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# A priori $L^\infty$ -estimates for degenerate complex Monge-Ampère equations

P. Eyssidieux, V. Guedj and A. Zeriahi

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**Abstract :** We study families of complex Monge-Ampère equations, focusing on the case where the cohomology classes degenerate to a non big class. We establish uniform a priori  $L^\infty$ -estimates for the normalized solutions, generalizing the recent work of S. Kolodziej and G. Tian. This has interesting consequences in the study of the Kähler-Ricci flow.

## 1 Introduction

Let  $\pi : X \rightarrow Y$  be a non degenerate holomorphic mapping between compact Kähler manifolds such that  $n := \dim_{\mathbb{C}} X \geq m := \dim_{\mathbb{C}} Y$ . Let  $\omega_X, \omega_Y$  Kähler forms on  $X$  and  $Y$  respectively. Let  $F : X \rightarrow \mathbb{R}^+$  be a non negative function such that  $F \in L^p(X)$  for some  $p > 1$ .

Set  $\omega_t := \pi^*(\omega_Y) + t\omega_X$ ,  $t > 0$ . We consider the following family of complex Monge-Ampère equations

$$(\star)_t \quad \begin{cases} (\omega_t + dd^c \varphi_t)^n & = c_t t^{n-m} F \omega_X^n \\ \max_X \varphi_t = 0 & = 0 \end{cases}$$

where  $\varphi_t$  is  $\omega_t$ -plurisubharmonic on  $X$  and  $c_t > 0$  is a constant given by

$$c_t t^{n-m} \int_X F \omega_X^n = \int_X \omega_t^n.$$

It follows from the seminal work of S.T. Yau [Y] and S. Kolodziej [K 1], [K 2] that the equation  $(\star)_t$  admits a unique continuous solution. (Observe that for  $t \in ]0, 1]$ ,  $\omega_t$  is a Kähler form).

Our aim here is to understand what happens when  $t \rightarrow 0^+$ , motivated by recent geometrical developpments [ST], [KT]. When  $n = m$ , the cohomology class  $\omega_0$  is big and semi-ample and this problem has been adressed by several authors recently (see [CN], [EGZ], [TZ], [To]).

We focus here on the case  $m < n$ . This situation is motivated by the study of the Kähler-Ricci flow on manifolds  $X$  of intermediate Kodaira dimension  $1 \leq \text{cod}(X) \leq n - 1$ . When  $n = 2$  this has been studied by J.Song and G.Tian [ST].

In a very recent and interesting paper [KT], S. Kolodziej and G. Tian were able to show, under a technical geometric assumption on the fibration  $\pi$ , that the solutions  $(\varphi_t)$  are uniformly bounded on  $X$  when  $t \searrow 0^+$ .

The purpose of this note is to (re)prove this result without any technical assumption and with a different method: we actually follow the strategy introduced by S. Kolodziej in [K] and further developed in [EGZ], [BGZ].

**THEOREM.** *There exists a uniform constant  $M = M(\pi, \|F\|_p) > 0$  such that the solutions to the Monge-Ampère equations  $(\star)_t$  satisfy*

$$\|\varphi_t\|_{L^\infty(X)} \leq M, \quad \forall t \in ]0, 1].$$

It follows from our result that Theorems 1 and 2 in [KT] hold without any technical assumption on the fibration (see condition 0.2 in [KT]).

This result has been announced by J-P. Demailly and N. Pali [DP].

## 2 Proof of the theorem

### 2.1 Preliminary remarks

**Uniform control of  $c_t$ .** Observe that  $\omega_0^k = 0$  for  $m < k \leq n$ , hence for all  $t \in ]0, 1]$ ,

$$\omega_t^n = \sum_{k=1}^m \binom{n}{k} t^{n-k} \omega_0^k \wedge \omega_X^{n-k}.$$

Note that  $]0, 1] \ni t \mapsto t^{m-n} \omega_t^n$  is increasing (hence decreases as  $t \searrow 0^+$ ) and satisfies for  $t \in ]0, 1]$

$$(1) \quad \binom{n}{m} \frac{\omega_0^m \wedge \omega_X^{n-m}}{\int_X \omega_0^m \wedge \omega_X^{n-m}} \leq \frac{\omega_t^n}{t^{n-m} \int_X \omega_0^m \wedge \omega_X^{n-m}} \leq \frac{\omega_1^n}{\int_X \omega_0^m \wedge \omega_X^{n-m}}.$$

In particular  $t \mapsto c_t$  is increasing in  $t \in ]0, 1]$  and

$$0 < \binom{n}{m} \frac{\int_X \omega_0^m \wedge \omega_X^{n-m}}{\int_X F \omega_X^n} =: c_0 \leq c_t \leq c_1.$$

**Uniform control of densities.** Let  $J_\pi$  denote the (modulus square) of the Jacobian of the mapping  $\pi$ , defined through

$$\omega_0^m \wedge \omega_X^{n-m} = J_\pi \omega_X^n.$$

Let us rewrite the equation  $(\star)_t$  as follows

$$(\omega_t + dd^c \varphi_t)^n = f_t \omega_t^n,$$

where for  $t \in ]0, 1]$

$$0 \leq f_t := c_t t^{n-m} F \frac{\omega_X^n}{\omega_t^n} \leq c_1 \frac{F}{J_\pi}.$$

Observe that

$$\int_X f_t \omega_t^n = c_t t^{n-m} \int_X F \omega_t^n = \int_X \omega_t^n =: Vol_{\omega_t}(X),$$

hence  $(f_t)$  is uniformly bounded in  $L^1(\omega_t/V_t)$ ,  $V_t := Vol_{\omega_t}(X)$ . We actually need a slightly stronger information.

**Lemma 2.1** *There exists  $p' > 1$  and a constant  $C = C(\pi, \|F\|_{L^p(X)}) > 0$  such that for all  $t \in ]0, 1]$*

$$\int_X f_t^{p'} \omega_t^n \leq C Vol_{\omega_t}(X).$$

**Proof of the lemma.** Set  $V_t := Vol_{\omega_t} = \int_X \omega_t^n$  and observe that

$$0 \leq f_t \frac{\omega_t^n}{V_t} \leq c_1 F \frac{\omega_X^n}{\int_X \omega_0^m \wedge \omega_X^{n-m}} = C_2 F \omega_X^n,$$

where  $C_2 := c_1 \int_X J_\pi \omega_X^n$ .

This shows that the densities  $f_t$  are uniformly in  $L^1$  w.r.t. the normalized volume forms  $\omega_t^n/V_t$ .

Since  $J_\pi$  is locally given as the square of the modulus of a holomorphic function which does not vanish identically, there exists  $\alpha \in ]0, 1[$  such that  $J_\pi^{-\alpha} \in L^1(X)$ . Fix  $\beta \in ]0, \alpha[$  satisfying the condition  $\beta/p + \beta/\alpha = 1$ . It follows from Hölder's inequality that

$$\int_X f_t^\beta \omega_X^n \leq \left( \int_X F^p \omega_X^n \right)^{\beta/p} \left( \int_X J_\pi^{-\alpha} \omega_X^n \right)^{\beta/\alpha}.$$

Setting  $\varepsilon := \beta/q$  and using Hölder's inequality again, we obtain

$$\int_X f_t^{1+\varepsilon} \frac{\omega_t^n}{V_t} \leq C_2 \int_X f_t^\varepsilon F \omega_X^n.$$

Now applying again Hölder inequality we get

$$\int_X f_t^{1+\varepsilon} \frac{\omega_t^n}{V_t} \leq C_2 \left( \int_X f_t^\beta \omega_X^n \right)^{1/q} \|F\|_{L^p(X)}.$$

Therefore denoting by  $p' := 1 + \varepsilon$ , we have the following uniform estimate

$$\int_X f_t^{p'} \frac{\omega_t^n}{V_t} \leq C(\pi, \|F\|_{L^p(X)}), \forall t \in ]0, 1],$$

where

$$C(\pi, \|F\|_{L^p(X)}) := C_2 \left( \int_X J_\pi^{-\alpha} \omega_X^n \right)^{\beta/\alpha q} \|F\|_{L^p(X)}^{1+\beta/q}.$$

►

## 2.2 Uniform domination by capacity

We now show that the measure  $\mu_t := f_t \omega_t^n / Vol_{\omega_t}$  are uniformly strongly dominated by the normalized capacity  $Cap_{\omega_t} / Vol_{\omega_t}(X)$ . It actually follows from a carefull reading of the no parameter proof given in [EGZ], [BGZ].

**Lemma 2.2** *There exists a constant  $C_0 = C_0(\pi, \|F\|_{L^p(\omega_X^n)}) > 0$  such that for any compact set  $K \subset X$  and  $t \in ]0, 1]$ ,*

$$\mu_t(K) \leq C_0^n \left( \frac{Cap_{\omega_t}(K)}{Vol_{\omega_t}(X)} \right)^2.$$

Proof: Fix a compact set  $K \subset X$ . Set  $V_t := Vol_{\omega_t}(X)$ . Hölder's inequality yields

$$\mu_t(K) \leq \left( \int_X f_t^{p'} \frac{\omega_t^n}{V_t} \right)^{1/p'} \left( \int_K \frac{\omega_t^n}{V_t} \right)^{1/q'}.$$

It remains to dominate uniformly the normalized volume forms  $\omega_t^n / V_t$  by the normalized capacities  $Cap_{\omega_t} / V_t$ . Fix  $\sigma > 0$  and observe that for any  $t \in ]0, 1]$ ,

$$\int_K \frac{\omega_t^n}{V_t} \leq \int_X e^{-\sigma(V_{K,\omega_t} - \max_X V_{K,\omega_t})} \frac{\omega_t^n}{V_t} T_{\omega_t}(K)^\sigma,$$

where

$$V_{K,\omega_t} := \sup\{\psi \in PSH(X, \omega_t); \psi \leq 0, \text{ on } K\}$$

is the  $\omega_t$ -extremal function of  $K$  and  $T_{\omega_t}(K) := \exp(-\sup_X V_{K,\omega_t})$  is the associated  $\omega_t$ -capacity of  $K$  (see [GZ 1] for their properties).

Observe that  $\omega_t^n / V_t \leq c_1 \omega_1^n$  and  $\omega_t \leq \omega_1$ , hence the family of functions  $V_{K,\omega_t} - \max_X V_{K,\omega_t}$  is a normalized family of  $\omega_1$ -psh functions. Thus there exists  $\sigma > 0$  which depends only on  $(X, \omega_1)$  and a constant  $B = B(\sigma, X, \omega_1)$  such that ([Z])

$$\int_X e^{-\sigma(V_{K,\omega_t} - \max_X V_{K,\omega_t})} \frac{\omega_t^n}{V_t} \leq B, \forall t \in ]0, 1].$$

The Alexander-Taylor comparison theorem (see Theorem 7.1 in [GZ 1]) now yields for a constant  $C_3 = C_3(\pi, \|F\|_{L^p(X)})$

$$\mu_t(K) \leq C_3 \exp \left[ -\sigma \left( \frac{V_t}{\text{Cap}_{\omega_t}(K)} \right)^{1/n} \right], \forall t \in ]0, 1].$$

We infer that there is a constant  $C_4 = C_4(\pi, \|F\|_{L^p(X)})$  such that

$$(2) \quad \mu_t(K) \leq C_4 \left( \frac{\text{Cap}_{\omega_t}(K)}{V_t} \right)^2, \forall t \in ]0, 1].$$

### 2.3 Uniform normalization

The comparison principle (see [K], [EGZ]) yields for any  $s > 0$  and  $\tau \in [0, 1]$

$$\tau^n \frac{\text{Cap}_{\omega_t}(\{\varphi_t \leq -s - \tau\})}{V_t} \leq \int_{\{\varphi_t \leq -s\}} \frac{(\omega_t + dd^c \varphi_t)^n}{V_t}.$$

It is now an exercise to derive from this inequality an a priori  $L^\infty$ -estimate,

$$\|\varphi_t\|_{L^\infty(X)} \leq C_5 + s_0(\omega_t),$$

where  $s_0(\omega_t)$  (see [EGZ], [BGZ]) is the smallest number  $s > 0$  satisfying the condition  $e^n C_0^n \text{Cap}_{\omega_t}(\{\psi \leq -s\})/V_t \leq 1$  for all  $\psi \in PSH(X, \omega_t)$  such that  $\sup_X \psi = 0$ . Recall from ([GZ 1], Prop. 3.6) that

$$\frac{\text{Cap}_{\omega_t}(\{\psi \leq -s - \tau\})}{V_t} \leq \frac{1}{s} \left( \int_X (-\psi) \frac{\omega_t^n}{V_t} + n \right).$$

Since  $\frac{\omega_t^n}{V_t} \leq C_1 \omega_1^n$ , it follows that

$$\frac{\text{Cap}_{\omega_t}(\{\psi \leq -s - \tau\})}{V_t} \leq \frac{1}{s} \left( C_1 \int_X (-\psi) \omega_1^n + n \right).$$

Since  $\psi$  is  $\omega_1$ -psh and normalized, we know that there is a constant  $A = A(X, \omega_1) > 0$  such that  $C_1 \int_X (-\psi) \omega_1^n \leq A$  for any such  $\psi$ . Therefore  $s_0(\omega_t) \leq s_0 := e^n C_0^n (A + n)$  for any  $t \in ]0, 1]$ . Finally we obtain the required uniform estimate for all  $t \in ]0, 1]$ .

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