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Self-similar solutions of the *p*-Laplace heat equation: the case p > 2.

Marie Françoise Bidaut-Véron*

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Abstract

We study the self-similar solutions of the equation

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$$u_t - div(|\nabla u|^{p-2} \nabla u) = 0,$$

in \mathbb{R}^N , when p > 2. We make a complete study of the existence and possible uniqueness of solutions of the form

$$u(x,t) = (\pm t)^{-\alpha/\beta} w((\pm t)^{-1/\beta} |x|)$$

of any sign, regular or singular at x = 0. Among them we find solutions with an expanding compact support or a shrinking hole (for t > 0), or a spreading compact support or a focussing hole (for t < 0). When t < 0, we show the existence of positive solutions oscillating around the particular solution $U(x,t) = C_{N,p}(|x|^p/(-t))^{1/(p-2)}$.

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1 Introduction and main results

Here we consider the self-similar solutions of the degenerate heat equation involving the p-Laplace operator

$$u_t - div(|\nabla u|^{p-2} \nabla u) = 0. \tag{E}_u$$

in \mathbb{R}^N , with p > 2. This study is the continuation of the work started in [4], relative to the case p < 2. It can be read independently. We set

$$\gamma = \frac{p}{p-2}, \qquad \eta = \frac{N-p}{p-1}, \tag{1.1}$$

thus $\gamma > 1$, $\eta < N$,

$$\frac{N+\gamma}{p-1} = \eta + \gamma = \frac{N-\eta}{p-2}.$$
(1.2)

If u is a solution, then for any $\alpha, \beta \in \mathbb{R}$, $u_{\lambda}(x, t) = \lambda^{\alpha} u(\lambda x, \lambda^{\beta} t)$ is a solution of (\mathbf{E}_u) if and only if

$$\beta = \alpha (p-2) + p = (p-2)(\alpha + \gamma);$$
 (1.3)

notice that $\beta > 0 \iff \alpha > -\gamma$. Given $\alpha \in \mathbb{R}$ such that $\alpha \neq -\gamma$, we search self-similar solutions, radially symmetric in x, of the form:

$$u = u(x,t) = (\varepsilon\beta t)^{-\alpha/\beta} w(r), \qquad r = (\varepsilon\beta t)^{-1/\beta} |x|, \qquad (1.4)$$

where $\varepsilon = \pm 1$. By translation, for any real T, we obtain solutions defined for any t > T when $\varepsilon \beta > 0$, or t < T when $\varepsilon \beta < 0$. We are lead to the equation

$$\left(|w'|^{p-2} w' \right)' + \frac{N-1}{r} |w'|^{p-2} w' + \varepsilon (rw' + \alpha w) = 0 \quad \text{in } (0, \infty).$$
 (**E**_w)

Our purpose is to give a complete description of all the solutions, with constant or changing sign. Equation (\mathbf{E}_w) is very interesting, because it is singular at any zero of w', since p > 2, implying a nonuniqueness phenomena.

For example, concerning the constant sign solutions near the origin, it can happen that

$$\lim_{r \to 0} w = a \neq 0, \qquad \lim_{r \to 0} w' = 0,$$

we will say that w is *regular*, or

$$\lim_{r \to 0} w = \lim_{r \to 0} w' = 0,$$

we say that w is *flat*. Or different kinds of singularities may occur, either at the level of w:

$$\lim_{r\to 0} w = \infty$$

or at the level of the gradient:

$$\lim_{r \to 0} w = a \in \mathbb{R}, \qquad \lim_{r \to 0} w' = \pm \infty, \qquad \text{when } p > N > 1,$$
$$\lim_{r \to 0} w = a \in \mathbb{R}, \qquad \lim_{r \to 0} w' = b \neq 0 \qquad \text{when } p > N = 1.$$

We first show that any local solution w of (\mathbf{E}_w) can be defined on $(0, \infty)$, thus any solution u of equation (\mathbf{E}_u) associated to w by (1.4) is defined on $\mathbb{R}^N \setminus \{0\} \times (0, \pm \infty)$. Then we prove the existence of regular solutions, flat ones, and of all singular solutions mentioned above.

Moreover, for $\varepsilon = 1$, there exist solutions w with a compact support $(0, \bar{r})$; then $u \equiv 0$ on the set

$$D = \left\{ (x,t) : x \in \mathbb{R}^N, \quad \beta t > 0, \quad |x| > (\beta t)^{1/\beta} \bar{r} \right\}.$$

For $\varepsilon = -1$, there exist solutions with a hole: $w(r) = 0 \iff r \in (0, \bar{r})$. Then $u \equiv 0$ on the set

$$H = \left\{ (x,t) : x \in \mathbb{R}^N, \quad \beta t < 0, \quad |x| < (-\beta t)^{1/\beta} \bar{r} \right\}.$$

The free boundary is of parabolic type for $\beta > 0$, of hyperbolic type for $\beta < 0$. This leads to four types of solutions, and we prove their existence:

• If t > 0, with $\varepsilon = 1, \beta > 0$, we say that u has an *expanding support*; the support increases from $\{0\}$ as t increases from 0.

• If t > 0, with $\varepsilon = -1, \beta < 0$, we say that u has a *shrinking hole*: the hole decreases from infinity as t increases from 0;

• If t < 0, with $\varepsilon = 1, \beta < 0$, we say that u has a spreading support: the support increases to be infinite as t increases to 0.

• If t < 0, with $\varepsilon = -1, \beta > 0$, we say that u has a *focussing hole*: the hole disappears as t increases to 0.

Up to our knowledge, some of them seem completely new, as for example the solutions with a shrinking hole or a spreading support. In particular we find again and improve some results of [8] concerning the existence of focusing type solutions.

Finally for t < 0 we also show the existence of positive solutions turning around the fundamental solution U given at (1.8) with a kind of periodicity, and also the existence of changing sign solutions doubly oscillating in |x| near 0 and infinity.

As in [4] we reduce the problem to dynamical systems.

When $\varepsilon = -1$, a critical negative value of α is involved:

$$\alpha^* = -\gamma + \frac{\gamma(N+\gamma)}{(p-1)(N+2\gamma)}.$$
(1.5)

1.1Explicit solutions

Obviously if w is a solution of (\mathbf{E}_w) , -w is also a solution. Some particular solutions are well-known.

The solution U. For any α such that $\varepsilon(\alpha + \gamma) < 0$, that means $\varepsilon\beta < 0$, there exist flat solutions of (\mathbf{E}_w) , given by

$$w(r) = \pm \ell r^{\gamma}, \tag{1.6}$$

where

$$\ell = \left(\frac{|\alpha + \gamma|}{\gamma^{p-1}(\gamma + N)}\right)^{1/(p-2)} > 0.$$
(1.7)

They correspond to a unique solution of (\mathbf{E}_u) called U, defined for t < 0, such that U(0,t) = 0, flat, blowing up at t = 0 for fixed $x \neq 0$:

$$U(x,t) = C\left(\frac{|x|^p}{-t}\right)^{1/(p-2)}, \qquad C = ((p-2)\gamma^{p-1}(\gamma+N))^{1/(2-p)}.$$
(1.8)

The case $\alpha = N$. Then $\beta = \beta_N = N(p-2) + p > 0$, and the equation has a first integral

$$w + \varepsilon r^{-1} |w'|^{p-2} w' = Cr^{-N}.$$
(1.9)

All the solutions corresponding to C = 0 are given by

$$w = w_{K,\varepsilon}(r) = \pm \left(K - \varepsilon \gamma^{-1} r^{p'}\right)_{+}^{(p-1)/(p-2)}, \quad K \in \mathbb{R},$$

$$u = \pm u_{K,\varepsilon}(x,t) = \pm (\varepsilon \beta_N t)^{-N/\beta_N} \left(K - \varepsilon \gamma^{-1} (\varepsilon \beta_N t)^{-p'/\beta_N} |x|^{p'}\right)_{+}^{(p-1)/(p-2)}.$$
 (1.10)

For $\varepsilon = 1, K > 0$, they are defined for t > 0, called *Barenblatt solutions*, regular with a compact support. Given c > 0, the function $u_{K,1}$, defined on $\mathbb{R}^N \times (0, \infty)$, is the unique solution of equation (\mathbf{E}_u) with initial data $u(0) = c\delta_0$, where δ_0 is the Dirac mass at 0, and K being linked by $\int u_K(x,t)dt = c$. The $u_{K,1}$ are the only nonnegative solutions defined on $\mathbb{R}^N \times (0,\infty)$, such that u(x,0) = 0 for any $x \neq 0$. For $\varepsilon = -1$, the $u_{K,-1}$ are defined for t < 0; for K > 0, w does not

vanish on $(0, \infty)$; for K < 0, w is flat with a hole near 0. For K = 0, we find again the function w given at (1.6).

The case $\alpha = \eta \neq 0$. We exhibit a family of solutions of (\mathbf{E}_w) :

$$w(r) = Cr^{-\eta}, \qquad u(t,x) = C |x|^{-\eta}, \qquad C \neq 0.$$
 (1.11)

The solutions u, independent of t, are p-harmonic in \mathbb{R}^N ; they are fundamental solutions when p < N. When p > N, w satisfies $\lim_{r\to 0} w = 0$, and $\lim_{r\to 0} w' = \infty$ for N > 1, $\lim_{r\to 0} w' = b$ for N = 1.

The case $\alpha = -p'$. Equation (\mathbf{E}_w) admits regular solutions of the form

$$w(r) = \pm K \left(N(Kp')^{p-2} + \varepsilon r^{p'} \right), \qquad u(x,t) = \pm K \left(N(Kp')^{p-2}t + |x|^{p'} \right), \qquad K > 0.$$
(1.12)

Here $\beta > 0$; in the two cases $\varepsilon = 1, t > 0$ and $\varepsilon = -1, t < 0, u$ is defined for any $t \in \mathbb{R}$ and of the form $\psi(t) + \Phi(|x|)$ with Φ nonconstant, and u(.,t) has a constant sign for t > 0 and changing sign for t < 0.

The case $\alpha = 0$. Equation (\mathbf{E}_w) can be explicitly solved: either $w' \equiv 0$, thus $w \equiv a \in \mathbb{R}$, u is a constant solution of (\mathbf{E}_u) , or there exists $K \in \mathbb{R}$ such that

$$\left|w'\right| = r^{-(\eta+1)} \left(K - \frac{\varepsilon}{\gamma+N} r^{N-\eta}\right)_{+}^{1/(p-2)}; \qquad (1.13)$$

and w follows by integration, up to a constant, and then $u(x,t) = w(|x|/(\varepsilon pt)^{1/p})$. If $\varepsilon = 1$, then t > 0, K > 0 and w' has a compact support; up to a constant, u has a compact support. If $\varepsilon = -1$, then t < 0; for K > 0, w is strictly monotone; for K < 0, w is flat, constant near 0; for K = 0, we find again (1.6). For $\varepsilon = \pm 1, K > 0$, observe that $\lim_{r\to 0} w = \pm \infty$ if $p \leq N$; and $\lim_{r\to 0} w = a \in \mathbb{R}$, $\lim_{r\to 0} w' = \pm \infty$ if p > N > 1; and $\lim_{r\to 0} w = a \in \mathbb{R}$, $\lim_{r\to 0} w' = K$ if p > N = 1. In particular we find solutions such that $w = cr^{|\eta|}(1 + o(1))$ near 0, with c > 0.

(v) Case N = 1 and $\alpha = -(p-1)/(p-2) < 0$. Here $\beta = 1$, and we find the solutions

$$w(r) = \pm \left(Kr + \varepsilon \left| \alpha \right|^{p-1} \left| K \right|^p \right)_+^{(p-1)/(p-2)}, \quad u(x,t) = \pm \left(K \left| x \right| + \left| \alpha \right|^{p-1} \left| K \right|^p t \right)_+^{(p-1)/(p-2)}, \quad (1.14)$$

If $\varepsilon = 1, t > 0$, then w has a singularity at the level of the gradient, and either K > 0, w > 0, or K < 0 and w has a compact support. If $\varepsilon = -1, t < 0$ then K > 0, w has a hole.

1.2 Main results

In the next sections we provide an exhaustive study of equation (\mathbf{E}_w) . Here we give the main results relative to the function u. Let us show how to return from w to u. Suppose that the behaviour of w is given by

$$\lim_{r \to 0} r^{\lambda} w(r) = c \neq 0, \qquad \lim_{r \to \infty} r^{\mu} w(r) = c' \neq 0, \qquad \text{where } \lambda, \mu \in \mathbb{R}.$$

(i) Then for fixed $t \neq 0$, the function u has a behaviour in $|x|^{-\lambda}$ near x = 0, and a behaviour in $|x|^{-\mu}$ for large |x|.

If $\lambda = 0$, then u is defined on $\mathbb{R}^N \times (0, \pm \infty)$. Either w is regular, then $u(., t) \in C^1 (\mathbb{R}^N \times (0, \infty))$; we will say that u is **regular**; nevertheless the regular solutions u presents a singularity at time t = 0 if and only if $\alpha < -\gamma$ or $\alpha > 0$. Or a singularity can appear for u at the level of the gradient.

If $\lambda < 0$, thus u is defined on $\mathbb{R}^N \times (0, \pm \infty)$ and u(0, t) = 0; either w is flat, we also say that u is **flat**, or a singularity appears at the level of the gradient.

If $0 < \lambda < N$, then $u(.,t) \in L^1_{loc}(\mathbb{R}^N)$ for $t \neq 0$, we say that x = 0 is a **weak** singularity. We will show that there exist no stronger singularity.

If $\lambda < N < \mu$; then $u(.,t) \in L^1(\mathbb{R}^N)$.

(ii) For fixed $x \neq 0$, the behaviour of u near t = 0, depends on the sign of β :

$$\lim_{t \to 0} |x|^{\mu} |t|^{(\alpha-\mu)/\beta} u(x,t) = C \neq 0 \quad \text{if} \quad \alpha > -\gamma,$$
$$\lim_{t \to 0} |x|^{\lambda} |t|^{(\alpha-\lambda)/\beta} u(x,t) = C \neq 0 \quad \text{if} \quad \alpha < -\gamma.$$

If $\mu < 0, \alpha > -\gamma$ or $\lambda < 0, \alpha < -\gamma$, then $\lim_{t\to 0} u(x,t) = 0$.

1.2.1 Solutions defined for t > 0

Here we look for solutions u of (\mathbf{E}_u) of the form (1.4) defined on $\mathbb{R}^N \setminus \{0\} \times (0, \infty)$. That means $\varepsilon \beta > 0$ or equivalently $\varepsilon = 1, -\gamma < \alpha$ (see Section 6) or $\varepsilon = -1, \alpha < -\gamma$ see (Section 7). We begin by the case $\varepsilon = 1$, treated at Theorem 6.1.

Theorem 1.1 Assume $\varepsilon = 1$, and $-\gamma < \alpha$.

(1) Let
$$\alpha < N$$
.

All regular solutions on $\mathbb{R}^N \setminus \{0\} \times (0, \infty)$ have a strict constant sign, in $|x|^{-\alpha}$ near ∞ for fixed t, with initial data $L |x|^{-\alpha} (L \neq 0)$ in \mathbb{R}^N ; thus $u(.,t) \notin L^1(\mathbb{R}^N)$, and u is unbounded when $\alpha < 0$. There exist nonnegative solutions such that near x = 0,

 $\begin{cases}
\text{for } p < N, & u \text{ has a weak singularity in } |x|^{-\eta}, \\
\text{for } p = N, & u \text{ has a weak singularity in } \ln |x|, \\
\text{for } p > N, & u \in C^0(\mathbb{R}^N \times (0, \infty), \quad u(0, t) = a > 0, \text{ with a singular gradient,}
\end{cases}$ (1.15)

and u has an expanding compact support for any t > 0, with initial data $L|x|^{-\alpha}$ in $\mathbb{R}^N \setminus \{0\}$.

There exist positive solutions with the same behaviour as $x \to 0$, in $|x|^{-\alpha}$ near ∞ for fixed t; and also solutions such that u has one zero for fixed $t \neq 0$, and the same behaviour. If p > N, there exist positive solutions satisfying (1.15), and also positive solutions such that

 $u \in C^0(\mathbb{R}^N \times (0,\infty), \quad u(0,t) = 0, \text{ in } |x|^{|\eta|} \text{ near } 0, \text{ with a singular gradient,}$ (1.16)

in $|x|^{-\alpha}$ near ∞ for fixed t, with and initial data $L|x|^{-\alpha}$ in $\mathbb{R}^N \setminus \{0\}$.

(2) Let
$$\alpha = N$$
.

All regular (Barenblatt) solutions are nonnegative, have a compact support for any t > 0. If $p \leq N$, all the other solutions have one zero for fixed t, satisfy (1.15) or (1.16) and have the same behaviour at ∞ .

(3) Let $N < \alpha$.

All regular solutions u have a finite number $m \ge 1$ of simple zeros for fixed t, and $u(.,t) \in L^1(\mathbb{R}^N)$. Either they are in $|x|^{-\alpha}$ near ∞ for fixed t, then there exist solutions with m zeros, compact support, satisfying (1.15); or they have a compact support. All the solutions have m or m + 1 zeros. There exist solutions satisfying (1.15) with m + 1 zeros, and in $|x|^{-\alpha}$ near ∞ . If p > N, there exist solutions satisfying (1.15) with m zeros; there exist also solutions with m zeros, u(0,t) = 0, and a singular gradient, in $|x|^{-\alpha}$ near ∞ .

Next we come to the case $\varepsilon = -1$, which is the subject of Theorem 7.1.

Theorem 1.2 Assume $\varepsilon = -1$ and $\alpha < -\gamma$.

All the solutions u on $\mathbb{R}^N \setminus \{0\} \times (0, \infty)$, in particular the regular ones, are **oscillating around** 0 for fixed t > 0 and large |x|, and $r^{-\gamma}w$ is asymptotically periodic in $\ln r$. Moreover there exist

solutions such that $r^{-\gamma}w$ is **periodic** in $\ln r$, in particular $C_1t^{-|\alpha/\beta|} \leq |u| \leq C_2t^{-|\alpha/\beta|}$ for some $C_1, C_2 > 0$;

solutions $u \in C^1(\mathbb{R}^N \times [0,\infty))$, $u(x,0) \equiv 0$, with a **shrinking hole**; **flat** solutions $u \in C^1(\mathbb{R}^N \times [0,\infty))$, in $|x|^{|\alpha|}$ near 0, with initial data $L |x|^{|\alpha|} (L \neq 0)$; solutions satisfying (1.15) near x = 0, and if p > N, solutions satisfying (1.16) near 0.

1.2.2 Solutions defined for t < 0

We look for solutions u of (\mathbf{E}_u) of the form (1.4) defined on $\mathbb{R}^N \setminus \{0\} \times (-\infty, 0)$. That means $\varepsilon \beta < 0$ or equivalently $\varepsilon = 1$, $\alpha < -\gamma$ (see Section 8, Theorem 8.1) or $\varepsilon = -1$, $\alpha > -\gamma$ (see Section 9). In the case $\varepsilon = 1$, we get the following:

Theorem 1.3 Assume $\varepsilon = 1$, and $\alpha < -\gamma$.

The function $U(x,t) = C\left(\frac{|x|^p}{-t}\right)^{1/(p-2)}$ is a positive **flat** solution on $\mathbb{R}^N \setminus \{0\} \times (-\infty, 0)$. All regular solutions have a constant sign, are unbounded in $|x|^{\gamma}$ near ∞ for fixed t, and blow up at t = 0 like $(-t)^{-|\alpha|/|\beta|}$ for fixed $x \neq 0$. There exist flat positive solutions $u \in C^1(\mathbb{R}^N \times (-\infty, 0])$, in $|x|^{\gamma}$ near ∞ for fixed t, with final data $L |x|^{|\alpha|}$ (L > 0).

There exist nonnegative solutions satisfying (1.15) near 0, with a spreading compact support, blowing up near t = 0 (like $|t|^{-(\eta+|\alpha|)/|\beta|}$ for p < N, or $|t|^{-|\alpha|/|\beta|} \ln |t|$ for p = N, or $(-t)^{-|\alpha|/|\beta|}$ for p > N).

There exist positive solutions with the same behaviour near 0, in $|x|^{\gamma}$ near ∞ , blowing up as above at t = 0, and solutions with one zero for fixed t, and the same behaviour. If p > N, there exist positive solutions satisfying (1.15) (resp. (1.16)) near 0, in $|x|^{\gamma}$ near ∞ for fixed t, blowing up at t = 0 like $|t|^{-|\alpha|/|\beta|}$ (resp. $|t|^{(|\eta|-|\alpha|)/|\beta|}$) for fixed x.

Up to a symmetry, all the solutions are described.

The most interesting case is $\varepsilon = -1, -\gamma < \alpha$. For simplicity we will assume that p < N. The case $p \ge N$ is much more delicate, and the complete results can be read in terms of w at Theorems 9.4, 9.6, 9.9, 9.10, 9.11 and 9.12. We discuss according to the position of α with respect to -p' and α^* defined at (1.5). Notice that $\alpha^* < -p'$.

Theorem 1.4 Assume $\varepsilon = -1$, and $-p' \leq \alpha \neq 0$. The function U is still a flat solution on $\mathbb{R}^N \setminus \{0\} \times (-\infty, 0)$.

(1) Let $0 < \alpha$.

All regular solutions have a strict constant sign, in $|x|^{\gamma}$ near ∞ for fixed t, blowing up at t = 0like $(-t)^{-1/(p-2)}$ for fixed $x \neq 0$.

There exist nonnegative solutions with a **focussing hole**: $u(x,t) \equiv 0$ for $|x| \leq C |t|^{1/\beta}$, t > 0, in $|x|^{\gamma}$ near ∞ for fixed t, blowing up at t = 0 like $(-t)^{-1/(p-2)}$ for fixed $x \neq 0$.

There exist positive solutions u with a (weak) singularity in $|x|^{-\eta} at x = 0$, in $|x|^{-\alpha}$ near ∞ for fixed t, with $u(.,t) \in L^1(\mathbb{R}^N)$ if $\alpha > N$, with final data $L|x|^{-\alpha}$ (L > 0) in $\mathbb{R}^N \setminus \{0\}$.

There exist positive solutions u in $|x|^{-\eta}$ at x = 0, in $|x|^{\gamma}$ near ∞ for fixed t, blowing up at t = 0 like $(-t)^{-1/(p-2)}$ for fixed $x \neq 0$; solutions with one zero and the same behaviour.

(2) Let $-p' < \alpha < 0$.

All regular solutions have one zero for fixed t, and the same behaviour. There exist solutions with one zero, in $|x|^{-\eta}at x = 0$, in $|x|^{|\alpha|}$ near ∞ for fixed t, with final data $L|x|^{-\alpha}$ (L > 0) in $\mathbb{R}^N \setminus \{0\}$. There exist solutions with one zero, u in $|x|^{-\eta}at x = 0$, in $|x|^{\gamma}$ near ∞ for fixed t, blowing up at t = 0 like $(-t)^{-1/(p-2)}$ for fixed $x \neq 0$; solutions with two zeros and the same behaviour. 3) Let $\alpha = -p'$.

All regular solutions have one zero and are in $|x|^{|\alpha|}$ near ∞ for fixed t, and with final data $L |x|^{|\alpha|}$ (L > 0). The other solutions have one or two zeros, are in $|x|^{-\eta}$ at x = 0, in $|x|^{\gamma}$ near ∞ for fixed t.

In any case, up to a symmetry, all the solutions are described.

Theorem 1.5 Assume $\varepsilon = -1, -\gamma < \alpha < -p'$. Then U is still a flat solution on $\mathbb{R}^N \setminus \{0\} \times (-\infty, 0)$.

(1)Let $\alpha \leq \alpha^*$.

Then there exist **positive flat solutions**, in $|x|^{\gamma}$ near 0, in $|x|^{|\alpha|}$ near ∞ for fixed t, with **final** data $L |x|^{-\alpha}$ (L > 0) in \mathbb{R}^N .

All the other solutions, among them the **regular ones**, have an **infinity of zeros**: u(t, .) is oscillating around 0 for large |x|. There exist solutions with a focussing hole, and solutions with a singularity in $|x|^{-\eta}$ at x = 0. There exist solutions **oscillating also for small** |x|, such that $r^{-\gamma}w$ is periodic in $\ln r$.

(2) There exist a critical unique value $\alpha_c \in (\max(\alpha^*, -p') \text{ such that for } \alpha = \alpha_c, \text{ there exists nonnegative solutions with a focussing hole near 0, in <math>|x|^{|\alpha|}$ near ∞ for fixed t, with final data $L |x|^{-\alpha}$ (L > 0) in \mathbb{R}^N . And $\alpha_c > -(p-1)/(p-2)$.

There exist positive flat solutions, such that $|x|^{-\gamma} u$ is bounded on \mathbb{R}^N for fixed t, blowing up at t = 0like $(-t)^{-1/(p-2)}$ for fixed $x \neq 0$. The regular solutions are oscillating around 0 as above. There exist solutions **oscillating around** 0, such that $r^{-\gamma}w$ is **periodic** in $\ln r$. There are solutions with a weak singularity in $|x|^{-\eta}$ at x = 0, and oscillating around 0 for large |x|.

(3) Let $\alpha^* < \alpha < \alpha_c$.

The regular solutions are as above. There exist solutions of the same types as above. Moreover there exist **positive** solutions, such that $r^{-\gamma}w$ is **periodic** in $\ln r$, thus there exist $C_1, C_2 > 0$ such that

$$C_1\left(\frac{|x|^p}{|t|}\right)^{1/(p-2)} \le u \le C_2\left(\frac{|x|^p}{|t|}\right)^{1/(p-2)}$$

There exist **positive** solutions, such that $r^{-\gamma}w$ is asymptotically periodic in $\ln r$ near 0 and in $|x|^{\gamma}$ near ∞ for fixed t; and also, solutions with a hole, and oscillating around 0 for large |x|. There exist solutions positive near 0, oscillating near ∞ , and $r^{-\gamma}w$ is **doubly asymptotically periodic** in $\ln r$.

4) Let
$$\alpha_c < \alpha < -p'$$
.

There exist nonnegative solutions with a focussing hole near 0, in $|x|^{\gamma}$ near ∞ for fixed t, blowing up at t = 0 like $(-t)^{-1/(p-2)}$ for fixed $x \neq 0$. Either the regular solutions have an **infinity** of zeros for fixed t, then the same is true for all the other solutions. Or they have a **finite** number $m \ge 2$ of zeros, and can be in $|x|^{\gamma}$ or $|x|^{|\alpha|}$ near ∞ (in that case they have a final data $L|x|^{|\alpha|}$); all the other solutions have m or m + 1 zeros.

In the case $\alpha = \alpha_c$, we find again the existence and uniqueness of the focussing solutions introduced in [8].

2 Different formulations of the problem

In all the sequel we assume

 $\alpha \neq 0,$

recalling that the solutions w are given explicitly by (1.13) when $\alpha = 0$. Defining

$$J_N(r) = r^N \left(w + \varepsilon r^{-1} \left| w' \right|^{p-2} w' \right), \qquad J_\alpha(r) = r^{\alpha - N} J_N(r), \tag{2.1}$$

equation (\mathbf{E}_w) can be written in an equivalent way under the forms

$$J'_{N}(r) = r^{N-1}(N-\alpha)w, \qquad J'_{\alpha}(r) = -\varepsilon(N-\alpha)r^{\alpha-2} |w'|^{p-2} w'.$$
(2.2)

If $\alpha = N$, then J_N is constant, so we find again (1.9).

We mainly use logarithmic substitutions; given $d \in \mathbb{R}$, setting

$$w(r) = r^{-d} y_d(\tau), \qquad Y_d = -r^{(d+1)(p-1)} \left| w' \right|^{p-2} w', \qquad \tau = \ln r, \tag{2.3}$$

we obtain the equivalent system:

$$\begin{cases}
y'_{d} = dy_{d} - |Y_{d}|^{(2-p)/(p-1)} Y_{d}, \\
Y'_{d} = (p-1)(d-\eta)Y_{d} + \varepsilon e^{(p+(p-2)d)\tau} (\alpha y_{d} - |Y_{d}|^{(2-p)/(p-1)} Y_{d}).
\end{cases}$$
(2.4)

At any point τ where $w'(\tau) \neq 0$, the functions y_d, Y_d satisfy the equations

$$y_d'' + (\eta - 2d)y_d' - d(\eta - d)y_d + \frac{\varepsilon}{p - 1}e^{((p - 2)d + p)\tau} \left| dy_d - y_d' \right|^{2 - p} \left(y_d' + (\alpha - d)y_d \right) = 0,$$
(2.5)

$$Y_d'' + (p-1)(\eta - 2d - p')Y_d' + \varepsilon e^{((p-2)d+p)\tau} |Y_d|^{(2-p)/(p-1)} (Y_d'/(p-1) + (\alpha - d)Y_d) - (p-1)^2(\eta - d)(p' + d)Y_d = 0,$$
(2.6)

The main case is $d = -\gamma$: setting $y = y_{-\gamma}$,

$$w(r) = r^{\gamma} y(\tau), \qquad Y = -r^{(-\gamma+1)(p-1)} |w'|^{p-2} w', \qquad \tau = \ln r,$$
 (2.7)

we are lead to the *autonomous* system

$$\begin{cases} y' = -\gamma y - |Y|^{(2-p)/(p-1)} Y, \\ Y' = -(\gamma + N)Y + \varepsilon(\alpha y - |Y|^{(2-p)/(p-1)} Y). \end{cases}$$
 (S)

Its study is fundamental: its phase portrait allows to study all the *signed* solutions of equation (\mathbf{E}_w) . Equation (2.5) takes the form

$$(p-1)y'' + (N+\gamma p)y' + \gamma(\gamma+N)y + \varepsilon \left|\gamma y + y'\right|^{2-p} \left(y' + (\alpha+\gamma)y\right) = 0, \qquad (\mathbf{E}_y)$$

Notice that $J_N(r) = r^{N+\gamma}(y(\tau) - \varepsilon Y(\tau)).$

Remark 2.1 Since (S) is autonomous, for any solution w of (\mathbf{E}_w) of the problem, all the functions $w_{\xi}(r) = \xi^{-\gamma} w(\xi r), \xi > 0$, are also solutions.

Notation 2.2 In the sequel we set $\varepsilon \infty := +\infty$ if $\varepsilon = 1$, $\varepsilon \infty := -\infty$ if $\varepsilon = -1$.

2.1 The phase plane of system (S)

In the phase plane (y, Y) we denote the four quadrants by

$$\mathcal{Q}_1 = (0,\infty) \times (0,\infty), \quad \mathcal{Q}_2 = (-\infty,0) \times (0,\infty), \quad \mathcal{Q}_3 = -\mathcal{Q}_1, \quad \mathcal{Q}_4 = -\mathcal{Q}_2.$$

Remark 2.3 The vector field at any point $(0,\xi)$, $\xi > 0$ satisfies $y' = -\xi^{1/(p-1)} < 0$, thus points to Q_2 ; moreover Y' < 0 if $\varepsilon = 1$. The field at any point $(\varphi, 0)$, $\varphi > 0$ satisfies $Y' = \varepsilon \alpha \varphi$, thus points to Q_1 if $\varepsilon \alpha > 0$ and to Q_4 if $\varepsilon \alpha < 0$; moreover $y' = -\gamma \varphi < 0$.

If $\varepsilon(\gamma + \alpha) \ge 0$, system (S) has a unique stationary point (0, 0). If $\varepsilon(\gamma + \alpha) < 0$, it admits three stationary points:

$$(0,0), M_{\ell} = (\ell, -(\gamma \ell)^{p-1}) \in \mathcal{Q}_4, M'_{\ell} = -M_{\ell} \in \mathcal{Q}_2, (2.8)$$

where ℓ is defined at (1.7). The point (0,0) is singular because p > 2; its study concern in particular the solutions w with a *double zero*. When $\varepsilon(\gamma + \alpha) < 0$, the point M_{ℓ} is associated to the solution $w \equiv \ell r^{\gamma}$ of equation (\mathbf{E}_w) given at (1.1).

Linearization around M_{ℓ} . Near the point M_{ℓ} , setting

$$y = \ell + \overline{y}, \qquad Y = -(\gamma \ell)^{p-1} + \overline{Y},$$
(2.9)

system (**S**) is equivalent in \mathcal{Q}_4 to

$$\overline{y}' = -\gamma \overline{y} - \varepsilon \nu(\alpha) \overline{Y} + \Psi(\overline{Y}), \qquad \overline{Y}' = \varepsilon \alpha \overline{y} - (\gamma + N + \nu(\alpha)) \overline{Y} + \varepsilon \Psi(\overline{Y}), \qquad (2.10)$$

where

$$\nu(\alpha) = -\frac{\gamma(N+\gamma)}{(p-1)(\gamma+\alpha)}, \text{ and } \Psi(\vartheta) = ((\gamma\ell)^{p-1} - \vartheta)^{1/(p-1)} - \gamma\ell + \frac{(\gamma\ell)^{2-p}}{p-1}\vartheta, \qquad \vartheta < (\gamma\ell)^{p-1}, \ (2.11)$$

thus $\varepsilon \nu(\alpha) > 0$. The linearized problem is given by

$$\overline{y}' = -\gamma \overline{y} - \varepsilon \nu(\alpha) \overline{Y}, \qquad \overline{Y}' = \varepsilon \alpha \overline{y} - (\gamma + N + \nu(\alpha)) \overline{Y}.$$

Its eigenvalues $\lambda_1 \leq \lambda_2$ are the solutions of equation

$$\lambda^2 + (2\gamma + N + \nu(\alpha))\lambda + p'(N + \gamma) = 0$$
(2.12)

The discriminant Δ of the equation (2.12) is given by

$$\Delta = (2\gamma + N + \nu(\alpha))^2 - 4p'(N + \gamma) = (N + \nu(\alpha))^2 - 4\nu(\alpha)\alpha.$$
(2.13)

For $\varepsilon = 1$, M_{ℓ} is a *sink*, and a node point, since $\nu(\alpha) > 0$, and $\alpha < 0$, thus $\Delta > 0$. For $\varepsilon = -1$, we have $\nu(\alpha) < 0$; the nature of M_{ℓ} depends on the critical value α^* defined at (1.5); indeed

$$\alpha = \alpha^* \Longleftrightarrow \lambda_1 + \lambda_2 = 0.$$

Then M_{ℓ} is a sink when $\alpha > \alpha^*$ and a source when $\alpha < \alpha^*$. Moreover α^* corresponds to a spiral point, and M_{ℓ} is a node point when $\Delta \ge 0$, that means $\alpha \le \alpha_1$, or $\gamma > N/2 + \sqrt{p'(N+\gamma)}$ and $\alpha_2 \le \alpha$, where

$$\alpha_1 = -\gamma + \frac{\gamma(N+\gamma)}{(p-1)(2\gamma+N+2(p'(N+\gamma))^{1/2})}, \qquad \alpha_2 = -\gamma + \frac{\gamma(N+\gamma)}{(p-1)(2\gamma+N-2(p'(N+\gamma))^{1/2})}.$$
(2.14)

When $\Delta > 0$, and $\lambda_1 < \lambda_2$, one can choose a basis of eigenvectors

$$e_1 = (-\varepsilon\nu(\alpha), \lambda_1 + \gamma)$$
 and $e_2 = (\varepsilon\nu(\alpha), -\gamma - \lambda_2).$ (2.15)

Remark 2.4 One verifies that $\alpha^* < -1$; and $\alpha^* < -(p-1)/(p-2)$ if and only if p > N. Also $\alpha_2 \leq 0$, and $\alpha_2 = 0 \iff N = p/((p-2)^2)$; and $\alpha_2 > -p' \iff \gamma^2 - 7\gamma - 8N < 0$, which is not always true.

As in [4, Theorem 2.16] we prove that the Hopf bifurcation point is not degenerate, which implies the existence of small cycles near α^* .

Proposition 2.5 Let $\varepsilon = -1$, and $\alpha = \alpha^* > -\gamma$. Then M_ℓ is a weak source. If $\alpha > \alpha^*$ and $\alpha - \alpha^*$ is small enough, there exists a unique limit cycle in \mathcal{Q}_4 , attracting at $-\infty$.

2.2 Other systems for positive solutions

When w has a constant sign, we define two functions associated to (y, Y):

$$\zeta(\tau) = \frac{|Y|^{(2-p)/(p-1)}Y}{y}(\tau) = -\frac{rw'(r)}{w(r)}, \qquad \sigma(\tau) = \frac{Y}{y}(\tau) = -\frac{|w'(r)|^{p-2}w'(r)}{rw(r)}.$$
(2.16)

Thus ζ describes the behaviour of w'/w and σ is the slope in the phase plane (y, Y). They satisfy the system

$$\zeta' = \zeta(\zeta - \eta) + \varepsilon |\zeta y|^{2-p} (\alpha - \zeta)/(p-1) = \zeta(\zeta - \eta + \varepsilon(\alpha - \zeta)/(p-1)\sigma),$$

$$\sigma' = \varepsilon(\alpha - N) + \left(|\sigma y|^{(2-p)/(p-1)} \sigma - N\right) (\sigma - \varepsilon) = \varepsilon(\alpha - \zeta) + (\zeta - N) \sigma.$$
(Q)

In particular, System (**Q**) provides a short proof of the local existence and uniqueness of the *regular* solutions: they correspond to its stationary point $(0, \varepsilon \alpha/N)$, see Section 3.1.

Moreover, if w and w' have a strict constant sign, that means in any quadrant Q_i , we can define

$$\psi = \frac{1}{\sigma} = \frac{y}{Y} \tag{2.17}$$

We obtain a new system relative to (ζ, ψ) :

$$\zeta' = \zeta(\zeta - \eta + \varepsilon(\alpha - \zeta)\psi/(p - 1)),$$

$$\psi' = \psi(N - \zeta + \varepsilon(\zeta - \alpha)\psi).$$
(P)

We are reduced to a polynomial system, thus with no singularity. System (**P**) gives the existence of singular solutions when p > N, corresponding to its stationary point $(\eta, 0)$, see Section 5.

We will also consider another system in any Q_i : setting

$$\zeta = -1/g, \qquad \sigma = -s, \qquad d\tau = gsd\nu = |Y|^{(p-2)/(p-1)} d\nu,$$
 (2.18)

we find

$$\frac{dg}{d\nu} = g(s(1+\eta g) + \varepsilon(1+\alpha g)/(p-1)), \\ ds/d\nu = -s(\varepsilon(1+\alpha g) + (1+Ng)s).$$
 (R)

System (**R**) allows to get the existence of solutions w with a hole or a compact support, and other solutions, corresponding to its stationary points $(0, -\varepsilon)$ and $(-1/\alpha, 0)$; it provides a complete study of the singular point (0, 0) of system (**S**), see Sections 3.3, 5; and of the focussing solutions, see Section 9.

Remark 2.6 The particular solutions can be found again in the different phase planes, where their trajectories are lines:

For $\alpha = N$, the solutions (1.10) correspond to $Y \equiv \varepsilon y$, that means $\sigma \equiv \varepsilon$. For $\alpha = \eta \neq 0$ the solutions (1.11) correspond to $\zeta \equiv \eta$. For $\alpha = -p'$, the solutions (1.12) are given by $\zeta + \varepsilon N \sigma \equiv \alpha$. For N = 1, $\alpha = -(p-2)/(p-1)$, the solutions (1.14) satisfy $\alpha g + \varepsilon s \equiv -1$.

3 Global existence

3.1 Local existence and uniqueness

Proposition 3.1 Let $r_1 > 0$ and $a, b \in \mathbb{R}$. If $(a, b) \neq (0, 0)$, there exists a unique solution w of equation (\mathbf{E}_w) in a neighborhood \mathcal{V} of r_1 , such that w and $|w'|^{p-2}w' \in C^1(\mathcal{V})$ and $w(r_1) = a$, $w'(r_1) = b$. It extends on a maximal interval I where $(w(r), w'(r)) \neq (0, 0)$.

Proof. If $b \neq 0$, the Cauchy theorem directly applies to system (S). If b = 0 the system is a priori singular on the line $\{Y = 0\}$ since p > 2. In fact it is only singular at (0,0). Indeed near any point $(\xi, 0)$ with $\xi \neq 0$, one can take Y as a variable, and

$$\frac{dy}{dY} = F(Y,y), \qquad F(Y,y) := \frac{\gamma y + |Y|^{(2-p)/(p-1)} Y}{(\gamma+N)Y + \varepsilon(|Y|^{(2-p)/(p-1)} Y - \alpha y)}$$

where F is continuous in Y and C^1 in y, hence local existence and uniqueness hold.

Notation 3.2 For any point $P_0 = (y_0, Y_0) \in \mathbb{R}^2 \setminus \{(0, 0)\}$, the unique trajectory in the phase plane (y, Y) of system (S) going through P_0 is denoted by $\mathcal{T}_{[P_0]}$. By symmetry, $\mathcal{T}_{[-P_0]} = -\mathcal{T}_{[P_0]}$.

Next we show the existence of regular solutions. Our proof is short, based on phase plane portrait, and not on a fixed point method, rather delicate because p > 2, see [3].

Theorem 3.3 For any $a \in \mathbb{R}$, $a \neq 0$, there exists a unique solution w = w(., a) of equation (\mathbf{E}_w) in an interval $[0, r_0)$, such that w and $|w'|^{p-2} w' \in C^1([0, r_0))$ and

$$w(0) = a, \qquad w'(0) = 0;$$
 (3.1)

and then $\lim_{r\to 0} |w'|^{p-2} w'/rw = -\varepsilon \alpha/N$. In other words in the phase plane (y, Y) there exists a unique trajectory \mathcal{T}_r such that $\lim_{\tau\to-\infty} y = \infty$, and $\lim_{\tau\to-\infty} Y/y = \varepsilon \alpha/N$.

Proof. We have assumed $\alpha \neq 0$ (when $\alpha = 0, w \equiv a$ from (1.13)). If such a solution w exists, then from (2.1) and (2.2), $J'_N(r) = r^{N-1}(N-\alpha)a(1+o(1))$ near 0. Thus $J_N(r) = r^{N-1}(1-\alpha/N)a(1+o(1))$, hence $\lim_{r\to 0} |w'|^{p-2} w'/rw = -\varepsilon\alpha/N$; in other words, $\lim_{\tau\to -\infty} \sigma = \varepsilon\alpha/N$. And

 $\lim_{\tau\to-\infty} y = \infty$, thus $\lim_{\tau\to-\infty} \zeta = 0$, and $\varepsilon \alpha \zeta > 0$ near $-\infty$. Reciprocally consider system (**Q**). The point $(0, \varepsilon \alpha/N)$ is stationary. Setting $\sigma = \varepsilon \alpha/N + \overline{\sigma}$, the linearized system near this point is given by

$$\zeta' = p'\zeta, \qquad \bar{\sigma}' = \varepsilon \zeta(\alpha - N)/N - N\bar{\sigma}.$$

One finds is a saddle point, with eigenvalues -N and p'. Then there exists a unique trajectory \mathcal{T}'_r in the phase-plane (ζ, σ) starting at $-\infty$ from $(0, \varepsilon \alpha/N)$ with the slope $\varepsilon(\alpha - N)/N(N + p') \neq 0$ and $\varepsilon \alpha \zeta > 0$. It corresponds to a unique trajectory \mathcal{T}_r in the phase plane (y, Y), and $\lim_{\tau \to -\infty} y = \infty$, since $y = |\sigma| |\zeta|^{1-p} p^{1/(p-2)}$. For any solution (ζ, σ) describing \mathcal{T}'_r , the function $w(r) = r^{\gamma} (|\sigma| |\zeta|^{1-p} (\tau))^{1/(p-2)}$ satisfies $\lim_{r \to 0} |w'|^{p-2} w'/rw = -\varepsilon \alpha/N$. As a consequence, $w^{(p-2)/(p-1)}$ has a finite nonzero limit, and $\lim_{r \to 0} w' = 0$; thus w is regular. Local existence and uniqueness follows for any $a \neq 0$, by Remark 2.1.

Definition 3.4 The trajectory \mathcal{T}_r in the plane (y, Y) and its opposite $-\mathcal{T}_r$ will be called regular trajectories. We shall say that y is regular. Observe that \mathcal{T}_r starts in \mathcal{Q}_1 if $\varepsilon \alpha > 0$, and in \mathcal{Q}_4 if $\varepsilon \alpha < 0$.

Remark 3.5 From Theorem 3.3 and Remark 2.1, all regular solutions are obtained from one one of them: $w(r, a) = aw(a^{-1/\gamma}r, 1)$. Thus they have the same behaviour near ∞ .

3.2 Sign properties

Next we give informations on the zeros of w or w', by using the monotonicity properties of the functions y_d, Y_d , in particular y, Y, and ζ and σ . At any extremal point τ , they satisfy respectively

$$y_d''(\tau) = y_d(\tau) \left(d(\eta - d) + \frac{\varepsilon(d - \alpha)}{p - 1} e^{((p-2)d + p)\tau} |dy_d(\tau)|^{2-p} \right),$$
(3.2)

$$Y_d''(\tau) = Y_d(\tau) \left((p-1)^2 (\eta - d)(p' + d) + \varepsilon (d - \alpha) e^{((p-2)d + p)\tau} |Y_d(\tau)|^{(2-p)/(p-1)} \right),$$
(3.3)

$$(p-1)y''(\tau) = \gamma^{2-p}y(\tau)\left(-\gamma^{p-1}(N+\gamma) - \varepsilon(\gamma+\alpha)|y(\tau)|^{2-p}\right) = -|Y(\tau)|^{(2-p)/(p-1)}Y'(\tau), \quad (3.4)$$

$$Y''(\tau) = Y(\tau) \left(-\gamma(N+\gamma) - \varepsilon(\gamma+\alpha) |Y(\tau)|^{(2-p)/(p-1)} \right) = \varepsilon \alpha y'(\tau), \tag{3.5}$$

$$(p-1)\zeta''(\tau) = -\varepsilon(p-2)((\alpha-\zeta)|\zeta|^{2-p}|y|^{-p}yy')(\tau) = \varepsilon(p-2)((\alpha-\zeta)(\gamma+\zeta)|\zeta y|^{2-p})(\tau), \quad (3.6)$$

$$(p-1)\sigma''(\tau) = -(p-2)((\sigma-\varepsilon)|\sigma|^{(2-p)/(p-1)}Y|y|^{(4-3p)/(p-1)}y')(\tau) = \zeta'(\tau)(\sigma(\tau)-\varepsilon).$$
(3.7)

Proposition 3.6 Let $w \neq 0$ be any solution of (\mathbf{E}_w) on an interval I.

(i) If $\varepsilon = 1$ and $\alpha \leq N$, then w has at most one simple zero; if $\alpha < N$ and w is regular, it has no zero. If $\alpha = N$ it has no simple zero and a compact support. If $\alpha > N$ and w is regular, it has at least one simple zero.

(ii) If $\varepsilon = -1$ and $\alpha \ge \min(0, \eta)$, then w has at most one simple zero. If $w \ne 0$ has a double zero, then it has no simple zero. If $\alpha > 0$ and w is regular, it has no zero.

(iii) If $\varepsilon = -1$ and $-p' \leq \alpha < \min(0, \eta)$, then w' has at most one simple zero, consequently w has at most two simple zeros, and at most one if w is regular. If $\alpha < -p'$, the regular solutions have at least two zeros.

Proof. (i) Let $\varepsilon = 1$. Consider two consecutive simple zeros $\rho_0 < \rho_1$ of w, with w > 0 on (ρ_0, ρ_1) ; hence $w'(\rho_1) < 0 < w'(\rho_0)$. If $\alpha \leq N$, we find from (2.1),

$$J_N(\rho_1) - J_N(\rho_0) = -\rho_1^{N-1} \left| w'(\rho_1) \right|^{p-2} - \rho_0^{N-1} w'(\rho_0)^{p-1} = (N-\alpha) \int_{\rho_0}^{\rho_1} s^{N-1} w ds$$

which is contradictory; thus w has at most one simple zero. The contradiction holds as soon as ρ_0 is simple, even if ρ_1 is not. If w is regular with w(0) > 0, and ρ_1 is a first zero, and $\alpha < N$,

$$J_N(\rho_1) = -\rho_1^{N-1} \left| w'(\rho_1) \right|^{p-1} = (N-\alpha) \int_0^{\rho_1} s^{N-1} w ds > 0$$

which is still impossible. If $\alpha = N$, the (Barenblatt) solutions are given by (1.10). Next suppose $\alpha > N$ and w regular. If w > 0, then $J_N < 0$, thus $w^{-1/(p-1)}w' + r^{1/(p-1)} < 0$. Then the function $r \mapsto r^{p'} + \gamma w^{(p-2)/(p-1)}$ is non increasing and we reach a contradiction for large r. Thus w has a first zero ρ_1 , and $J_N(\rho_1) < 0$, thus $w'(\rho_1) \neq 0$.

(ii) Let $\varepsilon = -1$ and $\alpha \ge \min(\eta, 0)$. Here we use the substitution (2.3) from some $d \ne 0$. If y_d has a maximal point, where it is positive, and is not constant, then (3.2) holds. Taking $d \in (0, \min(\alpha, \eta))$ if $\eta > 0$, $d = \eta$ if $\eta \le 0$, we reach a contradiction. Hence y_d has at most a simple zero, and no simple zero if it has a double one. Suppose w regular and $\alpha > 0$. Then w' > 0 near 0, from Theorem 3.3. As long as w stays positive, any extremal point r is a strict minimum, from (\mathbf{E}_w) , thus in fact w' stays positive.

(iii) Let $\varepsilon = -1$ and $-p' \leq \alpha < \min(0, \eta)$. Suppose that w' and has two consecutive zeros $\rho_0 < \rho_1$, and one of them is simple, and use again (2.3) with $d = \alpha$. Then the function Y_{α} has an extremal point τ , where it is positive and is not constant; from (3.3),

$$Y''_{\alpha}(\tau) = (p-1)^2 (\eta - \alpha)(p' + \alpha) Y_{\alpha}(\tau),$$
(3.8)

thus $Y''_{\alpha}(\tau) \geq 0$, which is contradictory. Next consider the regular solutions. They satisfy $Y_{\alpha}(\tau) = e^{(\alpha(p-1)+p)\tau}(|\alpha|a/N)(1+o(1) \text{ near } -\infty)$, from Theorem 3.3 and (2.3), thus $\lim_{\tau \to -\infty} Y_{\alpha} = 0$. As above Y_{α} cannot have any extremal point, then Y_{α} is positive and increasing. In turn w' < 0 from (2.3), hence w has at most one zero.

Proposition 3.7 Let $w \neq 0$ be any solution of (\mathbf{E}_w) on an interval I. If $\varepsilon = 1$, then w has a finite number of isolated zeros. If $\varepsilon = -1$, it has a finite number of isolated zeros in any interval $[m, M] \cap I$ with $0 < m < M < \infty$.

Proof. Let Z be the set of isolated zeros on I. If w has two consecutive isolated zeros $\rho_1 < \rho_2$, and $\tau \in (e^{\rho_1}, e^{\rho_2})$ is a maximal point of $|y_d|$, from (3.2), it follows that

$$\varepsilon e^{((p-2)d+p)\tau} |dy_d(\tau)|^{2-p} (d-\alpha) \leq (p-1)d(d-\eta).$$
 (3.9)

That means with $\rho = e^{\tau} \in (\rho_1, \rho_2)$,

$$\varepsilon \rho^{p} |w(\rho)|^{2-p} (d-\alpha) \leq (p-1)d^{p-1}(d-\eta).$$
 (3.10)

First suppose $\varepsilon = 1$ and fix $d > \alpha$. Consider the energy function

$$E(r) = \frac{1}{p'} |w'|^p + \frac{\alpha}{2} w^2.$$

It is nonincreasing since $E'(r) = -(N-1)r^{-1}|w'|^p - rw'^2$, thus bounded on $I \cap [\rho_1, \infty)$. Then w is bounded, ρ_2 is bounded, Z is a bounded set. If Z is infinite, there exists a sequence of zeros (r_n) converging to some point $\overline{r} \in [0,\infty)$, and a sequence (τ_n) of maximal points of $|y_d|$ converging to $\overline{\tau} = \ln \overline{r}$. If $\overline{r} > 0$, then $w(\overline{r}) = w'(\overline{r}) = 0$; we get a contradiction by taking $\rho = \rho_n = e^{\tau_n}$ in (3.10), because the left-hand side tends to ∞ . If $\overline{r} = 0$, fixing now $d < \eta$, there exists a sequence (τ_n) of maximal points of $|y_d|$ converging to $-\infty$. Then $w(\rho_n) = O(\rho_n^{p/(p-2)})$, and $w'(\rho_n) = -d\rho_n^{-1}w(\rho_n) = O(\rho_n^{2/(p-2)})$, thus $E(\rho_n) = o(1)$. Since E is monotone, it implies $\lim_{r\to 0} E(r) = 0$, hence $E \equiv 0$, and $w \equiv 0$, which is contradictory. Next suppose $\varepsilon = -1$ and fix $d < \alpha$. If $Z \cap [m, M]$ is infinite, we construct a sequence converging vers some $\overline{r} > 0$ and reach a contradiction as above.

Proposition 3.8 Let y be any non constant solution of (E_y) , on a maximal interval I where $(y, Y) \neq (0, 0)$, and s be an extremity of I.

(i) If y has a constant sign near s, then the same is true for Y.

(ii) If y > 0 is strictly monotone near s, then Y, ζ, σ are monotone near s.

(iii) If y > 0 is not strictly monotone near s, then $s = \pm \infty$, $\varepsilon(\gamma + \alpha) < 0$ and y oscillates around ℓ .

(iv) If y is oscillating around 0 near s, then $\varepsilon = -1$, $s = \pm \infty$, $\alpha < -p'$; if $\alpha > -\gamma$, then $|y| > \ell$ at the extremal points.

Proof. (i) The function w has at most one extremal point on I: at such a point, it satisfies $(|w'|^{p-2}w')' = -\varepsilon \alpha w$ with $\alpha \neq 0$. From (2.7), Y has a constant sign near s.

(ii) Suppose y strictly monotone near s. At any extremal point τ of Y, we find $Y''(\tau) = \varepsilon \alpha y'(\tau)$ from (3.5). Then $y'(\tau) \neq 0$, $Y''(\tau)$ has a constant sign. Thus τ is unique, and Y is strictly monotone near s. Next consider ζ . If there exists τ_0 such that $\zeta(\tau_0) = \alpha$, then $\zeta'(\tau_0) = \alpha(\alpha - \eta)$, from system (**Q**). If $\alpha \neq \eta$, then τ_0 is unique, thus $\alpha - \zeta$ has a constant sign near s. Then $\zeta''(\tau)$ has a constant

sign at any extremal point τ of ζ , from (3.6), thus ζ is strictly monotone near s. If $\alpha = \eta$, then $\zeta \equiv \alpha$. At last consider σ . If there exists τ_0 such that $\sigma(\tau_0) = \varepsilon$, then $\sigma'(\tau_0) = \varepsilon(\alpha - N)$ from System (Q). If $\alpha \neq N$, then τ_0 is unique, and $\sigma - \varepsilon$ has a constant sign near s. Thus $\sigma''(\tau)$ has a constant sign at any extremal point τ of σ , from (3.7) and assertion (i). If $\alpha = N$, then $\sigma \equiv \varepsilon$.

(iii) Let y be positive and not strictly monotone near s. There exists a sequence (τ_n) strictly monotone, converging to $\pm \infty$, such that $y'(\tau_n) = 0$, $y''(\tau_{2n}) > 0 > y''(\tau_{2n+1})$. Since $y(\tau_n) = \gamma^{-1} |Y|^{(2-p)/(p-1)} Y(\tau_n)$, we deduce Y < 0 near s, from (i). From (3.5),

$$-\varepsilon(\gamma+\alpha)y(\tau_{2n+1})^{2-p} \leq \gamma^{p-1}(N+\gamma) \leq -\varepsilon(\gamma+\alpha))y(\tau_{2n})^{2-p},$$
(3.11)

thus $\varepsilon(\gamma + \alpha) < 0$ and $y(\tau_{2n}) < \ell < y(\tau_{2n+1})$, and $Y(\tau_{2n+1}) < -(\gamma \ell)^{p-1} < Y(\tau_{2n})$. If s is finite, then y(s) = y'(s) = 0, which is impossible; thus $s = \pm \infty$.

(iv) If y is changing sign, then $\varepsilon = -1$ and $\alpha < -p'$, from Propositions 3.6 and 3.7. At any extremal point τ ,

$$(\alpha + \gamma) |y(\tau)|^{2-p} \leq \gamma^{p-1}(N + \gamma)$$

from (3.4); if $\alpha > -\gamma$ it means $|y| > \ell$.

3.3 Double zeros and global existence

Theorem 3.9 For any $\overline{r} > 0$, there exists a unique solution w of (E_w) defined in a interval $[\overline{r}, \overline{r} \pm h)$ such that

$$w > 0$$
 on $(\overline{r}, \overline{r} \pm h)$ and $w(\overline{r}) = w'(\overline{r}) = 0.$

Moreover $\varepsilon h < 0$ and

$$\lim_{r \to \overline{r}} |(\overline{r} - r)|^{(p-1)/(2-p)} \, \overline{r}^{1/(2-p)} w(r) = \pm ((p-2)/(p-1))^{(p-1)/(p-2)}.$$
(3.12)

In other words in the phase plane (y, Y) there exists a unique trajectory $\mathcal{T}_{\varepsilon}$ converging to (0, 0) at $\varepsilon \infty$. It has the slope ε and converges in finite time; it depends locally continuously of α .

Proof. Suppose that a solution $w \neq 0$ exists on $[\overline{r}, \overline{r} \pm h)$ with $w(\overline{r}) = w'(\overline{r}) = 0$. From Propositions 3.7 and 3.8, up to a symmetry, y > 0, |Y| > 0 near $\overline{\tau} = \ln \overline{r}$, and $\lim_{\tau \to \ln \overline{r}} y = 0$, and σ, ζ are monotone near $\ln r$. Let μ and λ be their limits. If $|\mu| = \infty$, then $|\lambda| = \infty$, because $\zeta = |Y|^{(2-p)/(p-1)} \sigma$, $|\zeta|^{p-2} \zeta = \sigma y^{2-p}$; then $f = 1/\zeta$ tends to 0; but

$$f' = -1 + \eta f + \varepsilon \frac{1 - \alpha f}{(p - 1)\sigma},\tag{3.13}$$

thus f' tends to -1, which is impossible. Thus μ is finite. If λ is finite, then $\mu = 0$, thus $\lambda = \alpha$, from system (**Q**), $\ln w$ is integrable at \overline{r} , which is not true. Then $\lambda = \varepsilon \infty$, hence

$$\mu = \lim_{\tau \to \ln \overline{r}} \sigma = \varepsilon$$

from system (**Q**). Then $\varepsilon Y > 0$ near $\overline{\tau}$, then $\varepsilon w' < 0$ near \overline{r} , thus $\varepsilon h < 0$. Consider system (**R**): as τ tends to $\overline{\tau}$, ν tends to $\pm \infty$, and (g, s) converges to the stationary point $(0, -\varepsilon)$.

Reciprocally, setting $s = -\varepsilon/\beta + h$, the linearized system of system (**R**) at this point is given by

$$rac{dg}{d
u} = -arepsilon rac{p-2}{p-1}g, \qquad rac{dh}{d
u} = (lpha - N)g + arepsilon h$$

The eigenvalues are $-\varepsilon(p-2)/(p-1)$ and ε , thus we find a saddle point. There are two trajectories converging to $(0, -\varepsilon)$. The first one satisfies $g \equiv 0$, it does not correspond to a solution of the initial problem. Then there exists a unique trajectory converging to $(0, -\varepsilon)$, as ν tends to $\varepsilon\infty$, with g > 0 near $\varepsilon\infty$. It is associated to the eigenvalue $-\varepsilon(p-2)/(p-1)$ and the eigenvector $((2p-3)/(p-1), \varepsilon(N-\alpha))$. It satisfies $dg/d\nu = -\varepsilon((p-2)/(p-1))g(1+o(1))$, thus $dg/d\tau =$ ((p-2)/(p-1))(1+o(1)). Then τ has a finite limit $\bar{\tau}$, and τ increases to $\bar{\tau}$ if $\varepsilon = 1$ and decreases to $\bar{\tau}$ if $\varepsilon = -1$. In turn $|Y|^{(p-2)/(p-1)} = gs$ tends to 0, and s tends to ε , thus (y, Y) tends to (0, 0)as τ tends to $\bar{\tau}$. Then w and w' converges to 0 at $\bar{r} = e^{\bar{\tau}}$. And $w'w^{-1/(p-1)} + (\varepsilon + o(1))r^{1/(p-1)} = 0$, which implies (3.12).

Corollary 3.10 Let $r_1 > 0$, and $a, b \in \mathbb{R}$ and w be any local solution such that $w(r_1) = a$, $w'(r_1) = b$.

(i) If (a,b) = (0,0), then w has a unique extension by 0 on (r_1,∞) if $\varepsilon = 1$, on $(0,r_1)$ if $\varepsilon = -1$. (ii) If $(a,b) \neq (0,0)$, w has a unique extension to $(0,\infty)$.

Proof. (i) Assume a = b = 0, the function $w \equiv 0$ is a solution. Let w be any local solution near r_1 , defined in an interval $(r_1 - h_1, r_1 + h_1)$ with $w(r_1) = w'(r_1) = 0$. Suppose that there exists $h_2 \in (0, h_1)$ such that $w(r_1 + \varepsilon h_1) \neq 0$. Let $\bar{h} = \inf \{h \in (0, h_1) : w(r_1 + \varepsilon h) \neq 0\}$, and $\bar{r} = r_1 + \varepsilon \bar{h}$, thus $w(\bar{r}) = w'(\bar{r}) = 0$, and for example w > 0 on some interval $(\bar{r}, \bar{r} + \varepsilon k))$ with k > 0. This contradicts theorem 3.9. Thus $w \equiv 0$ on $(r_1, r_1 + \varepsilon h_1)$.

(ii) From Theorems 3.9 and 3.3, w has no double zero for $\varepsilon(r-r_1) < 0$, and has a unique extension to a maximal interval with no double zero. From (i) it has a unique extension to $(0, \infty)$. In particular any local regular solution is defined on $[0, \infty)$.

4 Asymptotic behaviour

Next the function y is supposed to be monotone, thus w has a constant sign near 0 or ∞ , we can assume that w > 0.

Proposition 4.1 Let y be any solution of (E_y) strictly monotone and positive near $s = \pm \infty$.

(1) Then (ζ, σ) has a limit (λ, μ) near s, given by is some of the values

$$A_{\gamma} = \left(-\gamma, \varepsilon \frac{\alpha + \gamma}{N + \gamma}\right), \quad A_{r} = (0, \varepsilon \alpha/N), \quad A_{\alpha} = (\alpha, 0),$$

$$L_{\eta} = \eta (1, \infty) (if \ p \neq N), \quad L_{+} = (0, \infty) (if \ p \ge N), \quad L_{-} = (0, -\infty) (if \ p > N). \tag{4.1}$$

(2) More precisely,

(i) Either $\varepsilon(\gamma + \alpha) < 0$ and (y, Y) converges to $\pm M_{\ell}$. Then $(\lambda, \mu) = A_{\gamma}$ and $(\varepsilon = 1, s = \infty)$ or $(\varepsilon = -1, s = \infty \text{ for } \alpha \leq \alpha^*, s = \infty \text{ for } \alpha > \alpha^*).$

(ii) Or (y, Y) converges to (0, 0). Then $(s = \infty \text{ and } -\gamma < \alpha)$ or $(s = -\infty \text{ and } \alpha < -\gamma)$, or $(s = \varepsilon \infty \text{ and } \alpha = -\gamma)$ and $(\lambda, \mu) = A_{\alpha}$.

(iii) Or $\lim_{\tau \to s} y = \infty$. Then $s = -\infty$. If p < N, then $(\lambda, \mu) = A_r$ or L_η . If p = N, then $(\lambda, \mu) = A_r$ or L_+ . If p > N, then $(\lambda, \mu) = A_r, L_\eta, L_+$ or L_- .

Proof. (1) The functions Y, σ, ζ are also monotone, and by definition $\zeta \sigma > 0$. Thus ζ has a limit $\lambda \in [-\infty, \infty]$ and σ has a limit $\mu \in [-\infty, \infty]$, and $\lambda \mu \ge 0$.

(i) λ is finite. Indeed if $\lambda = \pm \infty$, then $f = 1/\zeta$ tends to 0. From (3.13), either $\mu = \pm \infty$, then f' tends to -1, which is imposible; or μ is finite, thus $\mu = \varepsilon$ from system (**Q**), then f' tends to (2-p)/(p-1), which is still contradictory.

(ii) Either μ is finite, thus (λ, μ) is a stationary point of system (Q), equal to A_{γ}, A_r or A_{α} .

(iii) Or $\mu = \pm \infty$ and $(\lambda, 0)$ is a stationary point of system (**P**).

• If $p \neq N$, either $\lambda = \eta \neq 0$ and $(\lambda, \mu) = L_{\eta}$; or $\lambda = 0$ and $(\lambda, \mu) = L_{+}$ or L_{-} . In the last case (ζ, ψ) converges to (0, 0), and $\zeta'/\psi' = -(\eta \zeta/N\psi)(1 + o(1))$, thus $\eta < 0$, that means p > N.

• If p = N, then again (ζ, ψ) converges to (0,0), thus $\mu = \pm \infty$, and $\psi' = N\psi(1 + o(1))$, and necessarily $s = -\infty$. We make the substitution (2.4) with d = 0. Then $y_0(\tau) = w(r)$, and y_0 satisfies

$$y_0' = -|Y_0|^{(2-p)/(p-1)} Y_0 = -\zeta y_0 = o(y_0), \qquad Y_0' = \varepsilon e^{p\tau} y_0(\alpha - \zeta) = \varepsilon e^{p\tau} y_0 \alpha (1 + o(1)).$$

Thus for any v > 0, we get $y_0 = O(e^{-v\tau})$ and $1/y_0 = O(e^{v\tau})$. Then Y'_0 is integrable, and Y_0 has a finite limit $|k|^{p-2} k$. Suppose that k = 0. Then $Y_0 = O(e^{(p-v)\tau})$, and y_0 has a finite limit $a \ge 0$. If $a \ne 0$, then $Y'_0 = \varepsilon \alpha a e^{p\tau} (1 + o(1))$; in turn $Y_0 = p^{-1} \varepsilon \alpha a e^{p\tau} (1 + o(1))$, and $\psi = e^{p\tau} y_0/Y_0$ does not tend to 0. If a = 0, then $y_0 = O(e^{p'\tau})$, which contradicts the estimate of $1/y_0$. Thus k > 0 and

$$y_0 = -k\tau(1+o(1)), \qquad Y_0 = k^{p-1}(1+o(1));$$
(4.2)

hence $(\lambda, \mu) = L_+$.

(2) Since y is monotone, we encounter one of the three following cases:

(i) (y, Y) converges to $\pm M_{\ell}$. Then $(\lambda, \mu) = A_{\gamma}$ and M_{ℓ} is a source (or a weak source) for $\alpha \leq \alpha^*$, a sink for $\alpha > \alpha^*$.

(ii) y tends to 0. Since λ is finite, (y, Y) converges to (0, 0). And $|\sigma| = |\zeta|^{p-1} y^{p-2}$ tends to 0, thus $(\lambda, \mu) = A_{\alpha}$. If $-\gamma < \alpha$, seeing that $y' = -y(\gamma + \zeta) < 0$ we find $s = \infty$. If $\alpha < -\gamma$, then $s = -\infty$. If $\alpha = -\gamma < 0$, then $\varepsilon (\gamma + \zeta) > 0$, from the first equation of **(Q)**, thus $\varepsilon y' < 0$, hence $s = \varepsilon \infty$.

(iii) y tends to ∞ . Either $\lambda \neq 0$, thus $|\sigma| = |\zeta|^{p-1} y^{p-2}$ tends to ∞ , and $\lambda = \eta$ from system (**Q**), thus $p \neq N$, $(\lambda, \mu) = L_{\eta}$. Or $\lambda = 0$ and μ is finite, thus $\mu = \varepsilon \alpha/N$, $(\lambda, \mu) = A_r$. Or $(\lambda, \mu) = L_0$; then either p = N, $L_0 = L_{\eta}$, or p > N. In any case, $y' = -y(\gamma + \zeta) < 0$, from (1.2), hence $s = -\infty$.

Next we apply these results to the functions w:

Proposition 4.2 We keep the assumptions of Proposition 4.1. Let w be the solution of (E_w) associated to y by (2.7).

(i) If $(\lambda, \mu) = A_{\gamma}$ (near 0 or ∞), then

$$\lim r^{-\gamma} w = \ell. \tag{4.3}$$

(ii) If $(\lambda, \mu) = A_{\alpha}$ (near 0 or ∞), then

$$\lim r^{\alpha}w = L > 0 \qquad \qquad if \ \alpha \neq -\gamma, \tag{4.4}$$

$$\lim r^{-\gamma} (\ln r)^{1/(p-2)} w = ((p-2)\gamma^{p-1}(N+\gamma))^{-1/(p-2)} \qquad if \ \alpha = -\gamma.$$
(4.5)

(iii) If p < N and $(\lambda, \mu) = L_{\eta}$, then

$$\lim_{r \to 0} r^{\eta} w = c > 0. \tag{4.6}$$

(iv) If p > N and $(\lambda, \mu) = L_{\eta}$, then

$$\lim_{r \to 0} r^{-|\eta|} w = c > 0.$$
(4.7)

(v) If p = N and $(\lambda, \mu) = L_+$, then

$$\lim_{r \to 0} |\ln r|^{-1} w = k > 0, \qquad \lim_{r \to 0} rw' = -k \qquad \text{if } p = N.$$
(4.8)

(vi) If p > N and $(\lambda, \mu) = L_+$, or L_- , then

$$\lim_{r \to 0} w = a > 0, \qquad \lim_{r \to 0} (-r^{(N-1)/(p-1)}w') = c > 0, \tag{4.9}$$

or

$$\lim_{r \to 0} w = a > 0, \qquad \lim_{r \to 0} (-r^{(N-1)/(p-1)}w') = c < 0.$$
(4.10)

Proof. (i) This follows directly from (2.7).

(ii) From (2.16), $rw'(r) = -\alpha w(r)(1 + o(1))$. We are lead to three cases.

• Either $-\gamma < \alpha$, and $s = \infty$. For any v > 0, we find $w = O(r^{-\alpha+v})$ and $1/w = O(r^{\alpha+v})$ near ∞ and $w' = O(r^{-\alpha-1+v})$. Then $J'_{\alpha}(r) = O(r^{\alpha(2-p)-p-1+v})$, hence J'_{α} is integrable, J_{α} has a limit L. And $\lim r^{\alpha}w = L$, seeing that $J_{\alpha}(r) = r^{\alpha}w(1+o(1))$. If L = 0, then $r^{\alpha}w = O(r^{\alpha(2-p)-p+v})$, which contradicts the estimate of $1/w = O(r^{\alpha+v})$ for v small enough. Thus L > 0.

• Or $\alpha < -\gamma$ and $s = -\infty$. For any v > 0, we find $w = O(r^{-\alpha-v})$ and $1/w = O(r^{\alpha+v})$ near 0 and $w' = O(r^{-\alpha-1-v})$. Then $J'_{\alpha}(r) = O(r^{\alpha(2-p)-p-1-v})$, and J'_{α} is still integrable, J_{α} has a limit L, and $\lim r^{\alpha}w = L$. If L = 0, then $r^{\alpha}w = O(r^{\alpha(2-p)-p-v})$, which contradicts the estimate of 1/w. Thus again L > 0.

• Or $\alpha = -\gamma$ and $s = \varepsilon \infty$. Then $Y = -\gamma^{p-1}y^{p-1}(1+o(1))$, and $\mu = 0$, thus $y - \varepsilon Y = y(1+o(1))$. From System (**S**),

$$(y - \varepsilon Y)' = \varepsilon (N + \gamma)Y = -\varepsilon (N + \gamma)\gamma^{p-1} (y - \varepsilon Y)^{p-1} (1 + o(1)).$$

Then $y = (N + \gamma)\gamma^{p-1}(p-2)|\tau|)^{-1/(p-2)}(1 + o(1))$, which is equivalent to (4.5).

(iii) From (2.16), we get $rw'(r) = -\eta w(r)(1+o(1))$. We use (2.3) with $d = \eta$, thus $y_{\eta} = r^{\eta}w$. We find $y_{\eta} = O(e^{-\upsilon\tau})$, $1/y_{\eta} = O(e^{-\upsilon\tau})$, in turn $Y_{\eta} = O(e^{-\upsilon\tau})$. From (2.4), $Y'_{\eta} = O(e^{(p+(p-2)\eta-\upsilon)\tau})$, thus Y'_{η} is integrable, hence Y_{η} has a finite limit. Now $(e^{-\eta\tau}y_{\eta})' = -e^{-\eta\tau}Y_{\eta}^{1/(p-1)}$, and $\eta > 0$, thus y_{η} has a limit c. If c = 0, then $Y_{\eta} = O(e^{(p+(p-2)\eta-\upsilon)\tau})$, $y_{\eta} = O(e^{((p+(p-2)\eta)/(p-1)-\upsilon)\tau})$, which contradicts $1/y_{\eta} = O(e^{-\upsilon\tau})$ for υ small enough. Then (4.6) holds.

(iv) As above, Y_{η} has a finite limit. In turn $r^{-|\eta|+1}w' = |Y_{\eta}|^{(2-p)/(p-1)}Y_{\eta}$ has a limit $c |\eta|$ and w has a limit $a \ge 0$. From (2.16), $rw' = |\eta| w(1 + o(1))$, hence a = 0. Then $c \ge 0$; if b = 0, then Y < 0, the function $v = -e^{(\gamma+N)\tau}Y > 0$ tends to 0 and

$$v' = -e^{(\gamma+N)\tau}\varepsilon(\alpha - \eta)y(1 + o(1)) = -\varepsilon(\alpha - \eta) |\eta| e^{-(\gamma+N)(p-2)/(p-1)\tau} v^{1/(p-1)};$$

we reach again a contradiction. Thus a = 0 and c > 0, and (4.7) holds.

(v) Assertion (4.8) follows from (4.2).

(vi) Here rw' = o(w), thus $w + |w'| = O(r^{-k})$ for any k > 0. Then J'_N is integrable, J_N has a limit at 0, and $\lim_{r\to 0} r^N w = 0$. Thus $\lim_{r\to 0} r^{(N-1)/(p-1)}w' = -c \in \mathbb{R}$, $\lim_{r\to 0} J_N = -\varepsilon |c|^{p-2}c$, $\lim_{r\to 0} w = a \ge 0$. If c = 0, then $J_N(r) = \int_0^r J'_N(s)ds$, implying that $\lim_{r\to 0} w' = 0$. Either a > 0

and then w is regular, then $\lim_{\tau \to -\infty} \sigma = \varepsilon$; or a = 0, then w' > 0 and $(w')^{p-1} = O(rw)$; in both cases we get a contradiction. Thus $c \neq 0$. If a = 0, we find $\lim_{\tau \to -\infty} \zeta = \eta$, which is not true, hence a > 0. In any case (4.9) or (4.10) holds.

Now we study the cases where y is not monotone, and eventually changing sign.

Proposition 4.3 Suppose $\varepsilon = -1$. Let $w \neq 0$ be any solution of (\mathbf{E}_w) .

(i) If $\alpha \leq -\gamma$, then w is oscillating near 0 at ∞ .

(ii) If $\alpha < 0$, then y and Y are bounded at ∞ .

Proof. (i) Suppose by contradiction that $w \ge 0$ for large r, then $y \ge 0$ for large τ . If y > 0 near ∞ , then from Proposition 3.8, either y is constant, which is impossible since (0,0) is the unique stationary point; or y is strictly monotone, which contradicts Proposition 4.1. Then there exists a sequence (τ_n) tending to ∞ such that $y(\tau_n) = y'(\tau_n) = 0$; from Theorem 3.10, $y \equiv 0$ on $(-\infty, \tau_n)$, thus $y \equiv 0$.

(ii) Consider the function

$$\tau \mapsto R(\tau) = \frac{y^2}{2} + \frac{|Y|^{p'}}{p' |\alpha|};$$

it satisfies

$$R'(\tau) = -\gamma y^2 + \frac{1}{|\alpha|} |Y|^{2/(p-1)} - \frac{N+\gamma}{|\alpha|} |Y|^{p'}.$$

From the Young inequality,

$$|\alpha| \left(R'(\tau) + \gamma R(\tau) \right) = |Y|^{2/(p-1)} - \left(N + \frac{1}{p-2} \right) |Y|^{p'} \le \left(\frac{2}{Np+\gamma} \right)^{(p-2)/2} \le 1$$

thus $R(\tau)$ is bounded for large τ , at least by $1/|\alpha|\gamma$. **Proof.**

Proposition 4.4 (i) Assume $\varepsilon = 1$, or $\varepsilon = -1$, $\alpha \notin (\alpha_2, \alpha_1)$. Then for any trajectory of system (S) in \mathcal{Q}_4 near $\pm \infty$, y is strictly monotone near $\pm \infty$.

(ii) Assume $\varepsilon = 1$, and $\alpha \leq \alpha^*$ or $-p' \leq \alpha$. Then system (S) admits no cycle in \mathcal{Q}_4 (or \mathcal{Q}_2).

Proof. (i) In any case M_{ℓ} is a node point. Following [4, Theorem 2.24], we use the linearization defined by (2.9). Consider the line L given by the equation $A\overline{y} + \overline{Y} = 0$, where A is a real parameter. The points of L are in \mathcal{Q}_4 whenever $\overline{Y} < (\gamma \ell)^{p-1}$ and $-\ell < \overline{y}$. We get

$$A\overline{y}' + \overline{Y}' = \left(\varepsilon\nu(\alpha)A^2 + (N+\nu(\alpha))A + \varepsilon\alpha\right)\overline{y} + (A+\varepsilon)\Psi(\overline{Y}).$$

From (2.13), apart from the case $\varepsilon = 1, \alpha = N$, we can find an A such that

$$\varepsilon\nu(\alpha)A^2 + (N + \nu(\alpha))A + \varepsilon\alpha = 0,$$

and $A + \varepsilon \neq 0$. Moreover $\Psi(\overline{Y}) \leq 0$ on $L \cap \mathcal{Q}_4$. Indeed $(p-1)\Psi'(t) = -((\gamma \ell)^{p-1} - t)^{(2-p)/(p-1)} + (\gamma \ell)^{2-p}$, thus Ψ has a maximum 0 on $(-\infty, (\delta \ell)^{p-1})$ at point 0. Then the orientation of the vector

field does not change along $L \cap Q_4$. In particular y cannot oscillate around ℓ , thus y is monotone, from Proposition 3.8. If $\varepsilon = 1, \alpha = N$, then $Y \equiv y \in (\ell, \infty)$ defines the trajectory \mathcal{T}_r , corresponding to the solutions given by (1.10) with K > 0. No solution y can oscillate around ℓ , since the trajectory cannot meet \mathcal{T}_r .

(ii) Suppose that there exists a cycle in \mathcal{Q}_4 .

• Assume $\alpha \leq \alpha^*$. Here M_ℓ is a source, or a weak source, from Proposition 2.5. Any trajectory starting from M_ℓ at $-\infty$ has a limit cycle in Q_1 , which is attracting at ∞ . Writing System (S) under the form $y' = f_1(y, Y), Y' = f_2(y, Y)$, the mean value of the Floquet integral on the period $[0, \mathcal{P}]$ is given by

$$I = \oint \left(\frac{\partial f_1}{\partial y}(y,Y) + \frac{\partial f_2}{\partial Y}(y,Y)\right)d\tau = \oint \left(\frac{|Y|^{(2-p)/(p-1)}}{p-1} - 2\gamma - N\right)d\tau.$$
(4.11)

Such a cycle is not unstable, thus $I \leq 0$. Now

$$\oint (\alpha y' - \gamma Y') d\tau = 0 = (\alpha + \gamma) \oint |Y|^{1/(p-1)} d\tau - \gamma (\gamma + N) \oint |Y| d\tau.$$

From the Jensen and Hölder inequalities, since 1/(p-1) < 1,

$$\gamma(\gamma+N)(\oint |Y|^{1/(p-1)} d\tau)^{p-2} \leq \alpha+\gamma,$$

$$1 \leq \left(\oint |Y|^{(2-p)/(p-1)})d\tau\right) \left(\oint |Y|^{1/(p-1)} d\tau\right)^{p-2} \leq \frac{(p-1)(2\gamma+N)}{\gamma(\gamma+N)}(\alpha+\gamma),$$

then $\alpha^* < \alpha$, which is contradictory.

• Assume $-p' \leq \alpha < 0$. Consider the functions $y_{\alpha} = e^{(\alpha+\gamma)\tau}y$ and $Y_{\alpha} = e^{(\alpha+\gamma)(p-1)\tau}Y$ defined by (2.3) with $d = \alpha$. They vary respectively from 0 to ∞ and from 0 to $-\infty$. They have no extremal point. Indeed at such a point, from (3.2) and (3.3) y''_{α} or Y''_{α} have a strict constant sign for $\alpha \neq \eta, p'$, which is contradictory. If $\alpha = \eta$ or p', from uniqueness y_{α} or Y_{α} is constant, thus y or Y is monotone, which is impossible. In any case $y'_{\alpha} > 0 > Y'_{\alpha}$ on $(-\infty, \infty)$. Next, from (2.5) and (2.6),

$$\frac{y_{\alpha}''}{y_{\alpha}'} + \eta - 2\alpha - \frac{1}{p-1} Y^{(2-p)/(p-1)} = \alpha(\eta - \alpha) \frac{y_{\alpha}}{y_{\alpha}'},\tag{4.12}$$

$$\frac{Y_{\alpha}''}{Y_{\alpha}'} + (p-1)(\eta - 2\alpha - p') - \frac{1}{p-1}Y^{(2-p)/(p-1)} = (p-1)^2(\eta - \alpha)(p' + \alpha)\frac{Y_{\alpha}}{Y_{\alpha}'}.$$
(4.13)

Let us integrate on the period \mathcal{P} . If $\eta \leq \alpha < 0$, then $\eta - N - 2(\alpha + \gamma) \geq 0$ from (4.12), which is contradictory. If $-p' \leq \alpha < \eta$, then $-2(\alpha + p' + \gamma) > 0$ from (4.13), still contradictory.

5 New local existence results

At Proposition 4.1 we gave all the *possible* behaviours of the positive solutions near $\pm \infty$. Next we prove their existence, and uniqueness or multiplicity. The case p > N is very delicate.

Theorem 5.1 (i) Suppose p < N. In the phase plane (y, Y) of system (S) there exist an infinity of trajectories \mathcal{T}_{η} such that $\lim_{\tau \to -\infty} (\zeta, \sigma) = L_{\eta}$; the corresponding w satisfy (4.6).

(ii) Suppose p > N. There exist a unique trajectory \mathcal{T}_u such that $\lim_{\tau \to -\infty} (\zeta, \sigma) = L_{\eta}$; in other words for any $c \neq 0$, there exists a unique solution w of equation (\mathbf{E}_w) such that (4.7) holds.

Proof. Suppose that such a trajectory exists in the plane (y, Y). In the phase plane (ζ, ψ) of System (**P**), ζ and ψ keep a strict constant sign, because the two axes $\zeta = 0$ and $\psi = 0$ contain particular trajectories, and (ζ, ψ) converges to $(\eta, 0)$ at $-\infty$. Reciprocally, setting $\zeta = \eta + \overline{\zeta}$, the linearized problem at point $(\eta, 0)$

$$\bar{\zeta}' = \eta \bar{\zeta} + \eta (\alpha - \eta) \varepsilon \psi / (p - 1), \qquad \psi' = (N - \eta) \psi,$$

admits the eigenvalues η and $N - \eta$. The trajectories linked to the eigenvalue η are tangent to the line $\psi = 0$.

(i) Case p < N. Then $\eta > 0$, and $(\eta, 0)$ is a source. In the plane (ζ, ψ) there exist an infinity of trajectories, starting from this point at $-\infty$, such that $\psi > 0$, and $\lim_{\tau \to -\infty} \zeta = \eta$, thus $\zeta > 0$. In the phase plane (y, Y), setting $y = (\psi |\zeta|^{p-2} \zeta)^{2-p}$ and $Y = y/\psi$, they correspond to an infinity of trajectories in the plane (y, Y) such that $\lim_{\tau \to -\infty} (\zeta, \sigma) = L_{\eta}$, and (4.6) holds from Proposition (4.2).

(ii) Case p > N. Then $\eta < 0$, and $(\eta, 0)$ is a saddle point. In the plane (ζ, ψ) , there exists a unique trajectory starting from $(\eta, 0)$, tangentially to the vector $(\eta(\alpha - \eta)\varepsilon/(p-1), N - \eta)$, with $\psi < 0$; it defines a unique trajectory \mathcal{T}_u in the plane (y, Y), and (4.7) holds. From Remark 2.1, we get a solution for any $c \neq 0$.

Theorem 5.2 (i) Suppose p = N. In the phase plane (y, Y), there exists an infinity of trajectories \mathcal{T}_+ such that $\lim_{\tau \to -\infty} (\zeta, \sigma) = L_+$; then w satisfies (4.8).

(ii) Suppose p > N. Then there exist an infinity of trajectories \mathcal{T}_+ (resp. \mathcal{T}_-) such $\lim_{\tau \to -\infty} (\zeta, \sigma) = L_+$ (resp. L_-); then the corresponding solutions w of (\mathbf{E}_w) satisfy (4.9) (resp. (4.10).

More precisely for any k > 0 (for p = N) or any a > 0 and $c \neq 0$ (for p > N) there exists a unique function w satisfying those conditions.

Proof. If $\lim_{\tau \to -\infty} (\zeta, \sigma) = L_{\pm}$, then $\lim_{\tau \to -\infty} (\zeta, \psi) = (0, 0)$, with $\zeta \psi > 0$ in case of L_+ , $\zeta \psi < 0$ in case of L_- . The linearization of System (**P**) near (0, 0) is given by

$$\zeta' = |\eta| \zeta, \qquad \psi' = N\psi$$

(i) Case p = N. The phase plane study is delicate because 0 is a center, thus we use a fixed method. Suppose that such a trajectory exists, and consider the substitution (2.3) with d = 0. From (4.2), there exists k > 0 such that $\zeta = |Y_0|^{(2-p)/(p-1)}/y_0 = -\tau^{-1}(1+o(1)) > 0$, and $\psi = -k^{2-p}\tau e^{N\tau}(1+o(1)) > 0$. Then $\zeta' = \tau^{-2}(1+o(1))$ from System (**P**). The function

$$V = \psi e^{-N/\zeta} \zeta$$

satisfies $\lim_{\tau \to -\infty} V = k^{2-p}$, and

$$V' = \frac{Ve^{N/\zeta}}{(N-1)\zeta^2} (\varepsilon \left(\alpha - \zeta\right) \left(N - (N-2)\zeta\right)V + 2N(N-1)\zeta^2 e^{-N/\zeta})$$

Thus $\varepsilon \alpha(V - k^{2-p}) > 0$ near $-\infty$. Moreover $\lim_{\tau \to -\infty} \zeta'/V' = 0$, so that ζ can be considered as a function of V near k^{2-p} , with $\lim_{V \to k^{2-p}} \zeta = 0$ and

$$\frac{d\zeta}{dV} = K(V,\zeta), \qquad K(V,\zeta) := \frac{\zeta^2}{V} \frac{\varepsilon \left(\alpha - \zeta\right) V + (N-1)\zeta^2 e^{-N/\zeta}}{\varepsilon \left(\alpha - \zeta\right) \left(N - (N-2)\zeta\right) V + 2N(N-1)\zeta^2 e^{-N/\zeta}}.$$

Reciprocally, extending the function $\zeta^2 e^{-N/\zeta}$ by 0 for $\zeta \leq 0$, the function K is of class C^1 near $(k^{2-p}, 0)$. For any k > 0, there exists a unique local solution $V \mapsto \zeta(V)$ on a interval \mathcal{V} where $\varepsilon \alpha(V - k^{2-p}) > 0$, such that $\zeta(k^{2-p}) = 0$. And $d\zeta/dV = (\zeta^2/Nk^{2-p})(1+o(1))$ near 0, thus $\zeta > 0$. In the plane (ζ, ψ) , taking one point P on the curve $\mathcal{C} = \{(\zeta(V), V\zeta(V)e^{N/\zeta(V)}) : v \in \mathcal{V}\}$, there exists a unique solution of System (**P**) issued from P at time 0. Its trajectory is on \mathcal{C} , thus it converges to (0,0), with $\zeta, \psi > 0$. It corresponds to a unique trajectory \mathcal{T}_+ in the plane (y,Y), and (ζ,σ) converges to L_+ , as τ tends to $-\infty$, from Proposition 4.1. The corresponding functions w satisfy (4.8) from Proposition (4.2).

(ii) Case p > N. Here (0,0) is a source for System (**P**). The lines $\zeta = 0$ and $\psi = 0$ contain trajectories. There exists an infinity of trajectories converging to (0,0), with $\zeta \psi \neq 0$; moreover, if $N \geq 2$, then $|\eta| < N$, thus $\lim_{\tau \to -\infty} (\psi/\zeta) = 0$. Our claim is more precise. Given a > 0 and $c \neq 0$, we look for a solution w of (**E**_w) such that $\lim_{r\to 0} w = a$, $\lim_{r\to 0} r^{\eta+1}w' = -c$. By scaling we can assume a = 1. If w_1 is a such a solution, then ζ and ψ have the sign of c near 0, and $\zeta(\tau) = ce^{|\eta|\tau}(1+o(1))$ and $|c|^{p-2}c\psi(\tau) = e^{N\tau}(1+o(1))$. The function

$$v = c(|c|^{p-2} c\psi)^{1/\kappa} / \zeta, \quad \text{with } \kappa = N/|\eta| > 1,$$

satisfies $\lim_{\tau \to -\infty} v = 1$, and can be expressed locally as a function of ζ , and

$$\frac{dv}{d\zeta} = H(\zeta, v), \qquad H(\zeta, v) := -\frac{v}{\kappa} \frac{(p-1)(\kappa+1) + \varepsilon(\kappa-p+1) \left|c\right|^{1-p-\kappa} \left(\zeta-\alpha\right) \left|\zeta\right|^{\kappa-1} v^{\kappa}}{(p-1)(\zeta-\eta) + \varepsilon \left|c\right|^{1-p-\kappa} \left(\alpha-\zeta\right) \left|\zeta\right|^{\kappa-1} \zeta v^{\kappa}}.$$

Reciprocally, there exists a unique solution $\zeta \mapsto v(\zeta)$ of this equation on a small interval [0, hc), with h > 0, such that v(0) = 1. Indeed H is locally continuous in ξ and C^1 in v. Taking one

point P on the curve $\mathcal{C}' = \left\{ (\zeta, |c|^{1-p-\kappa} |\zeta|^{\kappa-1} \zeta v(\zeta)) : \zeta \in [0, hc) \right\}$, there exists a unique solution of System (**P**) issued from P at time 0. Its trajectory is on \mathcal{C}' , thus converges to (0,0) with $\zeta \psi > 0$. It corresponds to a solution (y, Y) of System (**S**), such that (ζ, σ) converges to L_+ , as τ tends to $-\infty$, from Proposition 4.1. The corresponding function, called w_2 , satisfies $\lim_{r\to 0} r^{\eta+1} w_2^{\gamma^{-1}|\eta|-1} w_2' = -c$; thus w_2 has a limit a_2 , and $\lim_{r\to 0} r^{\eta-1} w_2' = a_2^{1-s}b$. Moreover $a_2 \neq 0$, because $a_2 = 0$ implies that $r^{-\gamma}w_2$ has a nonzero limit, thus (ζ, σ) converges to A_{γ} . The function $w(r) = a_2^{-1}w_2(a_2^{1/\gamma}r)$ satisfies $\lim_{r\to 0} w = 1$, and $\lim_{r\to 0} r^{\eta-1}w' = -c$, and the proof is done.

Theorem 5.3 (i) In the phase plane (y, Y), for any $\alpha \neq 0$ there exists at least a trajectory \mathcal{T}_{α} converging to (0,0) with y > 0, and $\lim(\zeta, \sigma) = A_{\alpha}$. The convergence holds at ∞ if $-\gamma < \alpha$, or $-\infty$ if $\alpha < -\gamma$, or $\varepsilon \infty$ if $\alpha = -\gamma$.

(ii) If $\varepsilon(\gamma + \alpha) < 0$, \mathcal{T}_{α} is unique, it is the unique trajectory converging to (0,0) at $-\varepsilon\infty$ with y > 0, and it depends locally continuously of α .

Proof. (i) Suppose that such a trajectory exists. Then τ tends to ∞ if $-\gamma < \alpha$, or $-\infty$ if $\alpha < -\gamma$, or $\varepsilon \infty$ if $\alpha = -\gamma$, from Proposition 4.1. Consider System (**R**), where g, s and ν are defined by (2.18). Then (g, s) converges to $(-1/\alpha, 0)$, with gs > 0, and ν tends to the same limits as τ , since Y converges to 0. Reciprocally, in the plane (g, s), let us show the existence of a trajectory converging to $(-1/\alpha, 0)$, different from the line s = 0. Setting $g = -1/\alpha + \overline{g}$, the linearized system at this point is

$$\frac{d\bar{g}}{d\nu} = -\frac{\varepsilon}{p-1}\bar{g} + \frac{\eta - \alpha}{\alpha^2}s, \qquad \frac{ds}{d\nu} = 0,$$

thus we find a center: the eigenvalues are 0 and $\lambda = \varepsilon/(p-1)$. Since the system is polynomial, it is known that System (**R**) admits a trajectory, depending locally continuously of α , such that sg > 0, and tangent to the eigenvector $((p-1)(\eta - \alpha), \varepsilon \alpha^2)$. It satisfies $ds/d\nu = (p-2)(\alpha + \gamma)s^2(1+o(1))$. Then $ds/d\tau = -(p-2)\alpha(\alpha + \gamma)s(1+o(1))$, thus τ tends to $\pm \infty$. And $|y|^{p-2} = |s| |g|^{1/(p-1)}$, then y tends to 0, (y, Y) converges to (0, 0), and $\lim(\zeta, \sigma) = A_{\alpha}$.

(ii) Suppose $\varepsilon(\gamma + \alpha) < 0$. Consider two trajectories $\mathcal{T}_1, \mathcal{T}_2$ in the plane (y, Y), converging to (0, 0) at $-\varepsilon\infty$, with y > 0. They are different from $\mathcal{T}_{\varepsilon}$ which converges at $\varepsilon\infty$, thus $\lim(\zeta_i, \sigma_i) = (\alpha, 0)$ from Proposition 4.1. Then ζ_1, ζ_2 can locally be expressed as a function of y, and

$$y\frac{d(\zeta_1 - \zeta_2)^2}{dy} = 2(F(\zeta_1, y) - F(\zeta_2, y))(\zeta_1 - \zeta_2)$$

near 0, where

$$F(\zeta, y) = \frac{1}{\gamma + \zeta} \left(-\zeta(\zeta - \eta) + \frac{\varepsilon}{p - 1} \left|\zeta y\right|^{2 - p} \left(\zeta - \alpha\right)\right).$$

Then $(\zeta_1 - \zeta_2)^2$ is nonincreasing, seeing that $\partial F / \partial \zeta(\zeta, y) = -((p-1)\varepsilon(\gamma+\alpha))^{-1} |\alpha y|^{2-p} (1+o(1))$. Hence $\zeta_1 \equiv \zeta_2$ near 0, and $\mathcal{T}_1 \equiv \mathcal{T}_2$.

6 The case $\varepsilon = 1, -\gamma \leq \alpha$

In that Section and in Sections 7, 8 and 9 we describe the solutions of (\mathbf{E}_w) . When we give a *uniqueness* result, we mean that w is unique, up to a scaling, from Remark 2.1.

Theorem 6.1 Assume $\varepsilon = 1, -\gamma \leq \alpha \ (\alpha \neq 0)$.

Any solution w of (\mathbf{E}_w) has a finite number of simple zeros, and satisfies (4.4) or (4.5) near ∞ or has a compact support. Either w is regular, or |w| satisfies (4.6),(4.8), (4.7),(4.9) or (4.10) near 0, and there exist solutions of each type.

(1) Case $\alpha < N$. All regular solutions have a strict constant sign, and satisfy (4.4) or (4.5) near ∞ . Moreover there exist (and exhaustively, up to a symmetry)

(i) a unique nonnegative solution with (4.6) or (4.8) or (4.9)) near 0, and compact support;

(ii) positive solutions with the same behaviour at 0 and (4.4) or (4.5) near ∞ ;

(iii) solutions with one simple zero, and |w| has the same behaviour at 0 and ∞ ;

(iv) for p > N, a unique positive solution with (4.7) near 0, and (4.4) or (4.5) near ∞ ;

(v) for p > N, positive solutions with (4.10) near 0, and (4.4) or (4.5) near ∞ .

(2) Case $\alpha = N$. Then the regular (Barenblatt) solutions have a constant sign with compact support. If $p \leq N$, all the other solutions are of type (iii). If p > N, there exist also solutions of type (iv) and (v).

(3) Case $\alpha > N$.

Either the regular solutions have m simple zeros and satisfy satisfies (4.4) near ∞ . Then there exist

(vi) a unique solution with m simple zeros, |w| satisfies (4.6), (4.8) or(4.9) near 0, with compact support;

(vii) solutions with m + 1 simple zeros, |w| satisfies (4.6), (4.8) or (4.9) near 0, and (4.4) or (4.5) near ∞ ;

(viii) for p > N, solutions with m simple zeros, |w| satisfies (4.9),(4.7) or (4.10) near 0, and (4.4) or (4.5) near ∞ .

Or the regular solutions have m simple zeros and a compact support. Then the other solutions are of type (vii) or (viii).

th 6.1, fig1: $\varepsilon = 1, N = 2, p = 3, \alpha = -2$ th 6.1, fig2: $\varepsilon = 1, N = 2, p = 3, \alpha = 1$

th 6.1,fig3: $\varepsilon = 1, N = 2, p = 3, \alpha = 2$ th 6.1,fig4: $\varepsilon = 1, N = 2, p = 3, \alpha = 50$

Proof. All the solutions w have a finite number of simple zeros, from Proposition 3.7 and Theorem 3.9. Either they have a compact support. Or y has a strict constant sign and is monotone near ∞ , and converge to (0,0) at ∞ , and (4.4) or (4.5) holds, from Propositions 3.8, 4.1.

In the phase plane (y, Y), system (**S**) admits only one stationary point (0, 0). The trajectory \mathcal{T}_r starts in \mathcal{Q}_4 when $\alpha < 0$, in \mathcal{Q}_1 when $\alpha > 0$, and $\lim_{\tau \to -\infty} y = \infty$, with an asymptotical direction of slope α/N . From Propositions 4.1 and 4.2 all the nonregular solutions $\pm w$ satisfy (4.6), (4.8),

(4.7), (4.9) or (4.10) near $-\infty$. The existence of solutions of any kind is proved at Theorems 5.1 and 5.2. When $p \leq N$, they correspond to trajectories $\pm \mathcal{T}_{\eta}$ such that \mathcal{T}_{η} starts in \mathcal{Q}_1 with an infinite slope, in any case above \mathcal{T}_r . When p > N, there is a unique trajectory \mathcal{T}_u satisfying (4.7), starting in \mathcal{Q}_4 , under \mathcal{T}_r ; the trajectories \mathcal{T}_+ start from \mathcal{Q}_1 , above \mathcal{T}_r ; the trajectories \mathcal{T}_- start in \mathcal{Q}_4 under \mathcal{T}_r . From Theorem 3.9, there exists a unique trajectory $\mathcal{T}_{\varepsilon}$ converging to (0,0) in \mathcal{Q}_1 at ∞ , with the slope 1.

(1) Case $\alpha < N$. From Proposition 3.6, all the solutions w have at most one simple zero.

The regular solutions stay positive, and \mathcal{T}_r stays in its quadrant, \mathcal{Q}_4 or \mathcal{Q}_1 , from Remark 2.3 (see figures 1 and 2). Then $\mathcal{T}_{\varepsilon}$ stays in \mathcal{Q}_1 , because it cannot meet \mathcal{T}_r for $\alpha > 0$, or the line $\{Y = 0\}$ for $\alpha < 0$, from Remark 2.3; and the corresponding w is of type (i).

Consider any trajectory $\mathcal{T}_{[P]}$ with $P \in \mathcal{Q}_1$ above $\mathcal{T}_{\varepsilon}$. It cannot stay in \mathcal{Q}_1 because it does not meet $\mathcal{T}_{\varepsilon}$ and converges to (0,0) with a slope 0. Thus it enters \mathcal{Q}_2 from Remark 2.3. Then y has a unique zero, and $\mathcal{T}_{[P]}$ stays in \mathcal{Q}_1 before P, and in $\mathcal{Q}_2 \cup \mathcal{Q}_3$ after P. Since $\mathcal{T}_{[P]}$ cannot meet $\pm \mathcal{T}_{\varepsilon}$, and $\lim_{\tau \to \infty} \zeta = \alpha$, $\mathcal{T}_{[P]}$ ends up in \mathcal{Q}_3 if $\alpha > 0$, in \mathcal{Q}_2 if $\alpha < 0$. It has the same behaviour as $\mathcal{T}_{\varepsilon}$ at $-\infty$, and w is of type (iii).

Next consider $\mathcal{T}_{[P]}$ for any $P \in \mathcal{Q}_1 \cup \mathcal{Q}_4$ between $\mathcal{T}_{\varepsilon}$ and \mathcal{T}_r . Then y stays positive, and $\mathcal{T}_{[P]}$ necessarily starts from \mathcal{Q}_1 , and w is of type (ii).

At least take any $P \in \mathcal{Q}_1 \cup \mathcal{Q}_4$ under \mathcal{T}_r . If $p \leq N$, $\mathcal{T}_{[P]}$ starts from \mathcal{Q}_3 and y has a unique zero, and -w is of type (iii). If p > N, either -w is of type (iii), or $\mathcal{T}_{[P]}$ stays in \mathcal{Q}_4 . From Theorems 5.1, 5.2, either $\mathcal{T}_{[P]}$ coincides with \mathcal{T}_u , and w is of type (iv), or with one of the trajectories \mathcal{T}_- , thus w is of type (v).

(2) Case $\alpha = N$. All the solutions are given by (1.9), which is equivalent to $J_N \equiv C$, where J_N is defined by (2.1). For C = 0, the regular (Barenblatt) solutions, given by (1.10), are nonnegative, with a compact support. In other words the trajectory $\mathcal{T}_{\varepsilon}$ given by Theorem 5.3 coincides with \mathcal{T}_r , it is given by $y \equiv Y, y > 0$ (see figure 3). The only change in the phase plane is the nonexistence of solutions of type (ii).

(3) Case $\alpha > N$.

The regular solutions have a number $m \ge 1$ of simple zeros, from Proposition 3.6 (see figure 4). As above, \mathcal{T}_r starts from \mathcal{Q}_1 with a finite slope α/N .

Either $\mathcal{T}_r \neq \mathcal{T}_{\varepsilon}$. Then the regular solutions satisfy $\lim_{r\to\infty} r^{\alpha}w = L \neq 0$. Since $\mathcal{T}_{\varepsilon}$ cannot meet \mathcal{T}_r , $\mathcal{T}_{\varepsilon}$ also cuts the line $\{y = 0\}$ at m points, and the corresponding w is of type (vi). For any $P \in \mathcal{Q}_1$ above \mathcal{T}_r , the trajectory $\mathcal{T}_{[P]}$ cuts the line $\{y = 0\}$ at m + 1 points and w is of type (vi). If p > N, there exist trajectories starting from \mathcal{Q}_1 between $\mathcal{T}_{\varepsilon}$ and \mathcal{T}_r , with (4.9), such that w has m simple zeros, and trajectories with (4.7) or (4.10), m zeros, and $\lim_{r\to\infty} r^{\alpha}w = L \neq 0$.

Or $\mathcal{T}_r = \mathcal{T}_{\varepsilon}$, the regular solutions have a compact support, and we only find solutions of type (vii), (viii).

Remark 6.2 In the case $\alpha = \eta < 0$, the solutions (iv) are given by (1.11). In the case N = 1, $\alpha = -(p-1)/(p-2)$, the solutions of types (i) and (v) are given by (1.14).

Remark 6.3 We conjecture that there exists an increasing sequence $(\bar{\alpha}_m)$, with $\bar{\alpha}_0 = N$ such that the regular solutions w have m simple zeros for $\alpha \in (\bar{\alpha}_{m-1}, \bar{\alpha}_m)$, with $\lim_{r\to\infty} r^{\alpha}w = L \neq 0$, and m simple zeros and a compact support for $\alpha = \bar{\alpha}_m$, in which case $\mathcal{T}_r = \mathcal{T}_{\varepsilon}$.

7 The case $\varepsilon = -1, \alpha \leq -\gamma$

Theorem 7.1 Assume $\varepsilon = -1$, $\alpha \leq -\gamma$. Then all the solutions w of (\mathbf{E}_w) , among them the regular ones, are ocillating near ∞ and $r^{-\gamma}w$ is asymptotically periodic in $\ln r$. There exist

- (i) solutions such that $r^{-\gamma}w$ is periodic in $\ln r$;
- (ii) a unique solution with a hole;
- (iii) flat solutions w with (4.4) or (4.5) near 0;
- (iv) solutions with (4.6) or (4.8) or (4.9) or also (4.10) near 0;
- (v) for p > N, a unique solution with (4.7) near 0.

th 7.1,fig5: $\varepsilon = -1, N = 1, p = 3, \alpha = -4$

Proof. Here again, (0,0) is the unique stationary point in the plane (y, Y). Any solution y of (\mathbf{E}_y) oscillates near ∞ , and (y, Y) is bounded from Proposition 4.3. From the strong form of the Poincaré-Bendixon theorem, see [7, p.239], all the trajectories have a limit cycle or are periodic. In particular \mathcal{T}_r starts in \mathcal{Q}_1 , since $\varepsilon \alpha > 0$, with the asymptotical direction $\varepsilon \alpha/N$. and it has a limit cycle \mathcal{O} . There exists a periodic trajectory of orbit \mathcal{O} , thus w is of type (i) (see figure 5).

From Theorem 5.2 there exists a unique trajectory $\mathcal{T}_{\varepsilon}$ starting from (0,0) with the slope -1, y > 0; it has a limit cycle $\mathcal{O}_{\varepsilon} \subset \mathcal{O}$, and w is of type (ii). For any P in the bounded domain delimitated by $\mathcal{O}_{\varepsilon}$, not located on $\mathcal{T}_{\varepsilon}$, the trajectory $\mathcal{T}_{[P]}$ does not meet $\mathcal{T}_{\varepsilon}$, and admits $\mathcal{O}_{\varepsilon}$ as limit

cycle; near $-\infty$, y has a constant sign, is monotone and converges to (0,0) from Propositions 3.8 and 4.1, and $\lim_{\tau \to -\infty} \zeta = \alpha$. This show again the existence of such trajectories, proved at Theorem 5.1, and there is an infinity of them; and w is if type (iii).

From Theorems 5.1 and 5.2, there exist trajectories starting from infinity, with \mathcal{O} as limit cycle, and w is of type (iv) or (v). If $\mathcal{O} = \mathcal{O}_{\varepsilon}$, all the solutions are described.

8 Case $\varepsilon = 1, \alpha < -\gamma$.

Theorem 8.1 Assume $\varepsilon = 1$, $\alpha < -\gamma$. Then $w \equiv \pm \ell r^{\gamma}$ is a solution of (\mathbf{E}_w) . All regular solutions have a strict constant sign, and satisfy (4.3) near ∞ . Moreover there exist (exhaustively, up to a symmetry)

(i) a unique positive flat solution with (4.4) near 0 and (4.3) near ∞ ;

(ii) a unique nonnegative solution with (4.6) or (4.8) or (4.9) near 0, and compact support;

(iii) positive solutions with the same behaviour near 0 and (4.3) near ∞ ;

(iv) solutions with one zero and the same behaviour near 0, and |w| satisfies (4.3) near ∞ ;

- (v) for p > N, positive solutions with (4.7) near 0 and (4.3) near ∞ ;
- (vi) for p > N, positive solutions with (4.10) near 0 and (4.3) near ∞ .

th 8.1, fig6: $\varepsilon = 1, N = 2, p = 3, \alpha = -6$

Proof. Here system (S) admits three stationary points in the plane (y, Y), given at (2.8), thus $w \equiv \pm \ell r^{\gamma}$ is a solution; and M_{ℓ} is a sink (see figure 6). Any solution y of (\mathbf{E}_y) has at most one zero, and is strictly monotone near $\pm \infty$, from Propositions 3.6 and 3.8.

From Theorems 3.9 and 5.3, there exists a unique trajectory $\mathcal{T}_{\varepsilon}$ converging to (0,0) in \mathcal{Q}_1 at ∞ , and a unique trajectory \mathcal{T}_{α} converging to (0,0) in \mathcal{Q}_4 at $-\infty$. The trajectory \mathcal{T}_r starts in \mathcal{Q}_4 with the asymptotical direction $-|\alpha|/N$. From Remark 2.3, \mathcal{Q}_4 is positively invariant, and \mathcal{Q}_1 negatively invariant. Then $\mathcal{T}_{\varepsilon}$ stays in \mathcal{Q}_1 , and \mathcal{T}_{α} and \mathcal{T}_r in \mathcal{Q}_4 . From Proposition 4.1, all the trajectories, apart from $\pm \mathcal{T}_{\varepsilon}$, converge to $\pm M_{\ell}$ at ∞ . Then \mathcal{T}_r converges to M_{ℓ} , and w satisfies (4.3) near ∞ . And \mathcal{T}_{α} also converges to M_{ℓ} , and w is of type (i).

From Propositions 4.1, Theorems 5.1 and 5.2, all the nonregular solutions which are positive near $-\infty$ satisfy (4.6), (4.8), (4.9), (4.10) or (4.7), and there exist such solutions. For p < N (resp. p = N), they correspond to trajectories \mathcal{T}_{η} (resp. \mathcal{T}_{+}) starting in \mathcal{Q}_1 . For p > N, there is a unique trajectory \mathcal{T}_u satisfying (4.7), starting in \mathcal{Q}_4 under \mathcal{T}_r ; and the trajectories \mathcal{T}_+ satisfying (4.9) start from \mathcal{Q}_1 ; the trajectories \mathcal{T}_- satisfying (4.10) and the unique trajectory \mathcal{T}_u satisfying (4.7) start from \mathcal{Q}_4 , under \mathcal{T}_r . Since $\mathcal{T}_{\varepsilon}$ stays in \mathcal{Q}_1 , it defines solutions w of type (ii).

Consider the basis of eigenvectors (e_1, e_2) defined at (2.15), where $\nu(\alpha) > 0$, associated to the eigenvalues $\lambda_1 < \lambda_2$. One verifies that $\lambda_1 < -\gamma < \lambda_2$; thus e_1 points towards Q_3 and e_2 points towards Q_4 . There exist unique trajectories \mathcal{T}_{e_1} and \mathcal{T}_{-e_1} converging to M_ℓ , tangentially to e_1 and $-e_1$. All the other trajectories converging to M_ℓ at ∞ are tangent to $\pm e_2$. Let

$$\mathcal{M} = \left\{ |Y|^{(2-p)/(p-1)} Y = -\gamma y \right\}, \qquad \mathcal{N} = \left\{ (N+\gamma)Y + \varepsilon |Y|^{(2-p)/(p-1)} Y = \varepsilon \alpha y \right\}$$

be the sets of extremal points of y and Y.

The trajectory \mathcal{T}_r starts above the curves \mathcal{M} and \mathcal{N} , thus y' < 0 and Y' > 0 near $-\infty$. And \mathcal{T}_r converges to M_ℓ at ∞ , tangentially to e_2 . Indeed if $\mathcal{T}_r = \mathcal{T}_{e_1}$, then y has a minimal point such that $y < \ell$ and $Y < -(\gamma \ell)^{p-1}$, then (y, Y) cannot be on \mathcal{M} . If $\mathcal{T}_r = \mathcal{T}_{-e_1}$, then Y has a maximal point such that $y > \ell$ and $Y < -(\gamma \ell)^{p-1}$, then also (y, Y) cannot be on \mathcal{N} . Finally \mathcal{T}_r cannot end up tangentially to $-e_2$, it would intersect \mathcal{T}_{e_1} or \mathcal{T}_{-e_1} .

The trajectory \mathcal{T}_{α} converge to M_{ℓ} tangentially to $-e_2$. Indeed if $\mathcal{T}_{\alpha} = \mathcal{T}_{e_1}$, then Y has a maximal point such that $y < \ell$ and $Y < -(\gamma \ell)^{p-1}$; if $\mathcal{T}_{\alpha} = \mathcal{T}_{-e_1}$, then y has a maximal point such that $y > \ell$ and $Y > -(\gamma \ell)^{p-1}$. In any case we reach a contradiction. Moreover \mathcal{T}_{e_1} does not stay in \mathcal{Q}_4 : y would have a minimal point such that $y < \ell$ and $Y < -(\gamma \ell)^{p-1}$, which is impossible; thus \mathcal{T}_{e_1} starts in \mathcal{Q}_3 , and enters \mathcal{Q}_4 at some point $(\xi_1, 0)$ with $\xi_1 < 0$. And -w is of type (iv).

Any trajectory $\mathcal{T}_{[P]}$, with P in the domain of $\mathcal{Q}_1 \cup \mathcal{Q}_4$ delimitated by $\mathcal{T}_r, \mathcal{T}_\alpha$ and \mathcal{T}_ε , comes from \mathcal{Q}_1 , and converges to M_ℓ in \mathcal{Q}_4 , in particular \mathcal{T}_{-e_1} ; the corresponding w are of type (iii).

Any trajectory $\mathcal{T}_{[P]}$, with P in the domain of $\mathcal{Q}_3 \cup \mathcal{Q}_4$ delimitated by $\mathcal{T}_{e_1}, \mathcal{T}_{\alpha}$ and $-\mathcal{T}_{\varepsilon}$, goes from \mathcal{Q}_3 to \mathcal{Q}_4 , and $\mathcal{T}_{[P]}$ converges to M_{ℓ} at ∞ , and -w is of type (iv). For any $\xi < \xi_1$, the trajectory $\mathcal{T}_{[(0,\xi)]}$ is of the same type. If $p \leq N$, any trajectory in the domain under \mathcal{T}_r , and \mathcal{T}_{e_1} is of the same type.

If p > N, moreover in this domain there exists a the unique trajectory \mathcal{T}_u and trajectories of the type \mathcal{T}_- corresponding to solutions w of type (v) and (vi), from Theorems 5.1 and 5.2. Up to a symmetry, all the solutions are described, and all of them do exist.

9 Case $\varepsilon = -1, -\gamma < \alpha$

Here again System (S) admits the three stationary points (2.8), thus $w \equiv \pm \ell r^{\gamma}$ is a solution of (\mathbf{E}_w) . The behaviour is very rich: it depends on the position of α with respect to α^* defined at (1.5), and 0, -p', and η (in case p > N), and also α_1, α_2 defined at (2.14). We start from some general remarks.

Remark 9.1 (i) There exists a unique trajectory $\mathcal{T}_{\varepsilon}$ starting from (0,0) in \mathcal{Q}_4 with the slope -1, from Theorem 3.9.

(ii) There exists a unique trajectory \mathcal{T}_{α} converging to (0,0) at ∞ , in \mathcal{Q}_1 if $\alpha > 0$, in \mathcal{Q}_4 if $\alpha < 0$, with a slope 0 at (0,0), and $\lim_{\tau \to \infty} \zeta = \alpha$, from Theorem 5.3.

(iii) From Remark 2.3, if $\alpha > 0$, \mathcal{Q}_4 is positively invariant and \mathcal{Q}_1 negatively invariant. If $\alpha < 0$, at any point $(0,\xi), \xi < 0$, the vector field points to \mathcal{Q}_4 , and at any point $(\varphi, 0), \varphi > 0$, it points to \mathcal{Q}_1 . Thus if $\mathcal{T}_{\varepsilon}$ does not stay in \mathcal{Q}_1 , then \mathcal{T}_{α} stays in the bounded domain delimitated by $\mathcal{Q}_4 \cap \mathcal{T}_{\varepsilon}$. If \mathcal{T}_{α} does not stay in \mathcal{Q}_4 , then $\mathcal{T}_{\varepsilon}$ stays in the bounded domain delimitated by $\mathcal{Q}_4 \cap \mathcal{T}_{\alpha}$. If $\mathcal{T}_{\varepsilon}$ is homoclinic, in other words $\mathcal{T}_{\varepsilon} = \mathcal{T}_{\alpha}$, it stays in \mathcal{Q}_4 .

Remark 9.2 From Propositions 4.1, Theorems 5.1 and 5.2, all the nonregular solutions positive near $-\infty$ satisfy (4.6) for p < N, (4.8) for p = N, corresponding to trajectories $\mathcal{T}_{\eta}, \mathcal{T}_{+}$ starting from \mathcal{Q}_1 ; and (4.9), (4.10) or (4.7) for p > N, corresponding to trajectories \mathcal{T}_{+} starting from \mathcal{Q}_1 , and $\mathcal{T}_{-}, \mathcal{T}_{u}$ starting from \mathcal{Q}_4 .

Remark 9.3 Any trajectory \mathcal{T} is bounded near ∞ from Proposition 4.3. From the strong form of the Poincaré-Bendixon theorem, any trajectory \mathcal{T} bounded at $\pm \infty$ converges to (0,0) or $\pm M_{\ell}$, or its limit set Γ_{\pm} at $\pm \infty$ is a cycle, or it is homoclinic, namely $\mathcal{T}_{\varepsilon} = \mathcal{T}_{\alpha}$. If there exists a limit cycle surrounding (0,0), it also surrounds the points $\pm M_{\ell}$, from Proposition 3.8.

The simplest case is $\alpha > 0$.

Theorem 9.4 Assume $\varepsilon = -1$, $\alpha > 0$.

Then $w \equiv \ell r^{\gamma}$ is a solution w of (\mathbf{E}_w) . All regular solutions have a strict constant sign; and satisfy (4.3) near ∞ . There exist (exhaustively, up to a symmetry)

(i) a unique nonnegative solution with a hole, and (4.3) near ∞ ;

(ii) a unique positive solution with (4.6), or (4.8) or (4.9), and (4.4) near ∞ ;

(iii) positive solutions with the same behaviour near 0, and (4.3) near ∞ ;

(iv) solutions with one zero, the same behaviour near 0, and |w| satisfies (4.3) near ∞ ; (v) for p > N, a unique positive solution with (4.7) near 0, and (4.3) near ∞ ; (vi) for p > N, positive solutions with (4.10) near 0, and (4.3) near ∞ .

th 9.4, fig7: $\varepsilon = -1, N = 1, p = 3, \alpha = 0.7$ th 9.4, fig8: $\varepsilon = -1, N = 1, p = 3, \alpha = 1$

Proof. Any solution y of (\mathbf{E}_y) has at most one zero, and y is strictly monotone near ∞ , from Propositions 3.6 and 4.4. The point M_ℓ is a sink and a node point, since $\alpha > 0 \geq \alpha_2$ (see figure 7). Consider the basis eigenvectors (e_1, e_2) , defined at (2.15), where $\nu(\alpha) < 0$, associated to the eigenvalues $\lambda_1 < \lambda_2 < 0$. One verifies that $\lambda_1 < -\gamma < \lambda_2$, thus e_1 points towards \mathcal{Q}_3 and e_2 points towards \mathcal{Q}_4 . There exist unique trajectories \mathcal{T}_{e_1} and \mathcal{T}_{-e_1} tangent to e_1 and $-e_1$ at ∞ . All the other trajectories which converge to M_ℓ end up tangentially to $\pm e_1$.

The trajectory \mathcal{T}_{α} stays in \mathcal{Q}_1 from Remark 9.1; near $-\infty$ it is of type \mathcal{T}_{η} for p < N, and \mathcal{T}_+ for $p \geq N$; it defines the solution of type (ii). Since \mathcal{T}_{α} is the unique trajectory converging to (0,0) at ∞ , all the trajectories, apart from $\pm \mathcal{T}_{\alpha}$, converge to $\pm M_{\ell}$ at ∞ , from Propositions 3.8 and 4.1.

The trajectories \mathcal{T}_r and $\mathcal{T}_{\varepsilon}$ start in \mathcal{Q}_4 , and stay in it from Remark 9.1, and both converge to M_ℓ at ∞ , then w satisfies (4.3); and \mathcal{T}_r starts with the asymptotical direction $-\alpha/N$. And $\mathcal{T}_{\varepsilon}$ defines the solution of type (i).

As in the proof of Theorem 8.1, \mathcal{T}_r ends up tangentially to e_2 , and $\mathcal{T}_{\varepsilon}$ tangentially to $-e_2$. Moreover \mathcal{T}_{e_1} does not stay in \mathcal{Q}_4 , it starts in \mathcal{Q}_3 , and converges to M_ℓ in \mathcal{Q}_4 , and -w is of type (iv). Any trajectory $\mathcal{T}_{[P]}$, with P in the domain of \mathcal{Q}_4 between \mathcal{T}_{e_1} , $\mathcal{T}_{\varepsilon}$, starts from \mathcal{Q}_3 , enters \mathcal{Q}_4 at some point $(0,\xi), \xi > \xi_1$, and has the same type as \mathcal{T}_{e_1} . Any trajectory $\mathcal{T}_{[(0,\xi)]}$ with $\xi < \xi_1$ is of the same type. Any trajectory $\mathcal{T}_{[P]}$, with P in the domain of $\mathcal{Q}_1 \cup \mathcal{Q}_4$ above $\mathcal{T}_r \cup \mathcal{T}_{\varepsilon}$, starts from \mathcal{Q}_1 , and converges to M_{ℓ} in \mathcal{Q}_4 , in particular \mathcal{T}_{-e_1} ; the corresponding w are of type (iii). If $p \leq N$, all the solutions are described. If p > N, moreover there exist trajectories staying in $\mathcal{Q}_4 : \mathcal{T}_u$ and the \mathcal{T}_- , starting under \mathcal{T}_r , corresponding to types (v) and (vi).

Remark 9.5 For $\alpha = N$, \mathcal{T}_r and $\mathcal{T}_{\varepsilon}$ are given by (1.10), respectively with K > 0 and K < 0. The trajectory $\mathcal{T}_{\varepsilon}$ describes the portion $0M_{\ell}$ of the line $\{Y = -y\}$, and \mathcal{T}_r the complementary half-line in \mathcal{Q}_4 (see figure 8).

Next we assume $-p' \leq \alpha < 0$. The case p > N is delicate: indeed the special value $\alpha = \eta$ is involved, because $\eta < 0$.

Theorem 9.6 Assume $\varepsilon = -1, p \leq N$, and $-p' \leq \alpha < 0$. Then $w \equiv \ell r^{\gamma}$ is a solution w of (\mathbf{E}_w) .

There exist a unique nonnegative solution with a hole, satisfying (4.3) at ∞ .

(1) If $\alpha \neq -p'$, all regular solutions have one zero, and |w| satisfies (4.3) near ∞ . There exist (exhaustively, up to a symmetry)

• for $p \leq N$,

(i) a unique solution with one zero, with (4.6) or (4.8) near 0, and (4.4) near ∞ ;

- (ii) solutions with one zero, with (4.6) or (4.8) near 0, and |w| satisfies (4.3) near ∞ ;
- (iii) solutions with two zeros, with (4.6) or (4.8) near 0, and (4.3) near ∞ ;
- for p > N, $\eta < \alpha$,
 - (iv) a unique positive solution, with (4.10) near 0, and (4.4) near ∞ ;
 - (v) a unique positive solution, with (4.7) near 0, and (4.3) near ∞ ;
 - (vi) positive solutions, with (4.10) near 0, and (4.3) near ∞ ;

(vii) solutions with one zero with (4.10) or (4.9) near 0, and (4.3) near ∞ ;

• for $p > N, \alpha < \eta$,

(viii) a unique solution with one zero, with (4.9) near 0, and (4.4) near ∞ ;

(ix) a unique solution with one zero, with (4.7) near 0, and |w| satisfies (4.3) near ∞ ;

(x) solutions with one zero, with (4.9) or (4.9) near 0, and |w| satisfies (4.3) near ∞ ;

(xi) solutions with two zeros, with (4.9) near 0, and (4.3) near ∞ .

• for $p > N, \alpha = \eta$, solutions of the form $w = cr^{|\eta|}$ (c > 0). The other solutions are of type (vii).

(2) If $\alpha = -p'$, all regular solutions have one zero and satisfy (4.4) near ∞ . The solutions without hole are of types (ii), (iii) for $p \leq N$, (ix), (x), (xi) for p > N.

th 9.6, fig
9: $\varepsilon = -1, N = 1, p = 3, \alpha = -0.7$ th 9.6, fig
10: $\varepsilon = -1, N = 1, p = 3, \alpha = -1.49$

th 9.6, fig11: $\varepsilon = -1, N = 1, p = 3, \alpha = -3/2$

Proof. Here again M_{ℓ} is a sink; but it is a node point only if $\alpha \geq \alpha_2$. The phase plane (y, Y) does not contain any cycle, from Proposition 4.4. From Proposition 3.6, any solution y has at most two zeros, and Y at most one.

The unique trajectory \mathcal{T}_{α} ends up in \mathcal{Q}_4 with the slope 0. From the uniqueness of \mathcal{T}_{α} and $\mathcal{T}_{\varepsilon}$, all the trajectories, apart from $\pm \mathcal{T}_{\alpha}$, converge to $\pm M_{\ell}$ at ∞ , from Proposition 4.1 and Remark 9.3. Since $\varepsilon \alpha > 0$, the trajectory \mathcal{T}_r starts in \mathcal{Q}_1 , and y has at most one zero. Then \mathcal{T}_r converges to $-M_{\ell}$ in \mathcal{Q}_2 , or $\mathcal{T}_r = -\mathcal{T}_{\alpha}$. The trajectory $\mathcal{T}_{\varepsilon}$ starts in \mathcal{Q}_4 with the slope -1, satisfies $y \geq 0$ from Proposition 3.6. If $\mathcal{T}_{\varepsilon}$ converge to (0,0), then $\mathcal{T}_{\varepsilon} = \mathcal{T}_{\alpha}$, thus it is homoclinic. Then M_{ℓ} is in the bounded component defined by $\mathcal{T}_{\varepsilon}$, and $\mathcal{T}_{\varepsilon}$ meets \mathcal{T}_r , which is impossible. Hence $\mathcal{T}_{\varepsilon}$ converges to M_{ℓ} in \mathcal{Q}_4 , and w is nonnegative with a hole and satisfies (4.3) near ∞ .

If $\alpha \neq -p'$, we claim that $\mathcal{T}_r \neq -\mathcal{T}_{\alpha}$. Indeed suppose $\mathcal{T}_r = -\mathcal{T}_{\alpha}$. Consider the functions y_{α}, Y_{α} , defined by (2.3) with $d = \alpha$. Then Y_{α} stays positive, and $Y_{\alpha} = O(e^{(\alpha(p-1)+p)\tau})$ at ∞ , thus

$$\lim_{\tau \to \infty} Y_{\alpha} = 0, \quad \lim_{\tau \to \infty} Y_{\alpha} = c > 0, \quad \lim_{\tau \to -\infty} y_{\alpha} = \infty, \quad \lim_{\tau \to \infty} y_{\alpha} = L < 0.$$

Moreover y_{α}, Y_{α} have no extremal point: at such a point, from (3.2), (3.3) the second derivatives have a strict constant sign; then $Y'_{\alpha} > 0 > y'_{\alpha}$. If $\alpha < \eta$ (in particular if $p \leq N$), from (4.13), near ∞ ,

$$(p-1)Y''_{\alpha}/Y'_{\alpha} \ge |Y|^{(2-p)/(p-1)} (1+o(1)),$$

thus $Y''_{\alpha} > 0$ near ∞ , which is contradictory; if $\alpha > \eta$, from (4.12)

$$(p-1)y''_{\alpha}/y'_{\alpha} \ge |Y|^{(2-p)/(p-1)} (1+o(1)),$$

thus $y''_{\alpha} < 0$ near ∞ , still contradictory. If $\alpha = \eta$, $\mathcal{T}_{\alpha} = \mathcal{T}_{u}$ from (1.11), thus again $\mathcal{T}_{r} \neq -\mathcal{T}_{\alpha}$.

If p > N and $\alpha \neq \eta$, we claim that $\mathcal{T}_{\alpha} \neq \mathcal{T}_{u}$. Indeed suppose $\mathcal{T}_{\alpha} = \mathcal{T}_{u}$. This trajectory stays \mathcal{Q}_{4} , the function ζ stays negative, and $\lim_{\tau \to -\infty} \zeta = \eta$, $\lim_{\tau \to \infty} \zeta = \alpha$. If ζ has an extremal point ϑ , then $\vartheta \in (\alpha, \eta)$ from System (**Q**), and ζ'' has a constant sign, the sign of $\alpha - \zeta$; it is impossible. Thus ζ is monotone; then $(\alpha - \eta)\zeta' > 0$, which contradicts System (**Q**).

(1) Case $\alpha \neq -p'$. Since $\mathcal{T}_r \neq -\mathcal{T}_{\alpha}$, \mathcal{T}_r converges to $-M_{\ell}$, and y has one zero, and |w| satisfies (4.3).

• Case $p \leq N$. All the other trajectories start in \mathcal{Q}_3 or \mathcal{Q}_1 , from Remarks 9.1 and 9.2. For any $\varphi > 0$, the trajectory $\mathcal{T}_{[(\varphi,0)]}$ goes from \mathcal{Q}_4 into \mathcal{Q}_1 , and converges to $-M_\ell$ in \mathcal{Q}_2 , since it cannot meet \mathcal{T}_r and $-\mathcal{T}_{\varepsilon}$; thus y has two zeros, and w is of type (iii). The trajectory \mathcal{T}_{α} cannot meet $\mathcal{T}_{[(\varphi,0)]}$, thus y has one zero, and it has the same behaviour at $-\infty$, and w is of type (i). All the trajectories $\mathcal{T}_{[P]}$ with P in the interior domain of \mathcal{Q}_1 delimitated by $-\mathcal{T}_{\varepsilon}$ and \mathcal{T}_r start from \mathcal{Q}_1 and converge to $-M_\ell$, y has precisely one zero, and has the same behaviour at $-\infty$, and w is of type (i).

• Case p > N, $\eta < \alpha$ (see figure 9). Any solution y has at most one simple zero. The trajectory \mathcal{T}_{α} stays in \mathcal{Q}_4 . Indeed if it started in \mathcal{Q}_3 , then for any trajectory $\mathcal{T}_{[(0,\xi)]}$ with $(0,\xi)$ above $-\mathcal{T}_{\alpha}$, the function y would have two zeros. Since $\mathcal{T}_{\alpha} \neq \mathcal{T}_u$, we have $\mathcal{T}_{\alpha} \in \mathcal{T}_-$, and w is of type (iv). The trajectory \mathcal{T}_u necessarily stays in \mathcal{Q}_4 and converges to M_ℓ , and w is of type (v). The trajectories $\mathcal{T}_{[P]}$, with P in the domain delimitated by $\mathcal{T}_u, \mathcal{T}_{\alpha}$ and $\mathcal{T}_{\varepsilon}$, are of type \mathcal{T}_- and converge in \mathcal{Q}_4 to M_ℓ , and w is of type (vi). The trajectories $\mathcal{T}_{[P]}$, with P in the domain delimitated by $\mathcal{T}_r, \mathcal{T}_\alpha$ and $-\mathcal{T}_{\varepsilon}$, are of type \mathcal{T}_- , and converge to $-M_\ell$, and y has one zero. The trajectories $\mathcal{T}_{[P]}$, with P in

the domain delimitated by \mathcal{T}_r and $-\mathcal{T}_u$, are of type \mathcal{T}_+ , converge to $-M_\ell$, and y has one zero. Both define solutions w of type (vii).

• Case $p > N, \alpha < \eta$ (see figure 10). We have seen that $\mathcal{T}_r \neq -\mathcal{T}_\alpha$. If $\mathcal{T}_\alpha \in \mathcal{T}_+$, then ζ decreases from 0 to α , which contradicts System (**Q**) at ∞ . Then \mathcal{T}_α does not stay in \mathcal{Q}_4 , it starts in \mathcal{Q}_3 and $-\mathcal{T}_\alpha \in \mathcal{T}_-$, hence y has a zero, and w is of type (viii). Then \mathcal{T}_u and the trajectories \mathcal{T}_- converge to $-M_\ell$, and y has one zero. The trajectories $\mathcal{T}_{[P]}$, with P in the domain delimitated by $\mathcal{T}_r, -\mathcal{T}_\alpha$ and $-\mathcal{T}_\varepsilon$, are of type \mathcal{T}_+ and converge to $-M_\ell$, y has one zero. They correspond to w is of type (ix) or (x). The trajectories $\mathcal{T}_{[P]}$, with P in \mathcal{Q}_4 above \mathcal{T}_r , cut the line $\{y = 0\}$ twice, and converge to M_ℓ , and w is of type (xi).

• Case $p > N, \alpha = \eta$. Then $\mathcal{T}_{\alpha} = \mathcal{T}_{u}$, the functions $w = cr^{-\eta}$ (c > 0) are particular solutions. The phase plane study is the same, and gives only solutions of type (vii).

(2) Case $\alpha = -p'$ (see figure 11). Here $\mathcal{T}_r = -\mathcal{T}_{\alpha}$, since the regular solutions are given by (1.12). Thus there exist no more solutions of type (ii) or (viii).

Next we study the behaviour of all the solutions when $\alpha < -p'$. In particular we prove the existence and uniqueness of an α_c for which there exists an homoclinic trajectory. Thus we find again some results obtained in [8], with new detailed proofs. We also improve the bounds for α_c , in particular $\alpha^* < \alpha_c$.

Lemma 9.7 Let

$$\alpha_p := -(p-1)/(p-2).$$

If N = 1, for $\alpha = \alpha_p$, then there exists an homoclinic trajectory in the phase plane (y, Y). If $N \ge 2$, for $\alpha = \alpha_p$, there is no homoclinic trajectory, moreover \mathcal{T}_{α} converges to M_{ℓ} at $-\infty$ or has a limit cycle in \mathcal{Q}_4 .

Proof. In the case N = 1, $\alpha = \alpha_p$, the explicit solutions (1.14) define an homoclinic trajectory in the phase plane (y, Y), namely $\mathcal{T}_{\varepsilon} = \mathcal{T}_{\alpha}$. In the phase plane (g, s) of System **(R)**, from Remark 2.6, they correspond to the line $s \equiv 1 + \alpha g$, joining the stationary points (0, 1) and $(-1/\alpha, 0)$.

Next assume $N \geq 2$ and consider the trajectory \mathcal{T}_{α} in the plane (y, Y). In the plane (g, s) of System (**R**), the corresponding trajectory \mathcal{T}'_{α} ends up at $(-1/\alpha, 0)$, as ν tends to ∞ from (2.18), with the slope $-k_p$. If \mathcal{T}_{α} is homoclinic, then \mathcal{T}'_{α} converges to (0, 1) as ν tends to $-\infty$. Consider the segment

$$T = \{(g, -k(g+1/\alpha_p) : g \in [0, 1/|\alpha_p|]\}, \quad \text{with} \quad k = p'\alpha_p^2/(N+2/(p-2)) > k_p.$$

Its extremity $(0, k/|\alpha_p|)$ is strictly under (0, 1). The domain \mathcal{R} delimitated by the axes, which are particular orbits, and T, is negatively invariant: indeed, at any point of T, we find

$$k\frac{dg}{d\nu} + \frac{ds}{d\nu} = (N-1)p'ks(g-\frac{1}{\gamma})^2.$$

The trajectory \mathcal{T}'_{α} ends up in \mathcal{R} , thus it stays in it, hence \mathcal{T}'_{α} cannot join (0, 1). In the phase plane $(y, Y), \mathcal{T}_{\alpha}$ is not homoclinic, and \mathcal{T}_{α} stays in \mathcal{Q}_4 , and Remark 9.3 applies.

Remark 9.8 Notice that $\alpha^* \leq \alpha_p \Leftrightarrow N \leq p$.

Theorem 9.9 Assume $\varepsilon = -1$, and $\alpha < -p'$. There exists a unique $\alpha_c < 0$ such that there exists an homoclinic trajectory in the plane (y, Y); in other words $\mathcal{T}_{\varepsilon} = \mathcal{T}_{\alpha}$. If N = 1, then $\alpha_c = \alpha_p$. If $N \ge 2$, then

$$max(\alpha^*, \alpha_p) < \alpha_c < \min(\alpha_2, -p').$$
(9.1)

Proof. In order to prove the existence of an homoclinic orbit for System (**S**), we could consider a Poincaré application as in [4], but it does not give uniqueness. Thus we consider the system (\mathbf{R}_{β}) obtained from (**R**) by setting $s = \beta S$:

$$\frac{dg}{d\nu} = gF(g,S), \qquad F(g,S) := \beta S(1+\eta g) - \frac{1}{p-1}(1+\alpha g), \\ \frac{dS}{d\nu} = SG(g,S), \qquad G(g,S) := 1 + \alpha g - \beta(1+Ng)S.$$
 (**R**_{\beta})

Its stationary points are

(0,0),
$$A' = (1/|\alpha|, 0), \quad B' = (0, 1/\beta), \quad M' = (1/\gamma, 1/(N+\gamma)(p-2)),$$

where M' corresponds to M_{ℓ} . The existence of homoclinic trajectory for System (**S**) resumes to the existence of a trajectory for System (**R**_{β}) in the plane (g, S), starting from B' and ending at A'.

(i) Existence. We can assume that $\alpha \in (\alpha_1, \min(\alpha_2, -p'))$, from Proposition 4.4. In the plane (g, S), consider the trajectories $\mathcal{T}'_{\varepsilon}$ and \mathcal{T}'_{α} corresponding to $\mathcal{T}_{\varepsilon} \cap \mathcal{Q}_4$ and $\mathcal{T}_{\alpha} \cap \mathcal{Q}_4$ in the plane (y, Y). Then $\mathcal{T}'_{\varepsilon}$ starts from B' and \mathcal{T}'_{α} ends up at A'. From Remark 9.1, for any $\alpha \in (\alpha_1, \alpha_2)$, with $\alpha \leq -p'$, we have three possibilities:

• $\mathcal{T}'_{\varepsilon}$ is converging to M' as ν tends to ∞ and turns around this point, since α is a spiral point, or it has a limit cycle in \mathcal{Q}_1 around M'. And \mathcal{T}'_{α} admits the line g = 0 as an asymptote as ν tends to $-\infty$, which means that \mathcal{T}_{α} does not stay in \mathcal{Q}_4 in the plane (y, Y). Then $\mathcal{T}'_{\varepsilon}$ meats the line

$$L := \{g = 1/\gamma\}$$

at a first point $(1/\gamma, S_0(\alpha))$. And \mathcal{T}'_{α} meats L at a last point $(1/\gamma, S_1(\alpha))$, such that $S_0(\alpha) - S_1(\alpha) < 0$;

• \mathcal{T}'_{α} is converging to M' at $-\infty$ or it has a limit cycle in \mathcal{Q}_1 around M'. And $\mathcal{T}'_{\varepsilon}$ admits the line S = 0 as an asymptote at ∞ , which means that $\mathcal{T}_{\varepsilon}$ does not stay in \mathcal{Q}_4 . Then with the same notations, $S_0(\alpha) - S_1(\alpha) > 0$.

•
$$T_{\varepsilon}' = T_{\alpha}'$$
, equivalently $S_0(\alpha) - S_1(\alpha) = 0$

The function $\alpha \mapsto \varphi(\alpha) = S_0(\alpha) - S_1(\alpha)$ is continuous, from Theorems 3.9 and 5.3. If $-p' < \alpha_2$, then $\varphi(-p')$ is well defined and $\varphi(-p') < 0$; indeed $\mathcal{T}_{\alpha} = -\mathcal{T}_r$, thus \mathcal{T}_{α} does not stay in \mathcal{Q}_4 from Theorem 9.6. If $\alpha_2 \leq -p'$, in the plane (y, Y), the trajectory \mathcal{T}_{α_2} leaves \mathcal{Q}_4 , from Proposition 4.4, because α_2 is a sink, and transversally from Remark 9.1. The same happens for \mathcal{T}_{α_2-v} for v > 0small enough, by continuity, thus $\varphi(\alpha_2 - v) < 0$. From Lemma 9.7, $\varphi(\alpha_p) > 0$ if $N \geq 2$, and $\varphi(\alpha_p) = 0$ if N = 1. In any case there exists at least an α_c satisfying (9.1), such that $\varphi(\alpha_c) = 0$.

(ii) Uniqueness. First observe that $1 + \eta g > 0$; indeed $1 + \eta / |\alpha| > (p' + \eta) / |\alpha| > 0$. Now

$$(p-1)F + G = p\beta S(1/\gamma - g) = (p-2)\beta S(1-\gamma g),$$

hence the curves $\{F = 0\}$ and $\{G = 0\}$ intersect at M' and A', $\{G = 0\}$ contains B' and is above $\{F = 0\}$ for $g \in (0, 1/\gamma)$ and under it for $g \in (1/\gamma, 1/|\alpha|)$. Moreover $\mathcal{T}'_{\varepsilon}$ has a negative slope at B', thus F > 0 > G near 0 from (\mathbf{R}_{β}) . And $\mathcal{T}'_{\varepsilon}$ cannot meet $\{G = 0\}$ for $(0, 1/\gamma)$, because on this curve the vector field is (gF, 0) and F > 0. Thus $\mathcal{T}'_{\varepsilon}$ satisfies F > 0 > G on $(0, 1/\gamma)$. In the same way \mathcal{T}'_{α} has a negative slope $-\theta \alpha^2/(p-1)(\eta + |\alpha|) < 0$ at $1/|\alpha|$, thus F > 0 > G near $1/|\alpha|$. And \mathcal{T}'_{α} cannot meet $\{F = 0\}$, because the vector field on this curve is (0, SG) and G < 0. Thus \mathcal{T}'_{α} satisfies F > 0 > G on $(1/\gamma, 1/|\alpha|)$.

Let $\alpha < \bar{\alpha}$. Then $\mathcal{T}'_{\varepsilon}$ is above $\bar{\mathcal{T}}'_{\varepsilon}$ near g = 0, and \mathcal{T}'_{α} is at the left of $\mathcal{T}'_{\bar{\alpha}}$ near S = 0. We show that $\varphi(\alpha) > \varphi(\bar{\alpha})$. First suppose that $\mathcal{T}'_{\varepsilon}$ and $\bar{\mathcal{T}}'_{\varepsilon}$ (or \mathcal{T}'_{α} and $\bar{\mathcal{T}}'_{\bar{\alpha}}$) intersect at a first point P_1 (or a last point) such $g \neq 1/\gamma$. Then at this point

$$\frac{1}{p-1}\frac{g}{S}\frac{dS}{dg} + 1 = \frac{(p-2)(1-\gamma g)S}{(p-1)S(1+\eta g) - \beta^{-1}(1+\alpha g)} = \frac{(p-2)(1-\gamma g)S}{h_S(g) - \beta^{-1}(1-\gamma g)}$$
(9.2)

with $h_S(g) = (p-1)S(1+\eta g) - g/(p-2)$. Thus the denominator, which is positive, is increasing in α on $(0, 1/\gamma)$, decreasing on $(1/\gamma, 1/|\alpha|)$; in any case $dS/dg > d\overline{S}/dg$ at P_1 , which is contradictory. Next suppose that there is an intersection on L. At such a point $P_1 = (1/\gamma, S_1) = (1/\gamma, \overline{S}_1)$ the derivatives are equal from (9.2), and P_1 is above M', because F > 0. At any points $(g, S(g)) \in \mathcal{T}'_{\varepsilon}$ (or \mathcal{T}'_{α}), $(g, \overline{S}(g)) \in \overline{\mathcal{T}}'_{\varepsilon}$ (or $\overline{\mathcal{T}}'_{\alpha}$), setting $g = 1/\gamma + u$,

$$\Phi(u) = \left(\frac{1}{p-1}\frac{g}{S}\frac{dS}{dg} + 1\right)\frac{1}{(p-2)S} = -\frac{\gamma}{h_S(1/\gamma)}u + \frac{1}{h_S^2(1/\gamma)}\left(\frac{\gamma}{\beta} + h_S'(1/\gamma)\right)u^2(1+o(1)),$$

$$\bar{\Phi}(u) = \left(\frac{1}{p-1}\frac{g}{\overline{S}}\frac{d\overline{S}}{dg} + 1\right)\frac{1}{(p-2)\overline{S}} = -\frac{\gamma}{h_{\overline{S}}(1/\gamma)}u + \frac{1}{h_{\overline{S}}^2(1/\gamma)}\left(\frac{\gamma}{\beta} + h_S'(1/\gamma)\right)u^2(1+o(1)),$$

And $h_S(1/\gamma) = h_{\overline{S}}(1/\gamma) > 0$, and $h'_S(1/\gamma) = h'_{\overline{S}}(1/\gamma)$, then

$$(\Phi - \bar{\Phi})(u) = \frac{\gamma u^2 (1/\beta - 1/\bar{\beta})}{h(1/\gamma)} (1 + o(1)).$$

This implies $d^2(S-\overline{S})/dg^2 = 0$ and $d^3(S-\overline{S})/dg^3 = 2S_1\gamma^2(p-1)(p-2)(1/\beta - 1/\overline{\beta}) > 0$, which is a contradiction. Then $\mathcal{T}'_{\varepsilon}$ and $\overline{\mathcal{T}}'_{\varepsilon}$ cannot intersect on this line, similarly for \mathcal{T}'_{α} and $\overline{\mathcal{T}}'_{\overline{\alpha}}$. Hence $\varphi(\alpha) > \varphi(\overline{\alpha})$, which proves the uniqueness.

As a consequence, for $\alpha < \alpha_c$, $\varphi(\alpha) > 0$, in the plane (y, Y), $\mathcal{T}_{\varepsilon}$ does not stay in \mathcal{Q}_4 ; for $\alpha > \alpha_c$, $\varphi(\alpha) < 0$, \mathcal{T}_{α} does not stay in \mathcal{Q}_4 . From Lemma 9.7, it follows that $\alpha_p < \alpha_c$ if $N \ge 2$. Moreover $\alpha^* < \alpha_c$. Indeed α^* is a weak source from Proposition 2.5, thus for $\alpha > \alpha^*$ small enough, there exists a unique cycle \mathcal{O} around M_{ℓ} , which is unstable. For such an α , $\mathcal{T}_{\varepsilon}$ cannot stay in \mathcal{Q}_4 : it would have \mathcal{O} as a limit cycle at ∞ , which contradicts the unstability.

Next we discuss according to the position of α with respect to α^* and α_c .

Theorem 9.10 Assume $\varepsilon = -1$, and $\alpha \leq \alpha^*$. Then

(i) there exist a unique flat positive solution w of (\mathbf{E}_w) with (4.3) near 0, and (4.4) near ∞ ; (ii) All the other solutions are oscillating at ∞ , among them the regular ones, and $r^{-\gamma}w$ is asymptotically periodic in $\ln r$. There exist solutions with a hole, also with (4.3), (4.6) or (4.9) or (4.9) or (4.7) near 0. There exist solutions such that $r^{-\gamma}w$ is periodic in $\ln r$.

th 9.10,fig 12: $\varepsilon = -1, N = 1, p = 3, \alpha = -2.53$ th 9.10, fig 13: $\varepsilon = -1, N = 1, p = 3, \alpha = -2.2$

Proof. Here $\alpha < \alpha_c$, from Theorem 9.9, and the trajectory \mathcal{T}_{α} stays in \mathcal{Q}_4 . From Proposition 4.4, it converges at $-\infty$ to M_{ℓ} , and w is of type (i).

The trajectory $\mathcal{T}_{\varepsilon}$ leaves \mathcal{Q}_4 , and cannot converge either to (0,0) since $\mathcal{T}_{\varepsilon} \neq \mathcal{T}_{\alpha}$, or to $\pm M_{\ell}$, because this point is a source, or a weak source. Recall that M_{ℓ} is a node point for $\alpha \leq \alpha_1$ (see figure 12,, where $\alpha_1 \cong -2.50$), or a spiral point (see figure 13). And $\mathcal{T}_{\varepsilon}$ is bounded at ∞ from Proposition 4.3. Then it has a limit cycle $\mathcal{O}_{\varepsilon}$ surrounding (0,0) from Proposition 4.4, and $\pm M_{\ell}$ from Remark 9.3. Thus w is oscillating around 0 near ∞ , $r^{-\gamma}w$ is asymptotically periodic in $\ln r$.

The solutions w corresponding to $\mathcal{O}_{\varepsilon}$ are oscillating and $r^{-\gamma}w$ is periodic in $\ln r$. Any trajectory $\mathcal{T}_{[P]}$ with P in the interior domain delimitated by $\mathcal{O}_{\varepsilon}$ converges to M_{ℓ} at $-\infty$ and has the same limit cycle at ∞ . The trajectory \mathcal{T}_r starts in \mathcal{Q}_1 , with $\lim_{\tau \to -\infty} y = \infty$ and cannot converge to any stationary point at ∞ . It is bounded, thus has a limit cycle \mathcal{O}_r surrounding \mathcal{O}_0 . For any $P \notin \mathcal{T}_r$ in the exterior domain to \mathcal{O}_r , the trajectory $\mathcal{T}_{[P]}$ admits \mathcal{O}_r as a limit cycle at ∞ , and y is necessarily monotone at $-\infty$, thus (4.6) or (4.9) or (4.9) or (4.7) near 0; all those solutions exist. The question of the uniqueness of the cycle ($\mathcal{O}_r = \mathcal{O}_{\varepsilon}$) is open.

Theorem 9.11 Let α_c be defined by Theorem 9.9.

(1) Let $\alpha^* < \alpha < \alpha_c$. Then all regular solutions w of (\mathbf{E}_w) are oscillating around 0 near ∞ , and $r^{-\gamma}w$ is asymptotically periodic in $\ln r$. There exist

(i) **positive** solutions, such that $r^{-\gamma}w$ is periodic in $\ln r$;

(ii) a unique positive solution such that $r^{-\gamma}w$ is asymptotically periodic in $\ln r$ near 0, with (4.4) near ∞ ;

(iii) positive solutions such that $r^{-\gamma}w$ is asymptotically periodic in $\ln r$ near 0, with (4.3) near ∞ ; (iv) solutions oscillating around 0 such that $r^{-\gamma}w$ is periodic in $\ln r$;

(v) solutions with a hole, oscillating near ∞ , such that $r^{-\gamma}w$ is asymptotically periodic in $\ln r$;

(vi) solutions satisfying (4.6) or (4.9) or (4.9) or (4.7) near 0, oscillating around 0 near ∞ , such that $r^{-\gamma}w$ is asymptotically periodic in $\ln r$;

(vii) solutions positive near 0, oscillating near ∞ , such that $r^{-\gamma}w$ is asymptotically periodic in $\ln r$ near 0 and ∞ .

(2) Let $\alpha = \alpha_c$.

(viii) There exist a unique nonnegative solution with a hole, with (4.4) near ∞ .

The regular solutions are as above. There exist solutions of types (iv), (vi), and

(ix) positive solutions such that $r^{-\gamma}w$ is bounded from above near 0, with (4.3) near ∞ .

th 9.11, fig 14: $\varepsilon = -1, N = 1, p = 3, \alpha = -2.1$ th 9.11, fig 15: $\varepsilon = -1, N = 1, p = 3, \alpha = -2$

Proof. (1) Let $\alpha^* < \alpha < \alpha_c$ (see figure 14). Then \mathcal{T}_{α} stays in \mathcal{Q}_4 , but cannot converge neither to M_{ℓ} which is a sink, nor to (0,0) since $\mathcal{T}_{\alpha} \neq \mathcal{T}_{\varepsilon}$. It has a limit cycle \mathcal{O}_{α} in \mathcal{Q}_4 at $-\infty$, surrounding M_{ℓ} , and w is of type (ii). The orbit \mathcal{O}_{α} corresponds to solutions of type (i). There exist positive solutions converging to M_{ℓ} at ∞ , with a limit cycle \mathcal{O}_{ℓ} at $-\infty$ surrounded by \mathcal{O}_{α} , and w is of type (iii). This cycle is unique ($\mathcal{O}_{\ell} = \mathcal{O}_{\alpha}$) for $\alpha - \alpha^*$ small enough, from Proposition 2.5. The trajectory $\mathcal{T}_{\varepsilon}$ still cannot stay in \mathcal{Q}_4 . As in the case $\alpha \leq \alpha^*$, $\mathcal{T}_{\varepsilon}$ has a limit cycle $\mathcal{O}_{\varepsilon}$ surrounding the three stationary points, w is of type (v), and \mathcal{T}_r is oscillating around 0, and there exist solutions of type (vi). Any trajectory $\mathcal{T}_{[P]}$ with $P \notin \mathcal{T}_{\varepsilon}$ in \mathcal{Q}_4 in the domain delimitated by \mathcal{O}_{α} and $\mathcal{O}_{\varepsilon}$ admits \mathcal{O}_{α} as a limit cycle at $-\infty$ and $\mathcal{O}_{\varepsilon}$ at ∞ , and w is of type (vi).

(2) Let $\alpha = \alpha_c$ (see figure 15). The homoclinic trajectory $\mathcal{T}_{\varepsilon} = \mathcal{T}_{\alpha}$ corresponds to the solution w of type (viii). The trajectory \mathcal{T}_r has a limit cycle \mathcal{O}_r surrounding the three points. Thus there exist solutions of types (iv) or (vi). Any trajectory ending up at M_{ℓ} at ∞ is bounded, contained in the domain delimitated by $\mathcal{T}_{\varepsilon}$, and its limit set at $-\infty$ is the homoclinic trajectory $\mathcal{T}_{\varepsilon}$, or a cycle around M_{ℓ} , and w is of type (ix).

Theorem 9.12 Assume $\varepsilon = -1$, and $\alpha_c < \alpha < -p'$.

There exist a unique nonnegative solution w of (\mathbf{E}_w) with a hole, with $r^{-\gamma}w$ bounded from above and below at ∞ . The regular solutions have at least two zeros.

(1) Either there exist oscillating solutions such that $r^{-\gamma}w$ is periodic in $\ln r$. Then the regular solutions have an infinity of zeros, and $r^{-\gamma}w$ is asymptotically periodic in $\ln r$. There exist

(i) solutions satisfying (4.6) or (4.9) or (4.9) or (4.7) near 0, oscillating near ∞ , such that $r^{-\gamma}w$ is asymptotically periodic in $\ln r$;

(ii) a unique solution oscillating near 0, such that $r^{-\gamma}w$ is asymptotically periodic in $\ln r$, and with (4.4) near ∞ ;

(iii) solutions positive near 0, with $r^{-\gamma}w$ bounded, and oscillating near ∞ , such that $r^{-\gamma}w$ is asymptotically periodic in $\ln r$.

(2) Or all the solutions have a finite number of zeros, and at least two. Two cases may occur:

• Either regular solutions have m zeros and $r^{-\gamma}w$ bounded from above and below at ∞ . Then there exist

(iv) solutions with m zeros, with (4.6) or (4.9), with (4.4) near ∞ ;

(v) solutions with m zeros with (4.6) or (4.9) and $r^{-\gamma}w$ bounded from above and below at ∞ ;

(vi) solutions with m + 1 zeros with (4.6) or (4.9) and $r^{-\gamma}w$ bounded from above and below at ∞ ;

(vii) (for p > N) a unique solution with m zeros, with (4.7) or (4.10) and $r^{-\gamma}w$ bounded from above and below at ∞ .

• Or regular solutions have m zeros and (4.4) holds near ∞ . Then there exist solutions of type (vi) or (vii).

th 9.12, fig 16: $\varepsilon = -1, N = 1, p = 3, \alpha = -1.98$ th 9.12, fig 17: $\varepsilon = -1, N = 1, p = 3, \alpha = -1.90$

Proof. Here $\mathcal{T}_{\varepsilon}$ stays in \mathcal{Q}_4 , converges to M_{ℓ} or has a limit cycle around M_{ℓ} , thus w has a hole and $r^{-\gamma}w$ bounded from above and below at ∞ . If $\alpha \geq \alpha_2$, there is no cycle in \mathcal{Q}_4 , from Proposition 4.4, thus $\mathcal{T}_{\varepsilon}$ converges to M_{ℓ} . (1) Either there exists a cycle surrounding (0,0) and $\pm M_{\ell}$, thus solutions w oscillating around 0, such that $r^{-\gamma}w$ is periodic in $\ln r$. Then \mathcal{T}_r has such a limit cycle \mathcal{O}_r , and w is oscillating around 0. The trajectory \mathcal{T}_{α} has a limit cycle at $-\infty$ of the same type $\mathcal{O}_{\alpha} \subset \mathcal{O}_r$, and w is of type (ii). For any $P \notin \mathcal{T}_{\varepsilon}$ in the interior domain in \mathcal{O}_{α} , $\mathcal{T}_{[P]}$ admits \mathcal{O}_{α} as a limit cycle at $-\infty$ and converges to M_{ℓ} at ∞ , or has a limit cycle in \mathcal{Q}_4 ; and w is of type (ii). For any $P \notin \mathcal{T}_r$, in the domain exterior to $\mathcal{O}_r, \mathcal{T}_{[P]}$ has \mathcal{O}_{α} as limit cycle at ∞ , and w is of type (i).

(2) Or no such cycle exists. Then any trajectory converges at ∞ , any trajectory, apart from $\pm \mathcal{T}_{\alpha}$, converges to $\pm M_{\ell}$ or has a limit cycle in \mathcal{Q}_1 . All the trajectories end up in \mathcal{Q}_2 or \mathcal{Q}_4 . Since \mathcal{T}_r starts in \mathcal{Q}_1 , y has at least one zero. Suppose that it is unique. Then \mathcal{T}_r converges to $-M_{\ell}$, thus Y stays positive. Consider the function $Y_{\alpha} = e^{(\alpha+\gamma)(p-1)\tau}Y$ defined by (2.3) with $d = \alpha$. From Theorem 3.3, $Y_{\alpha} = (a |\alpha| / N)e^{(\alpha(p-1)+p)\tau}(1+o(1))$ near $-\infty$; thus Y_{α} tends to ∞ , since $\alpha < p'$. And $Y_{\alpha} = (\gamma \ell)^{p-1}e^{(\alpha+\gamma)(p-1)\tau}$ near ∞ , thus also Y_{α} tends to ∞ ; then it has a minimum point τ , and from (2.6), $Y''_{\alpha}(\tau) = (p-1)^2(\eta - \alpha)(p' + \alpha)Y_{\alpha} < 0$, which is contradictory. Thus y has a number $m \geq 2$ of zeros.

Either $\mathcal{T}_r \neq \mathcal{T}_\alpha$. Since the slope of \mathcal{T}_α near $-\infty$ is infinite and the slope of \mathcal{T}_r is finite, \mathcal{T}_α cuts the line $\{y = 0\}$ at m points, starts from \mathcal{Q}_1 , and w is of type (iv). For any P in the domain of \mathcal{Q}_1 between \mathcal{T}_r and \mathcal{T}_α , $\mathcal{T}_{[P]}$ cuts $\{y = 0\}$ at m + 1 points, and w is of type (v). For any P in the domain of \mathcal{Q}_1 above \mathcal{T}_r , $\mathcal{T}_{[P]}$ cuts the line $\{y = 0\}$ at m + 1 points, and w is of type (vi). If p > N, the trajectories \mathcal{T}_- and \mathcal{T}_u cut the line $\{y = 0\}$ at m points, and w is of type (vi).

Or $\mathcal{T}_r = \mathcal{T}_{\alpha}$, and then we find only trajectories with w of type (vi) or (vii).

Remark 9.13 Consider the regular solutions in the range $\alpha_c < \alpha < -p'$. We conjecture that there exists a decreasing sequence $(\bar{\alpha}_n)$, with $\bar{\alpha}_0 = -p'$ and $\alpha_c < \bar{\alpha}_n$ such that for $\alpha \in (\bar{\alpha}_m, \bar{\alpha}_{m-1})$, y has m zeros and converges to $\pm M_\ell$; and for $\alpha = \bar{\alpha}_m$, y has m + 1 zeros and converges to (0,0), thus $T_r = T_\alpha$. We presume that $(\bar{\alpha}_m)$ has a limit $\bar{\alpha} > \alpha_c$. And for $\alpha < \bar{\alpha}$, y has an infinity of zeros, in other words there exists a cycle \mathcal{O}_r surrounding $\{0\}$ and $\pm M_\ell$.

Numerically, for $\alpha = \alpha_c$, the cycle \mathcal{O}_r seems to be the unique cycle surrounding the three points. But for $\alpha > \alpha_c$ and $\alpha - \alpha_c$ small enough, there exist **two different cycles** $\mathcal{O}_{\alpha} \subset \mathcal{O}_r$ (see figure 15). As α increases, we observe the coalescence of those cycles; they disappear after some value $\bar{\alpha}$ (see figure 16).

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