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Numerical Approximation of Continuous Traffic Congestion Equilibria

F. Benmansour, G. Carlier, G. Peyré, F. Santambrogio *

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Abstract

Starting from a continuous congested traffic framework recently introduced in [8], we present a consistent numerical scheme to compute equilibrium metrics. We show that equilibrium metric is the solution of a variational problem involving geodesic distances. Our discretization scheme is based on the Fast Marching Method. Convergence is proved and numerical results are given.

Keywords: traffic congestion, Wardrop equilibria, eikonal equation, subgradient descent, Fast Marching Method.

AMS subject classifications: 49M25, 65K10, 90C25.

1 Introduction

Congestion is an important issue in real-life applications such as road or communication networks. In the early 50's, Wardrop [15] considered the situation where a large number of vehicles have to go from one location to another, connected by a finite number of different roads. Each vehicle has to choose a pathway along the roads to minimize some transportation cost which depends not only on the chosen pathway but also on the number of vehicles (the traffic) along it. Naturally, shorter or wider roads are preferred by the vehicles. Nevertheless, the transportation "cost" on a given road depends increasingly on the traffic. To avoid the congestion some vehicles may choose a "longer" pathway to minimize the cost. Wardrop gave a minimal stability requirement for transportation strategies: the cost of every actually used pathway should be equal or less than that which would be experienced by a single vehicle on any other roads. In particular there is an equilibrium concept: all actually used pathways have the same cost, i.e. they compensate in term of congestion their differences given by length and other conditions. Beside the equilibrium concept there is a minimality concept as well: those pathways are minimal among all the possible ones. In other words, a Wardrop equilibrium is a situation where every traveler uses only shortest paths from his source to his destination, given the overall congestion pattern resulting from the individual strategies of all the users of the road network. This natural equilibrium concept, somehow similar to Nash's equilibrium, has been very popular since its introduction.

Few years later, Beckmann et al. [3] discovered that Wardrop equilibria can be obtained as minimizers of a certain convex optimization problem. This result is very useful for

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proving theoretical existence, uniqueness and stability results. The convex minimization formulation (or some dual one) can also be used to compute numerically traffic equilibria as long as the complexity of the problem is tractable which, in practice, often excludes the case of large networks. On the contrary, in the present paper, we consider a continuous case, where there is no network and every path is allowed. Our work is based on a continuous traffic equilibrium formulation recently introduced in [8]. We will see how this problem can be dealt without huge computational costs.

The study of Wardrop equilibria has mainly been restricted to the finite-dimensional case where the road network is a given finite graph. In [8], there is no network and congestion effects are captured by a metric that depends increasingly on the traffic intensity which gives a natural framework to study optimal transportation problems with congestion effects. It turns out that the problem addressed in [8] can also be interpreted as a continuous traffic equilibrium à la Wardrop. The equilibrium problem is then formulated in terms of the traffic intensity or the metric itself. As shown, in section 4, the equilibrium metric is obtained by minimizing some convex functional that is the difference of two terms. The first one is a standard convex integral functional. The second one is a weighted sum of geodesic distances between a given source and a given destination as a function of the metric. Building upon this continuous minimization problem, we introduce a discretized minimization scheme based on Sethian's FMM [12] (Fast Marching Method). We prove convergence of the discretized problems to the continuous one and we solve numerically the discretized problem by an iterative subgradient descent method. Each iteration of the discretized problem requires the computation of an element of the subgradient of the geodesic distance with respect to the current metric. In our companion paper [4], we show how to compute such a subgradient in the same routine of the FMM algorithm and develop other applications of this method.

The continuous traffic congestion model is introduced in section 2. In section 3, traffic equilibria are defined and characterized by means of some convex optimization problem. A more tractable dual formulation is given in Section 4. In section 5, we consider a discretization of this dual problem, prove that this discretization Γ-converges to the continuous problem and give a consistent numerical scheme to solve the discrete problem. Numerical results are presented in section 6.

2 The model

2.1 Notations and definitions

Given a separable and complete metric space X, $\mathcal{M}_+(X)$ and $\mathcal{M}_+^1(X)$ denote respectively the set of positive and finite Borel measures on X and the set of Borel probability measures on X. If X and Y are separable metric spaces, $\mu \in \mathcal{M}_+^1(X)$ and $f : X \to Y$ is a Borel map, $f\sharp\mu$ denotes the push forward of μ through f i.e. the element of $\mathcal{M}_+^1(Y)$ defined by $f\sharp\mu(B) = \mu(f^{-1}(B))$ for every Borel subset B of Y.

In the sequel, $\mathcal{L}^d$ denotes the d-dimensional Lebesgue measure. If μ and ν are in $\mathcal{M}_+(\mathbb{R}^d)$ then $\frac{d\mu}{d\nu}$ denotes the Radon-Nikodym derivative of μ with respect to ν. We write $\mu \ll \nu$ to express that μ is absolutely continuous with respect to ν, in which case, slightly abusing notations, μ is identified with the Radon-Nikodym derivative $\frac{d\mu}{d\nu}$.

The data of our problem are Ω (modelling a city, say), an open bounded convex subset of $\mathbb{R}^2$, two probability measures, μ_0 and μ_1 in $\mathcal{M}_+^1(\overline{\Omega})$, giving respectively the distribution of residents and services in the city $\overline{\Omega}$ and a *transport plan* γ i.e. a probability measure on $\overline{\Omega} \times \overline{\Omega}$ having μ_0 and μ_1 as marginals. The transport plan γ models the travelers'

everyday movements, more precisely for A and B Borel subsets of $\overline{\Omega}$, $\gamma(A \times B)$ represents the fraction of the travelers' population that commutes from a location in A to a location in B.

Introducing congestion naturally leads to consider spaces of paths, lengths of such paths and sets of probability measures on sets of paths. From now on, we shall denote:

- $C := W^{1,\infty}([0,1], \overline{\Omega})$, (endowed with the usual topology of $C^0([0,1], \mathbb{R}^2)$),
- $C^{x,y} := \{\sigma \in C : \ \sigma(0) = x, \ \sigma(1) = y\}$ (x, y in $\overline{\Omega}$),
- $l(\sigma) := \int_0^1 |\dot{\sigma}(t)| \, dt$, the length of $\sigma \in C$,
- for $\varphi \in C^0(\overline{\Omega}, \mathbb{R})$ and $\sigma \in C$, we define

$$L_\varphi(\sigma) := \int_0^1 \varphi(\sigma(t))|\dot{\sigma}(t)|dt,$$

- $e_0(\sigma) := \sigma(0)$, $e_1(\sigma) := \sigma(1)$, for all $\sigma \in C^0([0,1], \mathbb{R}^2)$.

Roughly speaking, a transportation strategy is a probability Q over paths ($Q(\Sigma)$ represents the number of travellers that use a path in Σ) which is compatible with the transport plan γ:

Definition 2.1. *A transportation strategy is a probability $Q \in \mathcal{M}_+^1(C)$ such that $(e_0, e_1)\sharp Q = \gamma$.*

The set of transportation strategies is denoted by

$$\mathcal{Q}(\gamma) := \{Q \in \mathcal{M}_+^1(C) \ : \ (e_0, e_1)\sharp Q = \gamma\}.$$

If $\xi \in C^0(\overline{\Omega}, \mathbb{R}_+)$, it is proven in Lemma 2.6 in [8] that the map $\sigma \mapsto L_\xi(\sigma)$ is l.s.c. on C hence Borel for the uniform topology. Since $0 \leq L_\xi(\sigma) \leq \|\xi\|_\infty l(\sigma)$, $L_\xi \in L^1(C, Q)$ as soon as:

$$\int_C l(\sigma) dQ(\sigma) < +\infty \tag{2.1}$$

which means that the average (with respect to Q) length is finite. If (2.1) holds then $L_\xi \in L^1(C, Q)$ for every $\xi \in C^0(\overline{\Omega}, \mathbb{R}_+)$. If $\xi \in C^0(\overline{\Omega}, \mathbb{R})$, let us write $\xi = \xi_+ - \xi_-$ with ξ_+ and ξ_- the positive and negative part of ξ. Since $L_\xi = L_{\xi_+} - L_{\xi_-}$ then L_ξ is Borel and in $L^1(C, Q)$ provided (2.1) holds.

Any $Q \in \mathcal{M}_+^1(C)$ induces a *traffic intensity* which is a nonnegative measure on Ω, intuitively the traffic intensity of a subregion of Ω is the cumulated traffic of this subregion (or the total amount of mass travelling through the subregion). For simplicity, in the next definition, to avoid the case of measures taking the value $+\infty$, we restrict to the case where Q satisfies (2.1).

Definition 2.2. *Let $Q \in \mathcal{M}_+^1(C)$ satisfy (2.1), the traffic intensity of Q is the measure $i_Q \in \mathcal{M}_+(\overline{\Omega})$ defined by:*

$$\int_{\overline{\Omega}} \varphi(x) di_Q(x) := \int_C L_\varphi(\sigma) dQ(\sigma) = \int_C \left(\int_0^1 \varphi(\sigma(t))|\dot{\sigma}(t)|dt \right) dQ(\sigma), \tag{2.2}$$

for all $\varphi \in C^0(\overline{\Omega}, \mathbb{R})$.

Let us remark that the total mass of i_Q is the average length with respect to Q.

2.2 On geodesic distances

For a continuous and nonnegative metric $\xi \in C^0(\overline{\Omega}, \mathbb{R}_+)$, the corresponding geodesic distance is denoted c_ξ

$$\begin{aligned} c_\xi(x,y) &:= \inf \{L_\xi(\sigma),\ \sigma \in C^{x,y}\} \\ &= \inf \left\{ \int_0^1 \xi(\sigma(t))|\dot{\sigma}(t)|dt,\ \sigma \in C^{x,y} \right\}, \forall (x,y) \in \overline{\Omega} \times \overline{\Omega}. \end{aligned}$$

A path σ such that $L_\xi(\sigma) = c_\xi(\sigma(0), \sigma(1))$ is called a geodesic for the metric ξ between $x = \sigma(0)$ and $y = \sigma(1)$.

In the sequel, we shall need to extend the definitions of c_ξ and L_ξ to the case where ξ only belongs to some L^p space. This is possible when $p > 2$ (we refer to [8] for details), by proceeding as follows. Let us assume that $p > 2$ and $\xi \geq 0$, $\xi \in L^p(\Omega)$, then the following is proved in [8].

Proposition 2.3. *Let us assume that $p > 2$ and define $\beta := 1 - 2/p$, then there exists a non-negative constant C such that for every $\xi \in C^0(\overline{\Omega}, \mathbb{R}^+)$ and every $(x_1, y_1, x_2, y_2) \in \Omega^4$, one has*

$$|c_\xi(x_1, y_1) - c_\xi(x_2, y_2)| \leq C\|\xi\|_{L^p(\Omega)} \left(|x_1 - x_2|^\beta + |y_1 - y_2|^\beta \right). \tag{2.3}$$

Consequently, if $(\xi_n)_n \in C^0(\overline{\Omega}, \mathbb{R}^+)^{\mathbb{N}}$ is bounded in L^p, then $(c_{\xi_n})_n$ admits a subsequence that converges in $C^0(\overline{\Omega} \times \overline{\Omega}, \mathbb{R}_+)$.

This allows to extend the definition of the geodesic distance c_ξ to the case $\xi \geq 0$ and $\xi \in L^p$ (with $p > 2$) as follows:

$$\overline{c}_\xi = \sup \left\{ c = \lim_n c_{\xi_n} \text{ in } C^0(\overline{\Omega} \times \overline{\Omega}) : (\xi_n)_n \in C^0(\overline{\Omega}),\ \xi_n \geq 0,\ \xi_n \to \xi \text{ in } L^p \right\}.$$

This definition really extends that of geodesic distance since when ξ is continuous and non-negative, then $\overline{c}_\xi = c_\xi$ (see [8] for details). Section 5.3 shows that this definition coincides with a different one, given in terms of pointwise a.e. subsolutions of an Eikonal Equation.

It remains to extend the definition of L_ξ. To do that, let us define for every $q \in [1, +\infty]$, the set:

$$\mathcal{Q}^q(\gamma) := \{Q \in \mathcal{Q}(\gamma) : i_Q \in L^q(\Omega)\} \tag{2.4}$$

where in the previous definition we have slightly abused notations to intend that $i_Q \ll \mathcal{L}^2$ and that the density of i_Q with respect to $\mathcal{L}^2$ is in L^q. If $p > 2$, $\xi \in L^p$, $\xi \geq 0$, $(\xi_n)_n$ is a sequence of non-negative continuous functions that converges to ξ in L^p and $Q \in \mathcal{Q}^{p^*}(\gamma)$ (p^* being the conjugate exponent of p), then the following holds (see [8]):

- $(L_{\xi_n})_n$ converges strongly in $L^1(C, Q)$ to some limit which is independent of the approximating sequence $(\xi_n)_n$ and which is, again, denoted as L_ξ.
- The following equality holds:

$$\int_\Omega \xi(x) i_Q(x)\ dx = \int_C L_\xi(\sigma)\ dQ(\sigma). \tag{2.5}$$

- The following inequality holds for Q-a.e. $\sigma \in C$:

$$L_\xi(\sigma) \geq \overline{c}_\xi(\sigma(0), \sigma(1)). \tag{2.6}$$

Having extended the definition of L_ξ and $\overline{c}_\xi$, we naturally call a geodesic any $\sigma \in C$ such that $L_\xi(\sigma) = \overline{c}_\xi(\sigma(0), \sigma(1))$.

2.3 Traffic congestion

Traffic congestion modelling has to capture the fact that travellers spend more time (or more effort) in regions where the traffic is dense i.e. where the traffic intensity is high. A natural way to do so (assuming for a moment that $i_Q \ll \mathcal{L}^2$ with a density with respect to $\mathcal{L}^2$ still denoted i_Q), is to associate to i_Q a metric of the form $g(.,i_Q(.))$ where g is a given congestion function. We shall always assume the following on the congestion function g:

- $g \in C^0(\Omega \times \mathbb{R}_+; \mathbb{R}_+)$,
- $g(x,.)$ is strictly increasing on $\mathbb{R}_+$ for every $x \in \Omega$,
- there exists strictly positive constants a and b and a constant $\alpha \in (0,1)$ such that

$$ai^\alpha \leq g(x,i) \leq b(i^\alpha + 1),\ \forall i \in \mathbb{R}_+\ \forall x \in \Omega. \tag{2.7}$$

Let us set $q := 1+\alpha$ and $q^* = 1+1/\alpha$ its conjugate exponent (note that $q^* > 2$ so that one can define $\overline{c}_\xi$ as in the previous paragraph whenever $\xi \in L^{q^*}$ and $\xi \geq 0$) and define $\mathcal{Q}^q(\gamma)$ by (2.4). Let us also remark that every $Q \in \mathcal{Q}^q(\gamma)$ satisfies (2.1). For $Q \in \mathcal{Q}^q(\gamma)$, the congested metric associated to Q is well-defined and given by:

Definition 2.4. *For $Q \in \mathcal{Q}^q(\gamma)$, the metric associated to Q, denoted ξ_Q is defined by*

$$\xi_Q(x) := g(x, i_Q(x)),\ x \in \Omega. \tag{2.8}$$

Note that our assumptions on g imply that $\xi_Q \in L^{q^*}$ whenever $Q \in \mathcal{Q}^q(\gamma)$. Since $q^* > 2$, by the results recalled in paragraph 2.2, $\overline{c}_{\xi_Q}$ is a well-defined continuous function, $L_{\xi_Q} \in L^1(C,Q)$ and (2.5) and (2.6) hold for $\xi = \xi_Q$.

3 Wardrop Equilibria as solutions of a convex minimization problem

3.1 Equilibria

An equilibrium is a transportation strategy Q such that the congested metric ξ_Q is well defined (i.e. $Q \in \mathcal{Q}^q(\gamma)$ with $q = \alpha+1 \in (1,2)$) and such that, given the congestion resulting from ξ_Q, travellers only use shortest paths i.e. geodesics. This stability requirement on the transportation strategy (all used routes connecting x to y have the same commuting time and a shorter commuting time than unused routes) was first introduced by Wardrop [15] in the case of a finite number of roads. The generalization to our continuous setting then reads as:

Definition 3.1. *A Wardrop equilibrium is a probability $Q \in \mathcal{Q}^q(\gamma)$ such that*

$$Q(\{\sigma \in C\ :\ L_{\xi_Q}(\sigma) = \overline{c}_{\xi_Q}(\sigma(0),\sigma(1))\}) = 1.$$

In other words, $Q \in \mathcal{Q}^q(\gamma)$ is a Wardrop equilibrium if and only if Q gives full mass to the set of geodesics for the metric ξ_Q defined by (2.8).

Of course, a necessary condition for the existence of a Wardrop equilibrium, in our model where congestion effects are very strong, is that $\mathcal{Q}^q(\gamma)$ is nonempty. As shown in section 3.2, this condition is also sufficient for existence of equilibria. At first glance however, the condition $\mathcal{Q}^q(\gamma) \neq \emptyset$ may seem difficult to check a priori if we only know γ. Yet, in the case when there is a finite number of sources and destinations (which is basically the situation considered in our numerical simulations), this condition is automatically fulfilled.

Proposition 3.2. *If γ is a discrete probability measure on $\Omega\times\Omega$ then, for every $p\in[1,2)$, $\mathcal{Q}^p(\gamma)\neq\emptyset$.*

Proof. It is of course enough to prove the claim when γ is a single Dirac mass $\gamma=\delta_{(x_0,y_0)}$ and up to a change of coordinates we may assume $x_0=(0,0)$ and $y_0=(1,0)$. Now let $\beta_0>0$ be such that for all $\beta\in[0,\beta_0]$, the curve σ_β defined below lies in Ω:

$$\sigma_\beta(t):=\begin{cases}(t,\beta t) & \text{if } t\in[0,1/2]\\ (t,\beta(1-t)) & \text{if } t\in[1/2,1].\end{cases}$$

We then define $dQ:=\delta_{\sigma_\beta}\otimes\mathbf{1}_{[0,\beta_0]}\frac{d\beta}{\beta_0}$, of course $Q\in\mathcal{Q}(\gamma)$ and Q satisfies (2.1). For $\varphi\in C^0(\overline{\Omega},\mathbb{R})$ we then have:

$$\begin{aligned}&\int_{\overline{\Omega}}\varphi(x)di_Q(x)=\frac{1}{\beta_0}\int_0^{\beta_0}\int_0^1\varphi(\sigma_\beta(t))\sqrt{1+\beta^2}\,dtd\beta\\&=\frac{1}{\beta_0}\int_0^{\beta_0}\int_0^{1/2}\varphi(t,\beta t)\sqrt{1+\beta^2}dtd\beta+\frac{1}{\beta_0}\int_0^{\beta_0}\int_{1/2}^1\varphi(t,\beta(1-t))\sqrt{1+\beta^2}dtd\beta\end{aligned}$$

Denoting by T_0 the triangle with vertices $(0,0)$, $(1/2,0)$, $(1/2,\beta_0/2)$ and performing the change of variables $(\beta,t)\to\sigma_\beta(t)$, the first integral in the expression above equals

$$\frac{1}{\beta_0}\int_{T_0}\varphi(x,y)\frac{1}{x}\sqrt{1+\frac{y^2}{x^2}}dxdy$$

and a similar expression gives the value of the second integral. Now remarking that $\sqrt{1+\frac{y^2}{x^2}}\leq\sqrt{1+\beta_0^2}$ on T_0 and that $1/x\in L^p(T_0,\mathcal{L}^2)$ for all $p\in[1,2)$ we get the desired result. □

Remark 3.3. Some techniques from incompressible fluid mechanics, as Yann Brenier pointed out to us, may be used to prove the existence of finite-energy Q's for more general fixed transport plan γ's. In fact, any incompressible measure Q realizing at any time a density μ and concentrated on $L-$Lipschitz curves gives rise to a traffic intensity i_Q such that $i_Q\leq L\mu$. This idea could be used to handle the case of generic transport plans γ from μ to μ or, composing with time-dependent diffeomorphism, even from a measure to another. It requires anyway L^p assumptions on the data to get L^p estimates on i_Q.

3.2 Existence and characterization of equilibria

Our study of equilibria relies on the following convex optimization problem:

$$(\mathcal{P})\ \inf\left\{\int_\Omega H(x,i_Q(x))dx\,:\,Q\in\mathcal{Q}^q(\gamma)\right\}\tag{3.1}$$

where the function H is defined by

$$H(x,i):=\int_0^i g(x,s)ds,\ \forall x\in\Omega,\ \forall i\in\mathbb{R}_+.\tag{3.2}$$

By the same arguments as in [8], one can prove the following.

Theorem 3.4. *Assume that $\mathcal{Q}^q(\gamma)\neq\emptyset$ ($q=1+\alpha\in(1,2)$) then $(\mathcal{P})$ admits at least one solution.*

Theorem 3.5. *Assume that $\mathcal{Q}^q(\gamma) \neq \emptyset$ ($q = 1 + \alpha \in (1,2)$) and let $Q \in \mathcal{Q}^q(\gamma)$, then Q is an equilibrium if and only if Q solves $(\mathcal{P})$.*

Actually, the problem addressed in [8] is slightly different from the one considered here, but all the proofs can be straightforwardly adapted. Indeed, in [8], the transport plan γ is not given a priori, only its marginals are, a situation slightly more general than the present one. Also, in [8], the congestion function g only depends on i but it is easy to check that the results of Theorems 3.4 and 3.5 extend to the case where g also depends on x under the assumptions of the present paper.

From now on, we assume that $\mathcal{Q}^q(\gamma) \neq \emptyset$ ($q = 1 + \alpha \in (1,2)$). In this case, equilibria exist and finding them amounts to solve $(\mathcal{P})$. Since $H(x,.)$ is strictly convex, if Q_1 and Q_2 are equilibria, then necessarily $i_{Q_1} = i_{Q_2}$. In other words, equilibria are not necessarily unique but they all induce the same intensity or equivalently the same metric ξ. The next section shows that finding the equilibrium metric amounts to solve another optimization problem which is dual to $(\mathcal{P})$. This dual problem turns out to be easier than $(\mathcal{P})$ to be numerically handled.

4 Dual formulation

For every $x \in \Omega$ and $\xi \geq 0$, let us define

$$H^*(x,\xi) := \sup\{\xi i - H(x,i),\ i \geq 0\},\ \xi_0(x) := g(x,0).$$

By our assumptions on g, one has $H^*(x,\xi) = 0$ for every $x \in \Omega$ and $\xi \leq \xi_0(x)$, and

$$\frac{1}{b^{1/\alpha}} \frac{(\xi - b)_+^{q^*}}{q^*} \leq H^*(x,\xi) \leq \frac{1}{a^{1/\alpha}} \frac{\xi^{q^*}}{q^*},\ \forall x \in \Omega,\ \forall \xi \geq \xi_0(x).$$

Let us recall Young's inequality:

$$H(x,i) + H^*(x,\xi) \geq \xi i,\ \forall i \geq 0, \forall \xi \geq \xi_0(x) \tag{4.1}$$

and that inequality (4.1) is strict unless $\xi = g(x,i) \geq \xi_0(x)$. In particular, for $Q \in \mathcal{Q}^q(\gamma)$, we have the identity

$$H(x, i_Q(x)) + H^*(x, \xi_Q(x)) = \xi_Q(x) i_Q(x) \tag{4.2}$$

and

$$H(x, i_Q(x)) + H^*(x,\xi) > \xi i_Q(x),\ \forall \xi \geq \xi_0(x),\ \xi \neq \xi_Q(x). \tag{4.3}$$

Let us now define the functional

$$J(\xi) = \int_\Omega H^*(x, \xi(x)) dx - \int_{\overline{\Omega} \times \overline{\Omega}} \overline{c}_\xi(x,y) d\gamma(x,y) \tag{4.4}$$

and consider:

$$(\mathcal{P}^*) \sup\left\{-J(\xi) :\ \xi \in L^{q^*}, \xi \geq \xi_0\right\} \tag{4.5}$$

Theorem 4.1. *Let us assume that $\mathcal{Q}^q(\gamma) \neq \emptyset$ ($q = 1 + \alpha \in (1,2)$), then*

$$\min(\mathcal{P}) = \max(\mathcal{P}^*) \tag{4.6}$$

and $\xi \in L^{q^*}$ solves $(\mathcal{P}^*)$ if and only if $\xi = \xi_Q$ for some $Q \in \mathcal{Q}^q(\gamma)$ solving $(\mathcal{P})$.

Proof. Let $Q \in \mathcal{Q}^q(\gamma)$ (so that $\xi_Q \geq \xi_0$) and $\xi \in L^{q^*}$ with $\xi \geq \xi_0$, from (4.1) and (2.5) we first get:

$$\begin{aligned}\int_\Omega H(x, i_Q(x))dx &\geq \int_\Omega \xi i_Q - \int_\Omega H^*(x, \xi(x))dx \\ &= \int_C L_\xi(\sigma)dQ(\sigma) - \int_\Omega H^*(x, \xi(x))dx.\end{aligned}$$

Using (2.6) and $Q \in \mathcal{Q}^q(\gamma)$ we then have

$$\int_C L_\xi(\sigma)dQ(\sigma) \geq \int_C \overline{c}(\sigma(0), \sigma(1))dQ(\sigma) = \int_{\overline{\Omega}\times\overline{\Omega}} \overline{c}_\xi(x, y)d\gamma(x, y).$$

Since $Q \in \mathcal{Q}^q(\gamma)$ and $\xi \in L^{q^*}$, $\xi \geq \xi_0$ are arbitrary and since we already know that the infimum of $(\mathcal{P})$ is attained we thus deduce

$$\min(\mathcal{P}) \geq \sup(\mathcal{P}^*). \tag{4.7}$$

Now let $Q \in \mathcal{Q}^q(\gamma)$ solve $(\mathcal{P})$ and let $\xi := \xi_Q$ (recall that ξ_Q does not depend on the choice of the minimizer Q), from Theorem 3.5, we know that

$$L_\xi(\sigma) = \overline{c}_\xi(\sigma(0), \sigma(1)) \text{ for } Q\text{-a.e. } \sigma \in C$$

with (2.5), integrating the previous identity and using $Q \in \mathcal{Q}(\gamma)$ we then get:

$$\int_\Omega \xi i_Q = \int_C L_\xi(\sigma)dQ(\sigma) = \int_{\overline{\Omega}\times\overline{\Omega}} \overline{c}_\xi(x, y)d\gamma(x, y).$$

Using (4.2), (4.7) and the fact that $Q \in \mathcal{Q}^q(\gamma)$ solves $(\mathcal{P})$ yields:

$$\begin{aligned}\sup(\mathcal{P}^*) \leq \min(\mathcal{P}) = \int_\Omega H(x, i_Q(x))dx &= \int_\Omega \xi i_Q - \int_\Omega H^*(x, \xi(x))dx \\ &= \int_{\overline{\Omega}\times\overline{\Omega}} \overline{c}_\xi(x, y)d\gamma(x, y) - \int_\Omega H^*(x, \xi(x))dx\end{aligned}$$

so that ξ solves $(\mathcal{P}^*)$ and (4.6) is satisfied. Finally if ξ solves $(\mathcal{P}^*)$ and $Q \in \mathcal{Q}^q(\gamma)$ solves $(\mathcal{P})$, then with (2.5) and (2.6), one has

$$\begin{aligned}\int_\Omega \xi i_Q - \int_\Omega H^*(x, \xi(x))dx &\geq \int_{\overline{\Omega}\times\overline{\Omega}} \overline{c}_\xi(x, y)d\gamma(x, y) - \int_\Omega H^*(x, \xi(x))dx \\ &= \max(\mathcal{P}^*) = \min(\mathcal{P}) = \int_\Omega H(x, i_Q(x))dx\end{aligned}$$

and thus we deduce from (4.1) and (4.3) that $\xi = \xi_Q$. □

In the sequel, we numerically approximate the unique equilibrium metric ξ_Q by a descent method on $(\mathcal{P}^*)$. One can recover the corresponding equilibrium intensity i_Q by inverting the relation $\xi(x) = g(x, i_Q(x))$.

5 Discrete Algorithm

The traffic intensity that we are looking for, which is both the optimal one and the unique equilibrium, is determined by solving the dual problem in the variable $\xi = H'(i)$. This section describes and justifies a numerical approximation of the optimal ξ in the dual problem. This procedure (approximating the solution of the dual problem instead of the primal one) is classical as far as discrete Wardrop problems on networks are concerned (see [1] and the references therein). Obviously our numerical procedure passes itself through a discretization, but this has nothing to do with the network formulation of discrete Wardrop problem. The discretization of ξ is indeed done on a square lattice and we prove the convergence of our scheme to the solution of the continuous problem.

5.1 A discrete functional via Fast Marching

This section is concerned with the discrete formulation of the dual problem. After setting the notations in this discrete setting, the convexity of the discrete functional, with respect to the discrete metric, is proven. The convergence of this discretization to its continuous counterpart is proven in the next section.

The metric is discretized as a vector $\xi = (\xi_{i,j})_{i,j} \in \mathbb{R}^N$ where $\xi_{i,j}$ represents the value of the metric at a point $x_{i,j} = (ih, jh) \in \Omega$ of a square lattice. The number of grid points is $N = h^{-2}$. The first term of the energy functional J defined in equation (4.4) is easily discretized as $h^2 \sum_{i,j} H^*(x_{i,j}, \xi_{i,j})$.

The second term of the energy J requires the discretization of the geodesic length c_ξ. This can be performed by computing the solution of a discretized Eikonal equation. For a fixed source S_0, the geodesic distance to S_0 is denoted as $\mathcal{U}_\xi(x) = c_\xi(S_0, x)$. The function $\mathcal{U}_\xi$ is the viscosity solution of the Eikonal equation

$$\|\nabla \mathcal{U}_\xi\| = \xi \quad \text{with} \quad \mathcal{U}_\xi(S_0) = 0. \tag{5.1}$$

The vector $(\mathcal{U}_{i,j})_{i,j}$ is a discretization of the values $\mathcal{U}_\xi(x_{i,j})$ of the geodesic distance on the lattice vertices, where the dependence on ξ is dropped to ease notations. A discrete version of (5.1) is solved in order to compute $\mathcal{U} = (\mathcal{U}_{i,j})$. For the Eikonal equation, classic finite difference schemes tend to overshoot and are unstable. Rouy and Tourin [11] showed that a correct scheme for approximating the viscosity solution for of (5.1) is given by the following first order accurate scheme:

$$(D_x \mathcal{U})^2_{i,j} + (D_y \mathcal{U})^2_{i,j} = h^2 (\xi_{i,j})^2, \tag{5.2}$$

where we denote

$$\begin{aligned} (D_x \mathcal{U})_{i,j} &:= \max\{(\mathcal{U}_{i,j} - \mathcal{U}_{i-1,j}), (\mathcal{U}_{i,j} - \mathcal{U}_{i+1,j}), 0\}/h, \\ (D_y \mathcal{U})_{i,j} &:= \max\{(\mathcal{U}_{i,j} - \mathcal{U}_{i,j-1}), (\mathcal{U}_{i,j} - \mathcal{U}_{i,j+1}), 0\}/h, \\ \text{and} \quad (D\mathcal{U})_{i,j} &:= \sqrt{(D_x \mathcal{U})^2_{i,j} + (D_y \mathcal{U})^2_{i,j}}. \end{aligned}$$

The Fast Marching Method (FMM) is a numerical method introduced by Sethian in [12] for efficiently solving the isotropic Eikonal equation (5.2) on a cartesian grid. Values of $\mathcal{U}$ may be regarded as the arrival times of wavefronts propagating from the source point S_0 with velocity $1/\xi$. The central idea behind the FMM is to visit grid points in an order consistent with the way wavefronts propagate, i.e. with the Huygens principle. The numerical complexity of the FMM is $O(N \log(N))$ operations for a grid with N points.

Let us call $c^h_\xi(S_0, T) = \mathcal{U}(T)$ the value of this solution, computed with FMM, at the target point T when the vanishing boundary datum is fixed at S_0 and the metric is ξ. The dependence with respect to the discretization parameter h will be sometimes omitted in this section, since the lattice is fixed.

The functional to be minimized is the following

$$J^h(\xi) = h^2 \sum_{i,j} H^*(x_{i,j}, \xi_{i,j}) - \sum_{\alpha,\beta} c^h_\xi(S_\alpha, T_\beta)\gamma_{\alpha,\beta}, \tag{5.3}$$

where the weights $\gamma_{\alpha,\beta}$ represent the coupling on the set of pairs sources S_α - targets T_β and $\sum_{\alpha,\beta} \gamma_{\alpha,\beta} = 1$.

Lemma 5.1. *The functional J^h is convex.*

Proof. The only non trivial point is proving the concavity of the FMM term. It is sufficient to prove that for fixed S_0 and T the function $\xi \mapsto \mathcal{U}_\xi(T)$ is concave in ξ and, thanks to the homogeneity, it is sufficient to prove super-additivity. We want to prove the inequality

$$\mathcal{U}_{\xi_1+\xi_2} \geq \mathcal{U}_{\xi_1} + \mathcal{U}_{\xi_2}.$$

Thanks to the comparison principle of Lemma 5.2 below, it is sufficient to prove that $\xi_1 + \xi_2 \geq D(\mathcal{U}_{\xi_1} + \mathcal{U}_{\xi_2})$. This is easily done if we notice that the operator D is convex (as it is a composition of the function $(x, y) \mapsto \sqrt{x^2 + y^2}$, which is convex and increasing in both x and y, and the operator D_x and D_y, which are convex since they are produced as a maximum of linear operators) and $1-$homogeneous, and hence it is subadditive. □

Lemma 5.2. *If $\xi \leq \eta$ then $\mathcal{U}_\xi \leq \mathcal{U}_\eta$.*

Proof. Let us suppose at first a strict inequality $\xi < \eta$. Take a minimum point for $\mathcal{U}_\eta - \mathcal{U}_\xi$ and suppose it is not the fixed point S_0. Computing D and using the previously noticed sub-additivity, we have

$$\eta = D\mathcal{U}_\eta \leq D(\mathcal{U}_\eta - \mathcal{U}_\xi) + D\mathcal{U}_\xi = D(\mathcal{U}_\eta - \mathcal{U}_\xi) + \xi,$$

which gives $D(\mathcal{U}_\eta - \mathcal{U}_\xi) \geq \eta - \xi > 0$. Yet, at minimum points we should have $D(\mathcal{U}_\eta - \mathcal{U}_\xi) = 0$ and this proves that the minimum is realized at S_0, i.e. that $\mathcal{U}_\eta - \mathcal{U}_\xi \geq 0$.

To handle the case $\xi \leq \eta$ without a strict inequality, just replace η by $(1 + \varepsilon)\eta$ and notice that the application $\eta \mapsto \mathcal{U}_\eta$ is continuous. □

5.2 Description of the algorithm and convergence of its iterations

The discrete function J^h, defined in equation (5.3) is convex and can be minimized using standard methods from convex optimization. The first term of the functional J^h is differentiable, while the second may not be. Hence a subgradient descent algorithm can be used for the discrete optimization problem.

In the following, the dependence of H^* on the position is omitted to ease notations and we write $H^*(x, \xi) = f(\xi)$. The functional to optimize is written in the form

$$J^h(\xi) = \sum_{i,j} f(\xi_{i,j}) + K(\xi),$$

for a regular function $f : \mathbb{R}^+ \to \mathbb{R}^+$ and a convex, but in general not C^1, function K which is not separable w.r.t. to the variables $\xi_{i,j}$.

The subgradient method corresponds (taking for simplicity $\xi_0 = g(.,0) = 0$) to the following iterative scheme

$$\xi^{(1)} = 1; \quad \xi^{(k+1)} = \max\{0,\, \xi^{(k)} - \rho_k w^{(k)}\} \tag{5.4}$$
$$\text{where} \quad (w^{(k)})_{i,j} = f'(\xi^{(k)}_{i,j}) + (v^{(k)})_{i,j} \in \partial J(\xi^{(k)})$$

where $v^{(k)} \in \partial K(\xi^{(k)})$ is a vector in the subdifferential of K at the previous point $\xi^{(k)}$ and ρ_k is a suitable sequence of steps.

A well-known result on subgradient algorithms (see for instance [5]) states that the sequence $(\xi^{(k)})_k$ converges to the unique minimizer of J (uniqueness comes from strict convexity) provided the following two constraints are enforced

- The step sizes ρ_k satisfy
$$\sum_k \rho_k = +\infty \quad \text{and} \quad \sum_k \rho_k^2 < +\infty.$$
This condition is satisfied for a sequence of steps such as $\rho_k = 1/k$.

- The sequence $(w^{(k)})_k$ stays bounded, which can be enforced by slightly modifying the penalty function f. Since K is $1-$homogeneous, it is Lipschitz continuous and hence its subgradients are bounded. However, the original penalty f is not necessarily Lipschitz, but, one can consider instead the following Lipschitz function $\tilde{f}$ defined as
$$\tilde{f}(t) = \begin{cases} f(t) & \text{if } t \le t_0, \\ f(t_0) + (t - t_0) f'(t_0) & \text{if } t \ge t_0. \end{cases}$$
It happens that, if t_0 is sufficiently large, the minimizers $\overline{\xi}$ of the modified energy $\tilde{J}$ and of the original one are the same and satisfy $\overline{\xi}_{i,j} \le t_0$. This choice of t_0 may always be performed as far as f is superlinear at infinity.

A computation of an element $v^{(k)}$ of the subdifferential of K is required for the gradient descent. Indeed, in [4] a recursive method that computes such a vector in parallel to the iteration of the FMM is detailed. This methods produces a vector which is the gradient of $\mathcal{U}$ w.r.t. ξ at differentiability points and a vector in the subdifferential at all other points. The complexity of this algorithm depends on the complexity of the original FMM, as it visits the points in the same order. Yet, since it must compute a $N-$dimensional gradient at each iteration, the result has complexity $O(N^2 \log(N))$ for a grid of N points.

Notice that, in the continuous framework, the gradient of $\mathcal{U}_\xi(T)$ with respect to ξ is concentrated along geodesics joining T to S_0. The discrete gradient which is computed with the method of [4] is supported on the grid points explored by the front propagation of the FMM in order to reach the point T (see figure 1, which gives sort of fattened geodesics).

The method of [4] allows us to compute with a fast algorithm subgradients of a single term $\xi \mapsto \mathcal{U}_\xi(T)$ computed by FFM. Since the functional K is a conic combination of terms of this kind, it requires to use this algorithm several times and combines the resulting subgradient in a subgradient of K (we recall that the sum of subdifferentials is always included in the subdifferential of the sum and, here, since every term in K is continuous, we actually have the converse inclusion).

The convergence of the subgradient descent (5.4) ensures that one can find with a fast iterative algorithm the minimizer of the discrete functional J^h. Section 5.3 relates this discrete solution to the original continuous solution and shows that they are close in a suitable sense. Section 6 shows numerical simulations that use the descent algorithm 5.4 for various examples of penalty functions $H^*(x, \xi)$.

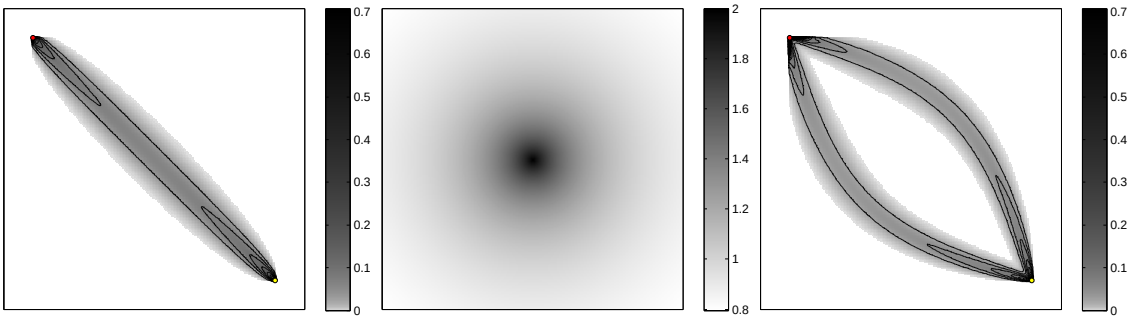


Figure 1: Examples of the subgradient computation using our algorithm presented in [4]. On the left, an element of the subdifferential of the geodesic with respect to the metric. The metric is constant, and the geodesic is a straight line. In the middle, we chose a non constant metric $\xi(x) = 1/(1.5 - \exp(-d(\mathcal{O}, x)))$, where $\mathcal{O}$ is the center of the image. On the right, an element of the subdifferential of the geodesic with respect to the metric given on the second figure.

5.3 $\Gamma-$convergence of the discrete functionals to the continuous one

This section considers the problem of the convergence of minima and minimizers of the discretized problems (5.3) to the original minimal value and minimizer of the functional (4.4). This allows to say that, as far as both the discretization step of the lattice is small and the number of iterations of the subgradient descent algorithm is large, one gets a good approximation of the desired optimal ξ.

The result is presented through a $\Gamma-$convergence proof, which implies the convergences we want (see [10]). Both the discrete problems and the continuous one are embedded in the L^{q^*} space. To do this, a vector $\xi \in \mathbb{R}^N$, with $N = h^{-2}$, is identified to the function $\xi \in L^{q^*}(\Omega)$ that takes the constant value $\xi_{i,j}$ on the axis-aligned square of width h centered around $x_{i,j}$. This actually means that the corresponding function are defined as piecewise constant on the dual lattice of the discretization grid.

The discrete optimization problem we have on $\mathbb{R}^N$ corresponds to solving

$$\min J_N(\xi) = \int H^*(x, \xi)\, d\mathcal{L}^d - \int \tilde{c}_\xi^N(S, T)\, \gamma_N(dS, dT), \quad \xi \in L_N^{q^*}(\Omega),$$

where the space $L_N^{q^*}(\Omega)$ is the space of L^{q^*} functions which are constant on every square of the $N-$th lattice. The measure γ_N is a discretization of γ on points of the lattice, and satisfies $\gamma_N \rightharpoonup \gamma$. The function $\tilde{c}_\xi^N$ is the piecewise affine extension of c_ξ^h on a triangular grid refining the square lattice (each square being divided into two triangles). One can notice that in order to compute integration, any extension with the same values on the support of γ_N would have given the same result. The same energy J_N may be extended to the whole $L^{q^*}(\Omega)$ by $+\infty$ outside $L_N^{q^*}(\Omega)$.

Before presenting the $\Gamma-$convergence result, we need to add some notions and equivalences concerning the meaning of c_ξ in the continuous case, when $\xi \in L^{q^*}$.

Section 2.2 introduces the quantity $\overline{c}_\xi$ by approximation with continuous functions. We are here concerned with the functions $T \mapsto \overline{c}_\xi(S_0, T)$ that is denoted $\mathcal{U}_\xi$ for simplicity when the dependence on S_0 is not ambiguous. Let us consider also the function v_ξ given by

$$v_\xi = \sup\{v \,:\, v \in W^{1,q}(\Omega),\, v(S_0) = 0,\, |\nabla v| \leq \xi \text{ a.e. }\},$$

which is the maximal a.e. subsolution of $|\nabla v| = \xi$. What we are looking for is very much linked to viscosity solution of the Hamilton-Jacobi equation $|\nabla v| = \xi$, when ξ is only L^{q^*} (see [6] for a definition via $W^{1,q}$ test functions and local a.e. inequality and [7] for some other notions and equivalences in the case $\xi \in L^\infty$). Yet, we are here interested only in the following two notions: the one presented in [8] (and in Section 2.2) and the maximal a.e. one.

Let us prove the following:

Lemma 5.3. *We have* $\mathcal{U}_\xi := \overline{c}_\xi(S_0, .) = v_\xi = \lim_{\varepsilon\to 0} c_{\xi * \rho_\varepsilon}(S_0, \cdot)$.

Proof. From Lemma 3.5 in [8] we know that there exists a sequence of continuous functions $(\xi_n)_n$ converging to ξ in L^{q^*} such that $c_{\xi_n} \to \overline{c}_\xi$ (in C^0 or weakly in W^{1,q^*}). Since for continuous functions the equality between $\overline{c}_\xi(S_0, \cdot)$ (which equals $c_\xi(S_0, \cdot)$, thanks to Lemma 3.4 in [8]) and v_ξ is known, we may infer

$$\xi_n \to \xi \text{ in } L^{q^*}(\Omega); \quad c_{\xi_n} \to \overline{c}_\xi \text{ in } C^0(\Omega); \quad |\nabla \overline{c}_{\xi_n}| \le \xi_n,$$

which implies at the limit $|\nabla \overline{c}_\xi| \le \xi$. This means that $\mathcal{U}_\xi$ is an a.e. subsolution of $|\nabla v| = \xi$. Consequently, $\mathcal{U}_\xi \le v_\xi$.

Let us now take an a.e. subsolution v, i.e. a function $v \in W^{1,q^*}(\Omega)$ with $|\nabla v| \le \xi$. Take a convolution kernel ρ_ε and define $v_\varepsilon = v * \rho_\varepsilon - (v * \rho_\varepsilon)(S_0)$. We have $\nabla v_\varepsilon = (\nabla v) * \rho_\varepsilon$ and consequently (it depends on the convexity of the norm) $|\nabla v_\varepsilon| \le \xi * \rho_\varepsilon := \xi_\varepsilon$. Moreover $v_\varepsilon(S_0) = 0$. From these facts we get $v_\varepsilon \le c_{\xi_\varepsilon}(S_0, \cdot)$, as this comes from the equality which is known for continuous functions (and ξ_ε is continuous). Since $(v * \rho_\varepsilon)(S_0) \to 0$, passing to the limit as $\varepsilon \to 0$ we get

$$v \le \lim_{\varepsilon\to 0} c_{\xi * \rho_\varepsilon}(S_0, \cdot) + (v * \rho_\varepsilon)(S_0) = \lim_{\varepsilon\to 0} c_{\xi * \rho_\varepsilon}(S_0, \cdot) \le \overline{c}_\xi(S_0, \cdot),$$

the last inequality being justified by the definition of $\overline{c}_\xi$. This proves the thesis, v being arbitrary. □

We can now go through the essential part of the $\Gamma-$convergence proof. We denote by $\tilde{\mathcal{U}}^N_{\xi_N}$ the functions $T \mapsto \tilde{c}^N_{\xi_N}(S_0, T)$.

Lemma 5.4. *Suppose* $\xi_N \rightharpoonup \xi$ *in* L^{q^*}, *with* $\xi_N \in L^{q^*}_N(\Omega)$. *Then for each* S_0 *the sequence* $(\tilde{\mathcal{U}}^N_{\xi_N})_N$ *is bounded in* W^{1,q^*} *and any weak limit* u *satisifes* $u \le v_\xi$.

Proof. Let us prove boundedness. We consider at first derivatives with respect to the component x of the variable: since $\tilde{\mathcal{U}}^N_{\xi_N}$ is an affine extension, its $\partial/\partial x$ derivative on a triangular cell coincides with the same derivative on the horizontal side of the cell. The slope on the side is bounded by the value of ξ_N at one of the vertices. If $\mathcal{T}$ is a triangle of the partition (its area being A), P_1 and P_2 are the two vertices we consider and Q_1 and Q_2 the two squares where ξ_N is constant (with area $2A$ each), respectively centered around P_1 and P_2, we get

$$\begin{aligned}\int_{\mathcal{T}} \left|\frac{\partial \tilde{\mathcal{U}}^N_{\xi_N}}{\partial x}\right|^{q^*} d\mathcal{L}^2 &\le A \max\{\xi_N(P_1)^{q^*}, \xi_N(P_2)^{q^*}\} \\ &\le A(\xi_N(P_1)^{q^*} + \xi_N(P_2)^{q^*}) = \frac{1}{2}\int_{Q_1\cup Q_2} \xi_N^{q^*}\, d\mathcal{L}^2.\end{aligned}$$

Summing up over all triangles any square is counted at most four times, and in the end we get

$$\int_\Omega \left| \frac{\partial \tilde{\mathcal{U}}^N_{\xi_N}}{\partial x} \right|^{q^*} dx \leq 2 \int_\Omega \xi_N^{q^*} \, dx.$$

This proves boundedness (since $(\xi_N)_N$ is bounded in L^{q^*} and the same argument may be performed for $\partial/\partial y$) and the existence of weak limits u (which are uniform limits as well, and hence the condition $u(S_0) = 0$ is conserved).

Let us now define

$$\xi_{N,x} = D_x \tilde{\mathcal{U}}^N_{\xi_N}, \quad \xi_{N,y} = D_y \tilde{\mathcal{U}}^N_{\xi_N}, \quad \text{with the property } \xi_N = \sqrt{\xi^2_{N,x} + \xi^2_{N,y}}.$$

Since ξ_N is bounded in L^{q^*}, the same is true for $\xi_{N,x}$ and $\xi_{N,y}$ and we denote by ξ_x and ξ_y their weak limits (up to subsequences). Let us fix an arbitrary rectangle R whose vertices belong to the $N-$th lattice. Let us now estimate the positive part of $\partial \tilde{\mathcal{U}}^N_{\xi_N}/\partial x$ on R: in this case it is sufficient to estimate all ascending slopes. Take an horizontal line in R: all these slopes coincide with the slopes on the horizontal boundaries of the triangles and are hence bounded by the values of $\xi_{N,x}$ at the right extremal points of the segments bounding the triangles. Fixing the ascending direction allows us to avoid superpositions in the choice of the extremal point. The same can be done for descending slopes and for vertical slopes. In the case of ascending horizontal slopes we get

$$\int_R \left(\frac{\partial \tilde{\mathcal{U}}^N_{\xi_N}}{\partial x} \right)^+ dx \leq \int_{R'_N} \xi_{N,x} \, dx \leq \int_R \xi_{N,x} \, dx + ||\xi_N||_{L^{q^*}} |R'_N \setminus R|^{1/q},$$

where R'_N is a rectangle a little bit larger than R, since it includes all the squares in the dual lattice around the points of ∂R. Yet, $|R'_N \setminus R| \to 0$ as $N \to \infty$. When $N \to \infty$ we can pass to the limit and obtain

$$\int_R \left(\frac{\partial u}{\partial x} \right)^+ dx \leq \int_R \xi_x \, dx,$$

which, R being arbitrary, implies

$$\left(\frac{\partial u}{\partial x} \right)^+ \leq \xi_x \ a.e.$$

and, taking the maximum over the positive and the negative part,

$$\left| \frac{\partial u}{\partial x} \right| \leq \xi_x \ a.e.$$

This proves $|\nabla u| \leq \sqrt{\xi_x^2 + \xi_y^2}$.

It is sufficient to prove $\sqrt{\xi_x^2 + \xi_y^2} \leq \xi$ to obtain that u is an a.e. subsolution of the Hamilton-Jacobi equation, and hence $u \leq v_\xi$. The inequality we want is a consequence of standard properties of weak convergence: if a sequence of vector-valued maps z_N (in this case, the pairs $(\xi_{N,x}, \xi_{N,y})$) weakly converges to a map z (in this case $z = (\xi_x, \xi_y)$), then, for any convex function f (in this case $f(x,y) = \sqrt{x^2 + y^2}$), $f(z_N) \rightharpoonup h$ implies $f(z) \leq h$ (to prove it, it is sufficient to express f as a supremum of affine functions). The thesis in hence proven. □

Theorem 5.5. *The sequence of functionals J_N previously defined $\Gamma-$converges with respect to the weak L^{q^*} convergence to the limit functional J. Moreover, as the sequence $(J_N)_N$ is equi-coercive and every functional, J included, is strictly convex, convergence of the unique minimizers and of the values of the minima is guaranteed.*

Proof. To prove $\Gamma-$convergence it is sufficient to prove that $\xi_N \rightharpoonup \xi$ implies $\liminf_N J_N(\xi_N) \geq J(\xi)$ and that for every ξ there exists a sequence $\xi_N \rightharpoonup \xi$ such that $J_N(\xi_N) \to J(\xi)$.

For proving the first part ($\Gamma-$liminf inequality), it is sufficient to notice that the first addend in the functionals is lower semicontinuous w.r.t. weak convergence, and that by Lemma 5.4, the second behaves the same way (but we have to use the equality provided by Lemma 5.3).

For the second part ($\Gamma-$limsup inequality), we know by [10] that it is sufficient to prove it for ξ in a class which is sufficiently dense ("dense in energy", i.e. so that we can guarantee approximation with convergence of the limit energy J). The class of regular functions satisfies this density property. Indeed, by convolution we know that both terms of the functional J converge (for the first it is dominated convergence and for the second use Lemma 5.3 again). For regular functions it is a well known fact that we can choose ξ_N by taking the values of ξ at the points of the lattice and we have uniform convergence of the solutions computed by the Fast Marching Method to the true solution of the Eikonal equation (this fact is mainly proved in [11], with techniques coming from [2]; for other convergence results on monotone discrete schemes for Hamilton-Jacobi equations see also [9] and [13]). □

6 Numerical results

In this section some numerical results are shown. Algorithm described in section 5.2 is used. The equilibrium states are visualized using the metric ξ computed by this algorithm.

The first example consists in a source-target pair (S, T) and a homogeneous functional H, which does not depend on the spatial location. As previously seen, $\xi = H'(i)$, and H' is strictly increasing function, thus (at least in the case where H does not depend on x), ξ and i are visualized as very similar. In the first example, we took $H^*(x, \xi) = \frac{1}{3}\xi^3$, so that $H(x, i) = \frac{2}{3}i^{\frac{3}{2}}$. On Figure 2 the equilibrium metric on a "desert" square is shown. Note that the equilibrium metric is symmetric as the initial configuration of the source and the target.

Next, we consider the case where the functional H is spatially varying. We take $H(x, i) = \frac{2}{3}\xi_0(x)i^{\frac{3}{2}}(x)$, where ξ_0 is a given fixed congestion metric. Thus $H^*(x, \xi) = \frac{1}{3}\xi^3/\xi_0^2$. The chosen congestion metric is

$$\begin{aligned}\xi_0(x) = 1 \quad &+A\exp\left(\tfrac{-|x-x_0|^2}{\sigma_1^2}\right) + A\exp\left(\tfrac{-|x-x_1|^2}{\sigma_1^2}\right)\\ &+B\exp\left(\tfrac{-|x-x_3|^2}{\sigma_2^2}\right) + B\exp\left(\tfrac{-|x-x_4|^2}{\sigma_2^2}\right).\end{aligned}$$

Points x_i for $i \in \{1, 2, 3, 4\}$ are chosen in a symmetric manner (see figure 3 (a)). We took $A = 10$, $B = 4$, $\sigma_1 = 15$ and $\sigma_2 = 7$ on a square grid of size 100×100. For figures 2 and 3, we used a single source-target pair, thus the traffic weight is $\gamma = 1$.

Let us now introduce obstacles and multiple sources and targets. In a symmetric configuration of two sources S_1 and S_2, and two targets T_1 and T_2; we consider a river where there is no traffic and a bridge linking the two sides of the river (see figure 4 (a)). We chose the traffic weights such that $\gamma_{1,1} + \gamma_{1,2} = 2(\gamma_{2,1} + \gamma_{2,2})$ and $\frac{\gamma_{2,2}}{\gamma_{2,1}} = \frac{\gamma_{1,1}}{\gamma_{1,2}} = 2$. The

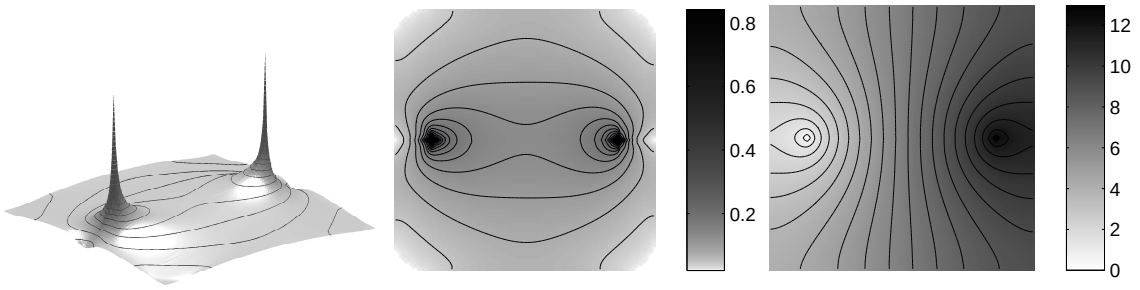
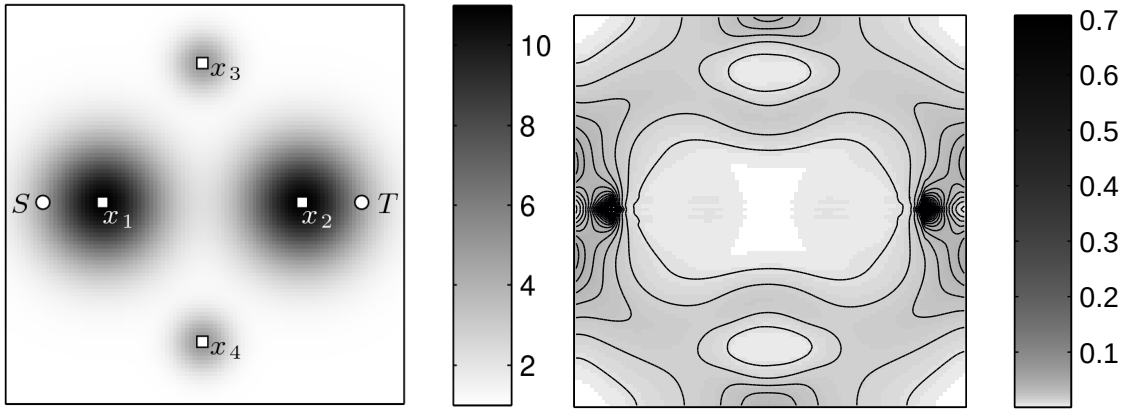


Figure 2: Traffic congestion equilibrium metric on a desert square between two points. On the left, the equilibrium metric ξ obtained via the subgradient descent is shown as a 3D surface. In the middle, a flat view of the same metric. On the right, the minimal action map $\mathcal{U}$ associated to ξ as well as its isolevels are shown. On the last image, one can see that the Wardrop conditions are satisfied.



(a) Congestion metric ξ_0

(b) Equilibrium traffic intensity $i_Q = \frac{\xi^2}{\xi_0^2}$

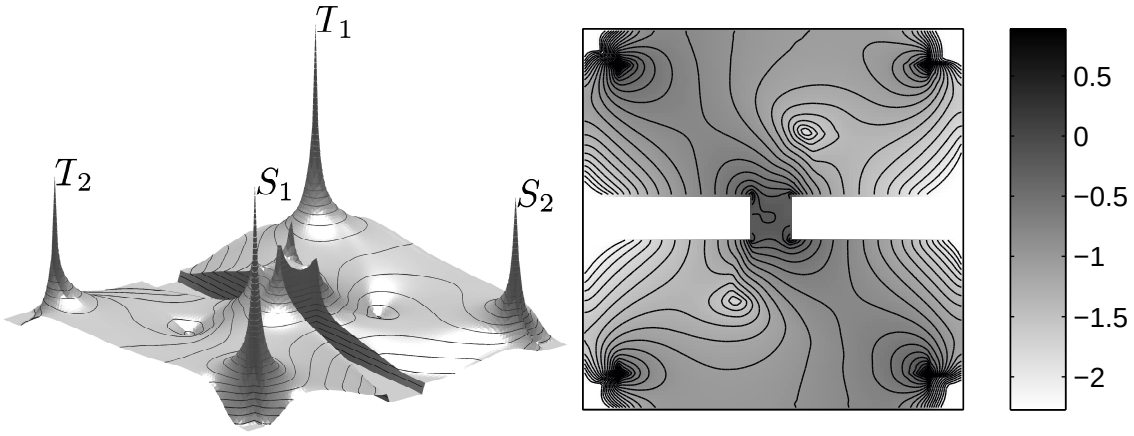
Figure 3: We chose parameters A, B, σ_1 and σ_2 such that the steady state traffic intensity is concentrated between and outside the bumps.

traffic intensity going out from S_1 is twice S_2's. One can note the two hollows on each side of the river appearing because of the inter-sides and intra-sides crossed traffics (see figure 4).

Finally, we solve with our algorithm an asymmetric and general problem with several sources and targets. In figure 5 (a), the chosen traffic weights before normalization are

$$(\gamma_{i,j}) = \begin{pmatrix} 0.5 & 0 & 0 & 0.5 \\ 0.3 & 0.3 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

Figure 5 (b), shows that the traffic concentrates on the shortest paths between the source and the target. For both last examples, the chosen function is homogenous and $H(\xi) = \frac{1}{3}\xi^3$.



(a) 3D view of ξ (b) Flat view of $\log(\xi)$

Figure 4: Two sources and two targets, with a river and a bridge on a symmetric configuration with asymmetric traffic weights.

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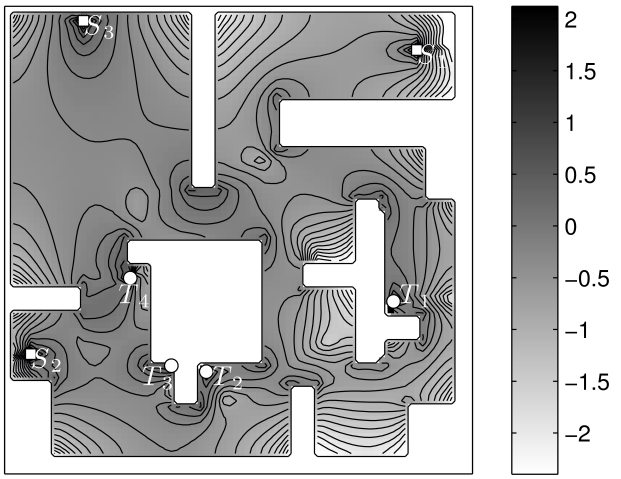
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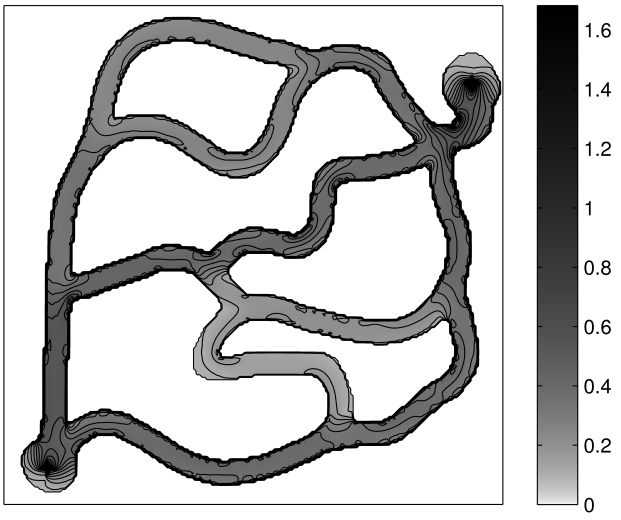
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(a) $\log(\xi)$, with obstacles and several sources and targets



(b) $\log(\xi)$ for a pair of source / target in a cavern

Figure 5: Traffic congestion equilibrium metric in the case of several sources and targets with obstacles. On figure (b), one can see that the traffic is higher on the shortest (in the Euclidian sense) pathways.

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