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# Asymptotic behavior of two-phase flows in heterogeneous porous media for capillarity depending only on space.

## I. Convergence to an entropy solution.

Clément Cancès<sup>\*,†</sup>

September 21, 2009

### Abstract

We consider an immiscible two-phase flow in a heterogeneous one-dimensional porous medium. We suppose particularly that the capillary pressure field is discontinuous with respect to the space variable. The dependence of the capillary pressure with respect to the oil saturation is supposed to be weak, at least for saturations which are not too close to 0 or 1. We study the asymptotic behavior when the capillary pressure tends to a function which does not depend on the saturation. In this paper, we show that if the capillary forces at the spacial discontinuities are oriented in the same direction that the gravity forces, or if the two phases move in the same direction, then the saturation profile with capillary diffusion converges toward the unique entropy solution to the hyperbolic scalar conservation law with discontinuous flux functions.

**key words.** *entropy solution, scalar conservation law, discontinuous porous media, capillarity*

**AMS subject classification.** *35L65, 76S05*

## 1 Presentation of the problem

The resolution of multi-phase flows in porous media are widely used in oil engineering to predict the motions of oil in subsoil. Their mathematical study is however difficult, then some physical assumptions have to be done, in order to get simpler problems (see e.g. [4, 9, 20]). A classical simplified model, so-called *dead-oil* approximation, consists in assuming that there is no gas, i.e. that the fluid is composed of two immiscible and incompressible phases and in neglecting all the different chemical species. The oil-phase and the water-phase are then both made of only one component.

### 1.1 the dead-oil problem in the one dimensional case

Suppose that  $\mathbb{R}$  represents a one dimensional homogeneous porous medium, with porosity  $\phi$  (which is supposed to be constant for the sake of simplicity). If  $u$  denotes the saturation of the water phase, and so  $(1 - u)$  the saturation of the oil phase thanks to the dead-oil approximation, writing the volume conservation of each phase leads to:

$$\phi \partial_t u + \partial_x V_w = 0, \quad (1)$$

$$-\phi \partial_t u + \partial_x V_o = 0, \quad (2)$$

where  $V_o$  (resp.  $V_w$ ) is the filtration speed of the oil phase (resp. water phase). Using the empirical diphasic Darcy law, we claim that

$$V_\beta = -K \frac{k_{r,\beta}(u)}{\mu_\beta} (\partial_x P_\beta - \rho_\beta g), \quad \beta = o, w, \quad (3)$$

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where  $K$  is the global permeability, only depending on the porous media,  $\mu_\beta, P_\beta, \rho_\beta$  are respectively the dynamical viscosity, the pressure and the density of the phase  $\beta$ ,  $g$  represents the effect of gravity,  $k_{r,\beta}$  denotes the relative permeability of the phase  $\beta$ . This last term comes from the interference of the two phases in the porous media.

There exists  $s_\star \in [0, 1)$  such that the function  $k_{r,w}$  is non-decreasing, with  $k_{r,w}(u) = 0$  if  $0 \leq u \leq s_\star < 1$ , and  $k_{r,w}$  is increasing on  $[s_\star, 1]$ . The function  $k_{r,o}$  is supposed to be non-increasing, with  $k_{r,o}(1) = 0$ . We suppose that there exists  $s^\star \in (s_\star, 1]$  such that  $k_{r,o}(s) = 0$  for  $s \in [s^\star, 1)$ , and  $k_{r,o}$  is decreasing on  $[0, s^\star)$ .

The pressures are supposed to be linked by the relation

$$P_{cap}(u) = P_w - P_o, \quad (4)$$

where  $P_{cap}$  is a smooth non-decreasing function called *capillary pressure*.

Adding (1) and (2), and using (3) and (4) yields

$$-\partial_x \left( \sum_{\beta=o,w} K \frac{k_{r,\beta}(u)}{\mu_\beta} (\partial_x P_\beta - \rho_\beta \mathbf{g}) \right) = 0,$$

and thus there exists  $q$ , called total flow-rate, only depending on time, such that

$$-\sum_{\beta=o,w} K \frac{k_{r,\beta}(u)}{\mu_\beta} (\partial_x P_\beta - \rho_\beta g) = q. \quad (5)$$

Using (3), (4) and (5), (1) can be rewritten

$$\begin{aligned} & \phi \partial_t u + \partial_x \left( \frac{q k_{r,w}(u)}{k_{r,w}(u) + \frac{\mu_w}{\mu_o} k_{r,o}(u)} \right) \\ & - K \partial_x \left( \frac{k_{r,w}(u) k_{r,o}(u)}{\mu_o k_{r,w}(u) + \mu_w k_{r,o}(u)} (\partial_x P_{cap}(u) - (\rho_w - \rho_o)g) \right) = 0. \end{aligned} \quad (6)$$

Supposing that the total flow rate  $q$  does not depend on times, and after a convenient rescaling, equation (6) becomes

$$\partial_t u + \partial_x (f(u) - \lambda(u) \partial_x \pi(u)) = 0, \quad (7)$$

where  $f$  is a Lipschitz continuous function, fulfilling  $f(0) = 0$ ,  $f(1) = q$ ,  $\lambda$  is a nonnegative Lipschitz continuous functions, with  $\lambda(0) = \lambda(1) = 0$ , and  $\pi$  is a non-decreasing function, also called capillary pressure. The effects of capillarity are often neglected, particularly in the case of reservoir simulation, and so (7) turns to a nonlinear hyperbolic equation called Buckley-Leverett equation, and we have to consider the initial-value problem

$$\begin{cases} \partial_t u + \partial_x f(u) = 0, \\ u(0) = u_0. \end{cases} \quad (\mathcal{BL})$$

## 1.2 discontinuous flux functions

We now consider heterogeneous one dimensional porous media, i.e. an apposition of several homogeneous porous media with different physical properties. This leads to discontinuous functions with respect to the spatial variable. For the sake of simplicity, we assume that the heterogeneous porous medium is made of only two homogeneous porous media represented by the open subsets  $\Omega_1 = \mathbb{R}_-^*$  and  $\Omega_2 = \mathbb{R}_+^*$ . Keeping the notations of (6),  $\phi, K, k_{r,\beta}(u, \cdot)$  and  $\pi(u, \cdot)$  are now discontinuous functions, i.e. piecewise constant functions, denoted  $\phi_i, K_i, k_{r,\beta,i}$  and  $\pi_i$  in  $\Omega_i$ . Thus the problem becomes

$$\begin{cases} \partial_t u + \partial_x (f_i(u) - \lambda_i(u) \partial_x \pi_i(u)) = 0, \\ u(0) = u_0, \\ + \text{transmission condition at } x = 0, \end{cases} \quad (8)$$

where  $f_i$  are Lipschitz continuous functions on  $[0, 1]$ , and can be decomposed in the following way:

$$f_i(u) = q r_i(u) + \lambda_i(u) (\rho_w - \rho_o) g, \quad (9)$$

where  $r_i$  is a non-decreasing Lipschitz continuous function fulfilling  $r_i(0) = 0$ ,  $r_i(1) = 1$ , and  $\lambda_i$  is a non-negative Lipschitz continuous function fulfilling  $\lambda_i(0) = 0$ ,  $\lambda_i(1) = 0$ . We stress here the fact that  $q$  and  $(\rho_w - \rho_o)g$  neither depend on the subdomain  $i$  nor on time.

We now have to give more details on this transmission conditions at  $x = 0$ . First neglect the effects of capillarity, so that (8) becomes the apposition of two Buckley-Leverett equations, linked by a transmission condition.

$$\begin{cases} \partial_t u + \partial_x f_i(u) = 0, \\ u(0) = u_0, \\ + \text{transmission condition at } x = 0, \end{cases} \quad (10)$$

We ask the conservation of mass at the interface between the two porous media, then we have to connect the flux. Denoting  $u_i$  the trace (if it exists) of  $u|_{\Omega_i}$  on  $\{x = 0\}$ , this means

$$f_1(u_1) = f_2(u_2). \quad (11)$$

The problem (10)-(11) has been widely studied recently (see [1, 2, 5, 7, 8, 21, 22, 23, 24, 26, 27, 28]).

It is particularly proven in [6] that the problem admits a unique entropy solution, defined below, with minimal assumptions on the functions  $f_i$ , i.e.  $f_1, f_2$  are Lipschitz continuous,  $f_1(0) = f_2(0) = 0$ ,  $f_1(1) = f_2(1) = q$ .

In the sequel,  $\text{sign}$  is the function defined by

$$\text{sign}(s) = \begin{cases} 0 & \text{if } s = 0, \\ 1 & \text{if } s > 0, \\ -1 & \text{if } s < 0. \end{cases}$$

Let  $\mathcal{U}$  be an open set, we denote by  $\mathcal{D}(\mathcal{U})$  the set of the smooth functions  $\psi$  compactly supported in  $\mathcal{U}$  i.e.  $\mathcal{C}_c^\infty(\mathcal{U})$ . The function  $\psi$  is supposed to belong to  $\mathcal{D}^+(\mathcal{U})$  if it furthermore fulfills  $\psi \geq 0$ .

**Definition 1.1 (entropy solution to (10)-(11))** Let  $u_0 \in L^\infty(\mathbb{R})$  with  $0 \leq u_0 \leq 1$ , and let  $T > 0$ . A function  $u$  is said to be an entropy solution to (10)-(11) if

1.  $u \in L^\infty(\mathbb{R} \times (0, T))$ ,  $0 \leq u \leq 1$  a.e.,
2. for all  $\psi \in \mathcal{D}^+(\mathbb{R} \times [0, T])$ , for all  $\kappa \in [0, 1]$ ,

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}} |u(x, t) - \kappa| \partial_t \psi(x, t) dx dt + \int_{\mathbb{R}} |u_0(x) - \kappa| \psi(x, 0) dx \\ & + \sum_{i=1,2} \int_0^T \int_{\Omega_i} \text{sign}(u(x, t) - \kappa) (f_i(u)(x, t) - f_i(\kappa)) \partial_x \psi(x, t) dx dt \\ & + |f_2(\kappa) - f_1(\kappa)| \int_0^T \psi(0, t) dt \geq 0. \end{aligned} \quad (12)$$

**Theorem 1.1 (Existence and uniqueness of entropy solution)** Let  $u_0 \in L^\infty(\mathbb{R})$  with  $0 \leq u_0 \leq 1$ , and let  $T > 0$ , then there exists a unique entropy solution to (10)-(11) in the sense of Definition 1.1.

Furthermore, the function  $u$  can be supposed to belong to  $C([0, T]; L_{loc}^1(\mathbb{R}))$ , and if  $u, v$  are two entropy solutions associated to initial data  $u_0, v_0$ , then, for all  $R > 0$ , the following comparison principle holds:  $\forall t \in [0, T]$ ,

$$\int_{-R}^R (u(x, t) - v(x, t))^\pm dx \leq \int_{-R-Ct}^{R+Ct} (u_0(x) - v_0(x))^\pm dx,$$

where  $C = \max_i(\text{Lip}(f_i))$ , with  $\text{Lip}(f_i) = \sup_{s \in (0,1)} |f_i'(s)|$ .

The existence of an entropy solution can be proven by the convergence of a finite volume approximation toward an entropy process solution [6, chapter 5], which is a kind of measure valued solution in the way of Di Perna [17]. The comparison principle is proven in [6, chapter 4], using the concept of kinetic solutions [25], and leads to an uniqueness result. The time continuity is proven in [14].

### 1.3 heterogeneities involving discontinuous capillarities

Let us now come back to the problem (8). Suppose for the sake of simplicity that the functions  $\pi_i$  are smooth and increasing on  $[0, 1]$ , and that  $\lambda_i(u) > 0$  if  $0 < u < 1$ . The problem is then a spatial coupling of two parabolic problems, and we will need to ask two transmission conditions: one for the trace, and one for the flux. Concerning the latter, the conservation of mass yields a relation analogous to (11), which can be written with rough notations:

$$f_1(u_1) - \lambda_1(u_1)\partial_x\pi_1(u_1) = f_2(u_2) - \lambda_2(u_2)\partial_x\pi_2(u_2). \quad (13)$$

Let us now focus on the trace transmission. It has been shown in [15] (but see also [10] and [18]) that the connection of the capillary pressures  $\pi_i(u_i)$  has to be done in a graphical sense, so that phenomena like oil trapping can appear. Thus we have to define the monotonous graphs  $\tilde{\pi}_i$ .

$$\tilde{\pi}_i(u) = \begin{cases} \pi_i(u) & \text{if } 0 < u < 1, \\ (-\infty, \pi_i(0)] & \text{if } u = 0, \\ [\pi_i(1), +\infty) & \text{if } u = 1. \end{cases}$$

It is shown in [15] and [13] that a natural way to connect the capillary pressures on the interface consists in asking:

$$\tilde{\pi}_1(u_1) \cap \tilde{\pi}_2(u_2) \neq \emptyset. \quad (14)$$

In order to state a convenient definition for the solution of (8)-(13)-(14), we need to introduce  $\varphi_i(u) = \int_0^u \lambda_i(s)\pi_i'(s)ds$ .

**Definition 1.2 (bounded flux solution)** *Let  $u_0 \in L^\infty(\mathbb{R})$ ,  $0 \leq u_0 \leq 1$ , and let  $T > 0$ . A function  $u$  is said to be a bounded flux solution to (8)-(13)-(14) if it fulfills:*

1.  $u \in L^\infty(\mathbb{R} \times (0, T))$ ,  $0 \leq u \leq 1$  a.e.,
2.  $\partial_x\varphi_i(u) \in L^\infty(\Omega_i \times (0, T))$ ,
3.  $\tilde{\pi}_1(u_1) \cap \tilde{\pi}_2(u_2) \neq \emptyset$  for a.e.  $t \in (0, T)$ ,
4.  $\forall \psi \in \mathcal{D}(\mathbb{R} \times [0, T])$ ,

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}} u(x, t)\partial_t\psi(x, t)dxdt + \int_{\mathbb{R}} u_0(x)\psi(x, 0)dx \\ & + \int_0^T \sum_{i=1,2} \int_{\Omega_i} (f_i(u)(x, t) - \partial_x\varphi_i(u)(x, t))\partial_x\psi(x, t)dxdt = 0. \end{aligned} \quad (15)$$

The bounded flux solution are so called since the point 2 of Definition 1.2 ensures that the flux  $f_i(u) - \partial_x\varphi_i(u)$  remains uniformly bounded. Such a condition will require assumptions on the initial data  $u_0$ , as it will be stated in theorem 1.2.

**Theorem 1.2 (existence of a bounded flux solution)** *Let  $f_1, f_2$  be Lipschitz continuous functions, and  $\varphi_1, \varphi_2$  be increasing Lipschitz continuous functions. Let  $u_0 \in L^1(\mathbb{R})$ ,  $0 \leq u_0 \leq 1$  fulfilling  $\partial_x\varphi_i(u_0) \in L^\infty(\Omega_i)$ , and  $\tilde{\pi}_1(u_{0,1}) \cap \tilde{\pi}_2(u_{0,2}) \neq \emptyset$ , where  $u_{0,i}$  denotes the trace on  $\{x = 0\}$  of  $u_0|_{\Omega_i}$ . Then there exists a bounded flux solution. Moreover,  $u$  belongs to  $\mathcal{C}([0, T]; L^1(\mathbb{R}))$ .*

The first part of this theorem is a straightforward adaptation to the case of unbounded domains and non-monotonous  $f_i$  of a result from [13] and [15]. This is based on a maximum principle on the fluxes  $(f_i(u) - \partial_x\varphi_i(u))$ . This particularly yields:

$$\|f_i(u) - \partial_x\varphi_i(u)\|_{L^\infty(\Omega_i \times (0, T))} \leq \max_{j=1,2} (\|f_j(u_0) - \partial_x\varphi_j(u_0)\|_{L^\infty(\Omega_j)}). \quad (16)$$

If  $u_0 \in L^1(\Omega)$ , then choosing  $\psi = \min(1, (1, R - |x|)^+)$  and letting  $R$  tend to  $\infty$  gives  $u \in L^\infty((0, T); L^1(\mathbb{R}))$ . Moreover, thanks to [14],  $u$  can be supposed to belong to  $\mathcal{C}([0, T]; L^1_{loc}(\mathbb{R}))$ . Then  $u$  belongs to  $\mathcal{C}([0, T]; L^1(\mathbb{R}))$ .

The choice of bounded flux solutions instead of more classical weak solution with  $\partial_x\varphi_i(u)$  only belonging to  $L^2((0, T); L^2_{loc}(\overline{\Omega}_i))$  has been motivated by the fact that it provides a comparison principle if we furthermore suppose that  $u \in \mathcal{C}([0, T]; L^1(\mathbb{R}))$ .

**Proposition 1.3** *Let  $u, v$  be two bounded flux solutions in the sense of Definition 1.2 associated to initial data  $u_0, v_0$ . Then, for all  $\psi \in \mathcal{D}^+(\mathbb{R} \times [0, T])$ ,*

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}} (u(x, t) - v(x, t))^{\pm} \partial_t \psi(x, t) dx dt + \int_{\mathbb{R}} (u_0(x) - v_0(x))^{\pm} \psi(x, 0) dx \\ & + \int_0^T \sum_{i=1,2} \int_{\Omega_i} \text{sign}_{\pm}(u(x, t) - v(x, t)) (f_i(u)(x, t) - f_i(v)(x, t)) \partial_x \psi(x, t) dx dt \\ & - \int_0^T \sum_{i=1,2} \int_{\Omega_i} \partial_x (\varphi_i(u)(x, t) - \varphi_i(v)(x, t)) \pm \partial_x \psi(x, t) dx dt \geq 0. \end{aligned} \quad (17)$$

This proposition is not sufficient to claim the uniqueness, but it will be very useful in the sequel. In order to obtain a uniqueness result, we have to ask furthermore that the initial data belongs to  $L^1(\mathbb{R})$ .

**Theorem 1.4 (uniqueness of bounded flux solution)** *Let  $u_0 \in L^1(\mathbb{R})$ ,  $0 \leq u_0 \leq 1$  a.e., with  $\partial_x \varphi_i(u_0) \in L^\infty(\Omega_i)$  and  $\tilde{\pi}_1(u_{0,1}) \cap \tilde{\pi}_2(u_{0,2}) \neq \emptyset$ . Then there exists a unique bounded flux solution  $u \in \mathcal{C}([0, T]; L^1(\mathbb{R}))$  in the sense of definition 1.2.*

This theorem is a straightforward consequence Proposition 1.3. Indeed, choosing  $\psi = \min(1, (1, R - |x|)^+)$  in (17), and letting  $\mathbb{R}$  tend to  $+\infty$  gives the comparison principle:  $\forall t \in [0, T]$ ,

$$\int_{\mathbb{R}} (u(x, t) - v(x, t))^{\pm} dx \leq \int_{\mathbb{R}} (u_0(x) - v_0(x))^{\pm} dx. \quad (18)$$

The uniqueness result follows.

## 1.4 capillary pressure independent of the saturation

In some cases, the dependence of the capillary pressure  $\pi_i$  with respect to the saturation seems to be weak, and some numerical simulation consider capillary pressures only depending on the porous medium, but not on the saturation. More precisely, we aim to consider graphs of capillary pressure on the form

$$\tilde{\pi}_i(u) = \begin{cases} P_i & \text{if } 0 < u < 1, \\ (-\infty, P_i] & \text{if } u = 0, \\ [P_i, +\infty) & \text{if } u = 1, \end{cases} \quad (19)$$

so that the capillary pressure would roughly speaking not depend on  $u$ .

If one considers an interface  $\{x = 0\}$  between two  $\Omega_i$ , where the  $\tilde{\pi}_i$  are on the form (19), we can give an orientation to the interface: the interface is said to be positively oriented if  $P_1 > P_2$ , and negatively oriented if  $P_1 < P_2$ . A positively oriented interface involve positive capillary forces, and a negatively oriented involve positive capillary forces. The gravity effects are also oriented by the sign of  $(\rho_w - \rho_o)g$  in (9). We have to make the assumption that

*“either the gravity effects and the interface are oriented in the same way, or the convective effects are larger than the gravity effects.”*

We also suppose that each  $f_i$  has a simple dynamic, i.e. it admits at most one extremum in  $(0, 1)$ .

We suppose in the sequel that  $P_1 < P_2$ . The assumption on the orientation of the forces and on the dynamic of  $f_i$  can be stated as follow :

$$\exists b_i \in [0, 1) \text{ s.t. } f_i \text{ is decreasing on } [0, b_i] \text{ and increasing on } [b_i, 1]. \quad (\mathcal{H})$$

With Assumption  $(\mathcal{H})$ , we particularly ensure that

$$f_i(1) = \max_{s \in [0, 1]} (f_i(s)).$$

The assumption on the dynamic on  $f_i$  is often fulfilled by the physical models, as it is stressed in [1] (see also [19]).

In order to simplify the study, we also suppose that  $q \geq 0$ , but we can also deal with  $q < 0$ .

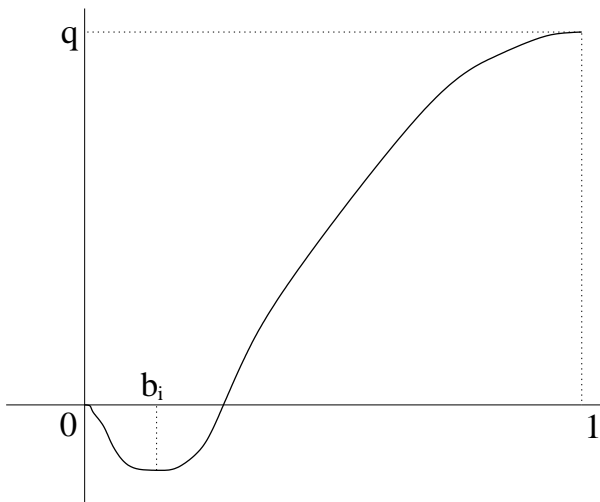


Figure 1: example of  $f_i$  fulfilling  $(\mathcal{H})$

We build a family of approximate problems  $(\mathcal{P}^\varepsilon)$  taking into account the capillary pressure: one suppose that  $\pi_i^\varepsilon(u) = P_i + \varepsilon u$ , where  $P_i$  is a constant depending only on the homogeneous subdomain  $\Omega_i$ . In fact, any  $\pi_i^\varepsilon$  converging uniformly to  $P_i$  on  $[0, 1]$  and such that  $u \mapsto \int_0^u \lambda_i(s) (\pi_i^\varepsilon)'(s) ds$  converges uniformly toward 0 would fit.

Up to a smoothing of the initial data, we obtain a resulting sequence  $(u^\varepsilon)_\varepsilon$  of bounded flux solutions for a problem of type (8)-(13)-(14). We will show that under Assumption  $(\mathcal{H})$ , this sequence tends almost everywhere to the unique entropy solution to (10)-(11).

## 1.5 organization of the paper

The paper is organized as follow: section 2 is devoted to the study of the approximate problem  $(\mathcal{P}^\varepsilon)$ . We first smooth the initial data in a convenient way, and then we give a  $L^2((0, T); H^1(\Omega_i))$ -estimate on the approximate solutions, and a family of  $BV$ -estimates. We also study the steady states which be useful to derive the entropy inequalities.

The convergence of the approximate solution  $u^\varepsilon$  toward the unique entropy solution  $u$  of (10)-(11) is proven in section 3. We first prove using the  $BV$ -estimates that up to a subsequence, the family of bounded flux solutions  $(u^\varepsilon)_\varepsilon$  converges almost everywhere, toward a weak solution  $u$ . We show that the limit is an entropy solution using the Rankine-Hugoniot relation, and the steady solution.

## 2 The approximate problems

In this section we will define the approximate problem  $(\mathcal{P}^\varepsilon)$ , and its solution  $u^\varepsilon$ . We will state a  $L^2((0, T); H_{loc}^1(\overline{\Omega}_i))$ -estimate and a family of  $BV$ -estimates, which will be the key points of the proof of convergence of  $u^\varepsilon$  toward a weak solution of the problem  $(\mathcal{P}^\varepsilon)$ .

In order to recover a family of entropy inequalities, we will build some steady solutions  $\kappa^\varepsilon$  to the problem  $(\mathcal{P}^\varepsilon)$ , and study their limit as  $\varepsilon \rightarrow 0$ . This last point will require strongly Assumption  $(\mathcal{H})$ .

### 2.1 smoothing the initial data

As it has already been stressed in Theorem 1.2, we need to assume some regularity on the initial data to ensure the existence of a bounded flux solution to problems of the type (8)-(13)-(14).

Let  $u_0$  belong to  $L^\infty(\mathbb{R})$ , with  $0 \leq u_0 \leq 1$ , we will build a family  $(u_0^\varepsilon)_\varepsilon$  of convenient approximate initial data.

**Lemma 2.1** *Let  $u_0 \in L^\infty(\mathbb{R})$ ,  $0 \leq u_0 \leq 1$ , then there exists a family  $(u_0^\varepsilon)_\varepsilon$  of approximate initial data such that:*

- $u_0^\varepsilon \in C_c^\infty(\mathbb{R}^*)$ ,  $0 \leq u_0^\varepsilon \leq 1$ ,

- $u_0^\varepsilon \rightarrow u_0$  a.e. in  $\mathbb{R}$ ,  $\|\varepsilon \partial_x u_0^\varepsilon\|_\infty \rightarrow 0$  as  $\varepsilon \rightarrow 0$ , and  $\|\varepsilon \partial_x u_0^\varepsilon\|_\infty \leq 1$  for all  $\varepsilon > 0$ ,
- If  $u_0 \in BV(\mathbb{R})$ , then  $\|\partial_x u_0^\varepsilon\|_{L^1(\mathbb{R})} \leq TV(u_0) + 4$ .

**Proof**

Let  $\alpha > 0$ , and let  $\rho_\alpha$  be a mollifier with support in  $(-\alpha, \alpha)$ . Let  $v^\alpha = (u_0 \circ \chi_{\alpha < |x| < 1/\alpha}) * \rho_\alpha$ , then it is clear that  $v^\alpha \in C_c^\infty(\mathbb{R}^*)$ , and that  $v^\alpha \rightarrow u_0$  a.e. in  $\mathbb{R}$  as  $\alpha \rightarrow 0$ . Choosing  $\varepsilon = \min\left(\alpha, \frac{\min(1, \sqrt{\alpha})}{\|\partial_x v^\alpha\|_\infty}\right)$ , and  $u_0^\varepsilon = v^\alpha$  ends the proof of Lemma 2.1.  $\square$

## 2.2 the problem ( $\mathcal{P}^\varepsilon$ )

Let  $P_1, P_2 \in \mathbb{R}$ , we define the functions  $\pi_i^\varepsilon$  by  $\pi_i^\varepsilon(u) = P_i + \varepsilon u$ , and

$$\tilde{\pi}_i^\varepsilon(u) = \begin{cases} P_i + \varepsilon u & \text{if } 0 < u < 1, \\ (-\infty, P_i] & \text{if } u = 0, \\ [P_i + \varepsilon, +\infty) & \text{if } u = 1. \end{cases}$$

If  $\varepsilon$  is small, the intersection of the ranges of the functions  $\pi_1^\varepsilon$  and  $\pi_2^\varepsilon$  is empty, and then the

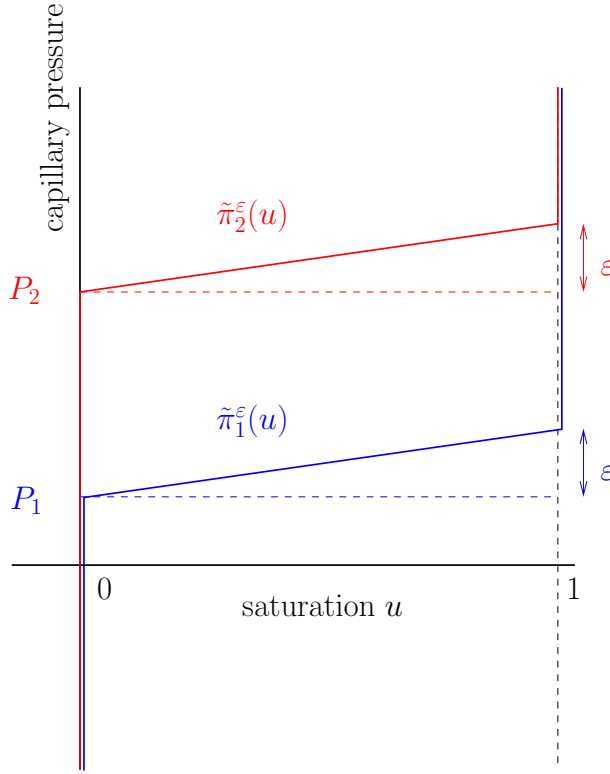


Figure 2: capillary pressures graphs  $\tilde{\pi}_i^\varepsilon$ .

graphical relation  $\tilde{\pi}_1(u_1) \cap \tilde{\pi}_2(u_2) \neq \emptyset$  connecting the capillary pressures at the interface becomes

$$(1 - u_1)u_2 = 0.$$

Let  $0 \leq u_0 \leq 1$ , and let  $(u_0^\varepsilon)_\varepsilon$  be built as in Lemma 2.1, let  $\varphi_i(u) = \int_0^u \lambda_i(s) ds$ . The approximate problem is: find  $u^\varepsilon$  s.t.

$$\begin{cases} \partial_t u^\varepsilon + \partial_x (f_i(u^\varepsilon) - \varepsilon \partial_x \varphi_i(u^\varepsilon)) = 0 & \text{in } \Omega_i \times (0, T), \\ \tilde{\pi}_1^\varepsilon(u^\varepsilon)(0^-, t) \cap \tilde{\pi}_2^\varepsilon(u^\varepsilon)(0^+, t) \neq \emptyset & \text{in } (0, T), \\ f_1(u^\varepsilon)(0^-, t) - \varepsilon \partial_x \varphi_1(u^\varepsilon)(0^-, t) = f_2(u^\varepsilon)(0^+, t) - \varepsilon \partial_x \varphi_2(u^\varepsilon)(0^+, t) & \text{in } (0, T), \\ u^\varepsilon(0) = u_0^\varepsilon & \text{in } \mathbb{R}. \end{cases} \quad (\mathcal{P}^\varepsilon)$$

This problems is of type (8)-(13)-(14), then the notion of bounded flux solution is a good frame to solve it.



**Definition 2.1 (solution to  $(\mathcal{P}^\varepsilon)$ )** A function  $u^\varepsilon$  is said to be a (bounded flux) solution to  $(\mathcal{P}^\varepsilon)$  if it fulfills

1.  $u^\varepsilon \in L^\infty(\mathbb{R} \times (0, T))$ ,  $0 \leq u^\varepsilon \leq 1$  a.e.,
2.  $\partial_x \varphi_i(u^\varepsilon) \in L^\infty(\Omega_i \times (0, T))$ ,
3.  $\forall \psi \in \mathcal{D}(\mathbb{R} \times [0, T])$ ,

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}} u^\varepsilon(x, t) \partial_t \psi(x, t) dx dt + \int_{\mathbb{R}} u_0^\varepsilon(x) \psi(x, 0) dx \\ & + \int_0^T \sum_{i=1,2} \int_{\Omega_i} (f_i(u^\varepsilon)(x, t) - \varepsilon \partial_x \varphi_i(u^\varepsilon)(x, t)) \partial_x \psi(x, t) dx dt = 0. \end{aligned} \quad (20)$$

We can use Theorem 1.2 to claim that there exists a family  $(u^\varepsilon)_\varepsilon$  of bounded flux solution to  $(\mathcal{P}^\varepsilon)$  in the sense of Definition 2.1. Moreover, this family of solution fulfills, thanks to (16) and Lemma 2.1: for all  $\varepsilon > 0$ ,

$$\|\varepsilon \partial_x \varphi_i(u^\varepsilon)\|_\infty \leq (\max_i \text{Lip}(\varphi_i) + \max_i \|f_i\|_{L^\infty(0,1)}). \quad (21)$$

Since  $u_0^\varepsilon$  belongs to  $L^1(\mathbb{R})$ , the solution  $u^\varepsilon$  is furthermore unique in  $\mathcal{C}([0, T], L^1(\mathbb{R}))$  thanks to Theorem 1.4.

### 2.3 a $L^2((0, T); H_{loc}^1(\overline{\Omega}_i))$ -estimate

All this subsection is devoted to prove the following estimate.

**Proposition 2.2** Let  $K$  be a compact subset of  $\overline{\Omega}_i$ , and let  $u^\varepsilon$  be a solution of  $(\mathcal{P}^\varepsilon)$  in the sense of Definition 2.1, then there exists  $C$  depending only on  $f_i, K, T$  (and not on  $\varepsilon$ ) such that

$$\sqrt{\varepsilon} \|\varphi_i(u^\varepsilon)\|_{L^2((0, T); H^1(K))} \leq C.$$

Particularly, this implies that  $\varepsilon \partial_x \varphi_i(u^\varepsilon) \rightarrow 0$  a.e. in  $\Omega_i \times (0, T)$  as  $\varepsilon \rightarrow 0$ .

#### Proof

We fix  $\varepsilon > 0$ . Since the functions  $\varphi_i^{-1}$  are not Lipschitz continuous, the problem  $(\mathcal{P}^\varepsilon)$  is not strictly parabolic, and the function  $u^\varepsilon$  is not a strong solution. In order to get more regularity on the approximate solution, we regularize the problem by adding an additional viscosity  $1/n$  ( $n \geq 1$ ), so that the so built approximate solution  $u_n^\varepsilon$  is regular enough to perform the calculation below.

Let  $n > 1$ , and let  $\varphi_{i,n}(u) = \varphi_i(u) + u/n$ , and let  $u_n^\varepsilon$  be a bounded flux solution of  $(\mathcal{P}^\varepsilon)$  with  $\varphi_{i,n}$  instead of  $\varphi_i$ . From (21), we know that  $\partial_x \varphi_{i,n}(u_n^\varepsilon)$  is uniformly bounded in  $L^\infty(\Omega_i \times (0, T))$ , and since  $\varphi_{i,n}^{-1}$  is a Lipschitz continuous function, one has  $\partial_x u_n^\varepsilon \in L^\infty(\Omega_i \times (0, T))$ . The following weak formulation holds:  $\forall \psi \in \mathcal{D}(\Omega \times [0, T])$ ,

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}} u_n^\varepsilon(x, t) \partial_t \psi(x, t) dx dt + \int_{\mathbb{R}} u_0^\varepsilon(x) \psi(x, 0) dx \\ & + \int_0^T \sum_{i=1,2} \int_{\Omega_i} (f_i(u_n^\varepsilon)(x, t) - \varepsilon \partial_x \varphi_{i,n}(u_n^\varepsilon)(x, t)) \partial_x \psi(x, t) dx dt = 0. \end{aligned} \quad (22)$$

Let  $(a, b) \subset \Omega_i$ . Let  $\zeta \in \mathcal{D}^+((a, b))$ , we deduce from (22) that

$$\left\langle \partial_t u_n^\varepsilon \mid u_n^\varepsilon \zeta^2 \right\rangle = \int_0^T \int_a^b f_i(u_n^\varepsilon) \partial_x (u_n^\varepsilon \zeta^2) dx dt - \varepsilon \int_0^T \int_a^b \partial_x \varphi_{i,n}(u_n^\varepsilon) \partial_x (u_n^\varepsilon \zeta^2) dx dt, \quad (23)$$

where  $\langle \cdot \mid \cdot \rangle$  is the duality bracket between  $L^2((0, T); H^{-1}(a, b))$  and  $L^2((0, T); H_0^1(a, b))$ . Since  $\varphi_{i,n}$  is a Lipschitz continuous function with  $(\|\lambda_i\|_\infty + 1/n)$  as Lipschitz constant, one has

$$\int_0^T \int_a^b \partial_x \varphi_{i,n}(u_n^\varepsilon) \partial_x (u_n^\varepsilon) \zeta^2 dx dt \geq \frac{1}{\|\lambda_i\|_\infty + 1/n} \int_0^T \int_a^b (\partial_x \varphi_{i,n}(u_n^\varepsilon))^2 \zeta^2 dx dt. \quad (24)$$

Let  $\Phi_i$  be a primitive of  $f_i$ , then:

$$\begin{aligned} \int_0^T \int_a^b f_i(u_n^\varepsilon) \partial_x (u_n^\varepsilon \zeta^2) dx dt &= \int_0^T \int_a^b \partial_x \Phi_i(u_n^\varepsilon) \zeta^2 dx dt + \int_0^T \int_a^b f_i(u_n^\varepsilon) u_n^\varepsilon \partial_x \zeta^2 dx dt \\ &= \int_0^T \int_a^b [f_i(u_n^\varepsilon) u_n^\varepsilon - \Phi_i(u_n^\varepsilon)] \partial_x \zeta^2 dx dt. \end{aligned} \quad (25)$$

Admit for the moment Lemma 2.3, stated and proven below. We deduce from (23), (24), (25) and Lemma 2.3 that

$$\begin{aligned} &\frac{\varepsilon}{\|\lambda_i\|_\infty + 1} \int_0^T \int_a^b (\partial_x \varphi_{i,n}(u_n^\varepsilon))^2 \zeta^2 dx dt \\ &\leq \frac{1}{2} \|\zeta\|_\infty^2 |b-a| + \int_0^T \int_a^b |u_n^\varepsilon (f_i(u_n^\varepsilon) - \varepsilon \partial_x \varphi_{i,n}(u_n^\varepsilon)) - \Phi_i(u_n^\varepsilon)| |\partial_x \zeta^2| dx dt. \end{aligned}$$

Using now the fact that  $u_n^\varepsilon$  is a bounded flux solution, we deduce from (16) that  $[u_n^\varepsilon (f_i(u_n^\varepsilon) - \varepsilon \partial_x \varphi_{i,n}(u_n^\varepsilon)) - \Phi_i(u_n^\varepsilon)]$  is uniformly bounded independently of  $\varepsilon$  and  $n$ , and so there exists  $C$  only depending on  $f_i$ ,  $|b-a|$ ,  $u_0$  and  $\lambda_i$  such that:

$$\varepsilon \int_0^T \int_a^b (\partial_x \varphi_{i,n}(u_n^\varepsilon))^2 \zeta^2 dx dt \leq C (\|\partial_x \zeta^2\|_{L^1((0,T); \mathcal{M}(\mathbb{R}))} + \|\zeta\|_\infty^2).$$

This estimate still holds for  $\zeta(x, t) = \chi_{(a,b)}(x)$ , for all  $(a, b) \in \overline{\Omega}_i^2$ , so we obtain

$$\varepsilon \int_0^T \int_a^b (\partial_x \varphi_{i,n}(u_n^\varepsilon))^2 dx dt \leq C(2T+1). \quad (26)$$

Classical compactness arguments provide the convergence, up to a subsequence, of  $(u_n^\varepsilon)_n$  to a solution  $u^\varepsilon$  of  $(\mathcal{P}^\varepsilon)$  in  $L^p(\mathbb{R} \times (0, T))$ ,  $1 \leq p < +\infty$ . This ensures particularly that, up to a subsequence,

$$\lim_{n \rightarrow +\infty} u_n^\varepsilon = u^\varepsilon \text{ a.e. in } \mathbb{R} \times (0, T).$$

Taking the limit w.r.t.  $n$  in (26) yields

$$\varepsilon \int_0^T \int_a^b (\partial_x \varphi_i(u^\varepsilon))^2 dx dt \leq C(2T+1).$$

□

**Lemma 2.3** *Let  $u_n^\varepsilon$  be an approximate solution of  $(\mathcal{P}^\varepsilon)$  with  $\varphi_{i,n}$  instead of  $\varphi_i$ , and let  $\psi \in \mathcal{D}^+((a, b))$ , then*

$$\langle \partial_t u_n^\varepsilon \mid u_n^\varepsilon \zeta^2 \rangle \geq -\frac{1}{2} \|\zeta\|_\infty^2 |b-a|.$$

**Proof**

Since  $(u_n^\varepsilon \zeta) \in L^2((0, T); H_0^1(a, b))$ , and  $\partial_t (u_n^\varepsilon \zeta) \in L^2((0, T); H^{-1}(a, b))$ , and so, up to a negligible set,  $(u_n^\varepsilon \zeta) \in C([0, T]; L^2(a, b))$ , and

$$\begin{aligned} \langle \partial_t u_n^\varepsilon \mid u_n^\varepsilon \zeta^2 \rangle &= \langle \partial_t u_n^\varepsilon \zeta \mid u_n^\varepsilon \zeta \rangle \\ &= \frac{1}{2} \int_a^b u_n^\varepsilon(x, T)^2 \zeta^2(x) dx - \frac{1}{2} \int_a^b u_0(x)^2 \zeta^2(x) dx \\ &\geq -\frac{1}{2} \|\zeta\|_\infty^2 |b-a|. \end{aligned}$$

□

## 2.4 the $BV$ -estimates

In this section we suppose that  $u_0 \in BV(\mathbb{R})$ . In order to avoid heavy notations which would lead to a good comprehension of the problem, the following proof will be formal. To establish the following estimates in a rigorous frame, one can introduce of a thin layer  $(-\eta, \eta)$  on which the pressure variates smoothly to replace the interface, and add some additional viscosity to obtain smooth strong solutions to the problem. This regularization of the problem has been performed in [15] and in [10], where a  $BV$ -estimate is also derived.

For  $a, b \in [0, 1]$ , we denote by

$$F_i(a, b) = \text{sign}(a - b)(f_i(a) - f_i(b)).$$

**Lemma 2.4** *There exists  $C$  depending only on  $f_i, T, u_0$  such that*

$$|\partial_t F_i(u^\varepsilon, \kappa)|_{\mathcal{M}_b(\Omega_i \times (0, T))} \leq C.$$

**Proof**

Suppose in the sequel that  $u^\varepsilon$  is a strong solution, i.e.

$$\partial_t u^\varepsilon + \partial_x [f_i(u^\varepsilon) - \varepsilon \partial_x \varphi_i(u^\varepsilon)] = 0$$

holds point-wise in  $\Omega_i \times (0, T)$ . Let  $h > 0$ , and let  $t \in (0, T - h)$ . Comparing  $u^\varepsilon(\cdot, \cdot + h)$  and  $u^\varepsilon$  with (18) yields

$$\int_{\mathbb{R}} |u^\varepsilon(x, t + h) - u^\varepsilon(x, t)| dx \leq \int_{\mathbb{R}} |u^\varepsilon(x, h) - u_0^\varepsilon(x)| dx.$$

Dividing by  $h$  and letting  $h$  tend to 0, one can claim using the fact that  $u^\varepsilon$  is supposed to be a strong solution

$$\begin{aligned} \int_{\mathbb{R}} |\partial_t u^\varepsilon(x, t)| dx &= \sum_{i=1,2} \int_{\Omega_i} |\partial_x [f_i(u^\varepsilon)(x, t) - \varepsilon \partial_x \varphi_i(u^\varepsilon)(x, t)]| dx \\ &\leq \int_{\mathbb{R}} |\partial_t u_0^\varepsilon(x)| dx \\ &\leq \sum_{i=1,2} \int_{\Omega_i} |\partial_x [f_i(u_0^\varepsilon)(x) - \varepsilon \partial_x \varphi_i(u_0^\varepsilon)(x)]| dx \end{aligned}$$

Lemma 2.1 then ensures that there exists  $C$  not depending on  $\varepsilon$  such that

$$\int_0^T \int_{\mathbb{R}} |\partial_t u^\varepsilon(x, t)| dx dt = \int_0^T \sum_{i=1,2} \int_{\Omega_i} |\partial_x [f_i(u^\varepsilon)(x, t) - \varepsilon \partial_x \varphi_i(u^\varepsilon)(x, t)]| dx dt \leq C. \quad (27)$$

Thanks to the regularity of  $f_i$ , this particularly ensures that, if we denote by  $\mathcal{M}_b(\Omega_i \times (0, T))$  the set of the bounded Radon measure on  $\Omega_i \times (0, T)$ , i.e. the dual space of  $\mathcal{C}_c(\overline{\Omega_i} \times [0, T], \mathbb{R})$  with the uniform norm, we obtain:  $\forall \kappa \in [0, 1]$ ,

$$|\partial_t F_i(u^\varepsilon, \kappa)|_{\mathcal{M}_b(\Omega_i \times (0, T))} \leq C \|f_i'\|_\infty.$$

□

**Lemma 2.5** *There exists  $C$  depending only on  $u_0, f_i$  and  $T$  such that*

$$\left| \partial_x \left( F_i(u^\varepsilon, \kappa) - \varepsilon \partial_x |\varphi_i(u^\varepsilon) - \varphi_i(\kappa)| \right) \right|_{\mathcal{M}_b(\Omega_i \times (0, T))} \leq C.$$

**Proof**

It follows from the work of Carillo (see e.g. [16]) that for all  $\kappa \in [0, 1]$ , for all  $\psi \in \mathcal{D}^+(\Omega_i \times [0, T])$ ,

$$\begin{aligned} &\int_0^T \int_{\Omega_i} |u^\varepsilon - \kappa| \partial_t \psi dx dt + \int_{\Omega_i} |u^\varepsilon - \kappa| \psi(0) dx \\ &\int_0^T \int_{\Omega_i} \left( F_i(u^\varepsilon, \kappa) - \varepsilon \partial_x |\varphi_i(u^\varepsilon) - \varphi_i(\kappa)| \right) \partial_x \psi dx dt \geq 0. \end{aligned} \quad (28)$$

Let  $\eta > 0$ , we denote by  $\omega_\eta(x) = (1 - |x|/\eta)^+$ , and suppose now that  $\psi$  belongs to  $\mathcal{D}^+(\overline{\Omega}_i \times [0, T])$ , i.e.  $\psi$  does not vanish on the interface  $\{x = 0\}$ . Estimate (28) still holds when we consider  $\psi_\eta = \psi(1 - \omega_\eta)$  as test function.

$$\begin{aligned} & \int_0^T \int_{\Omega_i} |u^\varepsilon - \kappa| (1 - \omega_\eta) \partial_t \psi dx dt + \int_{\Omega_i} |u^\varepsilon - \kappa| \psi(0) (1 - \omega_\eta) dx \\ & + \int_0^T \int_{\Omega_i} \left( F_i(u^\varepsilon, \kappa) - \varepsilon \partial_x |\varphi_i(u^\varepsilon) - \varphi_i(\kappa)| \right) (1 - \omega_\eta) \partial_x \psi dx dt \\ \geq & \int_0^T \int_{\Omega_i} \left( F_i(u^\varepsilon, \kappa) - \varepsilon \partial_x |\varphi_i(u^\varepsilon) - \varphi_i(\kappa)| \right) \psi \partial_x \omega_\eta dx dt. \end{aligned} \quad (29)$$

The fact that the flux induced by  $u^\varepsilon$  is uniformly bounded w.r.t.  $\varepsilon$  thanks to (21) implies that there exists  $C$  depending only on  $u_0, f_i$  such that

$$\|F_i(u^\varepsilon, \kappa) - \varepsilon \partial_x |\varphi_i(u^\varepsilon) - \varphi_i(\kappa)|\|_{L^\infty(\Omega_i \times (0, T))} \leq C. \quad (30)$$

Then we obtain the following estimate on the right-hand-side in (29):

$$\left| \int_0^T \int_{\Omega_i} \left( F_i(u^\varepsilon, \kappa) - \varepsilon \partial_x |\varphi_i(u^\varepsilon) - \varphi_i(\kappa)| \right) \psi \partial_x \omega_\eta dx dt \right| \leq CT \|\psi\|_\infty.$$

Letting  $\eta$  tend to 0 in inequality (29) gives:  $\forall \psi \in \mathcal{D}^+(\overline{\Omega}_i \times [0, T]), \forall \kappa \in [0, 1]$ ,

$$\begin{aligned} & \int_0^T \int_{\Omega_i} |u^\varepsilon - \kappa| \partial_t \psi dx dt + \int_{\Omega_i} |u^\varepsilon - \kappa| \psi(0) dx \\ & + \int_0^T \int_{\Omega_i} \left( F_i(u^\varepsilon, \kappa) - \varepsilon \partial_x |\varphi_i(u^\varepsilon) - \varphi_i(\kappa)| \right) \partial_x \psi dx dt \geq -CT \|\psi\|_\infty \end{aligned} \quad (31)$$

We introduce now a monotonous function  $\chi_\psi \in \mathcal{D}^+(\overline{\Omega}_i)$  equal to 1 on the support of  $\psi(\cdot, t)$  for all  $t \in [0, T]$  (so that  $\|\partial_x \chi_\psi\|_{L^1(\Omega_i)} = 1$ ), then  $\|\psi\|_\infty \chi_\psi \geq \psi$ . Choosing  $\|\psi\|_\infty \chi_\psi - \psi$  and  $\|\psi\|_\infty \chi_\psi + \psi$  as test function in (31) yields, using (30) once again

$$\left| \partial_t |u^\varepsilon - \kappa| - \partial_x \left( F_i(u^\varepsilon, \kappa) - \varepsilon \partial_x |\varphi_i(u^\varepsilon) - \varphi_i(\kappa)| \right) \right|_{\mathcal{M}_b(\Omega_i \times (0, T))} \leq 2CT. \quad (32)$$

Thanks to (27), there exists  $C'$  depending only on  $u_0, f_i$  and  $T$  such that

$$|\partial_t |u^\varepsilon - \kappa||_{\mathcal{M}_b(\Omega_i \times (0, T))} \leq C'. \quad (33)$$

Lemma 2.5 is so a consequence of (32) and (33).  $\square$

**Proposition 2.6** *Let  $u_0 \in BV(\mathbb{R})$  and let  $K = [a, b] \subset \overline{\Omega}_i$ . We introduce  $z_K^\varepsilon(x, t)$  defined on the whole space  $\mathbb{R}^2$  given by:*

$$z_{K, \kappa}^\varepsilon(x, t) = \begin{cases} F_i(u^\varepsilon, \kappa)(x, t) & \text{if } (x, t) \in K \times (0, T), \\ 0 & \text{otherwise.} \end{cases}$$

*There exists  $C$  depending only on  $u_0, f_i, T, K$  and a uniformly bounded function  $r_{K, \kappa}$ , with  $r_{K, \kappa}(\varepsilon)$  tends uniformly to 0 with respect to  $\kappa$  as  $\varepsilon \rightarrow 0$ , such that, for all  $(\xi, h) \in \mathbb{R}^2$ ,*

$$\iint_{\mathbb{R}^2} |z_{K, \kappa}^\varepsilon(x + \xi, t + h) - z_{K, \kappa}^\varepsilon(x, t)| dx dt \leq C(|\xi| + |h|) + r_{K, \kappa}(\varepsilon).$$

### Proof

Let  $h \in \mathbb{R}$ , one has

$$\begin{aligned} & \iint_{\mathbb{R}^2} |z_{K, \kappa}^\varepsilon(x, t + h) - z_{K, \kappa}^\varepsilon(x, t)| dx dt \leq |\partial_t z_{K, \kappa}^\varepsilon|_{\mathcal{M}_b(\mathbb{R}^2)} |h| \\ & \leq \left( |\partial_t F_i(u^\varepsilon, \kappa)|_{\mathcal{M}_b(\Omega_i \times (0, T))} + 2\|F_i(u^\varepsilon, \kappa)\|_{L^\infty(T + b - a)} \right) |h| \end{aligned}$$

It follows from Lemma 2.4 that there exists  $C_1$  depending only on  $u_0, f_i, K, T, \varphi_i$  such that

$$\iint_{\mathbb{R}^2} |z_{K,\kappa}^\varepsilon(x, t+h) - z_{K,\kappa}^\varepsilon(x, t)| dxdt \leq C_1|h|. \quad (34)$$

We also define

$$q_{K,\kappa}^\varepsilon(x, t) = \begin{cases} \varepsilon \partial_x |\varphi_i(u^\varepsilon)(x, t) - \varphi_i(\kappa)| & \text{if } (x, t) \in K \times (0, T), \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2.2 ensures that  $q_{K,\kappa}^\varepsilon(x, t)$  converges to 0 almost everywhere in  $\mathbb{R}^2$ , and the estimate (30) ensures us that  $q_{K,\kappa}^\varepsilon(x, t)$  stays uniformly bounded in  $L^\infty(\mathbb{R}^2)$  with respect to  $\varepsilon$ . Let  $\xi \in \mathbb{R}$ , then we have:

$$\begin{aligned} & \iint_{\mathbb{R}^2} |z_{K,\kappa}^\varepsilon(x+\xi, t) - q_{K,\kappa}^\varepsilon(x+\xi, t) - (z_{K,\kappa}^\varepsilon(x, t) - q_{K,\kappa}^\varepsilon(x, t))| dxdt \\ & \leq \left( \begin{array}{l} |\partial_x (F_i(u^\varepsilon, \kappa) - \varepsilon \partial_x |\varphi_i(u^\varepsilon)(x, t) - \varphi_i(\kappa)|)|_{\mathcal{M}_b(\Omega_i \times (0, T))} \\ + 2(T+b-a) (\|F_i(u^\varepsilon, \kappa) - \varepsilon \partial_x |\varphi_i(u^\varepsilon)(x, t) - \varphi_i(\kappa)|\|_{L^\infty}) \end{array} \right) |\xi|. \end{aligned} \quad (35)$$

Using (30) and Lemma 2.5 in (35) yields that there exists  $C_2$  depending only on  $u_0, f_i, K, T, \varphi_i$  such that

$$\iint_{\mathbb{R}^2} |z_{K,\kappa}^\varepsilon(x+\xi, t) - q_{K,\kappa}^\varepsilon(x+\xi, t) - (z_{K,\kappa}^\varepsilon(x, t) - q_{K,\kappa}^\varepsilon(x, t))| dxdt \leq C_2|\xi|.$$

This particularly ensures that:

$$\iint_{\mathbb{R}^2} |z_{K,\kappa}^\varepsilon(x+\xi, t) - z_{K,\kappa}^\varepsilon(x, t)| dxdt \leq C_2|\xi| + 2\|q_{K,\kappa}^\varepsilon(x, t)\|_{L^1(\mathbb{R}^2)}. \quad (36)$$

By choosing  $C = \max(C_1, C_2)$ , one we deduce from (34) and (36) that

$$\iint_{\mathbb{R}^2} |z_{K,\kappa}^\varepsilon(x+\xi, t+h) - z_{K,\kappa}^\varepsilon(x, t)| dxdt \leq C(|h| + |\xi|) + 2\|q_{K,\kappa}^\varepsilon(x, t)\|_{L^1(\mathbb{R}^2)}.$$

We conclude the proof of Proposition 2.6 by checking that  $\|q_{K,\kappa}^\varepsilon(x, t)\|_{L^1(\mathbb{R}^2)}$  converges uniformly to 0 with respect to  $\kappa$  as  $\varepsilon$  tends to 0.  $\square$

## 2.5 some steady solutions

The spatial discontinuities of the saturation and capillary pressure allow us to consider Kruřkov entropies  $|u - \kappa|$  only for non-negative test functions *vanishing on the interface*, i.e. in  $\mathcal{D}^+(\mathbb{R}^* \times [0, T])$ . This is not enough to obtain the convergence of  $u^\varepsilon$  toward an entropy solution  $u$  in the sense of Definition 1.1, and a relation has also to be derived at the level of the interface.

In order to deal with general test functions belonging to  $\mathcal{D}^+(\mathbb{R} \times [0, T])$ , we will so have to introduce some approximate Kruřkov entropies  $|u - \kappa^\varepsilon(x)|$ , where  $\kappa^\varepsilon$  are steady solutions of  $(\mathcal{P}^\varepsilon)$ . Letting  $\varepsilon \rightarrow 0$ , those steady solutions converges to piecewise constant function  $\tilde{\kappa}^j$  defined below. The limit  $u$  of approximate solutions  $u^\varepsilon$  will then be compared to this limit  $\tilde{\kappa}^j$  to prove that  $u$  is an entropy solution.

The building of convenient  $\kappa^\varepsilon$  strongly uses Assumption  $(\mathcal{H})$ . It will be shown in [12] that if  $(\mathcal{H})$  fails, non classical shocks can occur at the interface, and the limit  $\tilde{u}$  of the approximate solutions  $u^\varepsilon$  is thus not an entropy solution.

Recall that  $q \geq 0$ ,  $P_1 < P_2$ , and suppose that  $0 < \varepsilon < P_2 - P_1$ . Some simple adaptations can be done to cover the case  $q < 0$ . The transmission condition (14) can be summarized as follow: either  $u_1 = 1$ , or  $u_2 = 0$ .

We have to introduce the following sets:

$$\mathcal{E}_1 = \{\kappa_1 / \exists \kappa_2 \text{ with } f_1(\kappa_1) = f_2(\kappa_2)\},$$

$$\mathcal{E}_2 = \{\kappa_2 / \exists \kappa_1 \text{ with } f_1(\kappa_1) = f_2(\kappa_2)\}.$$

It follows from Assumption  $(\mathcal{H})$  that either  $\mathcal{E}_1 = [0, 1]$ , or  $\mathcal{E}_2 = [0, 1]$ , and so we are ensured that  $\kappa \in [0, 1]$  belongs either to  $\mathcal{E}_1$ , or to  $\mathcal{E}_2$  (or of course to both  $\mathcal{E}_1$  and  $\mathcal{E}_2$ ).

Check also that for all  $z \in (0, q]$ , it follows from  $(\mathcal{H})$  that there exists a unique  $\kappa_i(z)$  such that  $f_i(\kappa_i(z)) = z$ . On the contrary, if  $z \in \mathcal{E}_i$  with  $z \leq 0$ ,  $z$  has a priori not a unique antecedent through  $f_i$ . Let  $\kappa$  belong to  $\mathcal{E}_j$ , one denotes

$$\bar{\kappa}_i^j = \max_{\nu} \{f_i(\nu) = f_j(\kappa)\} \quad (37)$$

and

$$\underline{\kappa}_i^j = \min_{\nu} \{f_i(\nu) = f_j(\kappa)\}. \quad (38)$$

This section is devoted to establish the following proposition.

**Proposition 2.7** *For all  $\kappa \in \mathcal{E}_j$ , there exists a family of steady solutions  $(\kappa^\varepsilon)_\varepsilon$  to the problem  $(\mathcal{P}^\varepsilon)$  such that*

$$\kappa^\varepsilon \rightarrow \tilde{\kappa}^j \text{ a.e. in } \Omega_i, \quad (39)$$

where  $\tilde{\kappa}^j(x)$  can be chosen between:

$$i) \tilde{\kappa}^j(x) = \begin{cases} \bar{\kappa}_1^j & \text{if } x < 0, \\ \underline{\kappa}_2^j & \text{if } x > 0, \end{cases}$$

$$ii) \tilde{\kappa}^j(x) = \begin{cases} \bar{\kappa}_1^j & \text{if } x < 0, \\ \bar{\kappa}_2^j & \text{if } x > 0, \end{cases}$$

$$iii) \tilde{\kappa}^j(x) = \begin{cases} \underline{\kappa}_1^j & \text{if } x < 0, \\ \underline{\kappa}_2^j & \text{if } x > 0, \end{cases}$$

### Proof

Let  $\kappa \in \mathcal{E}_j$ .

- If  $\kappa = 1$ , the three limits  $\tilde{\kappa}^j$  are identically equal to 1, which is a steady solution fulfilling Proposition 2.7.
- We suppose now that  $\kappa < 1$ , and  $f_j(\kappa) > 0$ . Even in this case, the three reachable limit are the same. Thus we only have to build one sequence of converging steady solutions. Let  $y$  be a solution of:

$$\begin{cases} \frac{d}{dx} \varphi_1(y) = f_j(\kappa) - f_1(y), & \text{for } x > 0, \\ y(0) = 1. \end{cases} \quad (40)$$

The solution  $y(x)$  converges to  $\bar{\kappa}_1^j$  as  $x \rightarrow +\infty$ . The family  $(\kappa^\varepsilon)_\varepsilon$  defined by:  $\forall \varepsilon$

$$\kappa^\varepsilon(x) = \begin{cases} \bar{\kappa}_2^j \text{ or } \underline{\kappa}_2^j & \text{if } x > 0, \\ y(-x/\varepsilon) & \text{if } x < 0. \end{cases} \quad (41)$$

fulfills so the conclusion of Proposition 2.7.

- Suppose now  $f_j(\kappa) = 0$ . The solutions  $\kappa^\varepsilon(x)$  built with (40)-(41) converges toward the two reachable states  $i)$  and  $ii)$ . One can also choose  $\kappa^\varepsilon(x) = 0$ , which is of course a steady solution.
- It remains the case  $f_j(\kappa) < 0$ . The solutions  $\kappa^\varepsilon(x)$  built with (40)-(41) still converges toward the two reachable states  $i)$  and  $ii)$ .

Let  $w$  be a solution of

$$\begin{cases} \frac{d}{dx} \varphi_2(w) = f_2(w) - f_j(\kappa), & \text{for } x > 0, \\ w(0) = 0. \end{cases}$$

and

$$\kappa^\varepsilon(x) = \begin{cases} \underline{\kappa}_1^j & \text{if } x < 0, \\ w(x/\varepsilon) & \text{if } x > 0, \end{cases}$$

then  $\kappa^\varepsilon$  converges toward the third reachable state.

□

### 3 Convergence toward the entropy solution

#### 3.1 almost everywhere convergence

We state now a lemma which is an adaptation of Helly's selection theorem criterion, which states that if  $(v^\varepsilon)_\varepsilon$  is a family of measurable functions on an open subset  $\mathcal{U}$  of  $\mathbb{R}^k$  ( $k \geq 1$ ), uniformly bounded in  $L^\infty_{loc}(\mathcal{U})$  and  $BV_{loc}(\mathcal{U})$ , one can extract a subfamily still denoted by  $(v^\varepsilon)_\varepsilon$  that converges almost everywhere in  $\mathcal{U}$ , and the limit belongs to  $BV_{loc}(\mathcal{U})$ .

**Lemma 3.1** *Let  $\mathcal{U}$  be an open subset of  $\mathbb{R}^k$  ( $k \geq 1$ ). Let  $(v^\varepsilon)_\varepsilon$  be a family of functions uniformly bounded in  $L^\infty(\mathcal{U})$  with respect to  $\varepsilon$ . Let  $K$  be a compact subset of  $\mathcal{U}$ . For  $\zeta \in \mathbb{R}^k$ , we denote by*

$$K_\zeta = \{ x \in K \mid x + \zeta \in K \}.$$

*One assumes that there exists  $C > 0$  depending only on  $K$  (and thus not on  $\varepsilon$ ) and a function  $r$  fulfilling  $\lim_{\varepsilon \rightarrow 0} r(\varepsilon) = 0$ , such that for all  $\zeta \in \mathbb{R}^k$ ,*

$$\int_{K_\zeta} |v^\varepsilon(x + \zeta) - v^\varepsilon(x)| dx \leq C|\zeta| + r(\varepsilon). \quad (42)$$

*Then, there exists a sequence  $(\varepsilon_n)_n$  tending to 0, and  $v \in BV_{loc}(\mathcal{U})$  such that*

$$\lim_{n \rightarrow +\infty} v^{\varepsilon_n} = v \quad \text{a.e. in } \mathcal{U}.$$

#### Proof

Let  $K$  be a compact subset of  $\mathcal{U}$ . Estimate (42) says, roughly speaking, that  $v^\varepsilon$  is almost a  $BV$ -function on  $K$ , i.e. as close to  $BV(K)$  as wanted, provided that  $\varepsilon$  is supposed to be small enough. So we will build a family  $(w^\varepsilon)_\varepsilon$  of  $BV$ -functions, which will be close to the family  $(v^\varepsilon)_\varepsilon$ , at least for small  $\varepsilon$ , and we will show that  $(w^\varepsilon)_\varepsilon$  admits an adherence value  $v$  in  $BV(K)$  for the  $L^1(K)$ -topology, and that this  $v$  is also an adherence value of  $(v^\varepsilon)_\varepsilon$ . Another proof for the a.e. convergence toward a function  $v$  can be derived directly from Kolmogorov compactness criterion (see e.g. [11]). But the advantage of the following method is that it provides directly some regularity on the limit  $v \in BV_{loc}(\mathcal{U})$ .

Let  $(\rho^\varepsilon)_\varepsilon$  be a sequence of mollifiers, i.e. smooth, non negative and compactly supported functions with support included in the ball of center 0 and radius  $\varepsilon$ , and fulfilling  $\|\rho^\varepsilon\|_{L^1(\mathbb{R}^k)} = 1$  for all  $\varepsilon > 0$ . We define the smooth functions

$$w^\varepsilon = \tilde{v}^\varepsilon \star \rho^\varepsilon,$$

where  $\tilde{v}^\varepsilon(x) = v^\varepsilon(x)$  if  $x \in \mathcal{U}$  and  $\tilde{v}^\varepsilon(x) = 0$  if  $x \in \mathcal{U}^c$ .

Thanks to the regularity of  $w^\varepsilon$ ,

$$\int_{K_\zeta} |w^\varepsilon(x + \zeta) - w^\varepsilon(x)| dx \leq \|\nabla w^\varepsilon\|_{(L^1(K))^k} |\zeta|. \quad (43)$$

Suppose that  $\varepsilon < d(K, \partial\mathcal{U})$  (with the convention  $d(K, \emptyset) = \infty$ ). Thanks to (42), we have also

$$\int_{K_\zeta} |w^\varepsilon(x + \zeta) - w^\varepsilon(x)| dx \leq \int_{\{K_\zeta + \varepsilon\}} |v^\varepsilon(x + \zeta) - v^\varepsilon(x)| dx \leq C|\zeta| + r(\varepsilon), \quad (44)$$

where

$$\{K_\zeta + \varepsilon\} = \{ x \in \mathcal{U} \mid d(x, K) \leq \varepsilon \}.$$

Since  $r(\varepsilon)$  tends to 0 as  $\varepsilon \rightarrow 0$ , this particularly ensures

$$\limsup_{\varepsilon \rightarrow 0} \|\nabla w^\varepsilon\|_{(L^1(\mathbb{R}^k))^k} \leq C.$$

The family  $(w^\varepsilon)_\varepsilon$  is thus bounded in  $BV(K_\zeta)$  in the neighborhood of  $\varepsilon = 0$ , and thus, thanks to Helly's selection criterion, there exist  $v \in BV(K_\zeta)$ , and  $(\varepsilon_n)_n$  tending to 0 such that

$$w^{\varepsilon_n} \rightarrow v \quad \text{a.e. in } K \text{ as } n \rightarrow \infty.$$

Furthermore, for all  $n \in \mathbb{N}^*$ ,

$$\begin{aligned} \|w^{\varepsilon_n} - v^{\varepsilon_n}\|_{L^1(K)} &\leq \int_K \int_{B(0, \varepsilon_n)} |v^{\varepsilon_n}(x-y) - v^{\varepsilon_n}(x)| \rho^{\varepsilon_n}(y) dy dx \\ &\leq C\varepsilon_n + r(\varepsilon_n). \end{aligned}$$

This ensures that  $v^{\varepsilon_n}$  tends also almost everywhere toward  $v$  as  $n$  tends to  $+\infty$ .  $\square$

Lemma 3.1 will be used to prove the following convergence assertion.

**Proposition 3.2** *Suppose that  $u_0 \in BV(\mathbb{R})$ , and let  $u^\varepsilon$  be a solution to  $(\mathcal{P}^\varepsilon)$ . Up to an extraction, there exists  $u \in L^\infty(\mathbb{R} \times (0, T))$ ,  $0 \leq u \leq 1$  a.e. such that*

$$u^\varepsilon \rightarrow u \text{ a.e. in } \mathbb{R} \times (0, T).$$

Furthermore, there exists  $u_1, u_2 \in L^\infty(0, T)$ , such that

$$\begin{aligned} \lim_{\eta \rightarrow 0} \frac{1}{\eta} \int_0^T \int_{-\eta}^0 |u(x, t) - u_1(t)| dx dt &= 0, \\ \lim_{\eta \rightarrow 0} \frac{1}{\eta} \int_0^T \int_0^\eta |u(x, t) - u_2(t)| dx dt &= 0. \end{aligned}$$

**Proof**

Let  $K$  be a compact subset of  $\overline{\Omega}_i$ .

We define the function  $H_i : [0, 1] \mapsto \mathbb{R}$  by

$$H_i(u) = \int_0^1 (F_i(u, \sigma) - f_i(\sigma)) d\sigma, \quad (45)$$

so that, thanks to Proposition 2.6, there exists  $C$  depending on  $u_0, f_i, T, K$ , and a function  $r$  tending to 0 as  $\varepsilon$  tends to 0 such that for all  $\xi \in \mathbb{R}$ ,  $h \in (0, T)$ ,

$$\int_0^{T-h} \int_K |H_i(u^\varepsilon)(x + \xi, t + h) - H_i(u^\varepsilon)(x, t)| dx dt \leq C(|\xi| + |h|) + r(\varepsilon). \quad (46)$$

An integration by parts in (45) yields:  $\forall u \in [0, 1]$

$$\begin{aligned} H_i(u) &= - \int_0^1 (\sigma - b_i) \partial_\sigma (F_i(u, \sigma) - f_i(\sigma)) d\sigma + (2b_i - 1) f_i(u) \\ &= 2 \int_0^u (\sigma - b_i) f_i'(\sigma) d\sigma + (2b_i - 1) f_i(u) \end{aligned} \quad (47)$$

where, thanks to  $(\mathcal{H})$ ,  $f_i$  is decreasing on  $[0, b_i]$  and increasing on  $[b_i, 1]$ .

Using Proposition 2.6 with  $\kappa = 0$ ,

$$\int_0^{T-h} \int_K |f_i(u^\varepsilon)(x + \xi, t + h) - f_i(u^\varepsilon)(x, t)| dx dt \leq C(|\xi| + |h|) + r(\varepsilon), \quad (48)$$

where  $C$  and  $r$  have been updated. Denoting by  $A_i(u) = \int_0^u (\sigma - b_i) f_i'(\sigma) d\sigma$ , we obtain from (47) and (48)

$$\int_0^{T-h} \int_K |A_i(u^\varepsilon)(x + \xi, t + h) - A_i(u^\varepsilon)(x, t)| dx dt \leq C(|\xi| + |h|) + r(\varepsilon), \quad (49)$$

with a new update for  $C$  and  $r$ . Thus we deduce from Proposition 3.1 that, up to an extraction,  $A_i(u^\varepsilon)$  converges almost everywhere toward  $\overline{A}_i \in BV(K \times (0, T))$ . It follows from  $(\mathcal{H})$  that  $A_i$  is an increasing function, and so we obtain the convergence almost everywhere in  $K \times (0, T)$  of  $u^\varepsilon$  toward a measurable function  $v$ .

Since for all  $\varepsilon > 0$ ,  $0 \leq u^\varepsilon \leq 1$ , there exists  $u \in L^\infty(\mathbb{R} \times (0, T))$ ,  $0 \leq u \leq 1$  such that  $u^\varepsilon$  converges to  $u$  in the  $L^\infty(\mathbb{R} \times (0, T))$ - $\star$ -weak sense, we thus have, up to an extraction,

$$u^\varepsilon \rightarrow u \text{ a.e. in } K \times (0, T). \quad (50)$$



Since (50) holds for any compact subset  $K$  of  $\Omega_i$ , for  $i = 1, 2$ , we can claim that, up to an extraction,

$$u^\varepsilon \rightarrow u \text{ a.e. in } \mathbb{R} \times (0, T).$$

Moreover, since  $A_i(u)$  belongs to  $BV(\mathbb{R} \times (0, T))$ , we can claim that  $A_i(u)$  admits a strong trace on  $\{x = 0\} \times (0, T)$  (see for instance [3]). Using once again the fact that  $A_i^{-1}$  is a continuous function, we can claim that  $u$  admits also a strong trace on each side of the interface.  $\square$

### 3.2 convergence toward the entropy solution

In this section, it is proven that the limit value  $u$  for the family  $(u^\varepsilon)_\varepsilon$  exhibited previously is the entropy solution.

**Proposition 3.3** *Let  $u_0 \in BV(\mathbb{R})$ ,  $0 \leq u_0 \leq 1$  a.e., and let  $(u_0^\varepsilon)_\varepsilon$  a family of approximation of  $u_0$  given by Lemma 2.1, and let  $(u^\varepsilon)_\varepsilon$  be the induced sequence of bounded flux solution of (8)-(13)-(14). Then under Assumption (H)*

$$\lim_{\varepsilon \rightarrow 0} u^\varepsilon = u \text{ in } L_{loc}^p(\mathbb{R} \times [0, T]), \quad \forall p \in [1, \infty)$$

where  $u$  is the unique entropy solution to (10)-(11) associated to initial data  $u_0$ .

#### Proof

As already stated in section 2.5, either  $\mathcal{E}_1 = [0, 1]$ , or  $\mathcal{E}_2 = [0, 1]$ . We suppose that  $\mathcal{E}_j = [0, 1]$ , a straightforward adaptation of the following proof would allow to treat the other case.

Let  $\kappa \in [0, 1]$ . One can let  $\varepsilon$  tend to 0 in (28), and it follows from Propositions 2.2 and 3.2 and Lemma 2.1 that for all  $\psi \in \mathcal{D}^+(\mathbb{R}^* \times [0, T))$ ,

$$\begin{aligned} & \int_0^T \int_{\Omega_i} |u(x, t) - \kappa| \partial_t \psi(x, t) dx dt + \int_{\Omega_i} |u_0(x) - \kappa| \psi(0, x) dx \\ & \int_0^T \int_{\Omega_i} F_i(u, \kappa)(x, t) \partial_x \psi(x, t) dx dt \geq 0. \end{aligned} \quad (51)$$

Let  $\kappa^\varepsilon$  be a steady solution as built in Proposition 2.7. Since  $\kappa^\varepsilon$  is a bounded flux solution, one can compare  $u^\varepsilon$  and  $\kappa^\varepsilon$  thanks to Theorem 1.3. For all  $\kappa \in [0, 1]$ , for all  $\psi \in \mathcal{D}^+(\mathbb{R} \times [0, T))$ ,

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}} |u^\varepsilon(x, t) - \kappa^\varepsilon(x)| \partial_t \psi(x, t) dx dt + \int_{\mathbb{R}_-} |u_0^\varepsilon(x) - \kappa^\varepsilon(x)| \psi(x, 0) dx \\ & + \sum_{i=1,2} \int_0^T \int_{\Omega_i} \text{sign}(u^\varepsilon(x, t) - \kappa^\varepsilon(x)) (f_i(u^\varepsilon)(x, t) - f_i(\kappa^\varepsilon(x))) \partial_x \psi(x, t) dx dt \\ & - \sum_{i=1,2} \int_0^T \int_{\Omega_i} \varepsilon \partial_x |\varphi_i(u^\varepsilon)(x, t) - \varphi_i(\kappa^\varepsilon)(x)| \partial_x \psi(x, t) dx dt \geq 0. \end{aligned} \quad (52)$$

We let  $\varepsilon$  tend to 0 in (52). Thanks to Proposition 2.7, we can suppose that  $\kappa^\varepsilon(x)$  tends to the reachable steady state  $\tilde{\kappa}^j(x)$ . Letting  $\varepsilon$  tend to 0 in (52) yields: for all  $\psi \in \mathcal{D}^+(\mathbb{R} \times [0, T))$ ,

$$\begin{aligned} & \int_0^T \sum_{i=1,2} \int_{\Omega_i} |u(x, t) - \tilde{\kappa}_i^j| \partial_t \psi(x, t) dx dt + \sum_{i=1,2} \int_{\Omega_i} |u_0(x) - \tilde{\kappa}_i^j| \psi(0, x) dx \\ & \int_0^T \sum_{i=1,2} \int_{\Omega_i} F_i(u, \tilde{\kappa}_i^j)(x, t) \partial_x \psi(x, t) dx dt \geq 0. \end{aligned} \quad (53)$$

Since  $\tilde{\kappa}^j = 0$  and  $\tilde{\kappa}^j = 1$  are reachable states, we deduce from (53) that  $u$  is a weak solution to (10)-(11), i.e. :  $\forall \psi \in \mathcal{D}(\mathbb{R} \times [0, T))$ ,

$$\begin{aligned} & \int_0^T \int_{\Omega_i} u(x, t) \partial_t \psi(x, t) dx dt + \int_{\Omega_i} u_0(x) \psi(0, x) dx \\ & \int_0^T \int_{\Omega_i} f_i(u)(x, t) \partial_x \psi(x, t) dx dt \geq 0. \end{aligned} \quad (54)$$

Since  $u$  is a weak solution,  $u_1$  and  $u_2$  fulfill the Rankine-Hugoniot condition:

$$f_1(u_1) = f_2(u_2) \text{ a.e. in } (0, T). \quad (55)$$

Let  $\eta > 0$ , we define  $\omega_\eta(x) = (1 - |x|/\eta)^+$ . Let  $\psi \in \mathcal{D}^+(\mathbb{R} \times [0, T])$  taking  $\psi(1 - \omega_\eta)$  as test function in (51), and letting  $\eta$  tend to 0 yields

$$\begin{aligned} & \int_0^T \int_{\Omega_i} |u(x, t) - \kappa| \partial_t \psi(x, t) dx dt + \int_{\Omega_i} |u_0(x) - \kappa| \psi(0, x) dx \\ & \quad + \int_0^T \int_{\Omega_i} F_i(u, \kappa)(x, t) \partial_x \psi(x, t) dx dt \\ & \quad + \int_0^T [F_2(u_2, \kappa)(t) - F_1(u_1, \kappa)(t)] \psi(0, t) dt \geq 0. \end{aligned} \quad (56)$$

We will conclude the proof of the Proposition 3.3 by the following inequality:  $\forall \psi \in \mathcal{D}^+(\mathbb{R} \times [0, T])$ ,  $\forall \kappa \in [0, 1]$ ,

$$F_2(u_2, \kappa) - F_1(u_1, \kappa) \leq |f_2(\kappa) - f_1(\kappa)| \text{ a.e. in } (0, T). \quad (57)$$

Indeed, assume that (57) holds, then using (57) in (56) gives the entropy formulation (12), and  $u$  is an entropy solution in the sense of Definition 1.1. The convergence of the whole sequence is just a consequence of the uniqueness of the entropy solution, as stated in Theorem 1.1.  $\square$

We focus now on the last point to achieve the proof of Proposition 3.3, i.e. the proof of (57). This relation ensures that the limit  $u$  of the solutions  $u^\varepsilon$  of the approximate problem ( $\mathcal{P}^\varepsilon$ ) fulfills a entropy criterion at the level interface. As it was already stressed before, the relation (57) strongly uses Assumption ( $\mathcal{H}$ ). Indeed, the forthcoming proof uses the reachable steady states  $\tilde{\kappa}^j$ , whose building is based on ( $\mathcal{H}$ ).

*Proof of (57).* Choosing  $\psi \omega_\eta$  as test function in (53), and letting  $\eta$  tend to 0 yields

$$\int_0^T [F_1(u_1, \tilde{\kappa}_1^j)(t) - F_2(u_2, \tilde{\kappa}_2^j)(t)] \psi(0, t) dt \geq 0 \quad (58)$$

where  $\tilde{\kappa}_i^j$  is the value of  $\tilde{\kappa}^j$  in  $\Omega_i$ . Since this inequality holds for any non negative  $\psi$  and since  $F_i(u_i, \tilde{\kappa}_i^j)$  is uniformly bounded w.r.t.  $t \in (0, T)$ , a classical density argument provides

$$F_1(u_1, \tilde{\kappa}_1^j) - F_2(u_2, \tilde{\kappa}_2^j) \geq 0 \text{ a.e. in } (0, T). \quad (59)$$

This ensures that

$$F_2(u_2, \kappa) - F_1(u_1, \kappa) \leq A_1(u_1, \kappa) - A_2(u_2, \kappa) \quad (60)$$

with

$$\begin{aligned} A_1(u_1, \kappa) &= F_1(u_1, \tilde{\kappa}_1^j) - F_1(u_1, \kappa), \\ A_2(u_2, \kappa) &= F_2(u_2, \tilde{\kappa}_2^j) - F_2(u_2, \kappa), \end{aligned}$$

and the aim is now to show

$$A_1(u_1, \kappa) - A_2(u_2, \kappa) \leq |f_1(\kappa) - f_2(\kappa)|. \quad (61)$$

We will perform a case by case study to prove (61), and we will suppose for the sake of simplicity that  $j = 1$ , i.e.

$$f_i(\tilde{\kappa}_i^1) = f_1(\kappa), \quad i = 1, 2.$$

The proof in the case  $j = 2$  can be easily derived from the case  $j = 1$ . We also suppose in the sequel that  $q \geq 0$ . The case  $q < 0$  can also be derived, comparing a convenient choice of steady states  $\tilde{\kappa}^j$  with the solution  $u$ .

- If  $u_1 \geq \bar{\kappa}_1^1$  :  
since  $\kappa$  is equal either to  $\underline{\kappa}_1^1$  or to  $\bar{\kappa}_1^1$  (that are defined by (37) and (38)), one gets

$$u_1 \geq \bar{\kappa}_1^1 \geq \kappa \geq \underline{\kappa}_1^1$$

and  $f_1(\kappa) = f_1(\bar{\kappa}_1^1)$ . This ensures

$$A_1(u_1, \kappa) = 0. \quad (62)$$

One can furthermore claim thanks to Assumption ( $\mathcal{H}$ ) that  $f_1(u_1) \geq f_1(\kappa)$ . Using the definition of  $\bar{\kappa}_2^1$ , and (55), we obtain  $f_2(u_2) \geq f_2(\bar{\kappa}_2^1)$ , and so either  $u_2 \geq \bar{\kappa}_2^1$ , or  $u_2 \leq \underline{\kappa}_2^1$  and  $0 \geq f_2(u_2) \geq f_2(\underline{\kappa}_2^1) = f_1(\kappa)$ .

- If  $u_2 \geq \max(\kappa, \bar{\kappa}_2^1)$ ,

$$A_2(u_2, \kappa) = f_2(\bar{\kappa}_2^1) - f_2(\kappa) = f_1(\kappa) - f_2(\kappa). \quad (63)$$

- If  $u_2 \leq \min(\kappa, \bar{\kappa}_2^1)$ ,

$$A_2(u_2, \kappa) = f_2(\kappa) - f_2(\bar{\kappa}_2^1) = f_2(\kappa) - f_1(\kappa). \quad (64)$$

- If  $\bar{\kappa}_2^1 \leq u_2 \leq \kappa$ ,

$$A_2(u_2, \kappa) = 2f_2(u_2) - f_2(\kappa) - f_2(\bar{\kappa}_2^1).$$

Since one has  $f_2(u_2) \geq f_2(\bar{\kappa}_2^1) = f_1(\kappa)$ , we obtain

$$A_2(u_2, \kappa) \geq f_1(\kappa) - f_2(\kappa). \quad (65)$$

- If  $\kappa \leq u_2 \leq \underline{\kappa}_2^1$ ,

$$A_2(u_2, \kappa) = -2f_2(u_2) + f_2(\kappa) + f_2(\bar{\kappa}_2^1) \geq f_2(\kappa) - f_1(\kappa). \quad (66)$$

Estimate (61) follows from (62)-(63)-(64)-(65)-(66).

- If  $u_1 \leq \underline{\kappa}_1^1$  :  
then (62) still holds. In this case, ( $\mathcal{H}$ ) provides the fact that either  $0 \geq f_1(u_1) \geq f_1(\underline{\kappa}_1^1)$ , or  $f_1(u_1) \leq f_1(\underline{\kappa}_1^1)$ . Here again, the cases  $u_2 \geq \max(\kappa, \bar{\kappa}_2^1)$  and  $u_2 \leq \min(\kappa, \bar{\kappa}_2^1)$  will not lead to any difficulty.

- Suppose  $f_1(u_1) \leq f_1(\underline{\kappa}_1^1)$ , so that  $f_2(u_2) \leq f_2(\underline{\kappa}_2^1)$ , and so either  $u_2 \geq \bar{\kappa}_2^1$ , or  $u_2 \leq \underline{\kappa}_2^1$  and  $0 \geq f_2(u_2) \geq f_2(\underline{\kappa}_2^1) = f_1(\kappa)$ . We can reproduce the discussion already done to obtain (65) and (66).

- Suppose now that  $0 \geq f_1(u_1) \geq f_1(\underline{\kappa}_1^1)$ , then  $0 \geq f_2(u_2) \geq f_2(\underline{\kappa}_2^1)$  and so  $u_2$  does not belong to  $(\underline{\kappa}_2^1, \bar{\kappa}_2^1)$ . Estimate (65) or (66) holds, and thus (61) too.

- If  $u_1 \in (\underline{\kappa}_1^1, \bar{\kappa}_1^1)$  :

then it follows from ( $\mathcal{H}$ ) that  $0 \geq f_1(\kappa) \geq f_1(u_1)$ , and then  $0 \geq f_2(\bar{\kappa}_2^1) \geq f_2(u_2)$ . We can thus claim that  $\underline{\kappa}_2^1 \leq u_2 \leq \bar{\kappa}_2^1$ .

- If  $\kappa = \bar{\kappa}_1^1$ , choosing the reachable steady state  $\tilde{\kappa}^1(x) = \bar{\kappa}_1^1$  if  $x < 0$  and  $\tilde{\kappa}(x) = \bar{\kappa}_2^1$  if  $x > 0$  leads to

$$A_1(u_1, \kappa) = 0,$$

The case  $\kappa \geq u_2$  can be treated as in (64), and so we focus on the case  $\kappa \leq u_2 \leq \bar{\kappa}_2^1$ , then

$$A_2(u_2, \kappa) = -2f_2(u_2) + f_2(\kappa) + f_2(\bar{\kappa}_2^1) \geq f_2(\kappa) - f_1(\kappa).$$

- If  $\kappa = \underline{\kappa}_1^1$ , choosing once again  $\tilde{\kappa}_1^1 = \bar{\kappa}_1^1$  and  $\tilde{\kappa}_2^1 = \bar{\kappa}_2^1$ , we get

$$\begin{aligned} A_1(u_1, \kappa) &= 2(f_1(\kappa) - f_1(u_1)), \\ A_2(u_2, \kappa) &= -2f_2(u_2) + f_2(\kappa) + f_2(\bar{\kappa}_2^1), \end{aligned}$$

and so (55) provides

$$A_1(u_1, \kappa) - A_2(u_2, \kappa) = f_1(\kappa) - f_2(\kappa).$$

This discussion ends the proof of (57), and so the one of Proposition 3.3.  $\square$

We state now the main result, which is in fact the extension of Proposition 3.3 to a larger class of initial data.

**Theorem 3.4 (main result)** *Let  $u_0 \in L^\infty(\mathbb{R})$ ,  $0 \leq u_0 \leq 1$ , and let  $(u_0^\varepsilon)_\varepsilon$  a family of approximation of  $u_0$  given by Lemma 2.1, and let  $(u^\varepsilon)_\varepsilon$  be the induced sequence of bounded flux solution of  $(\mathcal{P}^\varepsilon)$ . Then under Assumption  $(\mathcal{H})$ ,*

$$\lim_{\varepsilon \rightarrow 0} u^\varepsilon = u \text{ in } L^p_{loc}(\mathbb{R} \times [0, T]), \quad \forall p \in [1, \infty)$$

where  $u$  is the unique entropy solution to (10)-(11) associated to initial data  $u_0$ .

**Proof**

Let  $u_0 \in L^\infty(\mathbb{R})$ ,  $0 \leq u_0 \leq 1$ , and let  $\nu > 0$ . There exists  $u_{0,\nu}$  in  $BV(\mathbb{R})$ ,  $0 \leq u_{0,\nu} \leq 1$  such that for all  $R > 0$ ,

$$\|u_{0,\nu} - u_0\|_{L^1(-R,R)} \leq C(R)\nu. \quad (67)$$

If one regularizes  $u_{0,\nu}$  into  $u_{0,\nu}^\varepsilon$  using Lemma 2.1, and if one denotes by  $u_\nu^\varepsilon$  the associated unique bounded flux solution, we have seen that  $u_\nu^\varepsilon$  converges almost everywhere to  $u_\nu$  as  $\varepsilon$  tends to 0. As previously, we denote by  $u_0^\varepsilon$  the regularization of  $u_0$  obtained via Lemma 2.1, and  $u^\varepsilon$  the unique associated bounded flux solution.

Let  $R > 0$ , then we have

$$\begin{aligned} \int_0^T \int_{-R}^R |u^\varepsilon - u| dx dt &\leq \int_0^T \int_{-R}^R |u^\varepsilon - u_\nu^\varepsilon| dx dt + \int_0^T \int_{-R}^R |u_\nu^\varepsilon - u_\nu| dx dt \\ &\quad + \int_0^T \int_{-R}^R |u_\nu - u| dx dt. \end{aligned} \quad (68)$$

The contraction principle stated in Theorem 1.1 yields

$$\int_0^T \int_{-R}^R |u_\nu(x,t) - u(x,t)| dx dt \leq T \int_{-R-MT}^{R+MT} |u_{0,\nu}(x) - u_0(x)| dx \quad (69)$$

where  $M \geq \max_i Lip(f_i)$ .

We denote by  $\zeta(x,t) = \min(1, (R+1+M(T-t)-|x|)^+)$ . One has  $\zeta = 1$  on  $(-R, R)$ ,  $\zeta \geq 0$  and  $\zeta \in L^1(\mathbb{R})$  thus

$$\int_0^T \int_{-R}^R |u^\varepsilon - u_\nu^\varepsilon| dx dt \leq \int_0^T \int_{\mathbb{R}} |u^\varepsilon - u_\nu^\varepsilon| \zeta dx dt. \quad (70)$$

It follows from Proposition 1.3 that for all  $\psi \in W^{1,1}(\mathbb{R} \times (0, T))$  with  $\psi \geq 0$  a.e. and  $\psi(\cdot, T) = 0$ , one has

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}} |u^\varepsilon(x,t) - u_\nu^\varepsilon(x,t)| \partial_t \psi(x,t) dx dt + \int_{\mathbb{R}} |u_0^\varepsilon(x) - u_{0,\nu}^\varepsilon(x)| \psi(x,0) dx \\ &+ \int_0^T \sum_{i=1,2} \int_{\Omega_i} \text{sign}(u^\varepsilon(x,t) - u_\nu^\varepsilon(x,t)) (f_i(u^\varepsilon)(x,t) - f_i(u_\nu^\varepsilon)(x,t)) \partial_x \psi(x,t) dx dt \\ &- \int_0^T \sum_{i=1,2} \int_{\Omega_i} \varepsilon \partial_x |\varphi_i(u^\varepsilon)(x,t) - \varphi_i(u_\nu^\varepsilon)(x,t)| \partial_x \psi(x,t) dx dt \geq 0. \end{aligned} \quad (71)$$

We denote by

$$\Lambda_1(t) = [-R-1-M(T-t), -R-M(T-t)],$$

$$\Lambda_2(t) = [R+M(T-t), R+1+M(T-t)],$$

$$\vartheta(x) = \begin{cases} 1 & \text{if } x < 0, \\ -1 & \text{if } x > 0, \end{cases}$$

so that

$$\partial_x \zeta(x,t) = \sum_{i=1,2} \vartheta(x) \chi_{\Lambda_i(t)}(x), \quad \partial_t \zeta(x,t) = -M \sum_{i=1,2} \chi_{\Lambda_i(t)}(x).$$

Taking  $\psi(x, t) = (T - t)\zeta(x, t)$  in (71) yields :

$$\begin{aligned}
& - \int_0^T \int_{\mathbb{R}} |u^\varepsilon(x, t) - u_\nu^\varepsilon(x, t)| \zeta(x, t) dx dt + T \int_{\mathbb{R}} |u_0^\varepsilon(x) - u_{0,\nu}^\varepsilon(x)| \zeta(x, 0) dx \\
& \quad - M \int_0^T (T - t) \sum_{i=1,2} \int_{\Lambda_i(t)} |u^\varepsilon(x, t) - u_\nu^\varepsilon(x, t)| dx dt \\
& + \int_0^T (T - t) \sum_{i=1,2} \int_{\Lambda_i(t)} \text{sign}(u^\varepsilon(x, t) - u_\nu^\varepsilon(x, t)) (f_i(u^\varepsilon)(x, t) - f_i(u_\nu^\varepsilon)(x, t)) dx dt \\
& \quad - \int_0^T (T - t) \sum_{i=1,2} \int_{\Lambda_i(t)} \varepsilon \vartheta(x) \partial_x |\varphi_i(u^\varepsilon)(x, t) - \varphi_i(u_\nu^\varepsilon)(x, t)| dx dt \geq 0.
\end{aligned}$$

Since  $M \geq \max_i \text{Lip}(f_i)$ , one has

$$\begin{aligned}
& \int_0^T (T - t) \sum_{i=1,2} \int_{\Lambda_i(t)} \text{sign}(u^\varepsilon(x, t) - u_\nu^\varepsilon(x, t)) (f_i(u^\varepsilon)(x, t) - f_i(u_\nu^\varepsilon)(x, t)) dx dt \\
& \leq M \int_0^T (T - t) \sum_{i=1,2} \int_{\Lambda_i(t)} |u^\varepsilon(x, t) - u_\nu^\varepsilon(x, t)| dx dt
\end{aligned}$$

and thus

$$\begin{aligned}
& \int_0^T \int_{\mathbb{R}} |u^\varepsilon(x, t) - u_\nu^\varepsilon(x, t)| \zeta(x, t) dx dt \leq T \int_{\mathbb{R}} |u_0^\varepsilon(x) - u_{0,\nu}^\varepsilon(x)| \zeta(x, 0) dx \\
& \quad - \int_0^T (T - t) \sum_{i=1,2} \int_{\Lambda_i(t)} \varepsilon \vartheta(x) \partial_x |\varphi_i(u^\varepsilon)(x, t) - \varphi_i(u_\nu^\varepsilon)(x, t)| dx dt. \tag{72}
\end{aligned}$$

We deduce, using (69) (70) and (72) in (68), that

$$\begin{aligned}
& \int_0^T \int_{-R}^R |u^\varepsilon(x, t) - u(x, t)| dx dt \leq T \int_{\mathbb{R}} |u_0^\varepsilon(x) - u_{0,\nu}^\varepsilon(x)| \zeta(x, 0) dx \\
& \quad - \int_0^T (T - t) \sum_{i=1,2} \int_{\Lambda_i(t)} \varepsilon \vartheta(x) \partial_x |\varphi_i(u^\varepsilon)(x, t) - \varphi_i(u_\nu^\varepsilon)(x, t)| dx dt \\
& \quad + T \int_{-R-MT}^{R+MT} |u_{0,\nu}(x) - u_0(x)| dx + \int_0^T \int_{-R}^R |u_\nu^\varepsilon(x, t) - u_\nu(x, t)| dx dt. \tag{73}
\end{aligned}$$

We can now let  $\varepsilon$  tend to 0. Thanks to Proposition 2.2, we can claim that

$$\lim_{\varepsilon \rightarrow 0} \int_0^T (T - t) \sum_{i=1,2} \int_{\Lambda_i(t)} \varepsilon \vartheta(x) \partial_x |\varphi_i(u^\varepsilon)(x, t) - \varphi_i(u_\nu^\varepsilon)(x, t)| dx dt = 0.$$

We also deduce from Proposition 3.3 that

$$\lim_{\varepsilon \rightarrow 0} \int_0^T \int_{-R}^R |u_\nu^\varepsilon(x, t) - u_\nu(x, t)| dx dt = 0.$$

Since  $\zeta(x, 0)$  is compactly supported, it follows from Lemma 2.1 and the dominated convergence theorem that

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}} |u_0^\varepsilon(x) - u_{0,\nu}^\varepsilon(x)| \zeta(x, 0) dx = \int_{\mathbb{R}} |u_0(x) - u_{0,\nu}(x)| \zeta(x, 0) dx.$$

Thus (73) becomes:

$$\begin{aligned}
\limsup_{\varepsilon \rightarrow 0} \int_0^T \int_{-R}^R |u^\varepsilon(x, t) - u(x, t)| dx dt & \leq T \int_{-R-MT}^{R+MT} |u_{0,\nu}(x) - u_0(x)| dx \\
& \quad + \int_{\mathbb{R}} |u_0(x) - u_{0,\nu}(x)| \zeta(x, 0) dx. \tag{74}
\end{aligned}$$

The inequality (74) holds for any  $\nu > 0$ , and letting  $\nu$  tend to 0 leads to

$$\lim_{\varepsilon \rightarrow 0} \int_0^T \int_{-R}^R |u^\varepsilon(x, t) - u(x, t)| dx dt = 0,$$

then  $u^\varepsilon$  tends to  $u$  in  $L^1_{loc}(\mathbb{R} \times [0, T])$  as  $\varepsilon \rightarrow 0$ . Since  $0 \leq u^\varepsilon \leq 1$  a.e., Hölder inequality gives the convergence in  $L^p_{loc}(\mathbb{R} \times [0, T])$  for all finite  $p$ .  $\square$

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