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Modes and quasi-modes on surfaces: variation on an idea of Andrew Hassell

Yves Colin de Verdière*

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1 Introduction

This paper is inspired from the nice idea of A. Hassell in [5]. From the classical paper of V. Arnol'd [1], we know that quasi-modes are not always close to exact modes. We will show that, for *almost all* Riemannian metrics on closed surfaces with an elliptic generic closed geodesic γ , there exists exact modes located on γ . Similar problems in the integrable case are discussed in several papers of J. Toth and S. Zelditch (see [8]).

2 Quasi-modes associated to an elliptic generic closed geodesic

2.1 Babich-Lazutkin and Ralston quasi-modes

Definition 2.1 *A periodic geodesic γ on a Riemannian surface (X, g) is said to be elliptic generic if the eigenvalues of the linearized Poincaré map of γ are of modulus 1 and are not roots of the unity.*

Theorem 2.1 (Babich-Lazutkin [2], Ralston [6, 7]) *If γ is an elliptic generic closed geodesic of period $T > 0$ on a closed Riemannian surface (X, g) , there exists a sequence of quasi-modes $(u_m)_{m \in \mathbb{N}}$ of $L^2(X, dx_g)$ norm equal to 1 which satisfies*

- $\|(\Delta_g - \lambda_m)u_m\|_{L^2(X, dx_g)} = O(m^{-\infty})$
- *There exists α so that¹*

$$\lambda_m = \left(\frac{2\pi m + \alpha}{T}\right)^2 + O(1)$$

- *For any compact K disjoint of γ , $\int_K |u_m|^2 = O(m^{-\infty})$.*

Corollary 2.1 *There exists a sub-sequence $(\mu_{j_m})_{m \in \mathbb{N}}$ of the spectrum $(\mu_j)_{j \in \mathbb{N}}$ of the Laplace operator so that $\mu_{j_m} = \lambda_m + O(m^{-\infty})$.*

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¹ α is given by $\alpha = (m_1 + \frac{1}{2})\theta + p\pi$ where $m_1 \in \mathbb{N}$ is a "transverse" quantum number, the linearized Poincaré map is a rotation of angle θ ($0 < \theta < 2\pi$) and $p = 0$ or 1 is a "Maslov index" of γ

3 Modes and quasi-modes following Arnol'd

Arnol'd [1] has observed that, given a quasi-mode $(u_m)_{m \in \mathbb{N}}$, there do not always exist a sequence $(\varphi_{j_m})_{m \in \mathbb{N}}$ of exact modes close to the quasi-mode $(u_m)_{m \in \mathbb{N}}$. His example is given by a planar domain with a symmetry of order 3.

A simpler example is given by a symmetric double well: let us give $V : \mathbb{R} \rightarrow [0, +\infty[$ a smooth even function with

- $\lim_{x \rightarrow \infty} V(x) = +\infty$
- $V^{-1}(0) = \{-a, a\}$ with $a > 0$
- $V(0) = b > 0$

. If $\hat{H} = -\hbar^2 d_x^2 + V(x)$ is the semi-classical Schrödinger operator, there exist quasi-modes located in the well $V := \{x \mid x > 0 \text{ and } V(x) < b\}$. The exact eigenfunctions are even or odd and hence are not localized in a single well.

The previous examples are in some sense *non generic*. They involve some symmetry of the operator.

4 The main result

Theorem 4.1 *Let us give a closed Riemannian surface (X, g_0) and a smooth non-zero function $f \geq 0$. Let us define the metric $g_t := \exp(-t.f)g_0$. Let us assume that there exist some intervals $I_m = [\lambda_m - l_m, \lambda_m + l_m]$, ($m \in \mathbb{N}$), **independent of t** , so that, for any $t \in [0, 1]$, there exists at least one eigenvalue of Δ_t inside I_m . Assume that $\lambda_m \rightarrow +\infty$ and $\sum_{m=1}^{\infty} l_m < \infty$. Choose a sequence $q_m \rightarrow 0$ so that $\sum_{m=1}^{\infty} l_m/q_m < \infty$.*

Then, for almost all $t \in [0, 1]$, for any sequence of exact modes $\varphi_m(t)$ of eigenvalues $\mu_m(t)$ with $\mu_m(t) \in I_m$, we have $\int_X f |\varphi_m(t)|^2 dx_t = o(q_m)$.

In particular, if $\Gamma = \text{support}(f)$, $\varphi_m(t) \rightarrow 0$ in $L_{\text{loc}}^2(X \setminus \Gamma)$.

Remark 4.1 *In applications, the interval I_m is provided from quasi-modes located in the support of f : if u_m is a quasi-mode for each values of t with*

$$\|(\Delta_t - \lambda_m)u_m\|_{L^2(X, dx_0)} \leq C_m \|u_m\|_{L^2(X, dx_0)}$$

with λ_m independent of t , we can take $l_m = cC_m$ with c large enough, depending only on bounds of f .

Remark 4.2 *The quasi-mode is only used in order to find a sequence of intervals I_m which contains at least one eigenvalue of Δ_t and is independent of t .*

Remark 4.3 *It works with (u_m) the quasi-modes of Theorem 2.1 with $\Gamma = \gamma$ an elliptic generic closed geodesic and f flat on γ , because the functions u_m satisfies an estimates*

$$u_m(x) = O\left(e^{-cd^2(x, \gamma)/\sqrt{\lambda_m}}\right).$$

We can then take $q_m = O(m^{-\infty})$.

We are unfortunately unable to prove that the modes φ_m are close to linear combinations of the quasi-modes given in Theorem 2.1 in the interval I_m .

The precise statement is

Corollary 4.1 *With the notations of Section 2, there exists a sequence $0 < l_m = o(m^{-\infty})$ so that, for any $t \in [0, 1]$, $\text{Spectrum}(\Delta_t) \cap [\lambda_m - l_m, \lambda_m + l_m] \neq \emptyset$ and a*

subset $Z \subset [0, 1]$ of measure 1, so that, for any sequence $\mu_{j_m}(t) \in [\lambda_m - l_m, \lambda_m + l_m]$ and for any $t \in Z$,

$$\int_X f |\varphi_{j_m}(t)|^2 dx_0 = o(m^{-\infty}) .$$

Moreover, for any compact $K \subset X$ with $K \cap \gamma = \emptyset$ and for any $k \in \mathbb{N}$, we have

$$\|\varphi_{j_m}(t)\|_{C^k(K)} = o(m^{-\infty}) .$$

Proof.–

The first part is a direct application of Theorem 4.1.

The second part comes from the Sobolev embeddings and the equations $\Delta_t^N \varphi_{j_m}(t) = \mu_{j_m}(t)^N \varphi_{j_m}(t)$ with $\mu_{j_m}(t) = o(m^2)$.

□

Remark 4.4 If we have only $l_m \rightarrow 0$, one can apply the previous result by taking first a sub-sequence m_k so that $\sum l_{m_k} < \infty$ and choosing then $q_{m_k} \rightarrow 0$. This does not work with $l_m = O(1)$ as in the paper [5].

5 Variation of the eigenvalues

With $g_t = e^{-tf} g_0$, we define $dx_t = e^{-tf} dx_0$ the Riemannian area of g_t and $\Delta_t = e^{tf} \Delta_0$ the Laplace operator. Let us denote by

$$\mu_1(t) = 0 < \mu_2(t) \leq \dots \leq \mu_j(t) \leq \dots$$

the eigenvalues of Δ_t and by $(\varphi_j(t))_{j \in \mathbb{N}}$ an associated orthonormal eigenbasis.

Lemma 5.1 • $\mu_j(t)$ is a continuous increasing function of t

- $\mu_j(t)$ is piecewise analytic and, at any regular point, the t -derivative of $\mu_j(t)$ is given by:

$$\dot{\mu}_j(t) = \mu_j(t) \int_X f \varphi_j(t)^2 dx_t . \quad (1)$$

Proof.–

- The Rayleigh quotient $R_t(\varphi)$ is given by

$$R_t(\varphi) = \int_X \|d\varphi\|_{g_0}^2 dx_0 / \int_X e^{-tf} \varphi^2 dx_0$$

which is an increasing function of t . Applying the min-max characterization of the eigenvalues, we get their monotonicity.

- Because Δ_t is an analytic function of t , we know that $\mu_j(t)$ is continuous and piecewise smooth as well as $\varphi_j(t)$. We can then compute formally the derivative of the eigenfunction's equation

$$e^{tf} \Delta_0 \varphi_j(t) = \mu_j(t) \varphi_j(t) ,$$

and get

$$f \Delta_t \varphi_j(t) + \Delta_t \dot{\varphi}_j(t) = \dot{\mu}_j(t) \varphi_j(t) + \mu_j(t) \dot{\varphi}_j(t) ,$$

and taking the t -scalar product with $\varphi_j(t)$, we get Equation (1).

□

6 The proof

The proof is an adaptation of the argument of [5]. Let us denote by $I_m = [\lambda_m - l_m, \lambda_m + l_m]$. From the Weyl law and the monotonicity of the μ_j 's, we deduce the

Lemma 6.1 *For any $t \in [0, 1]$, $\#\{j \mid \mu_j(t) \in I_m\} = O(\lambda_m)$ uniformly in t .*

In fact,

$$\#\{j \mid \mu_j(t) \in I_m\} \leq \#\{j \mid \mu_j(t) \leq \lambda_m + l_m\} \leq \#\{j \mid \mu_j(0) \leq \lambda_m + l_m\} !$$

We will also need the elementary

Lemma 6.2 *Let $F : [0, 1] \rightarrow \mathbb{R}$ be an increasing, continuous and piecewise C^1 function. Let us give a Borel set $Y \subset [0, 1]$ and I a compact interval of \mathbb{R} so so that $F'(t) \geq m > 0$ for almost all $t \in K = F^{-1}(I) \cap Y$. Then the Lebesgue measure $|K|$ of K satisfies $|K| \leq |I|/m$.*

Let us denote by

$$Z := \left\{ t \in [0, 1] \mid \lim_{m \rightarrow \infty} q_m^{-1} \left(\sup_{\mu_j(t) \in I_m} \int_X f |\varphi_j(t)|^2 dx_t \right) = 0 \right\},$$

(Z is well defined because there exists at least one $\mu_j(t) \in I_m$ for each m) and $Y = [0, 1] \setminus Z$. Let us denote also, for $\varepsilon > 0$, by

$$Y_\varepsilon^m := \left\{ t \mid \exists j \text{ with } \mu_j(t) \in I_m, \int_X f |\varphi_j(t)|^2 dx_t \geq \varepsilon q_m \right\}.$$

Using Lemma 6.1, the monotonicity of $t \rightarrow \mu_j(t)$ and the lower bound $\mu_j(t) \geq \mu_j(t) \varepsilon q_m$ in Lemma 6.2, we have

$$|Y_\varepsilon^m| \leq C \lambda_m \frac{|I_m|}{\varepsilon q_m (\lambda_m - C l_m)} = O\left(\frac{l_m}{\varepsilon q_m}\right).$$

Let us give a sequence $\varepsilon_m \rightarrow 0$, then, for any m_0 , $Y \subset \cup_{m \geq m_0} Y_{\varepsilon_m}^m$. But this implies that $|Y|$ is arbitrarily close to 0 by choosing ε_m so that $\sum_m l_m / \varepsilon_m q_m < \infty$. This proves that $|Z| = 1$ and the Theorem.

7 Null sets in Banach spaces

It is not clear what is a set of measure 0 in a infinite dimensional Banach space because there is no ‘‘Lebesgue measure’’ on it. There are several definitions of sets of measure 0 in a *separable Banach space* B . For Borel sets, it is shown in [4] that the notions of *cube* null sets and *Gaussian* null sets coincide.

Definition 7.1 • *A cube measure in B is defined as the distribution of a random variable $\sum_{i \in \mathbb{N}} t_i e_i$ where $\mathbf{t} = (t_i) \in [0, 1]^{\mathbb{N}}$ with the Lebesgue measure and the sequence $(e_i)_{i \in \mathbb{N}}$ span a dense subspace of B with $\sum_{i \in \mathbb{N}} \|e_i\| < \infty$.*

- *A cube null set is a Borel subset of B which is of measure 0 for every cube measure.*
- *A Gaussian measure on B is a Borel measure whose image by any continuous linear form on B is a (non-degenerate) Gaussian measure on \mathbb{R} (i.e. of the form $dm = A \cdot \exp(-(x - a)^2/b) dx$ with $A > 0$).*

- A Gaussian null set is a Borel set which is of measure 0 for every Gaussian measure.

It is proved in [4] that cube null sets and Gaussian null sets coincide in every separable Banach space.

We have the:

Lemma 7.1 *Let B be a separable Banach space and $C \subset B$ be a non empty open cone. Let us give a Borel set $Z \subset B$ so that, for any $x \in B$, $y \in C$,*

$$|\{t \mid x + ty \in Z\}| = 0 .$$

Then Z is a cube null and Gaussian null set.

Proof.-

Let us show that Z is of measure 0 for every cube measure given from sequence $(e_i)_{i \in \mathbb{N}}$. There exists $k \in \mathbb{N}$ and $(t_1, \dots, t_k) \in [0, 1]^k$ so that $e = \sum_{i=1}^k t_i e_i \in C$. Let us rewrite the Lebesgue measure on $[0, 1]^k$ as

$$\int_{[0,1]^k} f(t) dt = \int_X d\mu(d) \int_{d \cap [0,1]^k} f(t) ds \quad (2)$$

where X is the set of lines parallel to e cutting $[0, 1]^k$ and ds is the Lebesgue measure on the line d . Let us denote $t = (t', t'') \in [0, 1]^k \times [0, 1]^{\mathbb{N}}$ and denote by

$$Z_{t''} := \{t' \mid x + \sum_{i=1}^k t'_i e_i + \sum_{j>k} t''_j e_j \in Z\} .$$

Equation (2) shows that $Z_{t''}$ is of measure 0. We can then use Fubini Theorem on $[0, 1]^k \times [0, 1]^{\mathbb{N} \setminus \{1, \dots, k\}}$ in order to finish the proof.

□

8 From Theorem 4.1 to almost all metrics

We will apply the previous result to the following situation where (X, g_0) is our smooth closed surface and γ a closed geodesic; let us choose N large (and even) and define B as follows:

$$B = \{f \in C^N(X, \mathbb{R}) \mid \forall \alpha \text{ with } |\alpha| \leq N, D^\alpha f \text{ vanishes on } \gamma\}$$

and C the open cone of functions of B which satisfy

$$\exists c > 0 \text{ such that } f(x) \geq cd(x, \gamma)^N$$

with d the distance associated to g_0 .

Then Theorem 4.1 can be reformulated with *almost all metrics $e^f g_0$ with $f \in B$* instead of *almost all $t \in [0, 1]$* . Of course, we can only take l_m of the order of $m^{-N'}$, where N' depends on N .

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