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# On the growth of nonuniform lattices in pinched negatively curved manifolds

Françoise Dal'bo <sup>1</sup>,  
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## 1. Introduction

We study the relation between the exponential growth rate of volume in a pinched negatively curved manifold and the critical exponent of its lattices. These objects have a long and interesting story and are closely related to the geometry and the dynamical properties of the geodesic flow of the manifold (see e.g. [4], [9],[20] and references therein).

Throughout this paper,  $X$  will denote a complete and simply connected Riemannian manifold of dimension  $N \geq 2$  and we will assume that  $X$  has *pinched negative curvature*, that is its sectional curvature  $K_X$  is bounded between two negative constants  $-b^2 \leq -a^2 < 0$ . A *Kleinian group* of  $X$  is a torsion free and discrete subgroup  $\Gamma$  of  $Is(X)$ ; then,  $\Gamma$  operates freely and properly discontinuously on  $X$  and the quotient manifold  $M := X/\Gamma$  has a fundamental group which can be identified with  $\Gamma$ . The group  $\Gamma$  is called a *lattice* when the volume of  $M$  is finite; the lattice is said to be *uniform* if  $M$  is compact.

Recall that the exponential growth rate of  $X$ , also known as the *volume entropy* of  $X$ , is defined as

$$\omega(X) = \limsup_{R \rightarrow +\infty} \frac{1}{R} \ln v_X(\mathbf{x}, R)$$

where  $v_X(\mathbf{x}, R)$  is the volume of the open ball  $B_X(\mathbf{x}, R)$  of  $X$ , centered at the point  $\mathbf{x}$  and with radius  $R$ . By the triangular inequality, this quantity does not depend on the base point  $\mathbf{x}$ ; furthermore, under our pinching assumption, Bishop-Gunther's comparison theorem (see [14]) implies

$$(1) \quad (N - 1)a \leq \omega(X) \leq (N - 1)b.$$

The invariant  $\omega(X)$  has been intensively studied when  $Is(X)$  admits a *uniform* lattice  $\Gamma$ . It turns out that, in this case,  $\omega(X)$  is a true limit and equals the topological entropy of the geodesic flow of the compact manifold  $M$

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(see [17]). Furthermore, with a suitable normalization on the volume of  $M$ , it is a complete invariant of locally symmetric metrics on  $M$  (see [4]).

The second object of our interest in this paper is the *Poincaré series*  $P_\Gamma(s, \mathbf{x})$  of a Kleinian group  $\Gamma$ , defined by

$$P_\Gamma(s, \mathbf{x}) = \sum_{\gamma \in \Gamma} e^{-sd(\mathbf{x}, \gamma \mathbf{x})},$$

for  $\mathbf{x} \in X$  and  $s \in \mathbb{R}$ . Its abscissa of convergence, called the *critical exponent* of  $\Gamma$ , is equal to

$$\delta(\Gamma) = \limsup_{R \rightarrow \infty} \frac{1}{R} \ln v_\Gamma(\mathbf{x}, R),$$

where  $v_\Gamma(\mathbf{x}, R)$  is the cardinality of the "ball"  $B_\Gamma(\mathbf{x}, R) := \{\gamma \in \Gamma / d(\mathbf{x}, \gamma \mathbf{x}) \leq R\}$ ; again, by the triangular inequality,  $\delta(\Gamma)$  does not depend on  $\mathbf{x}$ .

A way to understand the dynamic significance of the volume entropy  $\omega(X)$  and its relation with  $\delta(\Gamma)$  is to consider the Laplace transform of the  $\Gamma$ -invariant volume form  $dv_X$  on  $X$ , namely

$$I_X(s) = \int_0^{+\infty} e^{-sr} v_X(\mathbf{x}, r) dr.$$

The abscissa of convergence of  $I_X(s)$  coincides with  $\omega(X)$ .

By a Fubini type argument, we also have  $I_X(s) = \frac{1}{s} \int_X e^{-sd(\mathbf{x}, y)} dv_X(y)$ . If  $D$  is a Borel fundamental domain for the action of  $\Gamma$  on  $X$ , we get, by invariance of  $dv_X$  :

$$sI_X(s) = \sum_{\gamma \in \Gamma} \int_{\gamma D} e^{-sd(\mathbf{x}, y)} dv_X(y) = \sum_{\gamma \in \Gamma} \int_D e^{-sd(\gamma^{-1} \mathbf{x}, y)} dv_X(y)$$

which, in turns, yields :

$$(2) \quad P_\Gamma(s, \mathbf{x}) \int_D e^{-sd(\mathbf{x}, y)} dv_X(y) \leq sI_X(s) \leq P_\Gamma(s, \mathbf{x}) \int_D e^{sd(\mathbf{x}, y)} dv_X(y)$$

From the left-hand side of (2) it immediately follows that we always have

$$(3) \quad \delta(\Gamma) \leq \omega(X).$$

Moreover, from the right-hand side of (2), we have  $\delta(\Gamma) = \omega(X)$  when  $\Gamma$  is a uniform lattice.

In this paper we shall investigate the case where  $X$  admits a *non-uniform lattice*  $\Gamma$ . Let us emphasize that, under this assumption, if  $X$  also admits a uniform lattice  $\Gamma_0$  then  $X$  is a symmetric space of non compact type (and rank 1). Actually, as the curvature does not vanish, the manifold  $X$  is not a Riemannian product; then (by [11], Corollary 9.2.2), either  $X$  is symmetric or the isometry group of  $X$  is discrete. But, in this last case,  $\Gamma_0$  would have finite index in  $Is(X)$  (see [11] 1.9.34) and, if  $\varphi$  is a parabolic isometry of  $X$ , then  $\varphi^n$  would belong to  $\Gamma_0$  for some  $n \geq 1$ , which contradicts the fact that a uniform lattice contains only axial elements.

Somewhat surprisingly, the equality  $\delta(\Gamma) = \omega(X)$  may fail for a non uniform lattice  $\Gamma$ ; actually, in the last section of this paper, we shall prove

**THEOREM 1.1.** *There exists a complete and simply connected Riemannian surface  $X$  with pinched negative curvature which admits a non uniform lattice  $\Gamma$  such that*

$$\delta(\Gamma) < \omega(X).$$

Our construction extends to any dimension. To explain it, recall that to each cuspidal end of the quotient manifold  $X/\Gamma$  corresponds a maximal parabolic subgroup  $\mathcal{P} \subset \Gamma$ , which has a *lower critical exponent* :

$$\delta^-(\mathcal{P}) = \liminf_{R \rightarrow \infty} \frac{1}{R} \ln v_{\mathcal{P}}(\mathbf{x}, R).$$

In strictly negative curvature, this exponent is nonzero, despite the fact that  $\mathcal{P}$  is virtually nilpotent (see [6]). The key point is that, in the variable curvature setting,  $\delta^-(\mathcal{P})$  may be distinct from  $\delta(\mathcal{P})$ , as was suggested a long time ago to the second author by B. Bowditch; in contrast, it is well known that the critical exponent of any non elementary Kleinian group always is a true limit [19]. We shall show in Section 5 that the inequality  $\omega(X) > \delta(\Gamma)$  may appear as soon as  $\delta^-(\mathcal{P}) < \delta(\mathcal{P})/2$ .

On the other hand, our example induces us to introduce a notion of pinching for non uniform lattices which ensures that  $\omega(X) = \delta(\Gamma)$ . Namely, we say that  $\Gamma$  is *parabolically 1/2-pinched* if for any maximal parabolic subgroup  $\mathcal{P} \subset \Gamma$ , we have

$$(4) \quad \frac{\delta(\mathcal{P})}{\delta^-(\mathcal{P})} \leq 2$$

We will prove

**THEOREM 1.2.** *Let  $X$  be a complete, simply connected Riemannian manifold with pinched negative curvature. Then for any lattice  $\Gamma \subset Is(X)$  which is parabolically 1/2-pinched, we have  $\delta(\Gamma) = \omega(X)$ .*

Moreover, we notice that, under the assumptions of this theorem, the invariant  $\omega(X)$  is a true limit; this follows from Corollary 4.5, combined with the fact that  $\delta(\Gamma)$  is a limit.

We shall see that Theorem 1.2 covers the case of lattices in any 1/4-pinched negatively curved manifold (i.e.  $\frac{b^2}{a^2} \leq 4$ ). As far as we know, even in the classical case of Riemannian negatively curved symmetric spaces of rank one (which are 1/4-pinched, cp. [15]), there does not exist an elementary proof of this result. Nevertheless, for those spaces, the equality  $\omega(X) = \delta(\Gamma)$  can be easily deduced from a general and deep result of A. Eskin and C. McMullen in [13] on lattices of affine symmetric spaces, obtained by algebraic methods. In contrast, the context of variable negative curvature forces us to use only elementary geometric arguments.

The equality  $\omega(X) = \delta(\Gamma)$  actually holds under a milder geometric assumption than 1/4-pinched curvature. Namely, we will say that a manifold  $M = X/\Gamma$  has *asymptotically 1/4-pinched curvature* when, for any  $\epsilon > 0$ , there exists a compact set  $C_\epsilon \subset M$ , such that the metric is  $(\frac{1}{4+\epsilon})$ -pinched on  $M \setminus C_\epsilon$ . A direct consequence of Theorem 1.2 is

**COROLLARY 1.3.** *Let  $X$  be a complete, simply connected Riemannian manifold with pinched negative curvature and let  $\Gamma$  be a lattice of  $X$ . If  $M := X/\Gamma$  has asymptotically  $1/4$ -pinched curvature, then  $\delta(\Gamma) = \omega(X)$ .*

We remark that the pinching constant  $\frac{1}{4}$  is optimal because, for every  $\epsilon > 0$ , the example we construct in Theorem 1.1 can be chosen so that the curvature is  $\frac{1}{4+\epsilon}$ -pinched.

The paper is organized as follows. Section 2 deals with elementary geometrical estimates inside horoballs. In Section 3, we relate the volume growth of balls inside a horoball  $\mathcal{H}$  with the critical exponent of ample parabolic subgroups preserving  $\mathcal{H}$ . In section 4, we first give an elementary proof of the equality  $\omega(X) = \delta(\Gamma)$  for  $\frac{1}{4}$ -pinched manifolds; this is of interest since the main idea about the behavior of a ball intersecting a horoball appears clearly in the proof. The proofs of Theorem 1.2 and Corollary 1.3 will follow. Section 5 is devoted to the construction of the example of Theorem 1.1; this relies on pretty technical results about convex functions, postponed to the Appendix.

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We fix here once and for all some notation about asymptotic behavior of functions :

**Notations.** We shall write  $f \stackrel{c}{\preceq} g$  (or simply  $f \preceq g$ ) when  $f(R) \leq cg(R)$  for some constant  $c > 0$  and  $R$  large enough. The notation  $f \stackrel{c}{\asymp} g$  (or simply  $f \asymp g$ ) means  $f \stackrel{c}{\preceq} g \preceq f$ .

Analogously, we shall write  $f \stackrel{c}{\sim} g$  (or simply  $f \sim g$ ) when  $|f(R) - g(R)| \leq c$  for some constant  $c > 0$  and  $R$  large enough.

The upper and lower exponential growth rates of a function  $f$  are denoted by  $\omega^+(f)$  (or simpler  $\omega(f)$ ) and  $\omega^-(f)$  respectively; namely we have

$$\omega^-(f) := \liminf_{R \rightarrow +\infty} \frac{\ln f(R)}{R} \quad \text{and} \quad \omega^+(f) = \omega(f) := \limsup_{R \rightarrow +\infty} \frac{\ln f(R)}{R}.$$

Finally, if  $f$  and  $g$  are two real functions, we denote by  $f * g$  the discrete convolution of  $f$  with  $g$ , defined by  $f * g(R) = \sum_{n=0}^{[R]} f(n)g(R-n)$  for any  $R \geq 0$ .

## 2. Radial flow and geometry of horoballs

As the curvature is bounded from above by  $-a^2 < 0$ , we have the following classical inequality :

**LEMMA 2.1.** *Let  $T$  be a geodesic triangle with different vertices  $\mathbf{x}, \mathbf{y}, \mathbf{z} \in X$  and angle at  $\mathbf{y}$  greater than  $\alpha > 0$ . Then there is a constant  $D = D(\alpha, a)$  such that*

$$d(\mathbf{x}, \mathbf{z}) \geq d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{z}) - D.$$

**Proof.** See [8].

□

Let  $X(\infty)$  be the boundary at infinity of  $X$ . Fix a point  $\xi$  in  $X(\infty)$  and consider its associated *radial semi-flow*,  $(\psi_{\xi,t})_{t \geq 0}$  defined as follows : for any  $\mathbf{x} \in X$ , the point  $\psi_{\xi,t}(\mathbf{x})$  lies on the geodesic ray  $[\mathbf{x}, \xi)$  at distance  $t$  from  $\mathbf{x}$ . For any horosphere  $\partial\mathcal{H}$  centered at  $\xi$ , we set  $\partial\mathcal{H}(t) = \psi_{\xi,t}(\partial\mathcal{H})$ , and we let  $d_t$  be the distance induced by  $d$  on the horosphere  $\partial\mathcal{H}(t)$ . For any points  $x, y \in \partial\mathcal{H}(t)$ , we have (see [16])

$$(5) \quad \frac{2}{a} \sinh\left(\frac{a}{2}d(x, y)\right) \leq d_t(x, y) \leq \frac{2}{b} \sinh\left(\frac{b}{2}d(x, y)\right).$$

By [16], the differential of the map  $\psi_{\xi,t} : \partial\mathcal{H} \rightarrow \partial\mathcal{H}(t)$  satisfies, for any vector  $v \in T(\partial\mathcal{H})$  and any  $t \geq 0$

$$(6) \quad e^{-bt} \|v\| \leq \|d\psi_{\xi,t}(v)\| \leq e^{-at} \|v\|.$$

This readily implies the estimates

$$(7) \quad e^{-b(N-1)t} \leq |Jac(\psi_{\xi,t})| \leq e^{-a(N-1)t}.$$

In particular, if  $\mu_t$  is the Riemannian measure induced on  $\partial\mathcal{H}(t)$  by the metric on  $X$ , we have, for any Borel set  $A \subset \partial\mathcal{H}$

$$(8) \quad e^{-b(N-1)t} \mu_0(A) \leq \mu_t(\psi_{\xi,t}(A)) = \int_A |Jac(\psi_{\xi,t})|(x) d\mu_0(x) \leq e^{-a(N-1)t} \mu_0(A).$$

If the points  $\mathbf{x}, \mathbf{y}$  belong to the horosphere  $\partial\mathcal{H}$ , we set

$$t_{\mathbf{x},\mathbf{y}} = \inf\{t \geq 0 / d_t(\psi_{\xi,t}(\mathbf{x}), \psi_{\xi,t}(\mathbf{y})) \leq 1\}.$$

The next lemma, which precises Lemma 4 in [9], will be of major importance in the following.

**LEMMA 2.2.** *There exists a constant  $c = c(a, b) > 0$ , only depending on the bounds on the curvature, such that, for any horosphere  $\partial\mathcal{H}$  and any  $\mathbf{x}, \mathbf{y} \in \partial\mathcal{H}$ , the arc  $\gamma_{\mathbf{x},\mathbf{y}}$  which is the ordered union of the three geodesic segments  $[\mathbf{x}, \psi_{\xi,t_{\mathbf{x},\mathbf{y}}}(\mathbf{x})]$ ,  $[\psi_{\xi,t_{\mathbf{x},\mathbf{y}}}(\mathbf{x}), \psi_{\xi,t_{\mathbf{x},\mathbf{y}}}(\mathbf{y})]$  and  $[\psi_{\xi,t_{\mathbf{x},\mathbf{y}}}(\mathbf{y}), \mathbf{y}]$  is a  $(1, c)$ -quasigeodesic. Furthermore, for any  $s, t \geq 0$ , we have*

$$d(\psi_{\xi,s}(\mathbf{x}), \psi_{\xi,t}(\mathbf{x})) \stackrel{c}{\sim} \varphi(s, t)$$

where  $\varphi$  is the function defined on  $\mathbb{R}_+ \times \mathbb{R}_+$  by

$$\varphi(s, t) = \begin{cases} 2t_{\mathbf{x},\mathbf{y}} - s - t & \text{when } s, t \leq t_{\mathbf{x},\mathbf{y}} \\ |s - t| & \text{otherwise.} \end{cases}$$

In particular, we have  $d(\mathbf{x}, \mathbf{y}) \stackrel{c}{\sim} 2t_{\mathbf{x},\mathbf{y}}$ .

**Proof.** If  $d_0(\mathbf{x}, \mathbf{y}) \leq 1$ , the arc  $\gamma_{\mathbf{x},\mathbf{y}}$  is the geodesic segment  $[\mathbf{x}, \mathbf{y}]$  and the lemma is obvious in this case. We now assume  $d_0(\mathbf{x}, \mathbf{y}) > 1$ . Let  $x = \psi_{\xi,t_{\mathbf{x},\mathbf{y}}}(\mathbf{x})$  and  $y = \psi_{\xi,t_{\mathbf{x},\mathbf{y}}}(\mathbf{y})$ . From the right hand side of (5), the distance  $d(x, y)$  is bounded from below by  $b' := \frac{2}{b} \sinh^{-1} \frac{b}{2}$ .

Let us now fix a point  $\xi'$  on the boundary at infinity of the space  $\mathbb{H}_a^N$  of constant curvature  $-a^2$ , and two points  $x', y'$  on the same horosphere centered at  $\xi'$ , and at distance  $b'$  each from the other on this space; comparing the triangles  $x y \xi$  and  $x' y' \xi'$  we deduce that  $\widehat{x y \xi} \leq \widehat{x' y' \xi'} \leq \frac{\pi}{2} - \theta$ , for some constant  $\theta > 0$  depending only on  $a$  and  $b$ . Since  $\widehat{\mathbf{x} x y} \geq \pi/2$ , we have  $\widehat{x y \mathbf{x}} \leq \pi/2$  and so  $\widehat{\mathbf{x} y \mathbf{y}} \geq \theta$ . Applying Lemma 2.1 successively to the triangles  $\mathbf{x} x y$

(with  $\alpha \geq \pi/2$ ) and  $\mathbf{x} \succ \mathbf{y}$  (with  $\alpha \geq \theta$ ) we obtain  $d(\mathbf{x}, \mathbf{y}) \sim d(\mathbf{x}, x) + d(y, \mathbf{y})$ . The second point follows from the first one, computing the distance between  $\psi_{\xi, s}(\mathbf{x})$  and  $\psi_{\xi, t}(\mathbf{y})$  along  $\gamma_{\mathbf{x}, \mathbf{y}}$ .  $\square$

Applying this lemma, we obtain the

**PROPOSITION 2.3.** *There exists a constant  $c = c(a, b) > 0$  such that for any point  $\xi$  in  $X(\infty)$ , any horoball  $\mathcal{H}$  centered at  $\xi$  and any  $\mathbf{x} \in \partial\mathcal{H}$  and  $R > 0$  we have*

$$B_X(\psi_{\xi, R/2}(\mathbf{x}), R/2) \subset B_X(\mathbf{x}, R) \cap \mathcal{H} \subset B_X(\psi_{\xi, R/2}(\mathbf{x}), R/2 + c).$$

**Proof.** We need only to prove the second inclusion, the first one being obvious. For  $\mathbf{z} \in B_X(\mathbf{x}, R) \cap \mathcal{H}$ , denote by  $\mathbf{y}$  the projection of  $\mathbf{z}$  on  $\partial\mathcal{H}$  and by  $\mathbf{z}_0$  the intersection of the horosphere centered at  $\xi$  and containing  $\mathbf{z}$  with the geodesic ray  $[\mathbf{x}, \xi)$ .

Assume first  $t_{\mathbf{x}, \mathbf{y}} \leq \max\{R/2, d(\mathbf{y}, \mathbf{z})\}$ ; setting  $s = R/2$  and  $t = d(\mathbf{y}, \mathbf{z})$  in the previous lemma, we get  $d(\psi_{\xi, R/2}(\mathbf{x}), \mathbf{z}) \sim |s - t| = d(\psi_{\xi, R/2}(\mathbf{x}), \mathbf{z}_0) \leq R/2$  (the last inequality following from the fact that  $d(\mathbf{x}, \mathbf{z}_0) \leq d(\mathbf{x}, \mathbf{z}) \leq R$ ).

Assume now  $t_{\mathbf{x}, \mathbf{y}} \geq \max\{R/2, d(\mathbf{y}, \mathbf{z})\}$ ; applying twice the previous lemma, we get in this case

$$\begin{cases} d(\mathbf{x}, \mathbf{z}) \sim 2t_{\mathbf{x}, \mathbf{y}} - d(\mathbf{z}, \mathbf{y}) & (\text{setting } s = 0 \text{ and } t = d(\mathbf{y}, \mathbf{z})) \\ d(\psi_{\xi, R/2}(\mathbf{x}), \mathbf{z}) \sim 2t_{\mathbf{x}, \mathbf{y}} - d(\mathbf{z}, \mathbf{y}) - R/2 & (\text{setting } s = R/2 \text{ and } t = d(\mathbf{y}, \mathbf{z})). \end{cases}$$

Since  $\mathbf{z} \in B_X(\mathbf{x}, R)$  there, thus exists  $c > 0$  such that  $d(\psi_{\xi, R/2}(\mathbf{x}), \mathbf{z}) \leq R/2 + c$ .  $\square$

In the next section, we will consider discrete parabolic subgroups of  $Is(X)$ ; any such group fixes one point  $\xi \in X(\infty)$  and preserves any horoball  $\mathcal{H}$  centered at  $\xi$ . We shall investigate the relation between the critical exponent of  $\mathcal{P}$  and the volume growth of  $X$ . Here we shall limit ourselves to remark :

**COROLLARY 2.4.** *If  $X$  is homogeneous, then for any discrete parabolic subgroup  $\mathcal{P}$  of  $Is(X)$ , we have*

$$\delta(\mathcal{P}) \leq \omega(X)/2.$$

This fact is well known when  $X$  is a rank one symmetric space; Proposition 2.3 allows to understand the geometrical reason of this inequality. Actually, let  $\mathcal{H}$  be an horoball preserved by  $\mathcal{P}$  and let  $\mathbf{x} \in \partial\mathcal{H}$ . As  $\mathcal{P}$  is discrete, we have  $d := \frac{1}{2} \inf_{p \in \mathcal{P}} d(\mathbf{x}, p\mathbf{x}) > 0$ , then

$$\bigsqcup_{p/d(\mathbf{x}, p\mathbf{x}) \leq R} B_X(p\mathbf{x}, d) \times [0, 1] \subset B_X(\mathbf{x}, R + d + 1) \cap \mathcal{H}.$$

By Proposition 2.3, we deduce  $v_{\mathcal{P}}(\mathbf{x}, R) \preceq \sup_{\mathbf{y} \in \mathcal{H}} v_X\left(\mathbf{y}, \frac{R + d + 1}{2} + c\right)$ . As  $X$  is homogeneous, for any  $\epsilon > 0$ , we have  $v_X(\mathbf{y}, r) \preceq e^{(\omega(X) + \epsilon)r}$  uniformly in  $\mathbf{y}$ . The Corollary follows.  $\square$

### 3. Growth of ample parabolic subgroups

Let be  $\mathcal{P}$  a parabolic subgroup of  $Is(X)$  fixing  $\xi \in X(\infty)$ . We shall say that  $\mathcal{P}$  is **ample** if it acts cocompactly on every horoball  $\partial\mathcal{H}$  centered at  $\xi$ . This holds in particular when  $\mathcal{P}$  is a maximal parabolic subgroup of a non uniform lattice of  $Is(X)$ .

We then fix a (relatively compact) Borel fundamental domain  $\mathcal{C} \subset \partial\mathcal{H}$  for the action of  $\mathcal{P}$  on  $\partial\mathcal{H}$ . For any  $t \geq 0$ , the set  $\mathcal{C}_t := \psi_{\xi,t}(\mathcal{C})$  is a fundamental domain for the action of  $\mathcal{P}$  on  $\partial\mathcal{H}(t)$ ; in the same way, the set  $\mathcal{E} := \cup_{t \geq 0} \mathcal{C}_t$ , which is canonically homeomorphic to  $\mathcal{C} \times \mathbb{R}^+$ , is a fundamental domain for the action of  $\mathcal{P}$  on the horoball  $\mathcal{H}$ .

We now associate to any ample parabolic group  $\mathcal{P}$  a function  $\mathcal{A}_{\mathcal{P}}$  which will play a crucial role in this paper :

**DEFINITION 3.1.** The **horospherical area** of  $\mathcal{P}$  is the function  $\mathcal{A}_{\mathcal{P}}(\mathbf{x}, t)$  defined by

$$\forall \mathbf{x} \in \partial\mathcal{H}, \forall t \geq 0 \quad \mathcal{A}_{\mathcal{P}}(\mathbf{x}, t) := \mu_t(\psi_{\xi,t}(\mathcal{C})).$$

The function  $t \mapsto \mathcal{A}_{\mathcal{P}}(\mathbf{x}, t)$  is decreasing and does not depend on the choice of the fundamental domain  $\mathcal{C}$ ; furthermore, by inequalities (8), for any  $R$  and  $R_0 > 0$ , we have

$$(9) \quad e^{-(N-1)bR_0} \leq \frac{\mathcal{A}_{\mathcal{P}}(\mathbf{x}, R + R_0)}{\mathcal{A}_{\mathcal{P}}(\mathbf{x}, R)} \leq e^{-(N-1)aR_0}.$$

The following proposition stresses the relation between the function  $\mathcal{A}_{\mathcal{P}}$  and the orbital counting function  $v_{\mathcal{P}}(\mathbf{x}, R)$  of  $\mathcal{P}$ .

**PROPOSITION 3.2.** *There exists a constant  $c = c(a, b, \text{diam}(\mathcal{C})) > 0$  such that for any  $\mathbf{x} \in X$*

$$v_{\mathcal{P}}(\mathbf{x}, R) \stackrel{c}{\asymp} \frac{1}{\mathcal{A}_{\mathcal{P}}(\mathbf{x}, \frac{R}{2})}.$$

*In particular, we have*

$$(10) \quad \delta(\mathcal{P}) = \omega\left(\frac{1}{\mathcal{A}_{\mathcal{P}}(\mathbf{x}, \frac{R}{2})}\right) \quad \text{and} \quad \delta^-(\mathcal{P}) = \omega^-\left(\frac{1}{\mathcal{A}_{\mathcal{P}}(\mathbf{x}, \frac{R}{2})}\right).$$

**Proof.** We recall that  $d_t$  denotes the horospherical distance on the horosphere  $\partial\mathcal{H}(t)$ . We let  $c$  be the constant of Lemma 2.2 such that  $d(\mathbf{x}, \mathbf{y}) \stackrel{c}{\sim} 2t_{\mathbf{x}, \mathbf{y}}$  for  $\mathbf{x}, \mathbf{y}$  on  $\partial\mathcal{H}$ . If  $d(\mathbf{x}, \mathbf{y}) = R$ , as  $t_{\mathbf{x}, \mathbf{y}} \stackrel{c/2}{\sim} \frac{R}{2}$ , we deduce

$$d_{\frac{R+c}{2}}\left(\mathbf{x}\left(\frac{R+c}{2}\right), \mathbf{y}\left(\frac{R+c}{2}\right)\right) \leq 1 \quad \text{and} \quad d_{\frac{R-c}{2}}\left(\mathbf{x}\left(\frac{R-c}{2}\right), \mathbf{y}\left(\frac{R-c}{2}\right)\right) \geq 1.$$

This implies that  $\psi_{\frac{R+c}{2}}(B_X(\mathbf{x}, R) \cap \partial\mathcal{H}) \subset B_1$  and  $\psi_{\frac{R-c}{2}}(B_X(\mathbf{x}, R) \cap \partial\mathcal{H}) \subset B_2$  with

$$B_1 := B_{\partial\mathcal{H}(\frac{R+c}{2})}\left(\mathbf{x}\left(\frac{R+c}{2}\right), 1\right) \quad \text{and} \quad B_2 := B_{\partial\mathcal{H}(\frac{R-c}{2})}\left(\mathbf{x}\left(\frac{R-c}{2}\right), 1\right).$$

Gauss equation implies that the sectional curvature of all horospheres for the induced metric is in between  $a^2 - b^2$  and  $2b(b-a)$  (see ([7], section 1.4, example (iii)). Therefore, there exist positive constants  $v^- = v^-(a, b, \mathbf{x})$  and  $v^+ = v^+(a, b, \mathbf{x})$  such that  $v^- \leq \text{vol}(B_i) \leq v^+$  for the induced volume form on the horospheres and  $i = 1, 2$ .

Now, there are at most  $v_{\mathcal{P}}(\mathbf{x}, R)$  distinct fundamental domains  $p(\mathcal{C})$  included in  $B_X(\mathbf{x}, R) \cap \partial\mathcal{H}$  and since the radial semi-flow  $(\psi_{\xi,t})_{t \geq 0}$  is equivariant with respect to the action of  $\mathcal{P}$  on the horospheres  $\partial\mathcal{H}(t)$ , there are also at most  $v_{\mathcal{P}}(\mathbf{x}, R)$  distinct fundamental domains  $p(\mathcal{C}(\frac{R+c}{2}))$  included in  $\psi_{\frac{R+c}{2}}(B_X(\mathbf{x}, R) \cap \partial\mathcal{H})$ . Therefore, we have  $v_{\mathcal{P}}(\mathbf{x}, R) \leq \frac{v^+}{\mathcal{A}_{\mathcal{P}}(\mathbf{x}, \frac{R+c}{2})}$  and by (9), this leads to

$$v_{\mathcal{P}}(\mathbf{x}, R) \leq \frac{1}{\mathcal{A}_{\mathcal{P}}(\mathbf{x}, \frac{R}{2})}.$$

On the other hand, we can cover the set  $B_X(\mathbf{x}, R) \cap \partial\mathcal{H}$  with  $v_{\mathcal{P}}(\mathbf{x}, R+d)$  distinct fundamental domains  $p(\mathcal{C})$ ; by the equivariance of  $(\psi_{\xi,t})_t$  we deduce again that  $\psi_{\frac{R-c}{2}}(B_X(\mathbf{x}, R) \cap \mathcal{H})$  can be covered by  $v_{\mathcal{P}}(\mathbf{x}, R+d)$  fundamental domains as well. Therefore, using (9) again

$$v_{\mathcal{P}}(\mathbf{x}, R) \geq \frac{v^-}{\mathcal{A}_{\mathcal{P}}(\mathbf{x}, \frac{R-c-d}{2})} \succeq \frac{1}{\mathcal{A}_{\mathcal{P}}(\mathbf{x}, \frac{R}{2})}.$$

□

We now estimate the volume of a ball of radius  $R$ , inside the horoball  $\mathcal{H}$ . We have

**PROPOSITION 3.3.** *There exists a constant  $c = c(a, b, \text{diam}(\mathcal{C})) > 0$  such that*

$$\text{vol}(B_X(\mathbf{x}, R) \cap \mathcal{H}) \stackrel{c}{\asymp} \int_0^R \frac{\mathcal{A}_{\mathcal{P}}(\mathbf{x}, t)}{\mathcal{A}_{\mathcal{P}}(\mathbf{x}, \frac{t+R}{2})} dt.$$

To get this result, we need the following refinement of Proposition 2.3.

**LEMMA 3.4.** *There exists a constant  $\Delta = \Delta(a, b, \text{diam}(\mathcal{C}))$  such that*

$$p(\mathcal{C}) \times \left[ (2t_p - R + \Delta)^+, (R - \Delta)^+ \right] \subset \left( p(\mathcal{E}) \cap B_X(\mathbf{x}, R) \right) \subset p(\mathcal{C}) \times \left[ (2t_p - R - \Delta)^+, R \right].$$

**Proof.** Let  $\Delta = c + \text{diam}(\mathcal{C})$ , where  $c$  is the constant of Lemma 2.2. We first prove the right hand side inclusion. Let  $\mathbf{z} = (\mathbf{z}_0, t) \in p(\mathcal{C}) \times \mathbb{R}^+$  and assume that this point belongs to  $B_X(\mathbf{x}, R)$ . Clearly  $t \leq R$  as  $t = B_{\xi}(\mathbf{x}, \mathbf{z}) \leq d(\mathbf{x}, \mathbf{z}) \leq R$ . If  $t_p \leq \frac{R+\Delta}{2}$  there is nothing left to prove; on the other hand, if  $t_p > \frac{R+\Delta}{2}$ , then  $2t_p - t \stackrel{c}{\lesssim} d(\mathbf{x}, \mathbf{z}) < R$  hence  $t \in [(2t_p - R + \Delta)^+, R]$ . Let us now consider the case where  $\mathbf{z} \in p(\mathcal{C}) \times [(2t_p - R + \Delta)^+, (R - \Delta)^+]$ . We may assume that  $R \geq \Delta$  and  $t_p \leq R - \Delta$ , otherwise there is nothing to prove. If  $t \geq t_p$  we have  $d(\mathbf{x}, \mathbf{z}) \stackrel{c}{\lesssim} t \leq R - \Delta$ , otherwise we have  $d(\mathbf{x}, \mathbf{z}) \stackrel{c}{\lesssim} 2t_p - t \leq 2t_p - (2t_p - R + \Delta)^+$ ; therefore, in both cases  $\mathbf{z} \in B_X(\mathbf{x}, R)$ . □

**Proof of Proposition 3.3 .** We simply write  $\mathcal{A}(R) = \mathcal{A}_{\mathcal{P}}(\mathbf{x}, R)$ . Recall that

$$B_X(\mathbf{x}, R) \cap \mathcal{H} = \bigsqcup_{p \in \mathcal{P}} B_X(\mathbf{x}, R) \cap p(\mathcal{E}).$$

By Lemma 3.4, we have  $B_X(\mathbf{x}, R) \cap p(\mathcal{E}) \subset p(\mathcal{C}) \times [(2t_p - R - \Delta)^+, R]$ . Then, we find

$$\begin{aligned} \sum_{p \in \mathcal{P}} \text{vol}\left(B_X(\mathbf{x}, R) \cap p(\mathcal{E})\right) &= \sum_{t_p \leq R + \frac{\Delta}{2}} \int_{(2t_p - R - \Delta)^+}^R \mathcal{A}(t) dt \\ &= \sum_{t_p \leq R + \frac{\Delta}{2}} \int_0^R \mathcal{A}(t) 1_{[(2t_p - R - \Delta)^+, +\infty[}(t) dt \end{aligned}$$

Now, as  $d(\mathbf{x}, p\mathbf{x}) \stackrel{c}{\sim} 2t_p \leq c \leq \Delta$ , for every fixed  $t \in [0, R]$  we have

$$\begin{aligned} \#\left\{p \in \mathcal{P} / t_p \leq R + \frac{\Delta}{2} \text{ and } 2t_p - R - \Delta \leq t\right\} &\leq v_{\mathcal{P}}\left(\mathbf{x}, \frac{t + R + \Delta}{2} + \Delta\right) \\ &\leq \frac{v^+}{\mathcal{A}\left(\frac{t + R + 3\Delta}{2}\right)} \\ &\preceq \frac{1}{\mathcal{A}\left(\frac{R + t}{2}\right)}, \end{aligned}$$

where we have successively used Proposition 3.2 and (9). This yields

$$\text{vol}(B_X(\mathbf{x}, R) \cap \mathcal{H}) \preceq \int_0^R \frac{\mathcal{A}(t)}{\mathcal{A}\left(\frac{t + R}{2}\right)} dt.$$

We now prove the converse inequality. Again, by Proposition 3.4, we deduce

$$B_X(\mathbf{x}, R) \cap p(\mathcal{E}) \supset p(\mathcal{C}) \times [(2t_p - R + \Delta)^+, R - \Delta].$$

We only consider those  $p$ 's such that  $\frac{R - \Delta}{2} \leq t_p \leq R - \Delta$ ; summing over these  $p$ 's, we find

$$\begin{aligned} \sum_{\frac{R - \Delta}{2} \leq t_p \leq R - \Delta} \text{vol}\left(B_X(\mathbf{x}, R) \cap p(\mathcal{E})\right) &= \sum_{\frac{R - \Delta}{2} \leq t_p \leq R - \Delta} \int_{2t_p - R - \Delta}^{R - \Delta} \mathcal{A}(t) dt \\ &\geq \sum_{\frac{R - \Delta}{2} \leq t_p \leq R - \Delta} \int_{R_0}^{R - \Delta} \mathcal{A}(t) 1_{[2t_p - R + \Delta, R - \Delta]}(t) dt \end{aligned}$$

for any  $R_0 \geq 0$ . Now, for every fixed  $t \in [R_0, R - \Delta]$ , we have

$$\begin{aligned} \#\left\{p \in \mathcal{P} / \frac{R - \Delta}{2} \leq t_p \leq R - \Delta \text{ and } 2t_p - R + \Delta \leq t\right\} &\geq v_{\mathcal{P}}\left(\mathbf{x}, t + R - 2\Delta\right) - v_{\mathcal{P}}\left(\mathbf{x}, R\right) \\ &\geq \frac{v^-}{\mathcal{A}\left(\frac{t + R - 2\Delta}{2}\right)} - \frac{v^+}{\mathcal{A}\left(\frac{R}{2}\right)} \\ &\geq \frac{1}{\mathcal{A}\left(\frac{t + R}{2}\right)} \left( v^- \frac{\mathcal{A}\left(\frac{t + R}{2}\right)}{\mathcal{A}\left(\frac{t + R - 2\Delta}{2}\right)} - v^+ \frac{\mathcal{A}\left(\frac{t + R}{2}\right)}{\mathcal{A}\left(\frac{R}{2}\right)} \right) \\ &\geq \frac{1}{\mathcal{A}\left(\frac{t + R}{2}\right)} \left( v^- e^{-b(N-1)\Delta} - v^+ e^{-a(N-1)R_0/2} \right) \end{aligned}$$

by Proposition 3.2 and (9). Therefore, if  $R_0$  is large enough, we find

$$\text{vol}\left(B_X(\mathbf{x}, R) \cap \mathcal{H}\right) \succeq \int_{R_0}^{R - \Delta} \frac{\mathcal{A}(t)}{\mathcal{A}\left(\frac{t + R}{2}\right)} dt.$$

We can replace this last integral by  $\int_0^R \frac{\mathcal{A}(t)}{\mathcal{A}(\frac{t+R}{2})} dt$  since,  $\int_{R-\Delta}^R \frac{\mathcal{A}(t)}{\mathcal{A}(\frac{t+R}{2})} dt$  is bounded in terms of  $a, b$  and  $\Delta$  and for  $R$  large enough

$$\int_{R_0}^{R-\Delta} \frac{\mathcal{A}(t)}{\mathcal{A}(\frac{t+R}{2})} dt \geq \int_{R_0}^{2R_0} \frac{\mathcal{A}(t)}{\mathcal{A}(\frac{t+R}{2})} dt \asymp \frac{1}{\mathcal{A}(R/2)} \asymp \int_0^{R_0} \frac{\mathcal{A}(t)}{\mathcal{A}(\frac{t+R}{2})} dt.$$

□

As a direct consequence of Propositions 3.2 and 3.3, we obtain

**COROLLARY 3.5.** *For any  $\epsilon > 0$  and  $\mathbf{x} \in \partial\mathcal{H}$ , we have*

*i) if  $\delta(\mathcal{P}) \geq 2\delta^-(\mathcal{P})$  then*

$$e^{(\delta^-(\mathcal{P})-\epsilon)R} \preceq \text{vol}\left(B_X(\mathbf{x}, R) \cap \mathcal{H}\right) \preceq e^{2\left(\delta(\mathcal{P})-\delta^-(\mathcal{P})+\epsilon\right)R}$$

*ii) if  $\delta(\mathcal{P}) < 2\delta^-(\mathcal{P})$  then*

$$e^{(\delta^-(\mathcal{P})-\epsilon)R} \preceq \text{vol}\left(B_X(\mathbf{x}, R) \cap \mathcal{H}\right) \preceq e^{2\left(\delta(\mathcal{P})+\epsilon\right)R}.$$

#### 4. Growth of nonuniform lattices

We suppose now that the manifold  $X$  admits a nonuniform lattice  $\Gamma$ . Let us recall some well known geometrical properties of  $\Gamma$  proved in the general context of geometrically finite groups in ([5]). Since the volume of  $M = X/\Gamma$  is finite, the *limit set* of  $\Gamma$  equals  $X(\infty)$  and is the disjoint union of its *radial* subset and of finitely many orbits  $\Gamma\xi_1, \dots, \Gamma\xi_l$  of points, called *bounded parabolic fixed points*. By definition, a point  $\xi_i$  corresponds to an end of the manifold  $M$  and is fixed by a parabolic subgroup of  $\Gamma$ . Denote  $\mathcal{P}_i$  the maximal parabolic subgroup fixing the point  $\xi_i$ . This group preserves any horoball  $\mathcal{H}$  centered at  $\xi_i$  and acts cocompactly on the horosphere  $\partial\mathcal{H}$ . By Margulis' lemma (see [20]), there exist closed horoballs  $\mathcal{H}_{\xi_1}, \dots, \mathcal{H}_{\xi_l}$  centered respectively at  $\xi_1, \dots, \xi_l$ , such that all the horoballs  $\gamma.\mathcal{H}_{\xi_i}$ , for  $1 \leq i \leq l$  and  $\gamma \in \Gamma$ , are disjoint or coincide. We fix an origin  $\mathbf{o} \in X$  and a convex Borel fundamental domain  $\mathcal{D}$  in  $X$  for the action of  $\Gamma$ , containing the geodesic rays  $[\mathbf{o}, \xi_1), \dots, [\mathbf{o}, \xi_l)$ . For each  $1 \leq i \leq l$ , we set  $\mathcal{E}_i = \mathcal{D} \cap \mathcal{H}_{\xi_i}$  and  $\mathcal{C}_i = \mathcal{D} \cap \partial\mathcal{H}_{\xi_i}$ . Those both sets are fundamental domains for the action of the group  $\mathcal{P}_i$  respectively on  $\mathcal{H}_{\xi_i}$  and  $\partial\mathcal{H}_{\xi_i}$ . Moreover, the set  $\mathcal{C}_0 = \mathcal{D} \setminus (\cup_{i=1}^l \mathcal{E}_i)$ , and hence each  $\mathcal{C}_i$ , is relatively compact. We may assume that  $\mathbf{o}$  belongs to the interior of  $\mathcal{C}_0$ .

The quotient manifold  $M$  is therefore decomposed into the disjoint union of a relatively compact set  $C_0$  and finitely many ends of finite volume  $E_i = \mathcal{H}_{\xi_i}/\mathcal{P}_i$ , which are the projections on  $M$  of the domains  $\mathcal{C}_0$  and  $\mathcal{E}_i$  respectively.

We first precise some bounds on the critical exponent  $\delta(\Gamma)$  in terms of bounds on the curvature of  $X$ .

**LEMMA 4.1.** *We have  $(N-1)a \leq \delta(\Gamma) \leq (N-1)b$ .*

In particular, when  $X$  is the real hyperbolic space  $\mathbb{H}_a^N$  of constant curvature  $-a^2$ , we have  $\delta(\Gamma) = (N-1)a$  and hence  $\delta(\Gamma) = \omega(\mathbb{H}_a^N)$ .

**Proof.** The inequality  $\delta(\Gamma) \leq (N-1)b$  follows from (3) and (1). It remains to prove the left hand side inequality of the Lemma

If  $\delta(\Gamma) = \omega(X)$ , the inequality follows from (1). Assume now  $\delta(\Gamma) < \omega(X)$  and consider  $s \in ]\delta(\Gamma), \omega(X)[$ . Inequality (2) implies

$$\int_{\mathcal{D}} e^{sd(\mathbf{o}, \mathbf{x})} dv_X(\mathbf{x}) = +\infty$$

which, by the decomposition  $\mathcal{D} = \mathcal{C}_0 \cup \left( \bigcup_{i=1}^l \mathcal{E}_i \right)$ , is equivalent to

$$(11) \quad \max_{i \in \{1, \dots, l\}} \int_{\mathcal{E}_i} e^{sd(\mathbf{o}, \mathbf{x})} dv_X(\mathbf{x}) = +\infty.$$

Note now that for  $\mathbf{x} \in \mathcal{E}_i$ , we have  $B_{\xi_i}(\mathbf{o}, \mathbf{x}) \leq d(\mathbf{o}, \mathbf{x}) \leq B_{\xi_i}(\mathbf{o}, \mathbf{x}) + \text{diam}(\mathcal{C}_i)$  where  $B_{\xi_i}(\cdot, \cdot)$  denotes the Busemann function centered at  $\xi_i$ . Therefore the integrals  $\int_{\mathcal{E}_i} e^{sd(\mathbf{o}, \mathbf{x})} dv_X(\mathbf{x})$  and  $\int_{\mathcal{E}_i} e^{sB_{\xi_i}(\mathbf{o}, \mathbf{x})} dv_X(\mathbf{x})$  are of the same nature.

By (8), we have

$$\int_{\mathcal{E}_i} e^{sB_{\xi_i}(\mathbf{o}, \mathbf{x})} dv_X(\mathbf{x}) = \int_{d(\mathbf{o}, \mathcal{C}_i)}^{+\infty} e^{st} \mu_t(\psi_{\xi_i, t}(\mathcal{C}_i)) dt \leq \mu_0(\mathcal{C}_i) \int_0^{+\infty} e^{t[s - (N-1)a]} dt$$

and the last integral must be divergent for all  $s \in ]\delta(\Gamma), \omega(X)[$ , so  $\delta(\Gamma) \geq (N-1)a$ .  $\square$

Recall that  $v_X(\mathbf{o}, R)$  denotes the volume of the open ball  $B_X(\mathbf{o}, R)$  and that  $v_\Gamma(\mathbf{o}, R)$  represents the cardinality of the intersection of this ball with  $\Gamma(\mathbf{o})$ . The following estimate will be used to obtain an upper bound for  $\delta(\Gamma)$ .

**PROPOSITION 4.2.** *There exists a constant  $\Delta = \Delta(a, b, \text{diam}(\mathcal{C}_0)) > 0$  such that, for all  $R > 0$ , we have*

$$(12) \quad v_X(\mathbf{o}, R - \Delta) \leq v_\Gamma(\mathbf{o}, R) + \sum_{i=1}^l \sum_{n=0}^{[R]} v_\Gamma(\mathbf{o}, n+1) \times \text{vol}\left(B_X(\mathbf{x}_i, R - n + \Delta) \cap \mathcal{H}_{\xi_i}\right)$$

where  $\mathbf{x}_i$  denotes the intersection of the geodesic ray  $[\mathbf{o}, \xi_i)$  with the horosphere  $\partial\mathcal{H}_{\xi_i}$ .

**Proof.** Set  $d_0 = \text{diam}(\mathcal{C}_0)$ . We have

$$(13) \quad B_X(\mathbf{o}, R) = \left( B_X(\mathbf{o}, R) \cap \Gamma \cdot \mathcal{C}_0 \right) \cup \left( \bigcup_{1 \leq i \leq l} \left( B_X(\mathbf{o}, R) \cap \Gamma \cdot \mathcal{H}_{\xi_i} \right) \right)$$

whence

$$B_X(\mathbf{o}, R) \cap \Gamma \cdot \mathcal{C}_0 \subset \bigcup_{\gamma \in B_\Gamma(\mathbf{o}, R + d_0)} \gamma(\mathcal{C}_0)$$

and

$$\text{vol}\left(B_X(\mathbf{o}, R) \cap \Gamma \cdot \mathcal{C}_0\right) \leq v_\Gamma(R + d_0).$$

Now, for each  $i \in \{1, \dots, l\}$  we define a map on  $\Gamma$  as follows : for any  $\gamma \in \Gamma$ , let  $x_{\gamma, i}$  be the intersection of the ray  $[\mathbf{o}, \gamma(\xi_i))$  with the horosphere  $\gamma(\partial\mathcal{H}_{\xi_i})$ . Since  $\mathcal{C}_i$  is a fundamental domain for the action of  $\mathcal{P}_i$  on  $\partial\mathcal{H}_{\xi_i}$  there exist a finite number of elements  $\bar{\gamma}$  in  $\gamma\mathcal{P}_i$  such that  $x_{\gamma, i} \in \bar{\gamma}(\mathcal{C}_i)$ . Choose one of those

elements and denote it by  $\bar{\gamma}_i$ . Let  $\bar{\Gamma}_i$  be the set of all  $\bar{\gamma}_i$  for  $\gamma$  in  $\Gamma$ . Since  $d(x_{\gamma,i}, \bar{\gamma}_i \mathbf{o}) \leq d_0$ , and since the angle at  $x_{\gamma,i}$  between the geodesic segments  $[x_{\gamma,i}, \mathbf{o}]$  and  $[x_{\gamma,i}, x]$  is greater than  $\pi/2$ , by lemma 2.1 there exists a constant  $d_1 > 0$  such that for every  $\gamma \in \Gamma$  and  $x \in \gamma \mathcal{H}_{\xi_i} \cap B_X(\mathbf{o}, R)$ , we have :

$$d(\mathbf{o}, \bar{\gamma}_i \mathbf{o}) + d(\bar{\gamma}_i \mathbf{o}, x) - d_1 \leq d(\mathbf{o}, x).$$

We have by (13)

$$B_X(\mathbf{o}, R) \cap \Gamma \mathcal{H}_{\xi_i} \subset \left( \bigcup_{0 \leq n \leq [R+d_0]} \bigcup_{\substack{\bar{\gamma} \in \bar{\Gamma}_i \\ n \leq d(\mathbf{o}, \bar{\gamma} \mathbf{o}) < n+1}} B_X(\bar{\gamma} \mathbf{o}, R - n + d_1) \cap \bar{\gamma} \mathcal{H}_{\xi_i} \right).$$

For each  $i$  denote  $\mathbf{x}_i$  the intersection of the geodesic ray  $[\mathbf{o}, \xi_i)$  with the horosphere  $\partial \mathcal{H}_{\xi_i}$ . One has

$$\text{vol} \left( B_X(\bar{\gamma} \mathbf{o}, R - n + d_1) \cap \bar{\gamma} \mathcal{H}_{\xi_i} \right) \leq \text{vol} \left( B_X(\mathbf{x}_i, R - n + d_1 + d_0) \cap \mathcal{H}_{\xi_i} \right),$$

while

$$\#\{\bar{\gamma} \in \bar{\Gamma}_i / n \leq d(\mathbf{o}, \bar{\gamma} \mathbf{o}) < n + 1\} \leq v_\Gamma(\mathbf{o}, n + 1),$$

so

$$v_X(\mathbf{o}, R - d_0) \leq v_\Gamma(\mathbf{o}, R) + \sum_{i=1}^l \sum_{n=0}^{[R]} v_\Gamma(\mathbf{o}, n + 1) \times \text{vol} \left( B_X(\mathbf{x}_i, R - n + d_1) \cap \mathcal{H}_{\xi_i} \right).$$

The lemma follows with  $\Delta \geq \max(d_0, d_1)$ .  $\square$

Proposition 4.2 is crucial to establish Theorem 1.2; we first give an elementary proof of this result, in the case where  $X$  is  $1/4$ -pinched.

**4.1. Proof of Theorem 1.2 : the  $\frac{1}{4}$ -pinched curvature case.** We prove here that if  $(X, g)$  is a complete, simply connected Riemannian manifold with  $1/4$ -pinched negative curvature, then for any lattice  $\Gamma \subset Is(X)$ , we have  $\delta(\Gamma) = \omega(X)$ .

We use the notations of Proposition 4.2.

By (3), we need only to show that  $\omega(X) \leq \delta(\Gamma)$ . By Proposition 2.3, we know that for  $r > 0$  the set  $B_X(\mathbf{x}_i, r) \cap \mathcal{H}_{\xi_i}$  is included in the ball of radius  $r/2 + c$  centered at the point  $\psi_{\xi_i, r/2}(\mathbf{x}_i)$ . Then, (12) leads to the following inequality

$$(14) \quad v_X(\mathbf{o}, R - \Delta) \leq v_\Gamma(\mathbf{o}, R) + \sum_{n=0}^{[R]} v_\Gamma(\mathbf{o}, n + 1) \times \sup_{\mathbf{x} / B_X(\mathbf{x}, \frac{R-n+\Delta}{2}) \subset \Theta} \text{vol} \left( B_X \left( \mathbf{x}, \frac{R-n+\Delta}{2} \right) \right).$$

From Bishop Gunther's theorem and the fact that  $b^2 \leq 4a^2$ , we have

$$\text{vol} \left( B_X(\mathbf{x}, r) \cap \Theta \right) \leq v_X(\mathbf{x}, r) \leq e^{b(N-1)r} \leq e^{2a(N-1)r},$$

for any  $\mathbf{x} \in X$  and  $r > 0$ . We conclude that  $\omega(X) \leq (N-1)a \leq \delta(\Gamma)$  using Lemma 4.1.  $\square$

**Remark -** The above proof uses in a crucial way Lemma 4.1 and it still works if we relax the pinching assumption as follows :

For any  $\epsilon > 0$ , there exists a compact set  $C_\epsilon \subset M$  such that the curvature on  $M \setminus C_\epsilon$  belongs to  $[-(4 + \epsilon)a^2, -a^2]$ .

However, this condition is much stronger than the  $\left(\frac{1}{4+\epsilon}\right)$ -pinching assumption and the proof of Corollary 1.3 requires the more precise estimates of the volume of balls obtained in the previous section.

**4.2. Proof of Theorem 1.2 : the general case.** We fix here a non uniform lattice  $\Gamma \subset Is(X)$  and apply the results of Section 3 to each maximal parabolic subgroup  $\mathcal{P}_i$  of  $\Gamma$ . We first set the

**DEFINITION 4.3.** Let  $M = X/\Gamma$  be a complete Riemannian manifold of finite volume with  $-b^2 \leq K_X \leq -a^2 < 0$  and with ends  $E_1, \dots, E_l$ . For  $1 \leq i \leq l$ , the **cuspidal function**  $\mathcal{F}_i$  associated with  $E_i$  is defined by

$$\forall \mathbf{x} \in X, \forall R > 0 \quad \mathcal{F}_i(\mathbf{x}, R) = \int_0^R \frac{\mathcal{A}_i(\mathbf{x}, t)}{\mathcal{A}_i\left(\mathbf{x}, \frac{t+R}{2}\right)} dt$$

where  $\mathcal{A}_i(\mathbf{x}, t)$  is the horospherical area function associated with  $E_i$ .

By (9), the growth rates  $\omega^\pm(\mathcal{F}_i(\mathbf{x}, \cdot))$  depend only on the ends  $E_i$  of  $M$  as for any points  $\mathbf{x}, \mathbf{y} \in X$  and any  $R_0 > 0$  fixed, we have  $\mathcal{F}_i(\mathbf{x}, R) \asymp \mathcal{F}_i(\mathbf{y}, R)$ . Those functions are of major importance in order to estimate  $v_X(\mathbf{x}, R)$ ; namely, we have the

**PROPOSITION 4.4.** *There exists  $\Delta = \Delta(a, b, \text{diam}(\mathcal{C}_0)) > 0$  such that*

$$(i) \quad v_X(\bullet, R + \Delta) \succeq v_\Gamma(\bullet, R) + \sum_{i=1}^l \mathcal{F}_i(\bullet, R)$$

$$(ii) \quad v_X(\bullet, R + \Delta) \preceq v_\Gamma(\bullet, R) + \sum_{i=1}^l v_\Gamma(\bullet, \cdot) * \mathcal{F}_i(\bullet, \cdot)(R)$$

which leads to the

**COROLLARY 4.5.** *We have  $\omega^\pm(X) = \max\left(\delta(\Gamma), \omega^\pm(\mathcal{F}_1), \dots, \omega^\pm(\mathcal{F}_l)\right)$ .*

**Proof of Proposition 4.4.** *Part (i).* We have

$$B_X(\mathbf{o}, R) \supset \bigsqcup_{\gamma \in B_\Gamma(\mathbf{o}, R-d_0)} \gamma(\mathcal{C}_0) \cup \bigcup_{i=1}^l (B_X(\mathbf{o}, R) \cap \mathcal{H}_i).$$

On the other hand  $B_X(\mathbf{o}, R) \cap \mathcal{H}_i \supset B_X(\mathbf{x}_i, R - d_0) \cap \mathcal{H}_i$ , and by Proposition 3.3, we have

$$v_X(\mathbf{o}, R) \succeq v_\Gamma(\mathbf{o}, R - d_0) + \sum_{i=1}^l \mathcal{F}_i(\mathbf{x}_i, R)$$

with  $\mathcal{F}_i(\mathbf{x}_i, R) \asymp \mathcal{F}_i(\mathbf{o}, R)$ ; the first inequality follows.

*Part (ii)* follows by plugging Proposition 3.3 in (12). □

**Proof of Theorem 1.2** By Corollary 4.5, it is enough to show that  $\omega(\mathcal{F}_i) \leq \delta(\Gamma)$  for  $1 \leq i \leq l$ . By Proposition 3.2, we have, for any  $\epsilon > 0$  :

$$\mathcal{A}_i(t) \preceq e^{-(2\delta^-(\mathcal{P})-\epsilon)t} \quad \text{and} \quad \mathcal{A}_i\left(\frac{t+R}{2}\right) \succeq e^{-(\delta(\mathcal{P})+\epsilon)(t+R)}.$$

So, we obtain  $\mathcal{F}_i(t) \preceq e^{(\delta(\mathcal{P})+\epsilon)R} \int_0^R e^{(\delta(\mathcal{P})-2\delta^-(\mathcal{P})+2\epsilon)t} dt \preceq e^{(\delta(\mathcal{P})+3\epsilon)R}$  as  $\delta(\mathcal{P}) - 2\delta^-(\mathcal{P}) \leq 0$ , therefore  $\omega(\mathcal{F}_i) \leq \delta(\mathcal{P}) \leq \delta(\Gamma)$ .  $\square$ .

**Proof of Corollary 1.3** Assume that  $M = X/\Gamma$  is asymptotically  $\frac{1}{4}$ -pinched. Then, for any fixed  $\epsilon > 0$  we know that outside a compact subset  $C_\epsilon$  the curvature of  $M$  is between  $-\beta^2$  and  $-\alpha^2$ , with  $\beta^2 \leq (4 + \epsilon)\alpha^2$ . Therefore we have

$$e^{-\beta(N-1)t} \preceq \mathcal{A}_i(t) \preceq e^{-\alpha(N-1)t}$$

hence, by Proposition 3.2, we deduce that

$$\frac{\delta(\mathcal{P})}{\delta^-(\mathcal{P})} \leq \frac{\beta}{\alpha} \leq 2 + \epsilon$$

for every maximal parabolic subgroup of  $\Gamma$ . As  $\epsilon$  is arbitrary, we deduce that  $M$  is parabolically  $\frac{1}{4}$ -pinched, and we conclude by Theorem 1.2.  $\square$

**Remark.** We have seen that, under the assumptions of Theorem 1.2, we have  $\omega(\mathcal{F}_i) \leq \delta(\Gamma)$  for all  $1 \leq i \leq l$ ; in particular,  $\omega(X)$  is a limit in this case.

## 5. An end with the leading role

We shall construct in this section a pinched, negatively curved surface  $S = X/\Gamma$  of finite volume such that  $\omega(X) > \delta(\Gamma)$ . The surface we exhibit is homeomorphic to a 3-punctured sphere, and we shall deform a hyperbolic metric on one end  $E$  of  $S$ .

Our construction rests on two main ideas :

i) we can deform the metric in the end  $E$  varying the sectional curvature from  $\alpha^2$  to  $\beta^2$  on different bands of  $E$ , in order that the function  $\mathcal{F}$  associated to  $E$  satisfies  $\omega(\mathcal{F}) > \delta(\mathcal{P})$ .

ii) we set  $\epsilon := \omega(\mathcal{F}) - \delta(\mathcal{P})$  and we show that the above deformation of the metric can be performed in such a way that  $\delta(\Gamma) < \delta(\mathcal{P}) + \epsilon$  also.

By Corollary 4.5 we conclude that  $\omega(X) > \delta(\Gamma)$ .

Fix positive real numbers  $\alpha$  and  $\beta$  such that  $\beta > 2\alpha$ . We can construct sequences of disjoint intervals  $[p_n, q_n]$ ,  $[r_n, s_n]$  included in  $[\Delta^{n-1}, \Delta^n]$  (for some  $\Delta > 1$ ), and a  $C^2$  convex, decreasing function  $\mathcal{A}(t)$  on  $[\Delta, +\infty[$  whose restrictions to  $[p_n, q_n]$  and  $[r_n, s_n]$  coincide respectively with  $e^{-\alpha t}$  and  $e^{-\beta t}$ . More precisely, we can arrange the points  $p_n, q_n, r_n$  and  $s_n$  in order that  $q_n \geq p_n + 1$  and  $t \in [p_n, q_n] \Leftrightarrow \frac{t+\Delta^n}{2} \in [r_n, s_n]$ , and we can choose  $\mathcal{A}$  such that  $e^{-\beta t} \leq \mathcal{A}(t) \leq e^{-\alpha t}$  and  $\frac{\mathcal{A}''(t)}{\mathcal{A}(t)} \in [\alpha^2 - \eta, \beta^2 + \eta]$  for all  $t \in [\Delta, +\infty[$  and some  $\eta > 0$ . The existence of such intervals and of the function  $\mathcal{A}$  is rather technical and we postponed the details of proof to the Appendix (Section 6).

By construction, the function  $\mathcal{F}(R) := \int_0^R \frac{\mathcal{A}(t)}{\mathcal{A}(\frac{t+R}{2})} dt$  satisfies :

$$\omega(\mathcal{F}) \geq \limsup_{n \rightarrow +\infty} \frac{1}{\Delta^n} \ln \int_{p_n}^{q_n} \frac{\mathcal{A}(t)}{\mathcal{A}(\frac{t+\Delta^n}{2})} dt > \beta/2.$$

We can now construct the surface of Theorem 1.1. Start from a 3-punctured sphere  $S$  with a metric  $g_0$  of finite volume and constant curvature  $-\alpha^2$ . Let  $\Gamma = \pi_1(S)$  and let  $\mathcal{P}$  be the maximal parabolic subgroup associated with the end  $E$  of  $S$ . Consider the horospherical parametrization  $\sigma : [0, 1[ \times \mathbb{R}^+ \rightarrow \mathcal{E}$  of  $\mathcal{E}$ ; with respect to these coordinates, the hyperbolic metric writes  $g = e^{-2\alpha t} dx^2 + dt^2$ . We now perturb  $g$  on  $E_n = \sigma([0, 1[ \times [p_n, +\infty[)$  to obtain a new  $C^2$ -metric  $g_n$  such that  $g_n = \mathcal{A}^2(t) dx^2 + dt^2$  on  $E_n$ , for  $\mathcal{A}$  defined above. We shall denote by  $d$  and  $d_n$  the distances on  $X$  associated respectively to  $g$  and  $g_n$  and we let  $\delta_n(\Gamma)$ ,  $\delta_n(\mathcal{P})$  be the critical exponents of  $\Gamma$  and  $\mathcal{P}$  relatively to the new metric  $g_n$ . Notice that  $K_X = -\frac{\mathcal{A}''}{\mathcal{A}}$  is pinched between  $-\beta^2 - \eta$  and  $-\alpha^2 + \eta$ ; furthermore  $\mathcal{A}(R)$  is precisely the horospherical area (length) function of  $\mathcal{P}$ , with respect to  $g_n$ , so  $\delta_n(\mathcal{P}) = \beta/2$  for all  $n$ , by Proposition 3.2 (while  $\delta_n^-(\mathcal{P}) \leq \alpha/2$ ). Since we know that  $\omega(\mathcal{F}) = \beta/2 + \epsilon$  for some  $\epsilon > 0$ , it will be enough to show that :

**PROPOSITION 5.1.** *For  $n$  large enough, we have  $\delta_n(\Gamma) \in ]\delta_n(\mathcal{P}), \delta_n(\mathcal{P}) + \epsilon[$ .*

**Proof.** Let  $p$  be a generator of  $\mathcal{P}$  and choose another parabolic element  $q \in \Gamma$  such that  $\Gamma$  is the free non abelian group over  $p$  and  $q$ . Fix  $N \geq 2$ ; each element  $\gamma \in \Gamma \setminus \{id\}$  can be written in a unique way as

$$(15) \quad \gamma = p^{l_1} q^{m_1} \dots p^{l_k} q^{m_k},$$

where  $l_i, m_i \in \mathbb{Z}^*$  except for  $l_1$  and  $m_k$  which may be zero. Given this decomposition, we select those  $l_i$  such that  $|l_i| \geq N$ , say  $l_{i_1}, \dots, l_{i_r}$ , and write

$$(16) \quad \gamma = Q_1 p^{l_{i_1}} Q_2 \dots p^{l_{i_r}} Q_r$$

where each  $Q_i$  is a subword of the expression (15), containing powers of  $q$  and powers of  $p$  not exceeding  $N$  in absolute value. Note that decomposition (16) is still unique. We denote by  $\mathcal{Q}_N$  the subset of elements  $\gamma \in \Gamma$  which write simply  $\gamma = Q_1$  in (16).

Now let  $\mathbf{o} \in X$  and  $\mathcal{D}$  be the Dirichlet domain for the action of  $\Gamma$ , centered at  $\mathbf{o}$ . Roughly speaking, the union of the geodesic segments

$$[\mathbf{o}, Q_1(\mathbf{o})], [Q_1(\mathbf{o}), Q_1 p^{l_{i_1}}(\mathbf{o})], \dots, [Q_1 \dots p^{l_{i_r}}(\mathbf{o}), \gamma(\mathbf{o})]$$

represents a quasigeodesic which stays close to  $[\mathbf{o}, \gamma(\mathbf{o})]$  and each of its subsegments corresponds to the excursion of the geodesic loop  $\gamma$  alternatively outside or inside the cusp  $E$ . We now precise this argument.

As  $K_X \leq -\alpha^2 + \eta$ , there exists a minimal angle  $\theta_0 > 0$  such that for all  $\mathbf{x} \in p^{\pm 2}(\mathcal{D})$  and all  $\mathbf{y} \in q^{\pm 1}(\mathcal{D})$ , we have  $\widehat{\mathbf{x} \mathbf{o} \mathbf{y}} \geq \theta_0$ . Then, when  $Q_1 \neq id$  in (16), by a ping-pong argument we deduce that  $\angle_{\mathbf{o}}(Q_1^{-1} \mathbf{o}, p^{l_{i_1}} Q_2 \dots Q_r \mathbf{o}) \geq \theta_0$ , as  $l_{i_1} \geq N \geq 2$ . Therefore, by Lemma 2.1, there exists a constant  $d = d(\alpha, \theta_0) > 0$  such that

$$d_n(\mathbf{o}, \gamma(\mathbf{o})) \geq d_n(\mathbf{o}, Q_1(\mathbf{o})) + d_n(\mathbf{o}, p^{l_{i_1}} Q_2 \dots p^{l_{i_r-1}} Q_r(\mathbf{o})) - d$$

Repeating this argument yields

$$d_n(\mathbf{o}, \gamma(\mathbf{o})) \geq \sum_{i=0}^r d_n(\mathbf{o}, Q_i(\mathbf{o})) + \sum_{j=1}^{r-1} d_n(\mathbf{o}, p^{l_{i,j}}(\mathbf{o})) - 2rd.$$

Consequently

$$(17) \quad \sum_{\gamma \in \Gamma} e^{-sd_n(\mathbf{o}, \gamma(\mathbf{o}))} \leq \sum_{\gamma \in \mathcal{Q}_N} e^{-sd_n(\mathbf{o}, \gamma(\mathbf{o}))} + \sum_{r \geq 1} \left( e^{2sd} \sum_{|k| \geq N} e^{-sd_n(\mathbf{o}, p^k(\mathbf{o}))} \sum_{\gamma \in \mathcal{Q}_N} e^{-sd_n(\mathbf{o}, \gamma(\mathbf{o}))} \right)^r$$

If  $n$  is large enough with respect to  $N$ , every element of  $\mathcal{Q}_N$  correspond to a geodesic loop staying in the part of  $S$  where the curvature is constant equal to  $-\alpha^2$ . For that choice of  $n$  and for  $s = \frac{\beta + \epsilon}{2}$ , we have

$$\sum_{\gamma \in \mathcal{Q}_N} e^{-sd_n(\mathbf{o}, \gamma(\mathbf{o}))} \leq \sum_{\gamma \in \Gamma} e^{-sd(\mathbf{o}, \gamma(\mathbf{o}))} := A.$$

The latter series converges because the value of the critical exponent of any lattice in the space of constant curvature case  $-\alpha^2$  is  $\alpha$  and  $\alpha < s$ .

Furthermore

$$\begin{aligned} \sum_{|k| \geq N} e^{-sd(\mathbf{o}, p^k(\mathbf{o}))} &\preceq \sum_{m \geq d(\mathbf{o}, p^N(\mathbf{o}))} v_{\mathcal{P}}(\mathbf{o}, m) e^{-sm} \\ &\preceq \sum_{m \geq d(\mathbf{o}, p^N(\mathbf{o}))} \frac{e^{-sm}}{\mathcal{A}(\frac{m}{2})} \\ &\preceq \sum_{m \geq d(\mathbf{o}, p^N(\mathbf{o}))} e^{-(s - \frac{\beta}{2})m} = \sum_{m \geq d(\mathbf{o}, p^N(\mathbf{o}))} e^{-\epsilon m/2} \end{aligned}$$

so that  $\sum_{|k| \geq N} e^{-sd(\mathbf{o}, p^k(\mathbf{o}))} \rightarrow 0$  when  $N \rightarrow +\infty$ . Then, we can choose  $N$  and  $n$  such that

$$\sum_{\gamma \in \mathcal{Q}_N} e^{-sd(\mathbf{o}, \gamma(\mathbf{o}))} \leq A < +\infty \quad \text{and} \quad \left( e^{2sd} \sum_{|k| \geq N} e^{-sd(\mathbf{o}, p^k(\mathbf{o}))} A \right) < 1.$$

For that choice, (17) implies that the Poincaré series associated with  $\Gamma$  converges at  $s$  and consequently :  $\delta(\Gamma) \leq s < \delta(\Gamma) + \epsilon$ .

**Remark.** Notice that the curvature of  $S$  is not asymptotically  $\frac{1}{4}$ -pinched as  $\beta > 2\alpha$ ; but, letting  $\alpha \rightarrow \beta/2$  and  $\eta \rightarrow 0$ , the metric can be chosen so that  $K_S$  is asymptotically  $(\frac{1}{4+\epsilon})$ -pinched, for any  $\epsilon > 0$ . □

## 6. Appendix

Let  $t_0, t_1, t_2, t_3$  be four real numbers satisfying  $t_0 < t_1 < t_2 < t_3$ . Denote by  $\varphi_1$  a  $C^2$  convex and decreasing function on  $[t_0, t_1]$  and  $\varphi_2$  a  $C^2$  convex

and decreasing function on  $[t_2, t_3]$ . A straightforward geometric argument on epigraphs of  $\varphi_1$  and  $\varphi_2$  shows that the following inequalities :

$$(18) \quad \varphi_1'(t_1)(t_2 - t_1) \underset{(a)}{<} \varphi_2(t_2) - \varphi_1(t_1) \underset{(b)}{<} \varphi_2'(t_2)(t_2 - t_1)$$

are necessary and sufficient for the existence of a  $C^2$  convex decreasing function  $\psi$  on  $[t_0, t_3]$  such that  $\psi|_{[t_0, t_1]} \equiv \varphi_1$  and  $\psi|_{[t_2, t_3]} \equiv \varphi_2$ .

LEMMA 6.1. *Let  $\alpha, \beta$  two positive reals such that  $\alpha < \beta$ .*

(I) *Inequalities (18) are satisfied for  $\varphi_1(t) = e^{-\alpha t}$  and  $\varphi_2(t) = e^{-\beta t}$  when  $t_2 - t_1 > \frac{1}{\alpha}$ .*

(II) *Inequalities (18) are satisfied for  $\varphi_1(t) = e^{-\beta t}$  and  $\varphi_2(t) = e^{-\alpha t}$  when  $t_2 > (\frac{\beta}{\alpha} + \epsilon)t_1$  for any  $\epsilon > 0$ .*

**Proof.** Case (I) :

$$(a) \Leftrightarrow e^{-\beta t_2 + \alpha t_1} + \alpha(t_2 - t_1) > 1$$

and the second inequality is satisfied when  $t_2 - t_1 > \frac{1}{\alpha}$ . Note that this condition is optimal if we want such an inequality to be satisfied for arbitrary large  $t_1$  because with  $u = t_2 - t_1$ , this inequality becomes

$$e^{(\alpha - \beta)t_1 - \beta u} + \alpha u > 1$$

and this inequality cant be satisfied for small  $u$  when  $t_1$  is too large.

With the previous notations,

$$(b) \Leftrightarrow e^{\beta u} e^{(\beta - \alpha)t_1} - \beta u - 1 > 0$$

and the latter inequality is always satisfied because  $e^x - x - 1 > 0$  for all  $x > 0$ .

Case (II) :

$$(a) \Leftrightarrow e^{-\alpha t_2 + \beta t_1} + \beta(t_2 - t_1) > 1$$

and this second inequality is satisfied when  $t_2 - t_1 > \frac{1}{\beta}$ . The same remark as in the case (I).

With the previous notations too,

$$(b) \Leftrightarrow e^{\alpha u} e^{(\alpha - \beta)t_1} - \alpha u - 1 > 0$$

with  $u = t_2 - t_1$ . If we set  $t_2 = (1 + x)t_1 + f(t_1)$  and substitute in the last term, a necessary condition in order to realise (b) is  $(x + 1) \geq \frac{\beta}{\alpha}$  and if we set  $(x + 1) = \frac{\beta}{\alpha}$  and replace, we get  $e^{\alpha f(t_1)} - (\beta - \alpha)t_1 - f(t_1) - 1 > 0$ . The conclusion follows. □

LEMMA 6.2. *Let  $t_0 < t_1 < t_2 < t_3$  and  $\eta > 0$ . There exists  $A = A(\eta, \alpha, \beta) > 0$  and  $B = B(\alpha, \beta) > 0$  such that if  $t_2 > A.t_1$  and  $t_0 > B$ ,*

(I) *There exists a  $C^2$  convex and decreasing function  $\psi$  on  $[t_0, t_3]$  satisfying :*

$$(C_1) \left\{ \begin{array}{l} \forall t \in [t_0, t_1], \quad \psi(t) = e^{-\alpha t} \\ \forall t \in [t_2, t_3], \quad \psi(t) = e^{-\beta t} \\ \forall t \in [t_0, t_3], \quad \alpha^2 - \eta \leq \frac{\psi''(t)}{\psi(t)} \leq \beta^2 - \eta \quad \text{and} \quad \psi(t) \geq e^{-\beta t} \end{array} \right.$$

(II) There exists a  $C^2$  convex and decreasing function  $\psi$  on  $[t_0, t_3]$  such that we have

$$(C_2) \begin{cases} \forall t \in [t_0, t_1], & \psi(t) = e^{-\beta t} \\ \forall t \in [t_2, t_3], & \psi(t) = e^{-\alpha t} \\ \forall t \in [t_0, t_3], & \alpha^2 - \eta \leq \frac{\psi''(t)}{\psi(t)} \leq \beta^2 + \eta \quad \text{and} \quad \psi(t) \geq e^{-\beta t} \end{cases}$$

**Proof.** By the previous remark, if we choose  $A > \frac{\beta}{\alpha}$  and  $B > \frac{1}{\beta - \alpha}$ , inequalities (18) are satisfied. In both cases, set

$$\psi(t) = e^{-t\varphi(t)} \quad t \in [t_0, t_3]$$

where  $\varphi$  is constant on  $[t_0, t_1]$  and  $[t_2, t_3]$  (depending in an obvious way on case I or II). Consider a  $C^2$  function  $\phi : [0, 1] \rightarrow [\alpha, \beta]$ ; set  $s = \lambda(t - t_1)$  where  $\lambda = \frac{1}{t_2 - t_1}$  and put  $\varphi(t) = \phi(s)$  for  $t \in [t_1, t_2]$ . A straightforward calculus gives, for  $s \in [0, 1]$  :

$$\begin{aligned} \frac{\psi''(t)}{\psi(t)} &= ((s\phi(s))' + \lambda t_1 \phi'(s))^2 - \lambda(2\phi'(s) + (s + \lambda t_1)\phi''(s)) \\ &= (k'(s))^2 + \lambda t_1(2k'(s)\phi'(s) + \lambda(t_1(\phi'(s))^2 - \phi''(s))) - \lambda(2\phi'(s) + s\phi''(s)) \\ &= (k'(s))^2 + \theta(\lambda) \end{aligned}$$

where  $k(s) := s\phi(s)$  and  $\theta$  is a function such that  $\theta(\lambda) \rightarrow 0$  when  $\lambda \rightarrow 0$ .

Set  $M_i = \sup_{s \in [0, 1]} |\phi^{(i)}(s)|$  for  $i = 1, 2$  (which depend only on  $(\alpha, \beta)$ ) and  $C = \frac{1}{8(\beta+1)(M_1+M_2+\beta)}$ . The previous equalities imply

$$(19) \quad (k'(s))^2 - \frac{\eta}{2} \leq \frac{\psi''(t)}{\psi(t)} \leq (k'(s))^2 + \frac{\eta}{2}$$

when  $\lambda t_1 < C\eta$  i.e. for  $t_2 > (1 + \frac{1}{C\eta})t_1 := A.t_1$ . We show in both cases that we can choose a  $C^2$  function  $\phi$  with values in  $[\alpha, \beta]$  such that for all  $s \in [0, 1]$  :

$$(20) \quad \alpha - \frac{\eta}{4} \leq k'(s) \leq \beta + \frac{\eta}{4}.$$

Case (I) : choose  $\phi : [0, 1] \rightarrow [\alpha, \beta]$  non decreasing satisfying  $\phi(0) = \alpha$ ,  $\phi(1) = \beta$  and  $\phi'(0) = \phi'(1) = \phi''(0) = \phi''(1) = 0$ . Then, the function  $\varphi$  can be extend on  $[t_0, t_3]$  in a  $C^2$  manner and on  $[0, 1]$ , we have  $k'(s) = (s\phi(s))' = \phi(s) + s\phi'(s) \geq \alpha$  and  $\phi(s) \leq \beta$  so that  $\psi''/\psi \geq \alpha^2 - \eta$  and  $\psi(t) \geq e^{-\beta t}$  are both satisfied on  $[t_0, t_3]$ . It implies in particular that the function  $\psi$  constructed is convex on  $[t_0, t_3]$ . Note that in this case, the inequality  $\lambda.t_1 < C\eta$  must be satisfied, for, in the second expression of  $\frac{\psi''}{\psi}$ , the term  $(t_1(\phi'(s))^2 - \phi''(s))$  is negative in the neighborhood of  $s_0 = \inf\{s; \phi'(s) = 0\}$ .

It is left to show that  $\phi$  or equivalently  $k$  can be chosen so that  $k'(s) = \phi(s) + s\phi'(s) \leq \beta + \frac{\eta}{4}$ . The boundary conditions for  $\phi$  up to the first order translate to  $k(0) = 0$ ,  $k(1) = \beta$ ,  $k'(0) = \alpha$  and  $k'(1) = \beta$ . For  $\epsilon_1 \in ]0, 1[$ , consider the  $C^0$ -piecewise affine function  $\bar{k}$  defined on  $[0, \epsilon_1]$  by  $\bar{k}(t) = \alpha.t$ , on  $[1 - \epsilon_1, 1]$  by  $\bar{k}(t) = \beta.t$  and affine on  $[\epsilon_1, 1 - \epsilon_1]$ . If we choose  $\epsilon_1$  small enough (depending on  $\eta$  and  $\alpha$ ), we can smooth  $\bar{k}$  to obtain a  $C^2$  function  $k$  on  $[0, 1]$  in such a way that the derivative  $k'$  satisfies

$$\begin{cases} k'(s) = \alpha & s \in [0, \epsilon_1/2] \\ k'(s) \leq \beta + \eta/(4\beta) & s \in [\epsilon_1/2, 1 - \epsilon_1/2] \\ k'(s) = \beta & s \in [1 - \epsilon_1/2, 1] \end{cases}$$

so that  $(k'(s))^2 \leq \beta^2 - \eta/2$ .

Case **(II)** : this case is similar. We choose  $\phi : [0, 1] \rightarrow [\alpha, \beta]$  non increasing satisfying  $\phi(0) = \beta$ ,  $\phi(1) = \alpha$  and  $\phi'(0) = \phi'(1) = \phi''(0) = \phi''(1) = 0$ , or equivalently (up to the first order), we choose  $k(s) = s\phi(s)$  satisfying  $k(0) = 0$ ,  $k(1) = \int_0^1 k'(s)ds = \alpha$ ,  $k'(0) = \beta$  and  $k'(1) = \alpha$ . The construction is symmetric to the previous one. In both cases, the desired inequalities : (20), (19) and  $e^{-\alpha t} \leq \psi(t) \leq e^{-\alpha t}$  are satisfied.  $\square$

Let us now construct the sequences of intervals  $[p_n, q_n]$ ,  $[r_n, s_n]$  and the function  $\mathcal{A}$  we used in Section 4. Let  $A > 1$  and  $B > 0$  given by Lemma 6.2. We set

$$\begin{cases} p_n = (1 - \lambda_0)\Delta^{n-1} + \lambda_0\Delta^n & \text{and } r_n = \frac{p_n + \Delta^n}{2} \\ q_n = (1 - \mu_0)\Delta^{n-1} + \mu_0\Delta^n & \text{and } s_n = \frac{q_n + \Delta^n}{2} \end{cases}$$

for  $\Delta, \lambda_0$  and  $\mu_0$  to be defined.

Fix  $(\lambda_0, \mu_0)$  in the (nonempty) set  $(]0, 1[ \cap \{(\lambda, \mu) ; 1 + \lambda - 2A\mu > 0 \wedge \mu > \lambda\})$ . The polynomial function  $P(x) = 2\lambda_0x^2 + ((2-A) - 2\lambda_0 - A\mu_0)x - A(1 - \mu_0)$  tends to infinity as  $x \rightarrow +\infty$ ; thus, we can choose a positive real number  $\Delta$  such that both inequalities

$$(21) \quad \Delta > \frac{2A - 1 + \lambda_0 - 2A\mu_0}{1 + \lambda_0 - 2A\mu_0}$$

$$(22) \quad P(q_0) > 0$$

are satisfied.

Inequality (21) insures that  $r_n > Aq_n$  and inequality (22) insures that  $p_{n+1} > As_n$ . By Lemma 6.2, there exists  $n_0 \in \mathbb{N}^*$  and a  $C^2$ -convex and decreasing function  $\mathcal{A}$  on  $[\Delta^{n_0-1}, +\infty[$  satisfying  $\frac{\mathcal{A}''(t)}{\mathcal{A}(t)} \geq \alpha^2 - \eta$  and  $\mathcal{A}(t) \geq e^{-\beta t}$  for all  $t \in [\Delta^{n_0-1}, +\infty[$ , and such that for  $n \geq n_0$ , we have :

$$\begin{cases} \mathcal{A}(t) = e^{-\alpha t} & \forall t \in [p_n, q_n] \\ \mathcal{A}(t) = e^{-\beta t} & \forall t \in [r_n, s_n]. \end{cases}$$

Note that by construction  $t \in [p_n, q_n] \Leftrightarrow \frac{t+R_n}{2} \in [r_n, s_n]$ .



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