw'(r) = -\eta w(r)(1 + o(1))$. We use (2.3) with $d = \eta$, thus $y_\eta = r^\eta w$. We find $y_\eta = O(e^{-v\tau})$, $1/y_\eta = O(e^{-v\tau})$, in turn $Y_\eta = O(e^{-v\tau})$. From (2.4), $Y'_\eta = O(e^{(p+(p-2)\eta-v)\tau})$, thus Y'_η is integrable, hence Y_η has a finite limit. Now $(e^{-\eta\tau}y_\eta)' = -e^{-\eta\tau}Y_\eta^{1/(p-1)}$, and $\eta > 0$, thus y_η has a limit c . If $c = 0$, then $Y_\eta = O(e^{(p+(p-2)\eta-v)\tau})$, $y_\eta = O(e^{((p+(p-2)\eta)/(p-1)-v)\tau})$, which contradicts $1/y_\eta = O(e^{-v\tau})$ for v small enough. Then (4.6) holds.

(iv) As above, Y_η has a finite limit. In turn $r^{-|\eta|+1}w' = |Y_\eta|^{(2-p)/(p-1)}Y_\eta$ has a limit $c|\eta|$ and w has a limit $a \geq 0$. From (2.16), $rw' = |\eta|w(1 + o(1))$, hence $a = 0$. Then $c \geq 0$; if $b = 0$, then $Y < 0$, the function $v = -e^{(\gamma+N)\tau}Y > 0$ tends to 0 and

$$v' = -e^{(\gamma+N)\tau}\varepsilon(\alpha - \eta)y(1 + o(1)) = -\varepsilon(\alpha - \eta)|\eta|e^{-(\gamma+N)(p-2)/(p-1)\tau}v^{1/(p-1)};$$

we reach again a contradiction. Thus $a = 0$ and $c > 0$, and (4.7) holds.

(v) Assertion (4.8) follows from (4.2).

(vi) Here $rw' = o(w)$, thus $w + |w'| = O(r^{-k})$ for any $k > 0$. Then J'_N is integrable, J_N has a limit at 0, and $\lim_{r \rightarrow 0} r^N w = 0$. Thus $\lim_{r \rightarrow 0} r^{(N-1)/(p-1)}w' = -c \in \mathbb{R}$, $\lim_{r \rightarrow 0} J_N = -\varepsilon|c|^{p-2}c$,

$\lim_{r \rightarrow 0} w = a \geq 0$. If $c = 0$, then $J_N(r) = \int_0^r J'_N(s)ds$, implying that $\lim_{r \rightarrow 0} w' = 0$. Either $a > 0$

and then w is regular, then $\lim_{\tau \rightarrow -\infty} \sigma = \varepsilon$; or $a = 0$, then $w' > 0$ and $(w')^{p-1} = O(rw)$; in both cases we get a contradiction. Thus $c \neq 0$. If $a = 0$, we find $\lim_{\tau \rightarrow -\infty} \zeta = \eta$, which is not true, hence $a > 0$. In any case (4.9) or (4.10) holds. \blacksquare

Now we study the cases where y is not monotone, and eventually changing sign.

Proposition 4.3 Suppose $\varepsilon = -1$. Let $w \neq 0$ be any solution of (\mathbf{E}_w) .

(i) If $\alpha \leq -\gamma$, then w is oscillating near 0 at ∞ .

(ii) If $\alpha < 0$, then y and Y are bounded at ∞ .

Proof. (i) Suppose by contradiction that $w \geq 0$ for large r , then $y \geq 0$ for large τ . If $y > 0$ near ∞ , then from Proposition 3.8, either y is constant, which is impossible since $(0, 0)$ is the unique stationary point; or y is strictly monotone, which contradicts Proposition 4.1. Then there exists a sequence (τ_n) tending to ∞ such that $y(\tau_n) = y'(\tau_n) = 0$; from Theorem 3.10, $y \equiv 0$ on $(-\infty, \tau_n)$, thus $y \equiv 0$.

(ii) Consider the function

$$\tau \mapsto R(\tau) = \frac{y^2}{2} + \frac{|Y|^{p'}}{p'|\alpha|};$$

it satisfies

$$R'(\tau) = -\gamma y^2 + \frac{1}{|\alpha|} |Y|^{2/(p-1)} - \frac{N + \gamma}{|\alpha|} |Y|^{p'}.$$

From the Young inequality,

$$|\alpha|(R'(\tau) + \gamma R(\tau)) = |Y|^{2/(p-1)} - (N + \frac{1}{p-2}) |Y|^{p'} \leq (\frac{2}{Np + \gamma})^{(p-2)/2} \leq 1$$

thus $R(\tau)$ is bounded for large τ , at least by $1/|\alpha|\gamma$. ■

Proof.

Proposition 4.4 (i) Assume $\varepsilon = 1$, or $\varepsilon = -1$, $\alpha \notin (\alpha_2, \alpha_1)$. Then for any trajectory of system (\mathbf{S}) in \mathcal{Q}_4 near $\pm\infty$, y is strictly monotone near $\pm\infty$.

(ii) Assume $\varepsilon = 1$, and $\alpha \leq \alpha^*$ or $-p' \leq \alpha$. Then system (\mathbf{S}) admits no cycle in \mathcal{Q}_4 (or \mathcal{Q}_2). ■

Proof. (i) In any case M_ℓ is a node point. Following [4, Theorem 2.24], we use the linearization defined by (2.9). Consider the line L given by the equation $A\bar{y} + \bar{Y} = 0$, where A is a real parameter. The points of L are in \mathcal{Q}_4 whenever $\bar{Y} < (\gamma\ell)^{p-1}$ and $-\ell < \bar{y}$. We get

$$A\bar{y}' + \bar{Y}' = (\varepsilon\nu(\alpha)A^2 + (N + \nu(\alpha))A + \varepsilon\alpha)\bar{y} + (A + \varepsilon)\Psi(\bar{Y}).$$

From (2.13), apart from the case $\varepsilon = 1, \alpha = N$, we can find an A such that

$$\varepsilon\nu(\alpha)A^2 + (N + \nu(\alpha))A + \varepsilon\alpha = 0,$$

and $A + \varepsilon \neq 0$. Moreover $\Psi(\bar{Y}) \leq 0$ on $L \cap \mathcal{Q}_4$. Indeed $(p-1)\Psi'(t) = -((\gamma\ell)^{p-1} - t)^{(2-p)/(p-1)} + (\gamma\ell)^{2-p}$, thus Ψ has a maximum 0 on $(-\infty, (\delta\ell)^{p-1})$ at point 0. Then the orientation of the vector

field does not change along $L \cap \mathcal{Q}_4$. In particular y cannot oscillate around ℓ , thus y is monotone, from Proposition 3.8. If $\varepsilon = 1, \alpha = N$, then $Y \equiv y \in (\ell, \infty)$ defines the trajectory \mathcal{T}_r , corresponding to the solutions given by (1.10) with $K > 0$. No solution y can oscillate around ℓ , since the trajectory cannot meet \mathcal{T}_r .

(ii) Suppose that there exists a cycle in \mathcal{Q}_4 .

• Assume $\alpha \leq \alpha^*$. Here M_ℓ is a source, or a weak source, from Proposition 2.5. Any trajectory starting from M_ℓ at $-\infty$ has a limit cycle in \mathcal{Q}_1 , which is attracting at ∞ . Writing System (S) under the form $y' = f_1(y, Y), Y' = f_2(y, Y)$, the mean value of the Floquet integral on the period $[0, \mathcal{P}]$ is given by

$$I = \oint \left(\frac{\partial f_1}{\partial y}(y, Y) + \frac{\partial f_2}{\partial Y}(y, Y) \right) d\tau = \oint \left(\frac{|Y|^{(2-p)/(p-1)}}{p-1} - 2\gamma - N \right) d\tau. \quad (4.11)$$

Such a cycle is not unstable, thus $I \leq 0$. Now

$$\oint (\alpha y' - \gamma Y') d\tau = 0 = (\alpha + \gamma) \oint |Y|^{1/(p-1)} d\tau - \gamma(\gamma + N) \oint |Y| d\tau.$$

From the Jensen and Hölder inequalities, since $1/(p-1) < 1$,

$$\gamma(\gamma + N) \left(\oint |Y|^{1/(p-1)} d\tau \right)^{p-2} \leq \alpha + \gamma,$$

$$1 \leq \left(\oint |Y|^{(2-p)/(p-1)} d\tau \right) \left(\oint |Y|^{1/(p-1)} d\tau \right)^{p-2} \leq \frac{(p-1)(2\gamma + N)}{\gamma(\gamma + N)} (\alpha + \gamma),$$

then $\alpha^* < \alpha$, which is contradictory.

• Assume $-p' \leq \alpha < 0$. Consider the functions $y_\alpha = e^{(\alpha+\gamma)\tau} y$ and $Y_\alpha = e^{(\alpha+\gamma)(p-1)\tau} Y$ defined by (2.3) with $d = \alpha$. They vary respectively from 0 to ∞ and from 0 to $-\infty$. They have no extremal point. Indeed at such a point, from (3.2) and (3.3) y''_α or Y''_α have a strict constant sign for $\alpha \neq \eta, p'$, which is contradictory. If $\alpha = \eta$ or p' , from uniqueness y_α or Y_α is constant, thus y or Y is monotone, which is impossible. In any case $y'_\alpha > 0 > Y'_\alpha$ on $(-\infty, \infty)$. Next, from (2.5) and (2.6),

$$\frac{y''_\alpha}{y'_\alpha} + \eta - 2\alpha - \frac{1}{p-1} Y^{(2-p)/(p-1)} = \alpha(\eta - \alpha) \frac{y_\alpha}{y'_\alpha}, \quad (4.12)$$

$$\frac{Y''_\alpha}{Y'_\alpha} + (p-1)(\eta - 2\alpha - p') - \frac{1}{p-1} Y^{(2-p)/(p-1)} = (p-1)^2(\eta - \alpha)(p' + \alpha) \frac{Y_\alpha}{Y'_\alpha}. \quad (4.13)$$

Let us integrate on the period \mathcal{P} . If $\eta \leq \alpha < 0$, then $\eta - N - 2(\alpha + \gamma) \geq 0$ from (4.12), which is contradictory. If $-p' \leq \alpha < \eta$, then $-2(\alpha + p' + \gamma) > 0$ from (4.13), still contradictory. ■

5 New local existence results

At Proposition 4.1 we gave all the *possible* behaviours of the positive solutions near $\pm\infty$. Next we prove their existence, and uniqueness or multiplicity. The case $p > N$ is very delicate.

Theorem 5.1 (i) Suppose $p < N$. In the phase plane (y, Y) of system (\mathbf{S}) there exist an infinity of trajectories \mathcal{T}_η such that $\lim_{\tau \rightarrow -\infty}(\zeta, \sigma) = L_\eta$; the corresponding w satisfy (4.6).

(ii) Suppose $p > N$. There exist a unique trajectory \mathcal{T}_u such that $\lim_{\tau \rightarrow -\infty}(\zeta, \sigma) = L_\eta$; in other words for any $c \neq 0$, there exists a unique solution w of equation (\mathbf{E}_w) such that (4.7) holds.

Proof. Suppose that such a trajectory exists in the plane (y, Y) . In the phase plane (ζ, ψ) of System (\mathbf{P}) , ζ and ψ keep a strict constant sign, because the two axes $\zeta = 0$ and $\psi = 0$ contain particular trajectories, and (ζ, ψ) converges to $(\eta, 0)$ at $-\infty$. Reciprocally, setting $\zeta = \eta + \bar{\zeta}$, the linearized problem at point $(\eta, 0)$

$$\bar{\zeta}' = \eta\bar{\zeta} + \eta(\alpha - \eta)\varepsilon\psi/(p - 1), \quad \psi' = (N - \eta)\psi,$$

admits the eigenvalues η and $N - \eta$. The trajectories linked to the eigenvalue η are tangent to the line $\psi = 0$.

(i) Case $p < N$. Then $\eta > 0$, and $(\eta, 0)$ is a source. In the plane (ζ, ψ) there exist an infinity of trajectories, starting from this point at $-\infty$, such that $\psi > 0$, and $\lim_{\tau \rightarrow -\infty} \zeta = \eta$, thus $\zeta > 0$. In the phase plane (y, Y) , setting $y = (\psi |\zeta|^{p-2} \zeta)^{2-p}$ and $Y = y/\psi$, they correspond to an infinity of trajectories in the plane (y, Y) such that $\lim_{\tau \rightarrow -\infty}(\zeta, \sigma) = L_\eta$, and (4.6) holds from Proposition (4.2).

(ii) Case $p > N$. Then $\eta < 0$, and $(\eta, 0)$ is a saddle point. In the plane (ζ, ψ) , there exists a unique trajectory starting from $(\eta, 0)$, tangentially to the vector $(\eta(\alpha - \eta)\varepsilon/(p - 1), N - \eta)$, with $\psi < 0$; it defines a unique trajectory \mathcal{T}_u in the plane (y, Y) , and (4.7) holds. From Remark 2.1, we get a solution for any $c \neq 0$. ■

Theorem 5.2 (i) Suppose $p = N$. In the phase plane (y, Y) , there exists an infinity of trajectories \mathcal{T}_+ such that $\lim_{\tau \rightarrow -\infty}(\zeta, \sigma) = L_+$; then w satisfies (4.8).

(ii) Suppose $p > N$. Then there exist an infinity of trajectories \mathcal{T}_+ (resp. \mathcal{T}_-) such $\lim_{\tau \rightarrow -\infty}(\zeta, \sigma) = L_+$ (resp. L_-); then the corresponding solutions w of (\mathbf{E}_w) satisfy (4.9) (resp. (4.10)).

More precisely for any $k > 0$ (for $p = N$) or any $a > 0$ and $c \neq 0$ (for $p > N$) there exists a unique function w satisfying those conditions.

Proof. If $\lim_{\tau \rightarrow -\infty}(\zeta, \sigma) = L_\pm$, then $\lim_{\tau \rightarrow -\infty}(\zeta, \psi) = (0, 0)$, with $\zeta\psi > 0$ in case of L_+ , $\zeta\psi < 0$ in case of L_- . The linearization of System (\mathbf{P}) near $(0, 0)$ is given by

$$\zeta' = |\eta|\zeta, \quad \psi' = N\psi.$$

(i) Case $p = N$. The phase plane study is delicate because 0 is a center, thus we use a fixed method. Suppose that such a trajectory exists, and consider the substitution (2.3) with $d = 0$. From (4.2), there exists $k > 0$ such that $\zeta = |Y_0|^{(2-p)/(p-1)}/y_0 = -\tau^{-1}(1 + o(1)) > 0$, and $\psi = -k^{2-p}\tau e^{N\tau}(1 + o(1)) > 0$. Then $\zeta' = \tau^{-2}(1 + o(1))$ from System **(P)**. The function

$$V = \psi e^{-N/\zeta} \zeta$$

satisfies $\lim_{\tau \rightarrow -\infty} V = k^{2-p}$, and

$$V' = \frac{V e^{N/\zeta}}{(N-1)\zeta^2} (\varepsilon(\alpha - \zeta)(N - (N-2)\zeta)V + 2N(N-1)\zeta^2 e^{-N/\zeta}).$$

Thus $\varepsilon\alpha(V - k^{2-p}) > 0$ near $-\infty$. Moreover $\lim_{\tau \rightarrow -\infty} \zeta'/V' = 0$, so that ζ can be considered as a function of V near k^{2-p} , with $\lim_{V \rightarrow k^{2-p}} \zeta = 0$ and

$$\frac{d\zeta}{dV} = K(V, \zeta), \quad K(V, \zeta) := \frac{\zeta^2}{V} \frac{\varepsilon(\alpha - \zeta)V + (N-1)\zeta^2 e^{-N/\zeta}}{\varepsilon(\alpha - \zeta)(N - (N-2)\zeta)V + 2N(N-1)\zeta^2 e^{-N/\zeta}}.$$

Reciprocally, extending the function $\zeta^2 e^{-N/\zeta}$ by 0 for $\zeta \leq 0$, the function K is of class C^1 near $(k^{2-p}, 0)$. For any $k > 0$, there exists a unique local solution $V \mapsto \zeta(V)$ on a interval \mathcal{V} where $\varepsilon\alpha(V - k^{2-p}) > 0$, such that $\zeta(k^{2-p}) = 0$. And $d\zeta/dV = (\zeta^2/Nk^{2-p})(1 + o(1))$ near 0, thus $\zeta > 0$. In the plane (ζ, ψ) , taking one point P on the curve $\mathcal{C} = \{(\zeta(V), V\zeta(V)e^{N/\zeta(V)}) : v \in \mathcal{V}\}$, there exists a unique solution of System **(P)** issued from P at time 0. Its trajectory is on \mathcal{C} , thus it converges to $(0, 0)$, with $\zeta, \psi > 0$. It corresponds to a unique trajectory \mathcal{T}_+ in the plane (y, Y) , and (ζ, σ) converges to L_+ , as τ tends to $-\infty$, from Proposition 4.1. The corresponding functions w satisfy (4.8) from Proposition (4.2).

(ii) Case $p > N$. Here $(0, 0)$ is a source for System **(P)**. The lines $\zeta = 0$ and $\psi = 0$ contain trajectories. There exists an infinity of trajectories converging to $(0, 0)$, with $\zeta\psi \neq 0$; moreover, if $N \geq 2$, then $|\eta| < N$, thus $\lim_{\tau \rightarrow -\infty} (\psi/\zeta) = 0$. Our claim is more precise. Given $a > 0$ and $c \neq 0$, we look for a solution w of **(E_w)** such that $\lim_{r \rightarrow 0} w = a$, $\lim_{r \rightarrow 0} r^{\eta+1} w' = -c$. By scaling we can assume $a = 1$. If w_1 is a such a solution, then ζ and ψ have the sign of c near 0, and $\zeta(\tau) = ce^{|\eta|\tau}(1 + o(1))$ and $|c|^{p-2} c\psi(\tau) = e^{N\tau}(1 + o(1))$. The function

$$v = c(|c|^{p-2} c\psi)^{1/\kappa} / \zeta, \quad \text{with } \kappa = N/|\eta| > 1,$$

satisfies $\lim_{\tau \rightarrow -\infty} v = 1$, and can be expressed locally as a function of ζ , and

$$\frac{dv}{d\zeta} = H(\zeta, v), \quad H(\zeta, v) := -\frac{v(p-1)(\kappa+1) + \varepsilon(\kappa-p+1)|c|^{1-p-\kappa}(\zeta-\alpha)|\zeta|^{\kappa-1}v^\kappa}{\kappa(p-1)(\zeta-\eta) + \varepsilon|c|^{1-p-\kappa}(\alpha-\zeta)|\zeta|^{\kappa-1}\zeta v^\kappa}.$$

Reciprocally, there exists a unique solution $\zeta \mapsto v(\zeta)$ of this equation on a small interval $[0, hc)$, with $h > 0$, such that $v(0) = 1$. Indeed H is locally continuous in ξ and C^1 in v . Taking one

point P on the curve $\mathcal{C}' = \left\{ (\zeta, |c|^{1-p-\kappa} |\zeta|^{\kappa-1} \zeta v(\zeta)) : \zeta \in [0, hc) \right\}$, there exists a unique solution of System **(P)** issued from P at time 0. Its trajectory is on \mathcal{C}' , thus converges to $(0, 0)$ with $\zeta \psi > 0$. It corresponds to a solution (y, Y) of System **(S)**, such that (ζ, σ) converges to L_+ , as τ tends to $-\infty$, from Proposition 4.1. The corresponding function, called w_2 , satisfies $\lim_{r \rightarrow 0} r^{\eta+1} w_2^{\gamma^{-1}|\eta|^{-1}} w_2' = -c$; thus w_2 has a limit a_2 , and $\lim_{r \rightarrow 0} r^{\eta-1} w_2' = a_2^{1-s} b$. Moreover $a_2 \neq 0$, because $a_2 = 0$ implies that $r^{-\gamma} w_2$ has a nonzero limit, thus (ζ, σ) converges to A_γ . The function $w(r) = a_2^{-1} w_2(a_2^{1/\gamma} r)$ satisfies $\lim_{r \rightarrow 0} w = 1$, and $\lim_{r \rightarrow 0} r^{\eta-1} w' = -c$, and the proof is done.

Theorem 5.3 (i) *In the phase plane (y, Y) , for any $\alpha \neq 0$ there exists at least a trajectory \mathcal{T}_α converging to $(0, 0)$ with $y > 0$, and $\lim(\zeta, \sigma) = A_\alpha$. The convergence holds at ∞ if $-\gamma < \alpha$, or $-\infty$ if $\alpha < -\gamma$, or $\varepsilon\infty$ if $\alpha = -\gamma$.*

(ii) *If $\varepsilon(\gamma + \alpha) < 0$, \mathcal{T}_α is unique, it is the unique trajectory converging to $(0, 0)$ at $-\varepsilon\infty$ with $y > 0$, and it depends locally continuously of α .*

Proof. (i) Suppose that such a trajectory exists. Then τ tends to ∞ if $-\gamma < \alpha$, or $-\infty$ if $\alpha < -\gamma$, or $\varepsilon\infty$ if $\alpha = -\gamma$, from Proposition 4.1. Consider System **(R)**, where g, s and ν are defined by (2.18). Then (g, s) converges to $(-1/\alpha, 0)$, with $gs > 0$, and ν tends to the same limits as τ , since Y converges to 0. Reciprocally, in the plane (g, s) , let us show the existence of a trajectory converging to $(-1/\alpha, 0)$, different from the line $s = 0$. Setting $g = -1/\alpha + \bar{g}$, the linearized system at this point is

$$\frac{d\bar{g}}{d\nu} = -\frac{\varepsilon}{p-1} \bar{g} + \frac{\eta - \alpha}{\alpha^2} s, \quad \frac{ds}{d\nu} = 0,$$

thus we find a center: the eigenvalues are 0 and $\lambda = \varepsilon/(p-1)$. Since the system is polynomial, it is known that System **(R)** admits a trajectory, depending locally continuously of α , such that $sg > 0$, and tangent to the eigenvector $((p-1)(\eta - \alpha), \varepsilon\alpha^2)$. It satisfies $ds/d\nu = (p-2)(\alpha + \gamma)s^2(1 + o(1))$. Then $ds/d\tau = -(p-2)\alpha(\alpha + \gamma)s(1 + o(1))$, thus τ tends to $\pm\infty$. And $|y|^{p-2} = |s| |g|^{1/(p-1)}$, then y tends to 0, (y, Y) converges to $(0, 0)$, and $\lim(\zeta, \sigma) = A_\alpha$.

(ii) Suppose $\varepsilon(\gamma + \alpha) < 0$. Consider two trajectories $\mathcal{T}_1, \mathcal{T}_2$ in the plane (y, Y) , converging to $(0, 0)$ at $-\varepsilon\infty$, with $y > 0$. They are different from \mathcal{T}_ε which converges at $\varepsilon\infty$, thus $\lim(\zeta_i, \sigma_i) = (\alpha, 0)$ from Proposition 4.1. Then ζ_1, ζ_2 can locally be expressed as a function of y , and

$$y \frac{d(\zeta_1 - \zeta_2)^2}{dy} = 2(F(\zeta_1, y) - F(\zeta_2, y)) (\zeta_1 - \zeta_2)$$

near 0, where

$$F(\zeta, y) = \frac{1}{\gamma + \zeta} (-\zeta(\zeta - \eta) + \frac{\varepsilon}{p-1} |\zeta y|^{2-p} (\zeta - \alpha)).$$

Then $(\zeta_1 - \zeta_2)^2$ is nonincreasing, seeing that $\partial F / \partial \zeta(\zeta, y) = -((p-1)\varepsilon(\gamma + \alpha))^{-1} |\alpha y|^{2-p} (1 + o(1))$. Hence $\zeta_1 \equiv \zeta_2$ near 0, and $\mathcal{T}_1 \equiv \mathcal{T}_2$. ■

6 The case $\varepsilon = 1$, $-\gamma \leq \alpha$

In that Section and in Sections 7, 8 and 9 we describe the solutions of (\mathbf{E}_w) . When we give a *uniqueness* result, we mean that w is unique, *up to a scaling*, from Remark 2.1.

Theorem 6.1 *Assume $\varepsilon = 1$, $-\gamma \leq \alpha$ ($\alpha \neq 0$).*

Any solution w of (\mathbf{E}_w) has a finite number of simple zeros, and satisfies (4.4) or (4.5) near ∞ or has a compact support. Either w is regular, or $|w|$ satisfies (4.6),(4.8), (4.7),(4.9) or (4.10) near 0, and there exist solutions of each type.

(1) *Case $\alpha < N$. All regular solutions have a strict constant sign, and satisfy (4.4) or (4.5) near ∞ . Moreover there exist (and exhaustively, up to a symmetry)*

- (i) *a unique nonnegative solution with (4.6) or (4.8) or (4.9) near 0, and compact support;*
- (ii) *positive solutions with the same behaviour at 0 and (4.4) or (4.5) near ∞ ;*
- (iii) *solutions with one simple zero, and $|w|$ has the same behaviour at 0 and ∞ ;*
- (iv) *for $p > N$, a unique positive solution with (4.7) near 0, and (4.4) or (4.5) near ∞ ;*
- (v) *for $p > N$, positive solutions with (4.10) near 0, and (4.4) or (4.5) near ∞ .*

(2) *Case $\alpha = N$. Then the regular (Barenblatt) solutions have a constant sign with compact support. If $p \leq N$, all the other solutions are of type (iii). If $p > N$, there exist also solutions of type (iv) and (v).*

(3) *Case $\alpha > N$.*

Either the regular solutions have m simple zeros and satisfy (4.4) near ∞ . Then there exist

- (vi) *a unique solution with m simple zeros, $|w|$ satisfies (4.6), (4.8) or (4.9) near 0, with compact support;*
- (vii) *solutions with $m + 1$ simple zeros, $|w|$ satisfies (4.6), (4.8) or (4.9) near 0, and (4.4) or (4.5) near ∞ ;*
- (viii) *for $p > N$, solutions with m simple zeros, $|w|$ satisfies (4.9),(4.7) or (4.10) near 0, and (4.4) or (4.5) near ∞ .*

Or the regular solutions have m simple zeros and a compact support. Then the other solutions are of type (vii) or (viii).

th 6.1,fig1: $\varepsilon = 1, N = 2, p = 3, \alpha = -2$ th 6.1,fig2: $\varepsilon = 1, N = 2, p = 3, \alpha = 1$

th 6.1,fig3: $\varepsilon = 1, N = 2, p = 3, \alpha = 2$ th 6.1,fig4: $\varepsilon = 1, N = 2, p = 3, \alpha = 50$

Proof. All the solutions w have a finite number of simple zeros, from Proposition 3.7 and Theorem 3.9. Either they have a compact support. Or y has a strict constant sign and is monotone near ∞ , and converge to $(0, 0)$ at ∞ , and (4.4) or (4.5) holds, from Propositions 3.8, 4.1.

In the phase plane (y, Y) , system **(S)** admits only one stationary point $(0, 0)$. The trajectory \mathcal{T}_r starts in \mathcal{Q}_4 when $\alpha < 0$, in \mathcal{Q}_1 when $\alpha > 0$, and $\lim_{\tau \rightarrow -\infty} y = \infty$, with an asymptotical direction of slope α/N . From Propositions 4.1 and 4.2 all the nonregular solutions $\pm w$ satisfy (4.6), (4.8),

(4.7), (4.9) or (4.10) near $-\infty$. The existence of solutions of any kind is proved at Theorems 5.1 and 5.2. When $p \leq N$, they correspond to trajectories $\pm \mathcal{T}_\eta$ such that \mathcal{T}_η starts in \mathcal{Q}_1 with an infinite slope, in any case above \mathcal{T}_r . When $p > N$, there is a unique trajectory \mathcal{T}_u satisfying (4.7), starting in \mathcal{Q}_4 , under \mathcal{T}_r ; the trajectories \mathcal{T}_+ start from \mathcal{Q}_1 , above \mathcal{T}_r ; the trajectories \mathcal{T}_- start in \mathcal{Q}_4 under \mathcal{T}_r . From Theorem 3.9, there exists a unique trajectory \mathcal{T}_ε converging to $(0,0)$ in \mathcal{Q}_1 at ∞ , with the slope 1.

(1) Case $\alpha < N$. From Proposition 3.6, all the solutions w have at most one simple zero.

The regular solutions stay positive, and \mathcal{T}_r stays in its quadrant, \mathcal{Q}_4 or \mathcal{Q}_1 , from Remark 2.3 (see figures 1 and 2). Then \mathcal{T}_ε stays in \mathcal{Q}_1 , because it cannot meet \mathcal{T}_r for $\alpha > 0$, or the line $\{Y = 0\}$ for $\alpha < 0$, from Remark 2.3; and the corresponding w is of type (i).

Consider any trajectory $\mathcal{T}_{[P]}$ with $P \in \mathcal{Q}_1$ above \mathcal{T}_ε . It cannot stay in \mathcal{Q}_1 because it does not meet \mathcal{T}_ε and converges to $(0,0)$ with a slope 0. Thus it enters \mathcal{Q}_2 from Remark 2.3. Then y has a unique zero, and $\mathcal{T}_{[P]}$ stays in \mathcal{Q}_1 before P , and in $\mathcal{Q}_2 \cup \mathcal{Q}_3$ after P . Since $\mathcal{T}_{[P]}$ cannot meet $\pm \mathcal{T}_\varepsilon$, and $\lim_{r \rightarrow \infty} \zeta = \alpha$, $\mathcal{T}_{[P]}$ ends up in \mathcal{Q}_3 if $\alpha > 0$, in \mathcal{Q}_2 if $\alpha < 0$. It has the same behaviour as \mathcal{T}_ε at $-\infty$, and w is of type (iii).

Next consider $\mathcal{T}_{[P]}$ for any $P \in \mathcal{Q}_1 \cup \mathcal{Q}_4$ between \mathcal{T}_ε and \mathcal{T}_r . Then y stays positive, and $\mathcal{T}_{[P]}$ necessarily starts from \mathcal{Q}_1 , and w is of type (ii).

At least take any $P \in \mathcal{Q}_1 \cup \mathcal{Q}_4$ under \mathcal{T}_r . If $p \leq N$, $\mathcal{T}_{[P]}$ starts from \mathcal{Q}_3 and y has a unique zero, and $-w$ is of type (iii). If $p > N$, either $-w$ is of type (iii), or $\mathcal{T}_{[P]}$ stays in \mathcal{Q}_4 . From Theorems 5.1, 5.2, either $\mathcal{T}_{[P]}$ coincides with \mathcal{T}_u , and w is of type (iv), or with one of the trajectories \mathcal{T}_- , thus w is of type (v).

(2) Case $\alpha = N$. All the solutions are given by (1.9), which is equivalent to $J_N \equiv C$, where J_N is defined by (2.1). For $C = 0$, the regular (Barenblatt) solutions, given by (1.10), are nonnegative, with a compact support. In other words the trajectory \mathcal{T}_ε given by Theorem 5.3 coincides with \mathcal{T}_r , it is given by $y \equiv Y$, $y > 0$ (see figure 3). The only change in the phase plane is the nonexistence of solutions of type (ii).

(3) Case $\alpha > N$.

The regular solutions have a number $m \geq 1$ of simple zeros, from Proposition 3.6 (see figure 4). As above, \mathcal{T}_r starts from \mathcal{Q}_1 with a finite slope α/N .

Either $\mathcal{T}_r \neq \mathcal{T}_\varepsilon$. Then the regular solutions satisfy $\lim_{r \rightarrow \infty} r^\alpha w = L \neq 0$. Since \mathcal{T}_ε cannot meet \mathcal{T}_r , \mathcal{T}_ε also cuts the line $\{y = 0\}$ at m points, and the corresponding w is of type (vi). For any $P \in \mathcal{Q}_1$ above \mathcal{T}_r , the trajectory $\mathcal{T}_{[P]}$ cuts the line $\{y = 0\}$ at $m + 1$ points and w is of type (vii). If $p > N$, there exist trajectories starting from \mathcal{Q}_1 between \mathcal{T}_ε and \mathcal{T}_r , with (4.9), such that w has m simple zeros, and trajectories with (4.7) or (4.10), m zeros, and $\lim_{r \rightarrow \infty} r^\alpha w = L \neq 0$.

Or $\mathcal{T}_r = \mathcal{T}_\varepsilon$, the regular solutions have a compact support, and we only find solutions of type (vii), (viii). ■

Remark 6.2 *In the case $\alpha = \eta < 0$, the solutions (iv) are given by (1.11). In the case $N = 1$, $\alpha = -(p-1)/(p-2)$, the solutions of types (i) and (v) are given by (1.14).*

Remark 6.3 We conjecture that there exists an increasing sequence $(\bar{\alpha}_m)$, with $\bar{\alpha}_0 = N$ such that the regular solutions w have m simple zeros for $\alpha \in (\bar{\alpha}_{m-1}, \bar{\alpha}_m)$, with $\lim_{r \rightarrow \infty} r^\alpha w = L \neq 0$, and m simple zeros and a compact support for $\alpha = \bar{\alpha}_m$, in which case $\mathcal{T}_r = \mathcal{T}_\varepsilon$.

7 The case $\varepsilon = -1, \alpha \leq -\gamma$

Theorem 7.1 Assume $\varepsilon = -1, \alpha \leq -\gamma$. Then all the solutions w of (\mathbf{E}_w) , among them the regular ones, are oscillating near ∞ and $r^{-\gamma}w$ is asymptotically periodic in $\ln r$. There exist

- (i) solutions such that $r^{-\gamma}w$ is periodic in $\ln r$;
- (ii) a unique solution with a hole;
- (iii) flat solutions w with (4.4) or (4.5) near 0;
- (iv) solutions with (4.6) or (4.8) or (4.9) or also (4.10) near 0;
- (v) for $p > N$, a unique solution with (4.7) near 0.

th 7.1,fig5: $\varepsilon = -1, N = 1, p = 3, \alpha = -4$

Proof. Here again, $(0, 0)$ is the unique stationary point in the plane (y, Y) . Any solution y of (\mathbf{E}_y) oscillates near ∞ , and (y, Y) is bounded from Proposition 4.3. From the strong form of the Poincaré-Bendixon theorem, see [7, p.239], all the trajectories have a limit cycle or are periodic. In particular \mathcal{T}_r starts in \mathcal{Q}_1 , since $\varepsilon\alpha > 0$, with the asymptotical direction $\varepsilon\alpha/N$. and it has a limit cycle \mathcal{O} . There exists a periodic trajectory of orbit \mathcal{O} , thus w is of type (i) (see figure 5).

From Theorem 5.2 there exists a unique trajectory \mathcal{T}_ε starting from $(0, 0)$ with the slope $-1, y > 0$; it has a limit cycle $\mathcal{O}_\varepsilon \subset \mathcal{O}$, and w is of type (ii). For any P in the bounded domain delimited by \mathcal{O}_ε , not located on \mathcal{T}_ε , the trajectory $\mathcal{T}_{[P]}$ does not meet \mathcal{T}_ε , and admits \mathcal{O}_ε as limit

cycle; near $-\infty$, y has a constant sign, is monotone and converges to $(0, 0)$ from Propositions 3.8 and 4.1, and $\lim_{\tau \rightarrow -\infty} \zeta = \alpha$. This shows again the existence of such trajectories, proved at Theorem 5.1, and there is an infinity of them; and w is of type (iii).

From Theorems 5.1 and 5.2, there exist trajectories starting from infinity, with \mathcal{O} as limit cycle, and w is of type (iv) or (v). If $\mathcal{O} = \mathcal{O}_\varepsilon$, all the solutions are described. ■

8 Case $\varepsilon = 1, \alpha < -\gamma$.

Theorem 8.1 *Assume $\varepsilon = 1, \alpha < -\gamma$. Then $w \equiv \pm lr^\gamma$ is a solution of (\mathbf{E}_w) . All regular solutions have a strict constant sign, and satisfy (4.3) near ∞ . Moreover there exist (exhaustively, up to a symmetry)*

- (i) a unique positive flat solution with (4.4) near 0 and (4.3) near ∞ ;
- (ii) a unique nonnegative solution with (4.6) or (4.8) or (4.9) near 0, and compact support;
- (iii) positive solutions with the same behaviour near 0 and (4.3) near ∞ ;
- (iv) solutions with one zero and the same behaviour near 0, and $|w|$ satisfies (4.3) near ∞ ;
- (v) for $p > N$, positive solutions with (4.7) near 0 and (4.3) near ∞ ;
- (vi) for $p > N$, positive solutions with (4.10) near 0 and (4.3) near ∞ .

th 8.1, fig6: $\varepsilon = 1, N = 2, p = 3, \alpha = -6$

Proof. Here system (\mathbf{S}) admits three stationary points in the plane (y, Y) , given at (2.8), thus $w \equiv \pm lr^\gamma$ is a solution; and M_ℓ is a sink (see figure 6). Any solution y of (\mathbf{E}_y) has at most one zero, and is strictly monotone near $\pm\infty$, from Propositions 3.6 and 3.8.

From Theorems 3.9 and 5.3, there exists a unique trajectory \mathcal{T}_ε converging to $(0,0)$ in \mathcal{Q}_1 at ∞ , and a unique trajectory \mathcal{T}_α converging to $(0,0)$ in \mathcal{Q}_4 at $-\infty$. The trajectory \mathcal{T}_r starts in \mathcal{Q}_4 with the asymptotical direction $-|\alpha|/N$. From Remark 2.3, \mathcal{Q}_4 is positively invariant, and \mathcal{Q}_1 negatively invariant. Then \mathcal{T}_ε stays in \mathcal{Q}_1 , and \mathcal{T}_α and \mathcal{T}_r in \mathcal{Q}_4 . From Proposition 4.1, all the trajectories, apart from $\pm\mathcal{T}_\varepsilon$, converge to $\pm M_\ell$ at ∞ . Then \mathcal{T}_r converges to M_ℓ , and w satisfies (4.3) near ∞ . And \mathcal{T}_α also converges to M_ℓ , and w is of type (i).

From Propositions 4.1, Theorems 5.1 and 5.2, all the nonregular solutions which are positive near $-\infty$ satisfy (4.6), (4.8), (4.9), (4.10) or (4.7), and there exist such solutions. For $p < N$ (resp. $p = N$), they correspond to trajectories \mathcal{T}_η (resp. \mathcal{T}_+) starting in \mathcal{Q}_1 . For $p > N$, there is a unique trajectory \mathcal{T}_u satisfying (4.7), starting in \mathcal{Q}_4 under \mathcal{T}_r ; and the trajectories \mathcal{T}_+ satisfying (4.9) start from \mathcal{Q}_1 ; the trajectories \mathcal{T}_- satisfying (4.10) and the unique trajectory \mathcal{T}_u satisfying (4.7) start from \mathcal{Q}_4 , under \mathcal{T}_r . Since \mathcal{T}_ε stays in \mathcal{Q}_1 , it defines solutions w of type (ii).

Consider the basis of eigenvectors (e_1, e_2) defined at (2.15), where $\nu(\alpha) > 0$, associated to the eigenvalues $\lambda_1 < \lambda_2$. One verifies that $\lambda_1 < -\gamma < \lambda_2$; thus e_1 points towards \mathcal{Q}_3 and e_2 points towards \mathcal{Q}_4 . There exist unique trajectories \mathcal{T}_{e_1} and \mathcal{T}_{-e_1} converging to M_ℓ , tangentially to e_1 and $-e_1$. All the other trajectories converging to M_ℓ at ∞ are tangent to $\pm e_2$. Let

$$\mathcal{M} = \left\{ |Y|^{(2-p)/(p-1)} Y = -\gamma y \right\}, \quad \mathcal{N} = \left\{ (N + \gamma)Y + \varepsilon |Y|^{(2-p)/(p-1)} Y = \varepsilon \alpha y \right\}$$

be the sets of extremal points of y and Y .

The trajectory \mathcal{T}_r starts above the curves \mathcal{M} and \mathcal{N} , thus $y' < 0$ and $Y' > 0$ near $-\infty$. And \mathcal{T}_r converges to M_ℓ at ∞ , tangentially to e_2 . Indeed if $\mathcal{T}_r = \mathcal{T}_{e_1}$, then y has a minimal point such that $y < \ell$ and $Y < -(\gamma\ell)^{p-1}$, then (y, Y) cannot be on \mathcal{M} . If $\mathcal{T}_r = \mathcal{T}_{-e_1}$, then Y has a maximal point such that $y > \ell$ and $Y < -(\gamma\ell)^{p-1}$, then also (y, Y) cannot be on \mathcal{N} . Finally \mathcal{T}_r cannot end up tangentially to $-e_2$, it would intersect \mathcal{T}_{e_1} or \mathcal{T}_{-e_1} .

The trajectory \mathcal{T}_α converge to M_ℓ tangentially to $-e_2$. Indeed if $\mathcal{T}_\alpha = \mathcal{T}_{e_1}$, then Y has a maximal point such that $y < \ell$ and $Y < -(\gamma\ell)^{p-1}$; if $\mathcal{T}_\alpha = \mathcal{T}_{-e_1}$, then y has a maximal point such that $y > \ell$ and $Y > -(\gamma\ell)^{p-1}$. In any case we reach a contradiction. Moreover \mathcal{T}_{e_1} does not stay in \mathcal{Q}_4 : y would have a minimal point such that $y < \ell$ and $Y < -(\gamma\ell)^{p-1}$, which is impossible; thus \mathcal{T}_{e_1} starts in \mathcal{Q}_3 , and enters \mathcal{Q}_4 at some point $(\xi_1, 0)$ with $\xi_1 < 0$. And $-w$ is of type (iv).

Any trajectory $\mathcal{T}_{[P]}$, with P in the domain of $\mathcal{Q}_1 \cup \mathcal{Q}_4$ delimited by $\mathcal{T}_r, \mathcal{T}_\alpha$ and \mathcal{T}_ε , comes from \mathcal{Q}_1 , and converges to M_ℓ in \mathcal{Q}_4 , in particular \mathcal{T}_{-e_1} ; the corresponding w are of type (iii).

Any trajectory $\mathcal{T}_{[P]}$, with P in the domain of $\mathcal{Q}_3 \cup \mathcal{Q}_4$ delimited by $\mathcal{T}_{e_1}, \mathcal{T}_\alpha$ and $-\mathcal{T}_\varepsilon$, goes from \mathcal{Q}_3 to \mathcal{Q}_4 , and $\mathcal{T}_{[P]}$ converges to M_ℓ at ∞ , and $-w$ is of type (iv). For any $\xi < \xi_1$, the trajectory $\mathcal{T}_{[(0,\xi)]}$ is of the same type. If $p \leq N$, any trajectory in the domain under \mathcal{T}_r , and \mathcal{T}_{e_1} is of the same type.

If $p > N$, moreover in this domain there exists a the unique trajectory \mathcal{T}_u and trajectories of the type \mathcal{T}_- corresponding to solutions w of type (v) and (vi), from Theorems 5.1 and 5.2. Up to a symmetry, all the solutions are described, and all of them do exist. ■

9 Case $\varepsilon = -1, -\gamma < \alpha$

Here again System (S) admits the three stationary points (2.8), thus $w \equiv \pm lr^\gamma$ is a solution of (\mathbf{E}_w) . The behaviour is very rich: it depends on the position of α with respect to α^* defined at (1.5), and 0, $-p'$, and η (in case $p > N$), and also α_1, α_2 defined at (2.14). We start from some general remarks.

Remark 9.1 (i) *There exists a unique trajectory \mathcal{T}_ε starting from $(0,0)$ in \mathcal{Q}_4 with the slope -1 , from Theorem 3.9.*

(ii) *There exists a unique trajectory \mathcal{T}_α converging to $(0,0)$ at ∞ , in \mathcal{Q}_1 if $\alpha > 0$, in \mathcal{Q}_4 if $\alpha < 0$, with a slope 0 at $(0,0)$, and $\lim_{\tau \rightarrow \infty} \zeta = \alpha$, from Theorem 5.3.*

(iii) *From Remark 2.3, if $\alpha > 0$, \mathcal{Q}_4 is positively invariant and \mathcal{Q}_1 negatively invariant. If $\alpha < 0$, at any point $(0, \xi), \xi < 0$, the vector field points to \mathcal{Q}_4 , and at any point $(\varphi, 0), \varphi > 0$, it points to \mathcal{Q}_1 . Thus if \mathcal{T}_ε does not stay in \mathcal{Q}_1 , then \mathcal{T}_α stays in the bounded domain delimited by $\mathcal{Q}_4 \cap \mathcal{T}_\varepsilon$. If \mathcal{T}_α does not stay in \mathcal{Q}_4 , then \mathcal{T}_ε stays in the bounded domain delimited by $\mathcal{Q}_4 \cap \mathcal{T}_\alpha$. If \mathcal{T}_ε is homoclinic, in other words $\mathcal{T}_\varepsilon = \mathcal{T}_\alpha$, it stays in \mathcal{Q}_4 .*

Remark 9.2 *From Propositions 4.1, Theorems 5.1 and 5.2, all the nonregular solutions positive near $-\infty$ satisfy (4.6) for $p < N$, (4.8) for $p = N$, corresponding to trajectories $\mathcal{T}_\eta, \mathcal{T}_+$ starting from \mathcal{Q}_1 ; and (4.9), (4.10) or (4.7) for $p > N$, corresponding to trajectories \mathcal{T}_+ starting from \mathcal{Q}_1 , and $\mathcal{T}_-, \mathcal{T}_u$ starting from \mathcal{Q}_4 .*

Remark 9.3 *Any trajectory \mathcal{T} is bounded near ∞ from Proposition 4.3. From the strong form of the Poincaré-Bendixon theorem, any trajectory \mathcal{T} bounded at $\pm\infty$ converges to $(0,0)$ or $\pm M_\ell$, or its limit set Γ_\pm at $\pm\infty$ is a cycle, or it is homoclinic, namely $\mathcal{T}_\varepsilon = \mathcal{T}_\alpha$. If there exists a limit cycle surrounding $(0,0)$, it also surrounds the points $\pm M_\ell$, from Proposition 3.8.*

The simplest case is $\alpha > 0$.

Theorem 9.4 *Assume $\varepsilon = -1, \alpha > 0$.*

Then $w \equiv lr^\gamma$ is a solution w of (\mathbf{E}_w) . All regular solutions have a strict constant sign; and satisfy (4.3) near ∞ . There exist (exhaustively, up to a symmetry)

- (i) *a unique nonnegative solution with a hole, and (4.3) near ∞ ;*
- (ii) *a unique positive solution with (4.6), or (4.8) or (4.9), and (4.4) near ∞ ;*
- (iii) *positive solutions with the same behaviour near 0, and (4.3) near ∞ ;*

- (iv) solutions with one zero, the same behaviour near 0, and $|w|$ satisfies (4.3) near ∞ ;
- (v) for $p > N$, a unique positive solution with (4.7) near 0, and (4.3) near ∞ ;
- (vi) for $p > N$, positive solutions with (4.10) near 0, and (4.3) near ∞ .

th 9.4, fig7: $\varepsilon = -1, N = 1, p = 3, \alpha = 0.7$ th 9.4, fig8: $\varepsilon = -1, N = 1, p = 3, \alpha = 1$

Proof. Any solution y of (\mathbf{E}_y) has at most one zero, and y is strictly monotone near ∞ , from Propositions 3.6 and 4.4. The point M_ℓ is a sink and a node point, since $\alpha > 0 \geq \alpha_2$ (see figure 7). Consider the basis eigenvectors (e_1, e_2) , defined at (2.15), where $\nu(\alpha) < 0$, associated to the eigenvalues $\lambda_1 < \lambda_2 < 0$. One verifies that $\lambda_1 < -\gamma < \lambda_2$, thus e_1 points towards \mathcal{Q}_3 and e_2 points towards \mathcal{Q}_4 . There exist unique trajectories \mathcal{T}_{e_1} and \mathcal{T}_{-e_1} tangent to e_1 and $-e_1$ at ∞ . All the other trajectories which converge to M_ℓ end up tangentially to $\pm e_1$.

The trajectory \mathcal{T}_α stays in \mathcal{Q}_1 from Remark 9.1; near $-\infty$ it is of type \mathcal{T}_η for $p < N$, and \mathcal{T}_+ for $p \geq N$; it defines the solution of type (ii). Since \mathcal{T}_α is the unique trajectory converging to $(0, 0)$ at ∞ , all the trajectories, apart from $\pm \mathcal{T}_\alpha$, converge to $\pm M_\ell$ at ∞ , from Propositions 3.8 and 4.1.

The trajectories \mathcal{T}_r and \mathcal{T}_ε start in \mathcal{Q}_4 , and stay in it from Remark 9.1, and both converge to M_ℓ at ∞ , then w satisfies (4.3); and \mathcal{T}_r starts with the asymptotical direction $-\alpha/N$. And \mathcal{T}_ε defines the solution of type (i).

As in the proof of Theorem 8.1, \mathcal{T}_r ends up tangentially to e_2 , and \mathcal{T}_ε tangentially to $-e_2$. Moreover \mathcal{T}_{e_1} does not stay in \mathcal{Q}_4 , it starts in \mathcal{Q}_3 , and converges to M_ℓ in \mathcal{Q}_4 , and $-w$ is of type (iv). Any trajectory $\mathcal{T}_{[P]}$, with P in the domain of \mathcal{Q}_4 between \mathcal{T}_{e_1} , \mathcal{T}_ε , starts from \mathcal{Q}_3 , enters \mathcal{Q}_4 at some point $(0, \xi)$, $\xi > \xi_1$, and has the same type as \mathcal{T}_{e_1} . Any trajectory $\mathcal{T}_{[(0, \xi)]}$ with $\xi < \xi_1$ is of the same type.

Any trajectory $\mathcal{T}_{[P]}$, with P in the domain of $\mathcal{Q}_1 \cup \mathcal{Q}_4$ above $\mathcal{T}_r \cup \mathcal{T}_\varepsilon$, starts from \mathcal{Q}_1 , and converges to M_ℓ in \mathcal{Q}_4 , in particular $\mathcal{T}_{-\varepsilon_1}$; the corresponding w are of type (iii). If $p \leq N$, all the solutions are described. If $p > N$, moreover there exist trajectories staying in \mathcal{Q}_4 : \mathcal{T}_u and the \mathcal{T}_- , starting under \mathcal{T}_r , corresponding to types (v) and (vi). ■

Remark 9.5 For $\alpha = N$, \mathcal{T}_r and \mathcal{T}_ε are given by (1.10), respectively with $K > 0$ and $K < 0$. The trajectory \mathcal{T}_ε describes the portion $0M_\ell$ of the line $\{Y = -y\}$, and \mathcal{T}_r the complementary half-line in \mathcal{Q}_4 (see figure 8).

Next we assume $-p' \leq \alpha < 0$. The case $p > N$ is delicate: indeed the special value $\alpha = \eta$ is involved, because $\eta < 0$.

Theorem 9.6 Assume $\varepsilon = -1$, $p \leq N$, and $-p' \leq \alpha < 0$. Then $w \equiv \ell r^\gamma$ is a solution w of (\mathbf{E}_w) .

There exist a unique nonnegative solution with a hole, satisfying (4.3) at ∞ .

(1) If $\alpha \neq -p'$, all regular solutions have one zero, and $|w|$ satisfies (4.3) near ∞ . There exist (exhaustively, up to a symmetry)

- for $p \leq N$,
 - (i) a unique solution with one zero, with (4.6) or (4.8) near 0, and (4.4) near ∞ ;
 - (ii) solutions with one zero, with (4.6) or (4.8) near 0, and $|w|$ satisfies (4.3) near ∞ ;
 - (iii) solutions with two zeros, with (4.6) or (4.8) near 0, and (4.3) near ∞ ;
- for $p > N$, $\eta < \alpha$,
 - (iv) a unique positive solution, with (4.10) near 0, and (4.4) near ∞ ;
 - (v) a unique positive solution, with (4.7) near 0, and (4.3) near ∞ ;
 - (vi) positive solutions, with (4.10) near 0, and (4.3) near ∞ ;
 - (vii) solutions with one zero with (4.10) or (4.9) near 0, and (4.3) near ∞ ;
- for $p > N$, $\alpha < \eta$,
 - (viii) a unique solution with one zero, with (4.9) near 0, and (4.4) near ∞ ;
 - (ix) a unique solution with one zero, with (4.7) near 0, and $|w|$ satisfies (4.3) near ∞ ;
 - (x) solutions with one zero, with (4.9) or (4.9) near 0, and $|w|$ satisfies (4.3) near ∞ ;
 - (xi) solutions with two zeros, with (4.9) near 0, and (4.3) near ∞ .
- for $p > N$, $\alpha = \eta$, solutions of the form $w = cr^{|\eta|}$ ($c > 0$). The other solutions are of type (vii).
 - (2) If $\alpha = -p'$, all regular solutions have one zero and satisfy (4.4) near ∞ . The solutions without hole are of types (ii), (iii) for $p \leq N$, (ix), (x), (xi) for $p > N$.

th 9.6, fig9: $\varepsilon = -1, N = 1, p = 3, \alpha = -0.7$ th 9.6, fig10: $\varepsilon = -1, N = 1, p = 3, \alpha = -1.49$

th 9.6, fig11: $\varepsilon = -1, N = 1, p = 3, \alpha = -3/2$

Proof. Here again M_ℓ is a sink; but it is a node point only if $\alpha \geq \alpha_2$. The phase plane (y, Y) does not contain any cycle, from Proposition 4.4. From Proposition 3.6, any solution y has at most two zeros, and Y at most one.

The unique trajectory \mathcal{T}_α ends up in \mathcal{Q}_4 with the slope 0. From the uniqueness of \mathcal{T}_α and \mathcal{T}_ε , all the trajectories, apart from $\pm\mathcal{T}_\alpha$, converge to $\pm M_\ell$ at ∞ , from Proposition 4.1 and Remark 9.3. Since $\varepsilon\alpha > 0$, the trajectory \mathcal{T}_r starts in \mathcal{Q}_1 , and y has at most one zero. Then \mathcal{T}_r converges to $-M_\ell$ in \mathcal{Q}_2 , or $\mathcal{T}_r = -\mathcal{T}_\alpha$.

The trajectory \mathcal{T}_ε starts in \mathcal{Q}_4 with the slope -1 , satisfies $y \geq 0$ from Proposition 3.6. If \mathcal{T}_ε converge to $(0, 0)$, then $\mathcal{T}_\varepsilon = \mathcal{T}_\alpha$, thus it is homoclinic. Then M_ℓ is in the bounded component defined by \mathcal{T}_ε , and \mathcal{T}_ε meets \mathcal{T}_r , which is impossible. Hence \mathcal{T}_ε converges to M_ℓ in \mathcal{Q}_4 , and w is nonnegative with a hole and satisfies (4.3) near ∞ .

If $\alpha \neq -p'$, we claim that $\mathcal{T}_r \neq -\mathcal{T}_\alpha$. Indeed suppose $\mathcal{T}_r = -\mathcal{T}_\alpha$. Consider the functions y_α, Y_α , defined by (2.3) with $d = \alpha$. Then Y_α stays positive, and $Y_\alpha = O(e^{(\alpha(p-1)+p)\tau})$ at ∞ , thus

$$\lim_{\tau \rightarrow \infty} Y_\alpha = 0, \quad \lim_{\tau \rightarrow \infty} Y'_\alpha = c > 0, \quad \lim_{\tau \rightarrow -\infty} y_\alpha = \infty, \quad \lim_{\tau \rightarrow \infty} y_\alpha = L < 0.$$

Moreover y_α, Y_α have no extremal point: at such a point, from (3.2), (3.3) the second derivatives have a strict constant sign; then $Y'_\alpha > 0 > y'_\alpha$. If $\alpha < \eta$ (in particular if $p \leq N$), from (4.13), near ∞ ,

$$(p-1)Y''_\alpha/Y'_\alpha \geq |Y|^{(2-p)/(p-1)}(1+o(1)),$$

thus $Y''_\alpha > 0$ near ∞ , which is contradictory; if $\alpha > \eta$, from (4.12)

$$(p-1)y''_\alpha/y'_\alpha \geq |Y|^{(2-p)/(p-1)}(1+o(1)),$$

thus $y''_\alpha < 0$ near ∞ , still contradictory. If $\alpha = \eta$, $\mathcal{T}_\alpha = \mathcal{T}_u$ from (1.11), thus again $\mathcal{T}_r \neq -\mathcal{T}_\alpha$.

If $p > N$ and $\alpha \neq \eta$, we claim that $\mathcal{T}_\alpha \neq \mathcal{T}_u$. Indeed suppose $\mathcal{T}_\alpha = \mathcal{T}_u$. This trajectory stays \mathcal{Q}_4 , the function ζ stays negative, and $\lim_{\tau \rightarrow -\infty} \zeta = \eta$, $\lim_{\tau \rightarrow \infty} \zeta = \alpha$. If ζ has an extremal point ϑ , then $\vartheta \in (\alpha, \eta)$ from System **(Q)**, and ζ'' has a constant sign, the sign of $\alpha - \zeta$; it is impossible. Thus ζ is monotone; then $(\alpha - \eta)\zeta' > 0$, which contradicts System **(Q)**.

(1) Case $\alpha \neq -p'$. Since $\mathcal{T}_r \neq -\mathcal{T}_\alpha$, \mathcal{T}_r converges to $-M_\ell$, and y has one zero, and $|w|$ satisfies (4.3).

- Case $p \leq N$. All the other trajectories start in \mathcal{Q}_3 or \mathcal{Q}_1 , from Remarks 9.1 and 9.2. For any $\varphi > 0$, the trajectory $\mathcal{T}_{[(\varphi, 0)]}$ goes from \mathcal{Q}_4 into \mathcal{Q}_1 , and converges to $-M_\ell$ in \mathcal{Q}_2 , since it cannot meet \mathcal{T}_r and $-\mathcal{T}_\varepsilon$; thus y has two zeros, and w is of type (iii). The trajectory \mathcal{T}_α cannot meet $\mathcal{T}_{[(\varphi, 0)]}$, thus y has one zero, and it has the same behaviour at $-\infty$, and w is of type (i). All the trajectories $\mathcal{T}_{[P]}$ with P in the interior domain of \mathcal{Q}_1 delimited by $-\mathcal{T}_\varepsilon$ and \mathcal{T}_r start from \mathcal{Q}_1 and converge to $-M_\ell$, y has precisely one zero, and has the same behaviour at $-\infty$, and w is of type (ii).

- Case $p > N$, $\eta < \alpha$ (see figure 9). Any solution y has at most one simple zero. The trajectory \mathcal{T}_α stays in \mathcal{Q}_4 . Indeed if it started in \mathcal{Q}_3 , then for any trajectory $\mathcal{T}_{[(0, \xi)]}$ with $(0, \xi)$ above $-\mathcal{T}_\alpha$, the function y would have two zeros. Since $\mathcal{T}_\alpha \neq \mathcal{T}_u$, we have $\mathcal{T}_\alpha \in \mathcal{T}_-$, and w is of type (iv). The trajectory \mathcal{T}_u necessarily stays in \mathcal{Q}_4 and converges to M_ℓ , and w is of type (v). The trajectories $\mathcal{T}_{[P]}$, with P in the domain delimited by $\mathcal{T}_u, \mathcal{T}_\alpha$ and \mathcal{T}_ε , are of type \mathcal{T}_- and converge in \mathcal{Q}_4 to M_ℓ , and w is of type (vi). The trajectories $\mathcal{T}_{[P]}$, with P in the domain delimited by $\mathcal{T}_r, \mathcal{T}_\alpha$ and $-\mathcal{T}_\varepsilon$, are of type \mathcal{T}_- , and converge to $-M_\ell$, and y has one zero. The trajectories $\mathcal{T}_{[P]}$, with P in

the domain delimited by \mathcal{T}_r and $-\mathcal{T}_u$, are of type \mathcal{T}_+ , converge to $-M_\ell$, and y has one zero. Both define solutions w of type (vii).

- Case $p > N, \alpha < \eta$ (see figure 10). We have seen that $\mathcal{T}_r \neq -\mathcal{T}_\alpha$. If $\mathcal{T}_\alpha \in \mathcal{T}_+$, then ζ decreases from 0 to α , which contradicts System **(Q)** at ∞ . Then \mathcal{T}_α does not stay in \mathcal{Q}_4 , it starts in \mathcal{Q}_3 and $-\mathcal{T}_\alpha \in \mathcal{T}_-$, hence y has a zero, and w is of type (viii). Then \mathcal{T}_u and the trajectories \mathcal{T}_- converge to $-M_\ell$, and y has one zero. The trajectories $\mathcal{T}_{[P]}$, with P in the domain delimited by $\mathcal{T}_r, -\mathcal{T}_\alpha$ and $-\mathcal{T}_\varepsilon$, are of type \mathcal{T}_+ and converge to $-M_\ell$, y has one zero. They correspond to w is of type (ix) or (x). The trajectories $\mathcal{T}_{[P]}$, with P in \mathcal{Q}_4 above \mathcal{T}_r , cut the line $\{y = 0\}$ twice, and converge to M_ℓ , and w is of type (xi).

- Case $p > N, \alpha = \eta$. Then $\mathcal{T}_\alpha = \mathcal{T}_u$, the functions $w = cr^{-\eta}$ ($c > 0$) are particular solutions. The phase plane study is the same, and gives only solutions of type (vii).

(2) Case $\alpha = -p'$ (see figure 11). Here $\mathcal{T}_r = -\mathcal{T}_\alpha$, since the regular solutions are given by (1.12). Thus there exist no more solutions of type (ii) or (viii). \blacksquare

Next we study the behaviour of all the solutions when $\alpha < -p'$. In particular we prove the existence and uniqueness of an α_c for which there exists an homoclinic trajectory. Thus we find again some results obtained in [8], with new detailed proofs. We also improve the bounds for α_c , in particular $\alpha^* < \alpha_c$.

Lemma 9.7 *Let*

$$\alpha_p := -(p-1)/(p-2).$$

If $N = 1$, for $\alpha = \alpha_p$, then there exists an homoclinic trajectory in the phase plane (y, Y) . If $N \geq 2$, for $\alpha = \alpha_p$, there is no homoclinic trajectory, moreover \mathcal{T}_α converges to M_ℓ at $-\infty$ or has a limit cycle in \mathcal{Q}_4 .

Proof. In the case $N = 1, \alpha = \alpha_p$, the explicit solutions (1.14) define an homoclinic trajectory in the phase plane (y, Y) , namely $\mathcal{T}_\varepsilon = \mathcal{T}_\alpha$. In the phase plane (g, s) of System **(R)**, from Remark 2.6, they correspond to the line $s \equiv 1 + \alpha g$, joining the stationary points $(0, 1)$ and $(-1/\alpha, 0)$.

Next assume $N \geq 2$ and consider the trajectory \mathcal{T}_α in the plane (y, Y) . In the plane (g, s) of System **(R)**, the corresponding trajectory \mathcal{T}'_α ends up at $(-1/\alpha, 0)$, as ν tends to ∞ from (2.18), with the slope $-k_p$. If \mathcal{T}_α is homoclinic, then \mathcal{T}'_α converges to $(0, 1)$ as ν tends to $-\infty$. Consider the segment

$$T = \{(g, -k(g + 1/\alpha_p)) : g \in [0, 1/|\alpha_p|]\}, \quad \text{with} \quad k = p'\alpha_p^2/(N + 2/(p-2)) > k_p.$$

Its extremity $(0, k/|\alpha_p|)$ is strictly under $(0, 1)$. The domain \mathcal{R} delimited by the axes, which are particular orbits, and T , is negatively invariant: indeed, at any point of T , we find

$$k \frac{dg}{d\nu} + \frac{ds}{d\nu} = (N-1)p'ks(g - \frac{1}{\gamma})^2.$$

The trajectory \mathcal{T}'_α ends up in \mathcal{R} , thus it stays in it, hence \mathcal{T}'_α cannot join $(0, 1)$. In the phase plane (y, Y) , \mathcal{T}_α is not homoclinic, and \mathcal{T}_α stays in \mathcal{Q}_4 , and Remark 9.3 applies. \blacksquare

Remark 9.8 Notice that $\alpha^* \leq \alpha_p \Leftrightarrow N \leq p$.

Theorem 9.9 Assume $\varepsilon = -1$, and $\alpha < -p'$. There exists a unique $\alpha_c < 0$ such that there exists an homoclinic trajectory in the plane (y, Y) ; in other words $\mathcal{T}_\varepsilon = \mathcal{T}_\alpha$. If $N = 1$, then $\alpha_c = \alpha_p$. If $N \geq 2$, then

$$\max(\alpha^*, \alpha_p) < \alpha_c < \min(\alpha_2, -p'). \quad (9.1)$$

Proof. In order to prove the existence of an homoclinic orbit for System **(S)**, we could consider a Poincaré application as in [4], but it does not give uniqueness. Thus we consider the system **(R $_\beta$)** obtained from **(R)** by setting $s = \beta S$:

$$\left. \begin{aligned} \frac{dg}{d\nu} &= gF(g, S), & F(g, S) &:= \beta S(1 + \eta g) - \frac{1}{p-1}(1 + \alpha g), \\ \frac{dS}{d\nu} &= SG(g, S), & G(g, S) &:= 1 + \alpha g - \beta(1 + Ng)S. \end{aligned} \right\} \quad (\mathbf{R}_\beta)$$

Its stationary points are

$$(0, 0), \quad A' = (1/|\alpha|, 0), \quad B' = (0, 1/\beta), \quad M' = (1/\gamma, 1/(N + \gamma)(p - 2)),$$

where M' corresponds to M_ℓ . The existence of homoclinic trajectory for System **(S)** resumes to the existence of a trajectory for System **(R $_\beta$)** in the plane (g, S) , starting from B' and ending at A' .

(i) *Existence.* We can assume that $\alpha \in (\alpha_1, \min(\alpha_2, -p'))$, from Proposition 4.4. In the plane (g, S) , consider the trajectories \mathcal{T}'_ε and \mathcal{T}'_α corresponding to $\mathcal{T}_\varepsilon \cap \mathcal{Q}_4$ and $\mathcal{T}_\alpha \cap \mathcal{Q}_4$ in the plane (y, Y) . Then \mathcal{T}'_ε starts from B' and \mathcal{T}'_α ends up at A' . From Remark 9.1, for any $\alpha \in (\alpha_1, \alpha_2)$, with $\alpha \leq -p'$, we have three possibilities:

- \mathcal{T}'_ε is converging to M' as ν tends to ∞ and turns around this point, since α is a spiral point, or it has a limit cycle in \mathcal{Q}_1 around M' . And \mathcal{T}'_α admits the line $g = 0$ as an asymptote as ν tends to $-\infty$, which means that \mathcal{T}_α does not stay in \mathcal{Q}_4 in the plane (y, Y) . Then \mathcal{T}'_ε meets the line

$$L := \{g = 1/\gamma\}$$

at a first point $(1/\gamma, S_0(\alpha))$. And \mathcal{T}'_α meets L at a last point $(1/\gamma, S_1(\alpha))$, such that $S_0(\alpha) - S_1(\alpha) < 0$;

- \mathcal{T}'_α is converging to M' at $-\infty$ or it has a limit cycle in \mathcal{Q}_1 around M' . And \mathcal{T}'_ε admits the line $S = 0$ as an asymptote at ∞ , which means that \mathcal{T}_ε does not stay in \mathcal{Q}_4 . Then with the same notations, $S_0(\alpha) - S_1(\alpha) > 0$.

- $\mathcal{T}'_\varepsilon = \mathcal{T}'_\alpha$, equivalently $S_0(\alpha) - S_1(\alpha) = 0$.

The function $\alpha \mapsto \varphi(\alpha) = S_0(\alpha) - S_1(\alpha)$ is continuous, from Theorems 3.9 and 5.3. If $-p' < \alpha_2$, then $\varphi(-p')$ is well defined and $\varphi(-p') < 0$; indeed $\mathcal{T}_\alpha = -\mathcal{T}_r$, thus \mathcal{T}_α does not stay in \mathcal{Q}_4 from Theorem 9.6. If $\alpha_2 \leq -p'$, in the plane (y, Y) , the trajectory \mathcal{T}_{α_2} leaves \mathcal{Q}_4 , from Proposition 4.4, because α_2 is a sink, and transversally from Remark 9.1. The same happens for \mathcal{T}_{α_2-v} for $v > 0$ small enough, by continuity, thus $\varphi(\alpha_2 - v) < 0$. From Lemma 9.7, $\varphi(\alpha_p) > 0$ if $N \geq 2$, and $\varphi(\alpha_p) = 0$ if $N = 1$. In any case there exists at least an α_c satisfying (9.1), such that $\varphi(\alpha_c) = 0$.

(ii) *Uniqueness.* First observe that $1 + \eta g > 0$; indeed $1 + \eta/|\alpha| > (p' + \eta)/|\alpha| > 0$. Now

$$(p-1)F + G = p\beta S(1/\gamma - g) = (p-2)\beta S(1 - \gamma g),$$

hence the curves $\{F = 0\}$ and $\{G = 0\}$ intersect at M' and A' , $\{G = 0\}$ contains B' and is above $\{F = 0\}$ for $g \in (0, 1/\gamma)$ and under it for $g \in (1/\gamma, 1/|\alpha|)$. Moreover \mathcal{T}'_ε has a negative slope at B' , thus $F > 0 > G$ near 0 from (\mathbf{R}_β) . And \mathcal{T}'_ε cannot meet $\{G = 0\}$ for $(0, 1/\gamma)$, because on this curve the vector field is $(gF, 0)$ and $F > 0$. Thus \mathcal{T}'_ε satisfies $F > 0 > G$ on $(0, 1/\gamma)$. In the same way \mathcal{T}'_α has a negative slope $-\theta\alpha^2/(p-1)(\eta + |\alpha|) < 0$ at $1/|\alpha|$, thus $F > 0 > G$ near $1/|\alpha|$. And \mathcal{T}'_α cannot meet $\{F = 0\}$, because the vector field on this curve is $(0, SG)$ and $G < 0$. Thus \mathcal{T}'_α satisfies $F > 0 > G$ on $(1/\gamma, 1/|\alpha|)$.

Let $\alpha < \bar{\alpha}$. Then \mathcal{T}'_ε is above $\bar{\mathcal{T}}'_\varepsilon$ near $g = 0$, and \mathcal{T}'_α is at the left of $\bar{\mathcal{T}}'_\alpha$ near $S = 0$. We show that $\varphi(\alpha) > \varphi(\bar{\alpha})$. First suppose that \mathcal{T}'_ε and $\bar{\mathcal{T}}'_\varepsilon$ (or \mathcal{T}'_α and $\bar{\mathcal{T}}'_\alpha$) intersect at a first point P_1 (or a last point) such $g \neq 1/\gamma$. Then at this point

$$\frac{1}{p-1} \frac{g}{S} \frac{dS}{dg} + 1 = \frac{(p-2)(1-\gamma g)S}{(p-1)S(1+\eta g) - \beta^{-1}(1+\alpha g)} = \frac{(p-2)(1-\gamma g)S}{h_S(g) - \beta^{-1}(1-\gamma g)} \quad (9.2)$$

with $h_S(g) = (p-1)S(1+\eta g) - g/(p-2)$. Thus the denominator, which is positive, is increasing in α on $(0, 1/\gamma)$, decreasing on $(1/\gamma, 1/|\alpha|)$; in any case $dS/dg > d\bar{S}/dg$ at P_1 , which is contradictory. Next suppose that there is an intersection on L . At such a point $P_1 = (1/\gamma, S_1) = (1/\gamma, \bar{S}_1)$ the derivatives are equal from (9.2), and P_1 is above M' , because $F > 0$. At any points $(g, S(g)) \in \mathcal{T}'_\varepsilon$ (or \mathcal{T}'_α), $(g, \bar{S}(g)) \in \bar{\mathcal{T}}'_\varepsilon$ (or $\bar{\mathcal{T}}'_\alpha$), setting $g = 1/\gamma + u$,

$$\Phi(u) = \left(\frac{1}{p-1} \frac{g}{S} \frac{dS}{dg} + 1 \right) \frac{1}{(p-2)S} = -\frac{\gamma}{h_S(1/\gamma)} u + \frac{1}{h_S^2(1/\gamma)} \left(\frac{\gamma}{\beta} + h'_S(1/\gamma) \right) u^2 (1 + o(1)),$$

$$\bar{\Phi}(u) = \left(\frac{1}{p-1} \frac{g}{\bar{S}} \frac{d\bar{S}}{dg} + 1 \right) \frac{1}{(p-2)\bar{S}} = -\frac{\gamma}{h_{\bar{S}}(1/\gamma)} u + \frac{1}{h_{\bar{S}}^2(1/\gamma)} \left(\frac{\gamma}{\beta} + h'_{\bar{S}}(1/\gamma) \right) u^2 (1 + o(1)),$$

And $h_S(1/\gamma) = h_{\bar{S}}(1/\gamma) > 0$, and $h'_S(1/\gamma) = h'_{\bar{S}}(1/\gamma)$, then

$$(\Phi - \bar{\Phi})(u) = \frac{\gamma u^2 (1/\beta - 1/\bar{\beta})}{h(1/\gamma)} (1 + o(1)).$$

This implies $d^2(S - \bar{S})/dg^2 = 0$ and $d^3(S - \bar{S})/dg^3 = 2S_1\gamma^2(p-1)(p-2)(1/\beta - 1/\bar{\beta}) > 0$, which is a contradiction. Then \mathcal{T}'_ε and $\bar{\mathcal{T}}'_\varepsilon$ cannot intersect on this line, similarly for \mathcal{T}'_α and $\bar{\mathcal{T}}'_\alpha$. Hence $\varphi(\alpha) > \varphi(\bar{\alpha})$, which proves the uniqueness.

As a consequence, for $\alpha < \alpha_c$, $\varphi(\alpha) > 0$, in the plane (y, Y) , \mathcal{T}_ε does not stay in \mathcal{Q}_4 ; for $\alpha > \alpha_c$, $\varphi(\alpha) < 0$, \mathcal{T}_α does not stay in \mathcal{Q}_4 . From Lemma 9.7, it follows that $\alpha_p < \alpha_c$ if $N \geq 2$. Moreover $\alpha^* < \alpha_c$. Indeed α^* is a weak source from Proposition 2.5, thus for $\alpha > \alpha^*$ small enough, there exists a unique cycle \mathcal{O} around M_ℓ , which is unstable. For such an α , \mathcal{T}_ε cannot stay in \mathcal{Q}_4 : it would have \mathcal{O} as a limit cycle at ∞ , which contradicts the instability. ■

Next we discuss according to the position of α with respect to α^* and α_c .

Theorem 9.10 *Assume $\varepsilon = -1$, and $\alpha \leq \alpha^*$. Then*

- (i) *there exist a unique flat positive solution w of (\mathbf{E}_w) with (4.3) near 0, and (4.4) near ∞ ;*
- (ii) *All the other solutions are oscillating at ∞ , among them the regular ones, and $r^{-\gamma}w$ is asymptotically periodic in $\ln r$. There exist solutions with a hole, also with (4.3), (4.6) or (4.9) or (4.9) or (4.7) near 0. There exist solutions such that $r^{-\gamma}w$ is periodic in $\ln r$.*

th 9.10,fig 12: $\varepsilon = -1, N = 1, p = 3, \alpha = -2.53$ th 9.10, fig 13: $\varepsilon = -1, N = 1, p = 3, \alpha = -2.2$

Proof. Here $\alpha < \alpha_c$, from Theorem 9.9, and the trajectory \mathcal{T}_α stays in \mathcal{Q}_4 . From Proposition 4.4, it converges at $-\infty$ to M_ℓ , and w is of type (i).

The trajectory \mathcal{T}_ε leaves \mathcal{Q}_4 , and cannot converge either to $(0, 0)$ since $\mathcal{T}_\varepsilon \neq \mathcal{T}_\alpha$, or to $\pm M_\ell$, because this point is a source, or a weak source. Recall that M_ℓ is a node point for $\alpha \leq \alpha_1$ (see

figure 12,, where $\alpha_1 \cong -2.50$), or a spiral point (see figure 13). And \mathcal{T}_ε is bounded at ∞ from Proposition 4.3. Then it has a limit cycle \mathcal{O}_ε surrounding $(0, 0)$ from Proposition 4.4, and $\pm M_\ell$ from Remark 9.3. Thus w is oscillating around 0 near ∞ , $r^{-\gamma}w$ is asymptotically periodic in $\ln r$.

The solutions w corresponding to \mathcal{O}_ε are oscillating and $r^{-\gamma}w$ is periodic in $\ln r$. Any trajectory $\mathcal{T}_{[P]}$ with P in the interior domain delimited by \mathcal{O}_ε converges to M_ℓ at $-\infty$ and has the same limit cycle at ∞ . The trajectory \mathcal{T}_r starts in \mathcal{Q}_1 , with $\lim_{\tau \rightarrow -\infty} y = \infty$ and cannot converge to any stationary point at ∞ . It is bounded, thus has a limit cycle \mathcal{O}_r surrounding \mathcal{O}_0 . For any $P \notin \mathcal{T}_r$ in the exterior domain to \mathcal{O}_r , the trajectory $\mathcal{T}_{[P]}$ admits \mathcal{O}_r as a limit cycle at ∞ , and y is necessarily monotone at $-\infty$, thus (4.6) or (4.9) or (4.9) or (4.7) near 0; all those solutions exist. The question of the uniqueness of the cycle ($\mathcal{O}_r = \mathcal{O}_\varepsilon$) is open.

Theorem 9.11 *Let α_c be defined by Theorem 9.9.*

(1) *Let $\alpha^* < \alpha < \alpha_c$. Then all regular solutions w of (\mathbf{E}_w) are oscillating around 0 near ∞ , and $r^{-\gamma}w$ is asymptotically periodic in $\ln r$. There exist*

- (i) **positive** solutions, such that $r^{-\gamma}w$ is periodic in $\ln r$;
- (ii) a unique positive solution such that $r^{-\gamma}w$ is asymptotically periodic in $\ln r$ near 0, with (4.4) near ∞ ;
- (iii) positive solutions such that $r^{-\gamma}w$ is asymptotically periodic in $\ln r$ near 0, with (4.3) near ∞ ;
- (iv) solutions oscillating around 0 such that $r^{-\gamma}w$ is periodic in $\ln r$;
- (v) solutions with a hole, oscillating near ∞ , such that $r^{-\gamma}w$ is asymptotically periodic in $\ln r$;
- (vi) solutions satisfying (4.6) or (4.9) or (4.9) or (4.7) near 0, oscillating around 0 near ∞ , such that $r^{-\gamma}w$ is asymptotically periodic in $\ln r$;
- (vii) solutions positive near 0, oscillating near ∞ , such that $r^{-\gamma}w$ is asymptotically periodic in $\ln r$ near 0 and ∞ .

(2) *Let $\alpha = \alpha_c$.*

(viii) *There exist a **unique nonnegative solution with a hole**, with (4.4) near ∞ .*

The regular solutions are as above. There exist solutions of types (iv), (vi), and

(ix) *positive solutions such that $r^{-\gamma}w$ is bounded from above near 0, with (4.3) near ∞ .*

■

th 9.11,fig 14: $\varepsilon = -1, N = 1, p = 3, \alpha = -2.1$ th 9.11,fig 15: $\varepsilon = -1, N = 1, p = 3, \alpha = -2$

Proof. (1) Let $\alpha^* < \alpha < \alpha_c$ (see figure 14). Then \mathcal{T}_α stays in \mathcal{Q}_4 , but cannot converge neither to M_ℓ which is a sink, nor to $(0, 0)$ since $\mathcal{T}_\alpha \neq \mathcal{T}_\varepsilon$. It has a limit cycle \mathcal{O}_α in \mathcal{Q}_4 at $-\infty$, surrounding M_ℓ , and w is of type (ii). The orbit \mathcal{O}_α corresponds to solutions of type (i). There exist positive solutions converging to M_ℓ at ∞ , with a limit cycle \mathcal{O}_ℓ at $-\infty$ surrounded by \mathcal{O}_α , and w is of type (iii). This cycle is unique ($\mathcal{O}_\ell = \mathcal{O}_\alpha$) for $\alpha - \alpha^*$ small enough, from Proposition 2.5. The trajectory \mathcal{T}_ε still cannot stay in \mathcal{Q}_4 . As in the case $\alpha \leq \alpha^*$, \mathcal{T}_ε has a limit cycle \mathcal{O}_ε surrounding the three stationary points, w is of type (v), and \mathcal{T}_r is oscillating around 0, and there exist solutions of type (vi). Any trajectory $\mathcal{T}_{[P]}$ with $P \notin \mathcal{T}_\varepsilon$ in \mathcal{Q}_4 in the domain delimited by \mathcal{O}_α and \mathcal{O}_ε admits \mathcal{O}_α as a limit cycle at $-\infty$ and \mathcal{O}_ε at ∞ , and w is of type (vii).

(2) Let $\alpha = \alpha_c$ (see figure 15). The homoclinic trajectory $\mathcal{T}_\varepsilon = \mathcal{T}_\alpha$ corresponds to the solution w of type (viii). The trajectory \mathcal{T}_r has a limit cycle \mathcal{O}_r surrounding the three points. Thus there exist solutions of types (iv) or (vi). Any trajectory ending up at M_ℓ at ∞ is bounded, contained in the domain delimited by \mathcal{T}_ε , and its limit set at $-\infty$ is the homoclinic trajectory \mathcal{T}_ε , or a cycle around M_ℓ , and w is of type (ix).

Theorem 9.12 *Assume $\varepsilon = -1$, and $\alpha_c < \alpha < -p'$.*

There exist a unique nonnegative solution w of (\mathbf{E}_w) with a hole, with $r^{-\gamma}w$ bounded from above and below at ∞ . The regular solutions have at least two zeros.

(1) *Either there exist oscillating solutions such that $r^{-\gamma}w$ is periodic in $\ln r$. Then the regular solutions have an infinity of zeros, and $r^{-\gamma}w$ is asymptotically periodic in $\ln r$. There exist*

- (i) solutions satisfying (4.6) or (4.9) or (4.9) or (4.7) near 0, oscillating near ∞ , such that $r^{-\gamma}w$ is asymptotically periodic in $\ln r$;
- (ii) a unique solution oscillating near 0, such that $r^{-\gamma}w$ is asymptotically periodic in $\ln r$, and with (4.4) near ∞ ;
- (iii) solutions positive near 0, with $r^{-\gamma}w$ bounded, and oscillating near ∞ , such that $r^{-\gamma}w$ is asymptotically periodic in $\ln r$.

(2) Or all the solutions have a finite number of zeros, and at least two. Two cases may occur:

- Either regular solutions have m zeros and $r^{-\gamma}w$ bounded from above and below at ∞ . Then there exist
 - (iv) solutions with m zeros, with (4.6) or (4.9), with (4.4) near ∞ ;
 - (v) solutions with m zeros with (4.6) or (4.9) and $r^{-\gamma}w$ bounded from above and below at ∞ ;
 - (vi) solutions with $m + 1$ zeros with (4.6) or (4.9) and $r^{-\gamma}w$ bounded from above and below at ∞ ;
 - (vii) (for $p > N$) a unique solution with m zeros, with (4.7) or (4.10) and $r^{-\gamma}w$ bounded from above and below at ∞ .
- Or regular solutions have m zeros and (4.4) holds near ∞ . Then there exist solutions of type (vi) or (vii).

■

th 9.12, fig 16: $\varepsilon = -1, N = 1, p = 3, \alpha = -1.98$ th 9.12, fig 17: $\varepsilon = -1, N = 1, p = 3, \alpha = -1.90$

Proof. Here \mathcal{T}_ε stays in \mathcal{Q}_4 , converges to M_ℓ or has a limit cycle around M_ℓ , thus w has a hole and $r^{-\gamma}w$ bounded from above and below at ∞ . If $\alpha \geq \alpha_2$, there is no cycle in \mathcal{Q}_4 , from Proposition 4.4, thus \mathcal{T}_ε converges to M_ℓ .

(1) Either there exists a cycle surrounding $(0, 0)$ and $\pm M_\ell$, thus solutions w oscillating around 0, such that $r^{-\gamma}w$ is periodic in $\ln r$. Then \mathcal{T}_r has such a limit cycle \mathcal{O}_r , and w is oscillating around 0. The trajectory \mathcal{T}_α has a limit cycle at $-\infty$ of the same type $\mathcal{O}_\alpha \subset \mathcal{O}_r$, and w is of type (ii). For any $P \notin \mathcal{T}_\varepsilon$ in the interior domain in \mathcal{O}_α , $\mathcal{T}_{[P]}$ admits \mathcal{O}_α as a limit cycle at $-\infty$ and converges to M_ℓ at ∞ , or has a limit cycle in \mathcal{Q}_4 ; and w is of type (iii). For any $P \notin \mathcal{T}_r$, in the domain exterior to \mathcal{O}_r , $\mathcal{T}_{[P]}$ has \mathcal{O}_α as limit cycle at ∞ , and w is of type (i).

(2) Or no such cycle exists. Then any trajectory converges at ∞ , any trajectory, apart from $\pm\mathcal{T}_\alpha$, converges to $\pm M_\ell$ or has a limit cycle in \mathcal{Q}_1 . All the trajectories end up in \mathcal{Q}_2 or \mathcal{Q}_4 . Since \mathcal{T}_r starts in \mathcal{Q}_1 , y has at least one zero. Suppose that it is unique. Then \mathcal{T}_r converges to $-M_\ell$, thus Y stays positive. Consider the function $Y_\alpha = e^{(\alpha+\gamma)(p-1)\tau}Y$ defined by (2.3) with $d = \alpha$. From Theorem 3.3, $Y_\alpha = (a|\alpha|/N)e^{(\alpha(p-1)+p)\tau}(1+o(1))$ near $-\infty$; thus Y_α tends to ∞ , since $\alpha < p'$. And $Y_\alpha = (\gamma\ell)^{p-1}e^{(\alpha+\gamma)(p-1)\tau}$ near ∞ , thus also Y_α tends to ∞ ; then it has a minimum point τ , and from (2.6), $Y_\alpha''(\tau) = (p-1)^2(\eta-\alpha)(p'+\alpha)Y_\alpha < 0$, which is contradictory. Thus y has a number $m \geq 2$ of zeros.

Either $\mathcal{T}_r \neq \mathcal{T}_\alpha$. Since the slope of \mathcal{T}_α near $-\infty$ is infinite and the slope of \mathcal{T}_r is finite, \mathcal{T}_α cuts the line $\{y = 0\}$ at m points, starts from \mathcal{Q}_1 , and w is of type (iv). For any P in the domain of \mathcal{Q}_1 between \mathcal{T}_r and \mathcal{T}_α , $\mathcal{T}_{[P]}$ cuts $\{y = 0\}$ at $m+1$ points, and w is of type (v). For any P in the domain of \mathcal{Q}_1 above \mathcal{T}_r , $\mathcal{T}_{[P]}$ cuts the line $\{y = 0\}$ at $m+1$ points, and w is of type (vi). If $p > N$, the trajectories \mathcal{T}_- and \mathcal{T}_u cut the line $\{y = 0\}$ at m points, and w is of type (vii).

Or $\mathcal{T}_r = \mathcal{T}_\alpha$, and then we find only trajectories with w of type (vi) or (vii). ■

Remark 9.13 Consider the regular solutions in the range $\alpha_c < \alpha < -p'$. We conjecture that there exists a decreasing sequence $(\bar{\alpha}_n)$, with $\bar{\alpha}_0 = -p'$ and $\alpha_c < \bar{\alpha}_n$ such that for $\alpha \in (\bar{\alpha}_m, \bar{\alpha}_{m-1})$, y has m zeros and converges to $\pm M_\ell$; and for $\alpha = \bar{\alpha}_m$, y has $m+1$ zeros and converges to $(0, 0)$, thus $\mathcal{T}_r = \mathcal{T}_\alpha$. We presume that $(\bar{\alpha}_m)$ has a limit $\bar{\alpha} > \alpha_c$. And for $\alpha < \bar{\alpha}$, y has an infinity of zeros, in other words there exists a cycle \mathcal{O}_r surrounding $\{0\}$ and $\pm M_\ell$.

Numerically, for $\alpha = \alpha_c$, the cycle \mathcal{O}_r seems to be the unique cycle surrounding the three points. But for $\alpha > \alpha_c$ and $\alpha - \alpha_c$ small enough, there exist **two different cycles** $\mathcal{O}_\alpha \subset \mathcal{O}_r$ (see figure 15). As α increases, we observe the coalescence of those cycles; they disappear after some value $\bar{\alpha}$ (see figure 16).

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