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Asymptotic behavior of two-phase flows in heterogeneous porous media for capillarity depending only on space.

I. Convergence to the optimal entropy solution.

Clément Cancès*†

November 6, 2009

Abstract

We consider an immiscible two-phase flow in a heterogeneous one-dimensional porous medium. We suppose particularly that the capillary pressure field is discontinuous with respect to the space variable. The dependence of the capillary pressure with respect to the oil saturation is supposed to be weak, at least for saturations which are not too close to 0 or 1. We study the asymptotic behavior when the capillary pressure tends to a function which does not depend on the saturation. In this paper, we show that if the capillary forces at the spacial discontinuities are oriented in the same direction that the gravity forces, or if the two phases move in the same direction, then the saturation profile with capillary diffusion converges toward the unique optimal entropy solution to the hyperbolic scalar conservation law with discontinuous flux functions.

key words. *entropy solution, scalar conservation law, discontinuous porous media, capillarity*

AMS subject classification. *35L65, 76S05*

1 Presentation of the problem

The resolution of multi-phase flows in porous media are widely used in oil engineering to predict the motions of oil in subsoil. Their mathematical study is however difficult, then some physical assumptions have to be done, in order to get simpler problems (see e.g. [6, 11, 30]). A classical simplified model, so-called *dead-oil* approximation, consists in assuming that there is no gas, i.e. that the fluid is composed of two immiscible and incompressible phases and in neglecting all the different chemical species. The oil-phase and the water-phase are then both made of only one component.

1.1 the dead-oil problem in the one dimensional case

Suppose that \mathbb{R} represents a one dimensional homogeneous porous medium, with porosity ϕ (which is supposed to be constant for the sake of simplicity). If u denotes the saturation of the water phase, and so $(1 - u)$ the saturation of the oil phase thanks to the dead-oil approximation, writing the volume conservation of each phase leads to:

$$\phi \partial_t u + \partial_x V_w = 0, \quad (1)$$

$$-\phi \partial_t u + \partial_x V_o = 0, \quad (2)$$

where V_o (resp. V_w) is the filtration speed of the oil phase (resp. water phase). Using the empirical diphasic Darcy law, we claim that

$$V_\beta = -K \frac{k_{r,\beta}(u)}{\mu_\beta} (\partial_x P_\beta - \rho_\beta g), \quad \beta = o, w, \quad (3)$$

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where K is the global permeability, only depending on the porous media, $\mu_\beta, P_\beta, \rho_\beta$ are respectively the dynamical viscosity, the pressure and the density of the phase β , g represents the effect of gravity, $k_{r,\beta}$ denotes the relative permeability of the phase β . This last term comes from the interference of the two phases in the porous media.

There exists $s_\star \in [0, 1)$ such that the function $k_{r,w}$ is non-decreasing, with $k_{r,w}(u) = 0$ if $0 \leq u \leq s_\star < 1$, and $k_{r,w}$ is increasing on $[s_\star, 1]$. The function $k_{r,o}$ is supposed to be non-increasing, with $k_{r,o}(1) = 0$. We suppose that there exists $s^\star \in (s_\star, 1]$ such that $k_{r,o}(s) = 0$ for $s \in [s^\star, 1)$, and $k_{r,o}$ is decreasing on $[0, s^\star)$.

The pressures are supposed to be linked by the relation

$$P_{cap}(u) = P_w - P_o, \quad (4)$$

where P_{cap} is a smooth non-decreasing function called *capillary pressure*.

Adding (1) and (2), and using (3) and (4) yields

$$-\partial_x \left(\sum_{\beta=o,w} K \frac{k_{r,\beta}(u)}{\mu_\beta} (\partial_x P_\beta - \rho_\beta \mathbf{g}) \right) = 0,$$

and thus there exists q , called total flow-rate, only depending on time, such that

$$-\sum_{\beta=o,w} K \frac{k_{r,\beta}(u)}{\mu_\beta} (\partial_x P_\beta - \rho_\beta g) = q. \quad (5)$$

Using (3), (4) and (5), (1) can be rewritten

$$\begin{aligned} & \phi \partial_t u + \partial_x \left(\frac{q k_{r,w}(u)}{k_{r,w}(u) + \frac{\mu_w}{\mu_o} k_{r,o}(u)} \right) \\ & - K \partial_x \left(\frac{k_{r,w}(u) k_{r,o}(u)}{\mu_o k_{r,w}(u) + \mu_w k_{r,o}(u)} (\partial_x P_{cap}(u) - (\rho_w - \rho_o)g) \right) = 0. \end{aligned} \quad (6)$$

Supposing that the total flow rate q does not depend on times, and after a convenient rescaling, equation (6) becomes

$$\partial_t u + \partial_x (f(u) - \lambda(u) \partial_x \pi(u)) = 0, \quad (7)$$

where f is a Lipschitz continuous function, fulfilling $f(0) = 0$, $f(1) = q$, λ is a nonnegative Lipschitz continuous functions, with $\lambda(0) = \lambda(1) = 0$, and π is a non-decreasing function, also called capillary pressure. The effects of capillarity are often neglected, particularly in the case of reservoir simulation, and so (7) turns to a nonlinear hyperbolic equation called Buckley-Leverett equation, and we have to consider the initial-value problem

$$\begin{cases} \partial_t u + \partial_x f(u) = 0, \\ u(0) = u_0. \end{cases} \quad (\mathcal{BL})$$

1.2 discontinuous flux functions and optimal entropy solution

We now consider heterogeneous one dimensional porous media, i.e. an apposition of several homogeneous porous media with different physical properties. This leads to discontinuous functions with respect to the spatial variable. For the sake of simplicity, we assume that the heterogeneous porous medium is made of only two homogeneous porous media represented by the open subsets $\Omega_1 = \mathbb{R}_-^*$ and $\Omega_2 = \mathbb{R}_+^*$. Keeping the notations of (6), $\phi, K, k_{r,\beta}(u, \cdot)$ and $\pi(u, \cdot)$ are now discontinuous functions, i.e. piecewise constant functions, denoted $\phi_i, K_i, k_{r,\beta,i}$ and π_i in Ω_i . Thus the problem becomes

$$\begin{cases} \partial_t u + \partial_x (f_i(u) - \lambda_i(u) \partial_x \pi_i(u)) = 0, \\ u(0) = u_0, \\ + \text{transmission condition at } x = 0, \end{cases} \quad (8)$$

where f_i are Lipschitz continuous functions on $[0, 1]$, and can be decomposed in the following way:

$$f_i(u) = q r_i(u) + \lambda_i(u) (\rho_w - \rho_o) g, \quad (9)$$

where r_i is a non-decreasing Lipschitz continuous function fulfilling $r_i(0) = 0$, $r_i(1) = 1$, and λ_i is a non-negative Lipschitz continuous function fulfilling $\lambda_i(0) = 0$, $\lambda_i(1) = 0$. We stress here the fact that q and $(\rho_w - \rho_o)g$ neither depend on the subdomain i nor on time.

We now have to give more details on this transmission conditions at $x = 0$. First neglect the effects of capillarity, so that (8) becomes the apposition of two Buckley-Leverett equations, linked by a transmission condition.

$$\begin{cases} \partial_t u + \partial_x f_i(u) = 0, \\ u(0) = u_0, \\ + \text{transmission condition at } x = 0, \end{cases} \quad (10)$$

We ask the conservation of mass at the interface between the two porous media, then we have to connect the flux. Denoting u_i the trace (if it exists) of $u|_{\Omega_i}$ on $\{x = 0\}$, this means that the following Rankine-Hugoniot condition has to be fulfilled:

$$f_1(u_1) = f_2(u_2). \quad (11)$$

Some assumptions has to be done on the flux functions f_i in order to carry out the study. Firstly, we suppose that the total flow-rate q is a non-negative constant. Dealing with non-positive q is also possible, since it suffices to change x by $-x$ and u by $(1 - u)$. Secondly, we suppose that each f_i has a simple dynamic on $[0, 1]$. More precisely,

$$\exists b_i \in [0, 1] \text{ s.t. } f_i \text{ is decreasing on } (0, b_i) \text{ and increasing on } (b_i, 1). \quad (12)$$

With Assumption (12), we particularly ensure that

$$q = f_i(1) = \max_{s \in [0, 1]} (f_i(s)).$$

The physical meaning of (12) is that buoyancy works on the oil-phase in the sense of decaying x . The case $b_i = 0$ can also correspond to situations where the total flow rate q is sufficiently strong for ensuring that both phases always move in the same direction. Indeed, The oil-flux, given in Ω_i by $f_i(u)$ has the same sign as the water-flux, given by $q - f_i(u)$. Note that the assumption on the dynamic on f_i is often fulfilled by the physical models, as it is stressed in [1] (see also [29]).

Thirdly, we assume

$$f_1 \text{ and } f_2 \text{ are not linear on any non-degenerate interval of } (0, 1). \quad (13)$$

This latter assumption allows us to claim, thanks to [38] (see also [44]) that a solution of

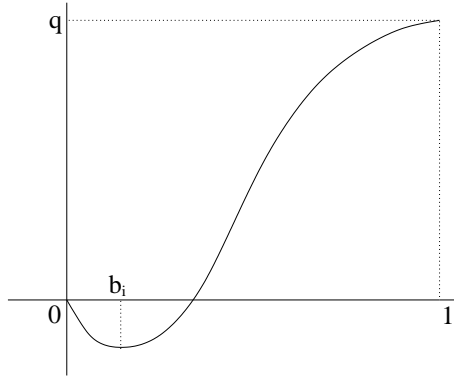


Figure 1: example of f_i fulfilling (12) and (13)

$$\partial_t u + \partial_x f_i(u) = 0$$

satisfying the entropy inequalities in $\Omega_i \times (0, T)$: $\forall \kappa \in [0, 1]$

$$\partial_t |u - \kappa| + \partial_x (\text{sign}(u - \kappa)(f_i(u) - f_i(\kappa))) \leq 0 \quad \text{in } \mathcal{D}'(\Omega_i \times [0, T]) \quad (14)$$

admits a strong trace u_i on $\{x = 0\} \times (0, T)$.

Remark. 1.1 As it is proven in Section 3.1, if $u_0 \in BV(\mathbb{R})$, the solution u we consider has strong traces on the interface without assuming (13). The assumption (13) provides the existence of strong traces for initial data in $L^\infty(\mathbb{R})$.

The problem (10)-(11) has been widely studied recently (see e.g. [1, 3, 5, 7, 8, 9, 22, 23, 24, 25, 31, 32, 33, 34, 35, 36, 39, 41, 42, 43]). It has been particularly shown by Adimurthi, Mishra and Veerappa Gowda [2] that there are infinitely many solutions satisfying (14). Additional entropy conditions has to be considered at the interface $\{x = 0\}$. We refer to [2] and [14] for a detailed discussion on the possible choices of entropy conditions at the interface. According to Kaasschieter [33] and Adimurthi, Jaffré and Veerappa Gowda [1], the relevant entropy condition at the interface for two-phase flows with *continuous capillary pressure field* is the so-called *optimal entropy condition* introduced in [2]. Assuming that both f_i are convex, the optimal entropy solution is characterized as follows: the discontinuity at the interface between u_1 and u_2 can not be undercompressive:

$$\min \{0, f_1'(u_1)\} \max \{0, f_2'(u_2)\} = 0. \quad (15)$$

Following the idea of Audusse and Perthame [5], the entropy condition at the interface can be derived by comparing the solution to steady states. Denoting by $\tilde{\kappa}(x) = \kappa_i$ if $x \in \Omega_1$, where $f_1(\kappa_1) = f_2(\kappa_2)$, then $\tilde{\kappa}$ has to satisfy the relation

$$\min \{0, f_1'(\kappa_1)\} \max \{0, f_2'(\kappa_2)\} = 0.$$

We denote by $\kappa_{\text{opt}}(x)$ be the steady state corresponding to the optimal entropy connection appearing in the work of Adimurthi, Mishra and Veerappa Gowda [2], described on Figure 2 and defined as follows:

- if $f_1(b_1) \leq f_2(b_2)$, then

$$\kappa_{\text{opt}}(x) = \begin{cases} b_2 & \text{if } x > 0, \\ \underline{b}_2 = \min \{\nu \mid f_1(\nu) = f_2(b_2)\} & \text{if } x < 0; \end{cases} \quad (16)$$

- if $f_1(b_1) \geq f_2(b_2)$, then

$$\kappa_{\text{opt}}(x) = \begin{cases} b_1 & \text{if } x < 0, \\ \bar{b}_1 = \max \{\nu \mid f_2(\nu) = f_1(b_1)\} & \text{if } x > 0. \end{cases} \quad (17)$$

It is easy to check that in both cases, the function $(x, t) \mapsto \kappa_{\text{opt}}(x)$ is a steady entropy solution in the sense of Definition 1.2. According to [2] and [14], the condition (15) can be replaced by the

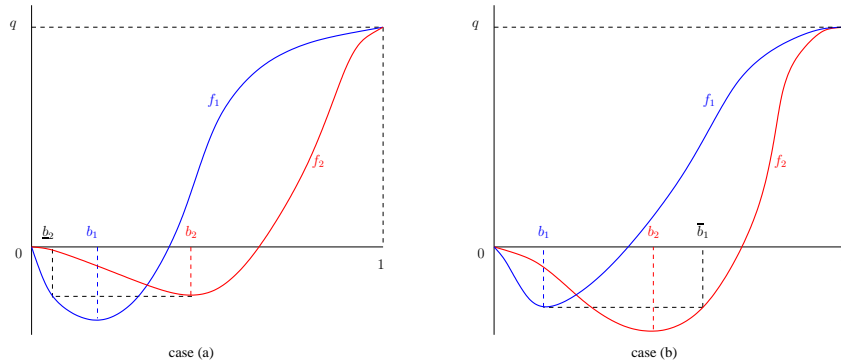


Figure 2: We exhibit particular steady states, which are so called *optimal connections* in the work of Adimurthi, Mishra and Veerappa Gowda [2]. The case (a) corresponds to the optimal connection (16) while the case (b) corresponds to the optimal connection (17).

formulation: $\forall \psi \in \mathcal{D}^+(\mathbb{R} \times [0, T])$,

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}} |u - \kappa_{\text{opt}}| \partial_t \psi \, dx dt + \int_{\mathbb{R}} |u_0 - \kappa_{\text{opt}}| \psi(\cdot, 0) \, dx \\ & + \int_0^T \sum_{i=1,2} \int_{\Omega_i} \text{sign}(u - \kappa_{\text{opt}}) (f_i(u) - f_i(\kappa_{\text{opt}})) \partial_x \psi \, dx dt \geq 0. \end{aligned} \quad (18)$$

This interface entropy condition does not require anymore that both f_i are convex to provide a L^1 -contraction semi-group, as it will be stated in Theorem 1.3 and shown in [14].

Definition 1.2 (entropy solution to (10)-(11)) Let $u_0 \in L^\infty(\mathbb{R})$ with $0 \leq u_0 \leq 1$, and let $T > 0$. A function u is said to be an entropy solution to (10)-(11) if

1. $u \in L^\infty(\mathbb{R} \times (0, T))$, $0 \leq u \leq 1$ a.e.;
2. for all $\psi \in \mathcal{D}^+(\mathbb{R} \times [0, T))$,

$$\int_0^T \int_{\mathbb{R}} u \partial_t \psi \, dx dt + \int_{\mathbb{R}} u_0 \psi(\cdot, 0) dx + \sum_{i=1,2} \int_0^T \int_{\Omega_i} f_i(u) \partial_x \psi \, dx dt = 0; \quad (19)$$

3. for $i = 1, 2$, for all $\psi \in \mathcal{D}^+(\Omega_i \times [0, T))$, for all $\kappa \in [0, 1]$,

$$\begin{aligned} & \int_0^T \int_{\Omega_i} |u - \kappa| \partial_t \psi \, dx dt + \int_{\Omega_i} |u_0 - \kappa| \psi(\cdot, 0) dx \\ & + \int_0^T \int_{\Omega_i} \text{sign}(u - \kappa) (f_i(u) - f_i(\kappa)) \partial_x \psi \, dx dt \geq 0; \end{aligned} \quad (20)$$

4. the inequalities (18) hold.

In the following theorem, we claim the existence and the uniqueness of the entropy solution to the problem (10)-(11).

Theorem 1.3 (Existence and uniqueness of the entropy solution) Let $u_0 \in L^\infty(\mathbb{R})$ with $0 \leq u_0 \leq 1$, and let $T > 0$, then there exists a unique entropy solution to (10)-(11) in the sense of Definition 1.2.

Furthermore, the function u can be supposed to belong to $C([0, T]; L^1_{loc}(\mathbb{R}))$, and if u, v are two entropy solutions associated to initial data u_0, v_0 , then, for all $R > 0$, the following comparison principle holds: $\forall t \in [0, T]$,

$$\int_{-R}^R (u(x, t) - v(x, t))^\pm dx \leq \int_{-R-Ct}^{R+Ct} (u_0(x) - v_0(x))^\pm dx,$$

where $C = \max_i(\text{Lip}(f_i))$, with $\text{Lip}(f_i) = \sup_{s \in (0,1)} |f'_i(s)|$.

The uniqueness and the L^1 -stability of the entropy solution stated above can be seen as a straightforward generalization of Theorem 3.1 in [14] to the case where $q \neq 0$. The existence of such an entropy solution is provided in [1] by showing the convergence of the discrete solution corresponding to a Godunov-type scheme, while a modified Engquist-Osher scheme is considered in [14].

In the particular case where $f_i(u) = k_i u(1 - u)$ for $k_1, k_2 \in \mathbb{R}_+$, then it is shown in [41] that entropy solutions can be obtained as the limit for $\mu \rightarrow 0$ and $\delta \rightarrow 0$ of the solutions $u^{\mu, \delta}$ to the problem

$$\partial_t u^{\mu, \delta} + \partial_x (k^\delta(x) u^{\mu, \delta} (1 - u^{\mu, \delta})) = \mu \partial_{xx} u^{\mu, \delta}$$

where k^δ is a smooth approximation of the piecewise constant function defined by $k(x) = k_i$ if $x \in \Omega_i$.

It has been proven in [33] that the entropy solution can also be obtained as limit for ε tends to 0 of weak solutions to regularized problems

$$\partial_t u^\varepsilon + \partial_x (f_i(u^\varepsilon) - \varepsilon \lambda_i(u^\varepsilon) \partial_x \pi_i(u^\varepsilon)) = 0$$

under the assumption that $\pi_1(0) = \pi_2(0)$ and $\pi_1(1) = \pi_2(1)$. This latter assumptions is relaxed in this paper and in [16].

The Godunov-type scheme proposed by Adimurthi, Jaffré and Veerappa Gowda [1] uses the fact that the fluxes are given by simple algebraic relations. Indeed, The flux $G_{\text{int}}(u_1, u_2) = f_1(u_1) = f_2(u_2)$ at the interface corresponding to traces u_1, u_2 is given by:

$$G_{\text{int}}(u_1, u_2) = \max \{G_1(u_1, 1), G_2(0, u_2)\}, \quad (21)$$

where G_i is the Godunov solver corresponding to f_i , that is

$$G_i(u, v) = \begin{cases} \min_{s \in [u, v]} f_i(s) & \text{if } u \leq v, \\ \max_{s \in [v, u]} f_i(s) & \text{if } u \geq v. \end{cases}$$

If the flux at the interface is given by

$$G_{\text{int}}(u_1, u_2) = f_1(u_1) = G_1(u_1, 1), \quad (22)$$

then the restriction $u|_{\Omega_1}$ of u to Ω_1 is the unique entropy solution to

$$\begin{cases} \partial_t u + \partial_x f_1(u) = 0 & \text{in } \Omega_1 \times (0, T), \\ u(0, \cdot) = \gamma & \text{in } (0, T), \\ u(\cdot, 0) = u_0 & \text{in } \Omega_1 \end{cases} \quad (23)$$

corresponding to $\gamma = 1$. Recall that the trace on $\{x = 0\}$ has to be understood in a weak sense (see [10, 37]). Since the solution to (23) depends in a non-decreasing way of the prescribed trace γ , we can claim that

$$u|_{\Omega_1} = \sup_{\gamma \in L^\infty((0, T); [0, 1])} \{v \text{ solution to (23)}\}. \quad (24)$$

Similarly, in the case where the flux at the interface is given by

$$G_{\text{int}}(u_1, u_2) = f_2(u_2) = G_2(0, u_2), \quad (25)$$

the restriction $u|_{\Omega_2}$ of u to Ω_2 is characterized by

$$u|_{\Omega_2} = \inf_{\gamma \in L^\infty((0, T); [0, 1])} \{v \text{ solution to (27)}\}, \quad (26)$$

where

$$\begin{cases} \partial_t u + \partial_x f_2(u) = 0 & \text{in } \Omega_2 \times (0, T), \\ u(0, \cdot) = \gamma & \text{in } (0, T), \\ u(\cdot, 0) = u_0 & \text{in } \Omega_2 \end{cases} \quad (27)$$

Since the problem is conservative, in both cases the solution u is entropic in both $\Omega_i \times (0, T)$, i.e. it satisfies (14), and minimizes the flux through the interface $\{x = 0\}$. It is shown in [16] (see also [18]) that this characterization still holds, but that the different physical assumptions lead to the selection of a solution to (10)-(11) which is not the entropy solution in the sense of Definition 1.2.

1.3 heterogeneities involving discontinuous capillarities

Let us now come back to the problem (8). Suppose for the sake of simplicity that the functions π_i are smooth and increasing on $[0, 1]$, and that $\lambda_i(u) > 0$ if $0 < u < 1$. The problem is then a spatial coupling of two parabolic problems, and we will need to ask two transmission conditions: one for the trace, and one for the flux. Concerning the latter, the conservation of mass yields a relation analogous to (11), which can be written with rough notations:

$$f_1(u_1) - \lambda_1(u_1) \partial_x \pi_1(u_1) = f_2(u_2) - \lambda_2(u_2) \partial_x \pi_2(u_2). \quad (28)$$

Let us now focus on the trace condition at the interface. In the case of heterogeneous media, the capillary pressure can be discontinuous at the interface. Numerical schemes for simulating such flows has been proposed in [17, 26, 27, 28]. It has been shown independently in [15] and [20] (but see also [12] and [26]) that the connection of the capillary pressures $\pi_i(u_i)$ has to be done in a graphical sense, so that phenomena like oil trapping can appear. Thus we have to define the monotonous graphs $\tilde{\pi}_i$.

$$\tilde{\pi}_i(u) = \begin{cases} \pi_i(u) & \text{if } 0 < u < 1, \\ (-\infty, \pi_i(0)] & \text{if } u = 0, \\ [\pi_i(1), +\infty) & \text{if } u = 1. \end{cases}$$

It is shown in [15] and [20] that a natural way to connect the capillary pressures on the interface consists in asking:

$$\tilde{\pi}_1(u_1) \cap \tilde{\pi}_2(u_2) \neq \emptyset. \quad (29)$$

In order to state a convenient definition for the solution of (8)-(28)-(29), we introduce the Kirchhoff transformation $\varphi_i(u) = \int_0^u \lambda_i(s) \pi_i'(s) ds$.

Definition 1.4 (bounded flux solution) Let $u_0 \in L^\infty(\mathbb{R})$, $0 \leq u_0 \leq 1$, and let $T > 0$. A function u is said to be a bounded flux solution to (8)-(28)-(29) if it fulfills:

1. $u \in L^\infty(\mathbb{R} \times (0, T))$, $0 \leq u \leq 1$ a.e.,
2. $\partial_x \varphi_i(u) \in L^\infty(\Omega_i \times (0, T))$,
3. $\tilde{\pi}_1(u_1) \cap \tilde{\pi}_2(u_2) \neq \emptyset$ for a.e. $t \in (0, T)$,
4. $\forall \psi \in \mathcal{D}(\mathbb{R} \times [0, T])$,

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}} u(x, t) \partial_t \psi(x, t) dx dt + \int_{\mathbb{R}} u_0(x) \psi(x, 0) dx \\ & + \int_0^T \sum_{i=1,2} \int_{\Omega_i} (f_i(u)(x, t) - \partial_x \varphi_i(u)(x, t)) \partial_x \psi(x, t) dx dt = 0. \end{aligned} \quad (30)$$

The bounded flux solution are so called since the point 2 of Definition 1.4 ensures that the flux $f_i(u) - \partial_x \varphi_i(u)$ remains uniformly bounded. Such a condition will require assumptions on the initial data u_0 , as it will be stated in the following theorem.

Theorem 1.5 (existence of a bounded flux solution) Let f_1, f_2 be Lipschitz continuous functions, and φ_1, φ_2 be increasing Lipschitz continuous functions. Let $u_0 \in L^1(\mathbb{R})$, $0 \leq u_0 \leq 1$ fulfilling $\partial_x \varphi_i(u_0) \in L^\infty(\Omega_i)$, and $\tilde{\pi}_1(u_{0,1}) \cap \tilde{\pi}_2(u_{0,2}) \neq \emptyset$, where $u_{0,i}$ denotes the trace on $\{x = 0\}$ of $u_0|_{\Omega_i}$. Then there exists a bounded flux solution. Moreover, u belongs to $\mathcal{C}([0, T]; L^1(\mathbb{R}))$.

The first part of this theorem is a straightforward adaptation to the case of unbounded domains and non-monotonous f_i of a result from [17] and [20] (see also [40]). This is based on a maximum principle on the fluxes $(f_i(u) - \partial_x \varphi_i(u))$. This particularly yields:

$$\|f_i(u) - \partial_x \varphi_i(u)\|_{L^\infty(\Omega_i \times (0, T))} \leq \max_{j=1,2} (\|f_j(u_0) - \partial_x \varphi_j(u_0)\|_{L^\infty(\Omega_j)}). \quad (31)$$

If $u_0 \in L^1(\Omega)$, then choosing $\psi = \min(1, (1, R - |x|)^+)$ and letting R tend to ∞ gives $u \in L^\infty((0, T); L^1(\mathbb{R}))$. Moreover, thanks to [19], u can be supposed to belong to $\mathcal{C}([0, T]; L^1_{loc}(\mathbb{R}))$. Then u belongs to $\mathcal{C}([0, T]; L^1(\mathbb{R}))$.

The choice of bounded flux solutions instead of more classical weak solution with $\partial_x \varphi_i(u)$ only belonging to $L^2((0, T); L^2_{loc}(\overline{\Omega}_i))$ has been motivated by the fact that it provides a comparison principle.

Proposition 1.6 Let u, v be two bounded flux solutions in the sense of Definition 1.4 associated to initial data u_0, v_0 . Then, for all $\psi \in \mathcal{D}^+(\mathbb{R} \times [0, T])$,

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}} (u(x, t) - v(x, t))^\pm \partial_t \psi(x, t) dx dt + \int_{\mathbb{R}} (u_0(x) - v_0(x))^\pm \psi(x, 0) dx \\ & + \int_0^T \sum_{i=1,2} \int_{\Omega_i} \text{sign}_\pm(u(x, t) - v(x, t)) (f_i(u)(x, t) - f_i(v)(x, t)) \partial_x \psi(x, t) dx dt \\ & - \int_0^T \sum_{i=1,2} \int_{\Omega_i} \partial_x (\varphi_i(u)(x, t) - \varphi_i(v)(x, t))^\pm \partial_x \psi(x, t) dx dt \geq 0. \end{aligned} \quad (32)$$

This proposition is not sufficient to claim the uniqueness, but it will be very useful in the sequel. In order to obtain a uniqueness result, we have to ask furthermore that the initial data belongs to $L^1(\mathbb{R})$.

Theorem 1.7 (uniqueness of bounded flux solution) Let $u_0 \in L^1(\mathbb{R})$, $0 \leq u_0 \leq 1$ a.e., with $\partial_x \varphi_i(u_0) \in L^\infty(\Omega_i)$ and $\tilde{\pi}_1(u_{0,1}) \cap \tilde{\pi}_2(u_{0,2}) \neq \emptyset$. Then there exists a unique bounded flux solution $u \in \mathcal{C}([0, T]; L^1(\mathbb{R}))$ in the sense of definition 1.4.

This theorem is a straightforward consequence Proposition 1.6. Indeed, choosing $\psi = \min(1, (1, R - |x|)^+)$ in (32), and letting R tend to $+\infty$ gives the comparison principle: $\forall t \in [0, T]$,

$$\int_{\mathbb{R}} (u(x, t) - v(x, t))^\pm dx \leq \int_{\mathbb{R}} (u_0(x) - v_0(x))^\pm dx. \quad (33)$$

The uniqueness result follows.

1.4 capillary pressure independent of the saturation

In some cases, the dependence of the capillary pressure π_i with respect to the saturation seems to be weak, and some numerical simulation consider capillary pressures only depending on the porous medium, but not on the saturation. More precisely, we aim to consider graphs of capillary pressure on the form

$$\tilde{\pi}_i(u) = \begin{cases} P_i & \text{if } 0 < u < 1, \\ (-\infty, P_i] & \text{if } u = 0, \\ [P_i, +\infty) & \text{if } u = 1, \end{cases} \quad (34)$$

so that the capillary pressure would roughly speaking not depend on u .

If one considers an interface $\{x = 0\}$ between two Ω_i , where the $\tilde{\pi}_i$ are on the form (34), we can give an orientation to the interface: the interface is said to be positively oriented if $P_1 > P_2$, and negatively oriented if $P_1 < P_2$. A positively oriented interface involve positive capillary forces, and a negatively oriented involve positive capillary forces. The gravity effects are also oriented by the sign of $(\rho_w - \rho_o)g$ in (9). We have to make the assumption that

“either the gravity effects and the interface are oriented in the same way, or the convective effects are larger than the gravity effects.”

Since we have supposed that gravity works in the sense of decaying x , we assume in the sequel that

$$P_1 < P_2. \quad (35)$$

We build a family of approximate problems (\mathcal{P}^ε) taking into account the capillary pressure: one suppose that $\pi_i^\varepsilon(u) = P_i + \varepsilon u$, where P_i is a constant depending only on the homogeneous subdomain Ω_i . In fact, any π_i^ε converging uniformly to P_i on $[0, 1]$ and such that $u \mapsto \int_0^u \lambda_i(s) (\pi_i^\varepsilon)'(s) ds$ converges uniformly toward 0 would fit.

Up to a smoothing of the initial data, we obtain a resulting sequence $(u^\varepsilon)_\varepsilon$ of bounded flux solutions for a problem of type (8)-(28)-(29). We will show that under Assumptions (12)-(13), this sequence tends almost everywhere to the unique entropy solution to (10)-(11).

This result has to be compared to the one presented in the associated paper [16], where it is shown that if Assumption (35) does not hold, non-classical shock can occur at the interface, representing *oil-trapping*.

1.5 organization of the paper

The paper is organized as follow: section 2 is devoted to the study of the approximate problem (\mathcal{P}^ε). We first smooth the initial data in a convenient way, and then we give a $L^2((0, T); H^1(\Omega_i))$ -estimate on the approximate solutions that shows in particular that if the approximate solution u^ε converges almost every where towards a function u , then u satisfies the points 1,2 and 3 of Definition 1.2. In order to prove that u^ε converges almost everywhere, we derive a family *BV*-estimates. In order to check that the last point of Definition 1.2 is fulfilled by the limit u of the approximate solutions $(u^\varepsilon)_\varepsilon$

2 The approximate problems

In this section we will define the approximate problem (\mathcal{P}^ε), and its solution u^ε . We will state a $L^2((0, T); H_{loc}^1(\overline{\Omega}_i))$ -estimate and a family of *BV*-estimates, which will be the key points of the proof of convergence of u^ε toward a weak solution of the problem (\mathcal{P}^ε).

In order to recover a family of entropy inequalities, we will build some steady solutions κ^ε to the problem (\mathcal{P}^ε), and study their limit as $\varepsilon \rightarrow 0$. This last point will require strongly Assumption (12).

2.1 smoothing the initial data

As it has already been stressed in Theorem 1.5, we need to assume some regularity on the initial data to ensure the existence of a bounded flux solution to problems of the type (8)-(28)-(29).

Let u_0 belong to $L^\infty(\mathbb{R})$, with $0 \leq u_0 \leq 1$, we will build a family $(u_0^\varepsilon)_\varepsilon$ of convenient approximate initial data.

Lemma 2.1 Let $u_0 \in L^\infty(\mathbb{R})$, $0 \leq u_0 \leq 1$, then there exists a family $(u_0^\varepsilon)_\varepsilon$ of approximate initial data such that:

- $u_0^\varepsilon \in C_c^\infty(\mathbb{R}^*)$, $0 \leq u_0^\varepsilon \leq 1$,
- $u_0^\varepsilon \rightarrow u_0$ a.e. in \mathbb{R} , $\|\varepsilon \partial_x u_0^\varepsilon\|_\infty \rightarrow 0$ as $\varepsilon \rightarrow 0$, and $\|\varepsilon \partial_x u_0^\varepsilon\|_\infty \leq 1$ for all $\varepsilon > 0$,
- If $u_0 \in BV(\mathbb{R})$, then $\|\partial_x u_0^\varepsilon\|_{L^1(\mathbb{R})} \leq TV(u_0) + 4$.

Proof: Let $\alpha > 0$, and let ρ_α be a mollifier with support in $(-\alpha, \alpha)$. Let $v^\alpha = (u_0 \circ \chi_{\alpha < |x| < 1/\alpha}) \star \rho_\alpha$, then it is clear that $v^\alpha \in C_c^\infty(\mathbb{R}^*)$, and that $v^\alpha \rightarrow u_0$ a.e. in \mathbb{R} as $\alpha \rightarrow 0$. Choosing $\varepsilon = \min\left(\alpha, \frac{\min(1, \sqrt{\alpha})}{\|\partial_x v^\alpha\|_\infty}\right)$, and $u_0^\varepsilon = v^\alpha$ ends the proof of Lemma 2.1. \square

2.2 the problem $(\mathcal{P}^\varepsilon)$

Let $P_1, P_2 \in \mathbb{R}$, we define the functions π_i^ε by $\pi_i^\varepsilon(u) = P_i + \varepsilon u$, and

$$\tilde{\pi}_i^\varepsilon(u) = \begin{cases} P_i + \varepsilon u & \text{if } 0 < u < 1, \\ (-\infty, P_i] & \text{if } u = 0, \\ [P_i + \varepsilon, +\infty) & \text{if } u = 1. \end{cases}$$

If ε is small, the intersection of the ranges of the functions π_1^ε and π_2^ε is empty, and then the

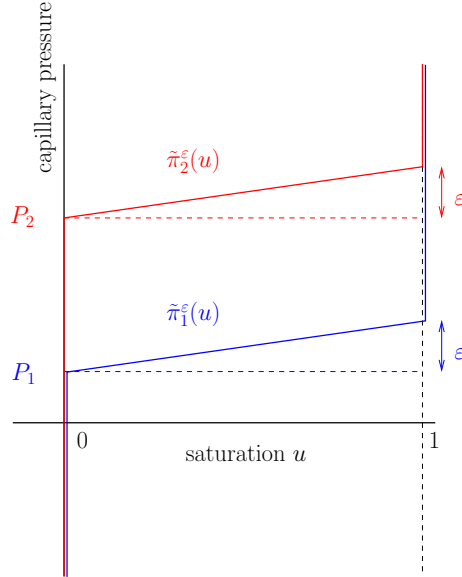


Figure 3: capillary pressures graphs $\tilde{\pi}_i^\varepsilon$.

graphical relation $\tilde{\pi}_1(u_1) \cap \tilde{\pi}_2(u_2) \neq \emptyset$ connecting the capillary pressures at the interface becomes

$$(1 - u_1)u_2 = 0.$$

Let $0 \leq u_0 \leq 1$, and let $(u_0^\varepsilon)_\varepsilon$ be built as in Lemma 2.1, let $\varphi_i(u) = \int_0^u \lambda_i(s) ds$. The approximate problem is: find u^ε s.t.

$$\begin{cases} \partial_t u^\varepsilon + \partial_x (f_i(u^\varepsilon) - \varepsilon \partial_x \varphi_i(u^\varepsilon)) = 0 & \text{in } \Omega_i \times (0, T), \\ \tilde{\pi}_1^\varepsilon(u^\varepsilon)(0^-, t) \cap \tilde{\pi}_2^\varepsilon(u^\varepsilon)(0^+, t) \neq \emptyset & \text{in } (0, T), \\ f_1(u^\varepsilon)(0^-, t) - \varepsilon \partial_x \varphi_1(u^\varepsilon)(0^-, t) = f_2(u^\varepsilon)(0^+, t) - \varepsilon \partial_x \varphi_2(u^\varepsilon)(0^+, t) & \text{in } (0, T), \\ u^\varepsilon(0) = u_0^\varepsilon & \text{in } \mathbb{R}. \end{cases} \quad (\mathcal{P}^\varepsilon)$$

This problem is of type (8)-(28)-(29), then the notion of bounded flux solution is a good frame to solve it.

Definition 2.2 (solution to $(\mathcal{P}^\varepsilon)$) A function u^ε is said to be a (bounded flux) solution to $(\mathcal{P}^\varepsilon)$ if it fulfills

1. $u^\varepsilon \in L^\infty(\mathbb{R} \times (0, T))$, $0 \leq u^\varepsilon \leq 1$ a.e.,
2. $\partial_x \varphi_i(u^\varepsilon) \in L^\infty(\Omega_i \times (0, T))$,
3. $\forall \psi \in \mathcal{D}(\mathbb{R} \times [0, T])$,

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}} u^\varepsilon(x, t) \partial_t \psi(x, t) dx dt + \int_{\mathbb{R}} u_0^\varepsilon(x) \psi(x, 0) dx \\ & + \int_0^T \sum_{i=1,2} \int_{\Omega_i} (f_i(u^\varepsilon)(x, t) - \varepsilon \partial_x \varphi_i(u^\varepsilon)(x, t)) \partial_x \psi(x, t) dx dt = 0. \end{aligned} \quad (36)$$

We can use Theorem 1.5 to claim that there exists a family $(u^\varepsilon)_\varepsilon$ of bounded flux solution to $(\mathcal{P}^\varepsilon)$ in the sense of Definition 2.2. Moreover, this family of solution fulfills, thanks to (31) and Lemma 2.1: for all $\varepsilon > 0$,

$$\|\varepsilon \partial_x \varphi_i(u^\varepsilon)\|_\infty \leq (\max_i \text{Lip}(\varphi_i)) + \max_i \|f_i\|_{L^\infty(0,1)}. \quad (37)$$

Since u_0^ε belongs to $L^1(\mathbb{R})$, the solution u^ε is furthermore unique in $\mathcal{C}([0, T], L^1(\mathbb{R}))$ thanks to Theorem 1.7.

2.3 a $L^2((0, T); H_{loc}^1(\overline{\Omega}_i))$ -estimate

All this subsection is devoted to prove the following estimate.

Proposition 2.3 *Let K be a compact subset of $\overline{\Omega}_i$, and let u^ε be a solution of $(\mathcal{P}^\varepsilon)$ in the sense of Definition 2.2, then there exists C depending only on f_i, K, T (and not on ε) such that*

$$\sqrt{\varepsilon} \|\varphi_i(u^\varepsilon)\|_{L^2((0, T); H^1(K))} \leq C.$$

Particularly, this implies that $\varepsilon \partial_x \varphi_i(u^\varepsilon) \rightarrow 0$ a.e. in $\Omega_i \times (0, T)$ as $\varepsilon \rightarrow 0$.

Proof: We fix $\varepsilon > 0$. Since the functions φ_i^{-1} are not Lipschitz continuous, the problem $(\mathcal{P}^\varepsilon)$ is not strictly parabolic, and the function u^ε is not a strong solution. In order to get more regularity on the approximate solution, we regularize the problem by adding an additional viscosity $1/n$ ($n \geq 1$), so that the so built approximate solution u_n^ε is regular enough to perform the calculation below.

Let $n > 1$, and let $\varphi_{i,n}(u) = \varphi_i(u) + u/n$, and let u_n^ε be a bounded flux solution of $(\mathcal{P}^\varepsilon)$ with $\varphi_{i,n}$ instead of φ_i . From (37), we know that $\partial_x \varphi_{i,n}(u_n^\varepsilon)$ is uniformly bounded in $L^\infty(\Omega_i \times (0, T))$, and since $\varphi_{i,n}^{-1}$ is a Lipschitz continuous function, one has $\partial_x u_n^\varepsilon \in L^\infty(\Omega_i \times (0, T))$. The following weak formulation holds: $\forall \psi \in \mathcal{D}(\Omega \times [0, T])$,

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}} u_n^\varepsilon(x, t) \partial_t \psi(x, t) dx dt + \int_{\mathbb{R}} u_0^\varepsilon(x) \psi(x, 0) dx \\ & + \int_0^T \sum_{i=1,2} \int_{\Omega_i} (f_i(u_n^\varepsilon)(x, t) - \varepsilon \partial_x \varphi_{i,n}(u_n^\varepsilon)(x, t)) \partial_x \psi(x, t) dx dt = 0. \end{aligned} \quad (38)$$

Let $(a, b) \subset \Omega_i$. Let $\zeta \in \mathcal{D}^+((a, b))$, we deduce from (38) that

$$\left\langle \partial_t u_n^\varepsilon \mid u_n^\varepsilon \zeta^2 \right\rangle = \int_0^T \int_a^b f_i(u_n^\varepsilon) \partial_x (u_n^\varepsilon \zeta^2) dx dt - \varepsilon \int_0^T \int_a^b \partial_x \varphi_{i,n}(u_n^\varepsilon) \partial_x (u_n^\varepsilon \zeta^2) dx dt, \quad (39)$$

where $\langle \cdot \mid \cdot \rangle$ is the duality bracket between $L^2((0, T); H^{-1}(a, b))$ and $L^2((0, T); H_0^1(a, b))$. Since $\varphi_{i,n}$ is a Lipschitz continuous function with $(\|\lambda_i\|_\infty + 1/n)$ as Lipschitz constant, one has

$$\int_0^T \int_a^b \partial_x \varphi_{i,n}(u_n^\varepsilon) \partial_x (u_n^\varepsilon \zeta^2) dx dt \geq \frac{1}{\|\lambda_i\|_\infty + 1/n} \int_0^T \int_a^b (\partial_x \varphi_{i,n}(u_n^\varepsilon))^2 \zeta^2 dx dt. \quad (40)$$

Let Φ_i be a primitive of f_i , then:

$$\begin{aligned} \int_0^T \int_a^b f_i(u_n^\varepsilon) \partial_x (u_n^\varepsilon \zeta^2) dx dt &= \int_0^T \int_U \partial_x \Phi_i(u_n^\varepsilon) \zeta^2 dx dt + \int_0^T \int_a^b f_i(u_n^\varepsilon) u_n^\varepsilon \partial_x \zeta^2 dx dt \\ &= \int_0^T \int_a^b [f_i(u_n^\varepsilon) u_n^\varepsilon - \Phi_i(u_n^\varepsilon)] \partial_x \zeta^2 dx dt. \end{aligned} \quad (41)$$

Admit for the moment Lemma 2.4, stated and proven below. We deduce from (39), (40), (41) and Lemma 2.4 that

$$\begin{aligned} &\frac{\varepsilon}{\|\lambda_i\|_\infty + 1} \int_0^T \int_a^b (\partial_x \varphi_{i,n}(u_n^\varepsilon))^2 \zeta^2 dx dt \\ &\leq \frac{1}{2} \|\zeta\|_\infty^2 |b-a| + \int_0^T \int_a^b |u_n^\varepsilon (f_i(u_n^\varepsilon) - \varepsilon \partial_x \varphi_{i,n}(u_n^\varepsilon)) - \Phi_i(u_n^\varepsilon)| |\partial_x \zeta^2| dx dt. \end{aligned}$$

Using now the fact that u_n^ε is a bounded flux solution, we deduce from (31) that $[u_n^\varepsilon (f_i(u_n^\varepsilon) - \varepsilon \partial_x \varphi_{i,n}(u_n^\varepsilon)) - \Phi_i(u_n^\varepsilon)]$ is uniformly bounded independently of ε and n , and so there exists C only depending on f_i , $|b-a|$, u_0 and λ_i such that:

$$\varepsilon \int_0^T \int_a^b (\partial_x \varphi_{i,n}(u_n^\varepsilon))^2 \zeta^2 dx dt \leq C (\|\partial_x \zeta^2\|_{L^1((0,T); \mathcal{M}(\mathbb{R}))} + \|\zeta\|_\infty^2).$$

This estimate still holds for $\zeta(x, t) = \chi_{(a,b)}(x)$, for all $(a, b) \in \overline{\Omega}_i^2$, so we obtain

$$\varepsilon \int_0^T \int_a^b (\partial_x \varphi_{i,n}(u_n^\varepsilon))^2 dx dt \leq C(2T+1). \quad (42)$$

Classical compactness arguments provide the convergence, up to a subsequence, of $(u_n^\varepsilon)_n$ to a solution u^ε of $(\mathcal{P}^\varepsilon)$ in $L^p(\mathbb{R} \times (0, T))$, $1 \leq p < +\infty$. This ensures particularly that, up to a subsequence,

$$\lim_{n \rightarrow +\infty} u_n^\varepsilon = u^\varepsilon \text{ a.e. in } \mathbb{R} \times (0, T).$$

Taking the limit w.r.t. n in (42) yields

$$\varepsilon \int_0^T \int_a^b (\partial_x \varphi_i(u^\varepsilon))^2 dx dt \leq C(2T+1).$$

□

Lemma 2.4 *Let u_n^ε be an approximate solution of $(\mathcal{P}^\varepsilon)$ with $\varphi_{i,n}$ instead of φ_i , and let $\psi \in \mathcal{D}^+((a, b))$, then*

$$\langle \partial_t u_n^\varepsilon \mid u_n^\varepsilon \zeta^2 \rangle \geq -\frac{1}{2} \|\zeta\|_\infty^2 |b-a|.$$

Proof: Since $(u_n^\varepsilon \zeta) \in L^2((0, T); H_0^1(a, b))$, and $\partial_t (u_n^\varepsilon \zeta) \in L^2((0, T); H^{-1}(a, b))$, and so, up to a negligible set, $(u_n^\varepsilon \zeta) \in C([0, T]; L^2(a, b))$, and

$$\begin{aligned} \langle \partial_t u_n^\varepsilon \mid u_n^\varepsilon \zeta^2 \rangle &= \langle \partial_t u_n^\varepsilon \zeta \mid u_n^\varepsilon \zeta \rangle \\ &= \frac{1}{2} \int_a^b u_n^\varepsilon(x, T)^2 \zeta^2(x) dx - \frac{1}{2} \int_a^b u_0(x)^2 \zeta^2(x) dx \\ &\geq -\frac{1}{2} \|\zeta\|_\infty^2 |b-a|. \end{aligned}$$

□

2.4 the BV-estimates

In this section we suppose that $u_0 \in BV(\mathbb{R})$. In order to avoid heavy notations which would not lead to a good comprehension of the problem, the following proof will be formal. To establish the following estimates in a rigorous frame, one can introduce of a thin layer $(-\eta, \eta)$ on which the pressure varies smoothly to replace the interface, and add some additional viscosity to obtain smooth strong solutions to the problem. This regularization of the problem has been performed in [12] and in [20].

For $a, b \in [0, 1]$, we denote by

$$F_i(a, b) = \text{sign}(a - b)(f_i(a) - f_i(b)).$$

Lemma 2.5 *There exists C depending only on f_i, T, u_0 such that*

$$|\partial_t F_i(u^\varepsilon, \kappa)|_{\mathcal{M}_b(\Omega_i \times (0, T))} \leq C.$$

Proof: Suppose in the sequel that u^ε is a strong solution, i.e.

$$\partial_t u^\varepsilon + \partial_x [f_i(u^\varepsilon) - \varepsilon \partial_x \varphi_i(u^\varepsilon)] = 0$$

holds point-wise in $\Omega_i \times (0, T)$. Let $h > 0$, and let $t \in (0, T - h)$. Comparing $u^\varepsilon(\cdot, \cdot + h)$ and u^ε with (33) yields

$$\int_{\mathbb{R}} |u^\varepsilon(x, t + h) - u^\varepsilon(x, t)| dx \leq \int_{\mathbb{R}} |u^\varepsilon(x, h) - u_0^\varepsilon(x)| dx.$$

Dividing by h and letting h tend to 0, one can claim using the fact that u^ε is supposed to be a strong solution

$$\begin{aligned} \int_{\mathbb{R}} |\partial_t u^\varepsilon(x, t)| dx &= \sum_{i=1,2} \int_{\Omega_i} |\partial_x [f_i(u^\varepsilon)(x, t) - \varepsilon \partial_x \varphi_i(u^\varepsilon)(x, t)]| dx \\ &\leq \int_{\mathbb{R}} |\partial_t u_0^\varepsilon(x)| dx \\ &\leq \sum_{i=1,2} \int_{\Omega_i} |\partial_x [f_i(u_0^\varepsilon)(x) - \varepsilon \partial_x \varphi_i(u_0^\varepsilon)(x)]| dx \end{aligned}$$

Lemma 2.1 then ensures that there exists C not depending on ε such that

$$\int_0^T \int_{\mathbb{R}} |\partial_t u^\varepsilon(x, t)| dx dt = \int_0^T \sum_{i=1,2} \int_{\Omega_i} |\partial_x [f_i(u^\varepsilon)(x, t) - \varepsilon \partial_x \varphi_i(u^\varepsilon)(x, t)]| dx dt \leq C. \quad (43)$$

Thanks to the regularity of f_i , this particularly ensures that, if we denote by $\mathcal{M}_b(\Omega_i \times (0, T))$ the set of the bounded Radon measure on $\Omega_i \times (0, T)$, i.e. the dual space of $C_c(\overline{\Omega_i} \times [0, T], \mathbb{R})$ with the uniform norm, we obtain: $\forall \kappa \in [0, 1]$,

$$|\partial_t F_i(u^\varepsilon, \kappa)|_{\mathcal{M}_b(\Omega_i \times (0, T))} \leq C \|f_i'\|_\infty.$$

□

Lemma 2.6 *There exists C depending only on u_0, f_i and T such that*

$$\left| \partial_x \left(F_i(u^\varepsilon, \kappa) - \varepsilon \partial_x |\varphi_i(u^\varepsilon) - \varphi_i(\kappa)| \right) \right|_{\mathcal{M}_b(\Omega_i \times (0, T))} \leq C.$$

Proof: It follows from the work of Carillo (see e.g. [21]) that for all $\kappa \in [0, 1]$, for all $\psi \in \mathcal{D}^+(\Omega_i \times [0, T])$,

$$\begin{aligned} &\int_0^T \int_{\Omega_i} |u^\varepsilon - \kappa| \partial_t \psi dx dt + \int_{\Omega_i} |u^\varepsilon - \kappa| \psi(0) dx \\ &\int_0^T \int_{\Omega_i} \left(F_i(u^\varepsilon, \kappa) - \varepsilon \partial_x |\varphi_i(u^\varepsilon) - \varphi_i(\kappa)| \right) \partial_x \psi dx dt \geq 0. \end{aligned} \quad (44)$$

Let $\eta > 0$, we denote by $\omega_\eta(x) = (1 - |x|/\eta)^+$, and suppose now that ψ belongs to $\mathcal{D}^+(\overline{\Omega}_i \times [0, T])$, i.e. ψ does not vanish on the interface $\{x = 0\}$. Estimate (44) still holds when we consider $\psi_\eta = \psi(1 - \omega_\eta)$ as test function.

$$\begin{aligned} & \int_0^T \int_{\Omega_i} |u^\varepsilon - \kappa| (1 - \omega_\eta) \partial_t \psi dx dt + \int_{\Omega_i} |u^\varepsilon - \kappa| \psi(0) (1 - \omega_\eta) dx \\ & + \int_0^T \int_{\Omega_i} \left(F_i(u^\varepsilon, \kappa) - \varepsilon \partial_x |\varphi_i(u^\varepsilon) - \varphi_i(\kappa)| \right) (1 - \omega_\eta) \partial_x \psi dx dt \\ \geq & \int_0^T \int_{\Omega_i} \left(F_i(u^\varepsilon, \kappa) - \varepsilon \partial_x |\varphi_i(u^\varepsilon) - \varphi_i(\kappa)| \right) \psi \partial_x \omega_\eta dx dt. \end{aligned} \quad (45)$$

The fact that the flux induced by u^ε is uniformly bounded w.r.t. ε thanks to (37) implies that there exists C depending only on u_0, f_i such that

$$\|F_i(u^\varepsilon, \kappa) - \varepsilon \partial_x |\varphi_i(u^\varepsilon) - \varphi_i(\kappa)|\|_{L^\infty(\Omega_i \times (0, T))} \leq C. \quad (46)$$

Then we obtain the following estimate on the right-hand-side in (45):

$$\left| \int_0^T \int_{\Omega_i} \left(F_i(u^\varepsilon, \kappa) - \varepsilon \partial_x |\varphi_i(u^\varepsilon) - \varphi_i(\kappa)| \right) \psi \partial_x \omega_\eta dx dt \right| \leq CT \|\psi\|_\infty.$$

Letting η tend to 0 in inequality (45) gives: $\forall \psi \in \mathcal{D}^+(\overline{\Omega}_i \times [0, T]), \forall \kappa \in [0, 1]$,

$$\begin{aligned} & \int_0^T \int_{\Omega_i} |u^\varepsilon - \kappa| \partial_t \psi dx dt + \int_{\Omega_i} |u^\varepsilon - \kappa| \psi(0) dx \\ & + \int_0^T \int_{\Omega_i} \left(F_i(u^\varepsilon, \kappa) - \varepsilon \partial_x |\varphi_i(u^\varepsilon) - \varphi_i(\kappa)| \right) \partial_x \psi dx dt \geq -CT \|\psi\|_\infty \end{aligned} \quad (47)$$

We introduce now a monotonous function $\chi_\psi \in \mathcal{D}^+(\overline{\Omega}_i)$ equal to 1 on the support of $\psi(\cdot, t)$ for all $t \in [0, T]$ (so that $\|\partial_x \chi_\psi\|_{L^1(\Omega_i)} = 1$), then $\|\psi\|_\infty \chi_\psi \geq \psi$. Choosing $\|\psi\|_\infty \chi_\psi - \psi$ and $\|\psi\|_\infty \chi_\psi + \psi$ as test function in (47) yields, using (46) once again

$$\left| \partial_t |u^\varepsilon - \kappa| - \partial_x \left(F_i(u^\varepsilon, \kappa) - \varepsilon \partial_x |\varphi_i(u^\varepsilon) - \varphi_i(\kappa)| \right) \right|_{\mathcal{M}_b(\Omega_i \times (0, T))} \leq 2CT. \quad (48)$$

Thanks to (43), there exists C' depending only on u_0, f_i and T such that

$$|\partial_t |u^\varepsilon - \kappa||_{\mathcal{M}_b(\Omega_i \times (0, T))} \leq C'. \quad (49)$$

Lemma 2.6 is so a consequence of (48) and (49). \square

Proposition 2.7 *Let $u_0 \in BV(\mathbb{R})$ and let $K = [a, b] \subset \overline{\Omega}_i$. We introduce $z_{K, \kappa}^\varepsilon(x, t)$ defined on the whole space \mathbb{R}^2 given by:*

$$z_{K, \kappa}^\varepsilon(x, t) = \begin{cases} F_i(u^\varepsilon, \kappa)(x, t) & \text{if } (x, t) \in K \times (0, T), \\ 0 & \text{otherwise.} \end{cases}$$

There exists C depending only on u_0, f_i, T, K and a uniformly bounded function $r_{K, \kappa}$, with $r_{K, \kappa}(\varepsilon)$ tends uniformly to 0 with respect to κ as $\varepsilon \rightarrow 0$, such that, for all $(\xi, h) \in \mathbb{R}^2$,

$$\iint_{\mathbb{R}^2} |z_{K, \kappa}^\varepsilon(x + \xi, t + h) - z_{K, \kappa}^\varepsilon(x, t)| dx dt \leq C(|\xi| + |h|) + r_{K, \kappa}(\varepsilon).$$

Proof: Let $h \in \mathbb{R}$, one has

$$\begin{aligned} & \iint_{\mathbb{R}^2} |z_{K, \kappa}^\varepsilon(x, t + h) - z_{K, \kappa}^\varepsilon(x, t)| dx dt \leq |\partial_t z_{K, \kappa}^\varepsilon|_{\mathcal{M}_b(\mathbb{R}^2)} |h| \\ & \leq \left(|\partial_t F_i(u^\varepsilon, \kappa)|_{\mathcal{M}_b(\Omega_i \times (0, T))} + 2\|F_i(u^\varepsilon, \kappa)\|_{L^\infty(T + b - a)} \right) |h| \end{aligned}$$

It follows from Lemma 2.5 that there exists C_1 depending only on $u_0, f_i, K, T, \varphi_i$ such that

$$\iint_{\mathbb{R}^2} |z_{K,\kappa}^\varepsilon(x, t+h) - z_{K,\kappa}^\varepsilon(x, t)| dxdt \leq C_1|h|. \quad (50)$$

We also define

$$q_{K,\kappa}^\varepsilon(x, t) = \begin{cases} \varepsilon \partial_x |\varphi_i(u^\varepsilon)(x, t) - \varphi_i(\kappa)| & \text{if } (x, t) \in K \times (0, T), \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2.3 ensures that $q_{K,\kappa}^\varepsilon(x, t)$ converges to 0 almost everywhere in \mathbb{R}^2 , and the estimate (46) ensures us that $q_{K,\kappa}^\varepsilon(x, t)$ stays uniformly bounded in $L^\infty(\mathbb{R}^2)$ with respect to ε . Let $\xi \in \mathbb{R}$, then we have:

$$\begin{aligned} & \iint_{\mathbb{R}^2} |z_{K,\kappa}^\varepsilon(x+\xi, t) - q_{K,\kappa}^\varepsilon(x+\xi, t) - (z_{K,\kappa}^\varepsilon(x, t) - q_{K,\kappa}^\varepsilon(x, t))| dxdt \\ & \leq \left(\begin{array}{l} |\partial_x (F_i(u^\varepsilon, \kappa) - \varepsilon \partial_x |\varphi_i(u^\varepsilon)(x, t) - \varphi_i(\kappa)|)|_{\mathcal{M}_b(\Omega_i \times (0, T))} \\ + 2(T+b-a) (\|F_i(u^\varepsilon, \kappa) - \varepsilon \partial_x |\varphi_i(u^\varepsilon)(x, t) - \varphi_i(\kappa)|\|_{L^\infty}) \end{array} \right) |\xi|. \end{aligned} \quad (51)$$

Using (46) and Lemma 2.6 in (51) yields that there exists C_2 depending only on $u_0, f_i, K, T, \varphi_i$ such that

$$\iint_{\mathbb{R}^2} |z_{K,\kappa}^\varepsilon(x+\xi, t) - q_{K,\kappa}^\varepsilon(x+\xi, t) - (z_{K,\kappa}^\varepsilon(x, t) - q_{K,\kappa}^\varepsilon(x, t))| dxdt \leq C_2|\xi|.$$

This particularly ensures that:

$$\iint_{\mathbb{R}^2} |z_{K,\kappa}^\varepsilon(x+\xi, t) - z_{K,\kappa}^\varepsilon(x, t)| dxdt \leq C_2|\xi| + 2\|q_{K,\kappa}^\varepsilon(x, t)\|_{L^1(\mathbb{R}^2)}. \quad (52)$$

By choosing $C = \max(C_1, C_2)$, one we deduce from (50) and (52) that

$$\iint_{\mathbb{R}^2} |z_{K,\kappa}^\varepsilon(x+\xi, t+h) - z_{K,\kappa}^\varepsilon(x, t)| dxdt \leq C(|h| + |\xi|) + 2\|q_{K,\kappa}^\varepsilon(x, t)\|_{L^1(\mathbb{R}^2)}.$$

We conclude the proof of Proposition 2.7 by checking that $\|q_{K,\kappa}^\varepsilon(x, t)\|_{L^1(\mathbb{R}^2)}$ converges uniformly to 0 with respect to κ as ε tends to 0. \square

2.5 some steady solutions

The spatial discontinuities of the saturation and capillary pressure allow us to consider Kruřkov entropies $|u - \kappa|$ only for non-negative test functions *vanishing on the interface*, i.e. in $\mathcal{D}^+(\mathbb{R}^* \times [0, T])$. This is not enough to obtain the convergence of u^ε toward an entropy solution u in the sense of Definition 1.2, and a relation has also to be derived at of the interface.

In order to deal with general test functions belonging to $\mathcal{D}^+(\mathbb{R} \times [0, T])$, we will so have to introduce some approximate Kruřkov entropies $|\cdot - \kappa^\varepsilon(x)|$, where κ^ε are steady solutions of $(\mathcal{P}^\varepsilon)$. Letting $\varepsilon \rightarrow 0$, those steady solutions converge to piecewise constant function $\tilde{\kappa}^j$ defined below. The functions $|\cdot - \tilde{\kappa}^j(x)|$ correspond to the so called *partially adapted entropies* introduced by Audusse and Perthame [5]. We will then be able to compare the limit u of approximate solutions u^ε to this limit $\tilde{\kappa}^j$, and then to prove that u is the unique κ entropy solution.

The building of convenient κ^ε strongly uses Assumption (12). It will be shown in [16] that if (12) fails, non classical shocks can occur at the interface, and the limit \tilde{u} of the approximate solutions u^ε is thus not an entropy solution.

Recall that $q \geq 0$, $P_1 < P_2$, and suppose that $0 < \varepsilon < P_2 - P_1$. Some simple adaptations can be done to cover the case $q < 0$. The transmission condition (29) can be summarized as follow: either $u_1 = 1$, or $u_2 = 0$.

We have to introduce the following sets:

$$\mathcal{E}_1 = \{\kappa_1 / \exists \kappa_2 \text{ with } f_1(\kappa_1) = f_2(\kappa_2)\},$$

$$\mathcal{E}_2 = \{\kappa_2 / \exists \kappa_1 \text{ with } f_1(\kappa_1) = f_2(\kappa_2)\}.$$

It follows from Assumption (12) that either $\mathcal{E}_1 = [0, 1]$, or $\mathcal{E}_2 = [0, 1]$, and so we are ensured that $\kappa \in [0, 1]$ belongs either to \mathcal{E}_1 , or to \mathcal{E}_2 (or of course to both). Check also that for all $z \in (0, q]$, it follows from (12) that there exists a unique $\kappa_i(z)$ such that $f_i(\kappa_i(z)) = z$. On the contrary, if $z \in \mathcal{E}_i$ with $z \leq 0$, z has two antecedents through f_i . Let κ belong to \mathcal{E}_j , one denotes

$$\bar{\kappa}_i^j = \max\{\nu \mid f_i(\nu) = f_j(\kappa)\} \quad (53)$$

and

$$\underline{\kappa}_i^j = \min\{\nu \mid f_i(\nu) = f_j(\kappa)\}. \quad (54)$$

Definition 2.8 (reachable steady state) *A function $\kappa(x)$ is said to be a reachable steady state if there exists a steady entropy solution $\kappa^\varepsilon(x)$ to the problem $(\mathcal{P}^\varepsilon)$ in the sense of Definition 2.2 converging to $\kappa(x)$ in $L^1(\mathbb{R})$ as ε tends to 0.*

This section is devoted to establish the following proposition, that exhibits all the reachable steady states. In particular, all the steady states that are not undercompressible are reachable.

Proposition 2.9 *For all $\kappa \in \mathcal{E}_j$, there exists a family of steady solutions $(\kappa^\varepsilon)_\varepsilon$ to the problem $(\mathcal{P}^\varepsilon)$ such that*

$$\kappa^\varepsilon \rightarrow \tilde{\kappa}^j \text{ a.e. in } \Omega_i, \quad (55)$$

where $\tilde{\kappa}^j(x)$ can be chosen between:

- i) $\tilde{\kappa}^j(x) = \bar{\kappa}_1^j$ if $x < 0$ and $\underline{\kappa}_2^j$ if $x > 0$;
- ii) $\tilde{\kappa}^j(x) = \bar{\kappa}_1^j$ if $x < 0$ and $\bar{\kappa}_2^j$ if $x > 0$;
- iii) $\tilde{\kappa}^j(x) = \underline{\kappa}_1^j$ if $x < 0$ and $\underline{\kappa}_2^j$ if $x > 0$.

In particular, κ_{opt} defined in (16)-(17) is a reachable steady state.

Proof: Let $\kappa \in \mathcal{E}_j$.

- If $\kappa = 1$, the three limits $\tilde{\kappa}^j$ are identically equal to 1, which is a steady solution fulfilling Proposition 2.9.
- We suppose now that $\kappa < 1$, and $f_j(\kappa) > 0$. Even in this case, the three reachable limit are the same. Thus we only have to build one sequence of converging steady solutions. Let y be a solution of:

$$\begin{cases} \frac{d}{dx} \varphi_1(y) = f_j(\kappa) - f_1(y), & \text{for } x > 0, \\ y(0) = 1. \end{cases} \quad (56)$$

The solution $y(x)$ converges to $\bar{\kappa}_1^j$ as $x \rightarrow +\infty$. The family $(\kappa^\varepsilon)_\varepsilon$ defined by: $\forall \varepsilon$

$$\kappa^\varepsilon(x) = \begin{cases} \bar{\kappa}_2^j \text{ or } \underline{\kappa}_2^j & \text{if } x > 0, \\ y(-x/\varepsilon) & \text{if } x < 0. \end{cases} \quad (57)$$

fulfills so the conclusion of Proposition 2.9.

- Suppose now $f_j(\kappa) = 0$. The solutions $\kappa^\varepsilon(x)$ built with (56)-(57) converges toward the two reachable steady states i) and ii). One can also choose $\kappa^\varepsilon(x) = 0$, which is of course a steady solution.
- It remains the case $f_j(\kappa) < 0$. The solutions $\kappa^\varepsilon(x)$ built with (56)-(57) still converges toward the two reachable steady states i) and ii).

Let w be a solution of

$$\begin{cases} \frac{d}{dx} \varphi_2(w) = f_2(w) - f_j(\kappa), & \text{for } x > 0, \\ w(0) = 0. \end{cases}$$

and

$$\kappa^\varepsilon(x) = \begin{cases} \underline{\kappa}_1^j & \text{if } x < 0, \\ w(x/\varepsilon) & \text{if } x > 0, \end{cases}$$

then κ^ε converges toward the third reachable steady state. □

3 Convergence toward the entropy solution

3.1 almost everywhere convergence

We state now a lemma which is an adaptation of Helly's selection theorem criterion, which states that if $(v^\varepsilon)_\varepsilon$ is a family of measurable functions on an open subset \mathcal{U} of \mathbb{R}^k ($k \geq 1$), uniformly bounded in $L^\infty_{loc}(\mathcal{U})$ and $BV_{loc}(\mathcal{U})$, one can extract a subfamily still denoted by $(v^\varepsilon)_\varepsilon$ that converges almost everywhere in \mathcal{U} , and the limit belongs to $BV_{loc}(\mathcal{U})$.

Lemma 3.1 *Let \mathcal{U} be an open subset of \mathbb{R}^k ($k \geq 1$). Let $(v^\varepsilon)_\varepsilon$ be a family of functions uniformly bounded in $L^\infty(\mathcal{U})$ with respect to ε . Let K be a compact subset of \mathcal{U} . For $\zeta \in \mathbb{R}^k$, we denote by*

$$K_\zeta = \{ x \in K \mid x + \zeta \in K \}.$$

One assumes that there exists $C > 0$ depending only on K (and thus not on ε) and a function r fulfilling $\lim_{\varepsilon \rightarrow 0} r(\varepsilon) = 0$, such that for all $\zeta \in \mathbb{R}^k$,

$$\int_{K_\zeta} |v^\varepsilon(x + \zeta) - v^\varepsilon(x)| dx \leq C|\zeta| + r(\varepsilon). \quad (58)$$

Then, there exists a sequence $(\varepsilon_n)_n$ tending to 0, and $v \in BV_{loc}(\mathcal{U})$ such that

$$\lim_{n \rightarrow +\infty} v^{\varepsilon_n} = v \quad \text{a.e. in } \mathcal{U}.$$

Proof: Let K be a compact subset of \mathcal{U} . Estimate (58) says, roughly speaking, that v^ε is almost a BV -function on K , i.e. as close to $BV(K)$ as wanted, provided that ε is supposed to be small enough. So we will build a family $(w^\varepsilon)_\varepsilon$ of BV -functions, which will be close to the family $(v^\varepsilon)_\varepsilon$, at least for small ε , and we will show that $(w^\varepsilon)_\varepsilon$ admits an adherence value v in $BV(K)$ for the $L^1(K)$ -topology, and that this v is also an adherence value of $(v^\varepsilon)_\varepsilon$. Another proof for the a.e. convergence toward a function v can be derived directly from Kolmogorov compactness criterion (see e.g. [13]). But the advantage of the following method is that it provides directly some regularity on the limit $v \in BV_{loc}(\mathcal{U})$.

Let $(\rho^\varepsilon)_\varepsilon$ be a sequence of mollifiers, i.e. smooth, non negative and compactly supported functions with support included in the ball of center 0 and radius ε , and fulfilling $\|\rho^\varepsilon\|_{L^1(\mathbb{R}^k)} = 1$ for all $\varepsilon > 0$. We define the smooth functions

$$w^\varepsilon = \tilde{v}^\varepsilon \star \rho^\varepsilon,$$

where $\tilde{v}^\varepsilon(x) = v^\varepsilon(x)$ if $x \in \mathcal{U}$ and $\tilde{v}^\varepsilon(x) = 0$ if $x \in \mathcal{U}^c$.

Thanks to the regularity of w^ε ,

$$\int_{K_\zeta} |w^\varepsilon(x + \zeta) - w^\varepsilon(x)| dx \leq \|\nabla w^\varepsilon\|_{(L^1(K))^k} |\zeta|. \quad (59)$$

Suppose that $\varepsilon < d(K, \partial\mathcal{U})$ (with the convention $d(K, \emptyset) = \infty$). Thanks to (58), we have also

$$\int_{K_\zeta} |w^\varepsilon(x + \zeta) - w^\varepsilon(x)| dx \leq \int_{\{K_\zeta + \varepsilon\}} |v^\varepsilon(x + \zeta) - v^\varepsilon(x)| dx \leq C|\zeta| + r(\varepsilon), \quad (60)$$

where

$$\{K_\zeta + \varepsilon\} = \{ x \in \mathcal{U} \mid d(x, K) \leq \varepsilon \}.$$

Since $r(\varepsilon)$ tends to 0 as $\varepsilon \rightarrow 0$, this particularly ensures

$$\limsup_{\varepsilon \rightarrow 0} \|\nabla w^\varepsilon\|_{(L^1(\mathbb{R}^k))^k} \leq C.$$

The family $(w^\varepsilon)_\varepsilon$ is thus bounded in $BV(K_\zeta)$ in the neighborhood of $\varepsilon = 0$, and thus, thanks to Helly's selection criterion, there exist $v \in BV(K_\zeta)$, and $(\varepsilon_n)_n$ tending to 0 such that

$$w^{\varepsilon_n} \rightarrow v \quad \text{a.e. in } K \text{ as } n \rightarrow \infty.$$

Furthermore, for all $n \in \mathbb{N}^*$,

$$\begin{aligned} \|w^{\varepsilon_n} - v^{\varepsilon_n}\|_{L^1(K)} &\leq \int_K \int_{B(0, \varepsilon_n)} |v^{\varepsilon_n}(x-y) - v^{\varepsilon_n}(x)| \rho^{\varepsilon_n}(y) dy dx \\ &\leq C\varepsilon_n + r(\varepsilon_n). \end{aligned}$$

This ensures that v^{ε_n} tends also almost everywhere toward v as n tends to $+\infty$. \square

Lemma 3.1 will be used to prove the following convergence assertion.

Proposition 3.2 *Suppose that $u_0 \in BV(\mathbb{R})$, and let u^ε be a solution to $(\mathcal{P}^\varepsilon)$. Up to an extraction, there exists $u \in L^\infty(\mathbb{R} \times (0, T))$, $0 \leq u \leq 1$ a.e. such that*

$$u^\varepsilon \rightarrow u \text{ a.e. in } \mathbb{R} \times (0, T).$$

Furthermore, there exists $u_1, u_2 \in L^\infty(0, T)$, such that

$$\begin{aligned} \lim_{\eta \rightarrow 0} \frac{1}{\eta} \int_0^T \int_{-\eta}^0 |u(x, t) - u_1(t)| dx dt &= 0, \\ \lim_{\eta \rightarrow 0} \frac{1}{\eta} \int_0^T \int_0^\eta |u(x, t) - u_2(t)| dx dt &= 0. \end{aligned}$$

Proof: Let K be a compact subset of $\overline{\Omega}_i$. We define the function $H_i : [0, 1] \mapsto \mathbb{R}$ by

$$H_i(u) = \int_0^1 (F_i(u, \sigma) - f_i(\sigma)) d\sigma, \quad (61)$$

so that, thanks to Proposition 2.7, there exists C depending on u_0, f_i, T, K , and a function r tending to 0 as ε tends to 0 such that for all $\xi \in \mathbb{R}, h \in (0, T)$,

$$\int_0^{T-h} \int_K |H_i(u^\varepsilon)(x + \xi, t + h) - H_i(u^\varepsilon)(x, t)| dx dt \leq C(|\xi| + |h|) + r(\varepsilon). \quad (62)$$

An integration by parts in (61) yields: $\forall u \in [0, 1]$

$$\begin{aligned} H_i(u) &= - \int_0^1 (\sigma - b_i) \partial_\sigma (F_i(u, \sigma) - f_i(\sigma)) d\sigma + (2b_i - 1) f_i(u) \\ &= 2 \int_0^u (\sigma - b_i) f'_i(\sigma) d\sigma + (2b_i - 1) f_i(u) \end{aligned} \quad (63)$$

where, thanks to (12), f_i is decreasing on $[0, b_i]$ and increasing on $[b_i, 1]$. Using Proposition 2.7 with $\kappa = 0$,

$$\int_0^{T-h} \int_K |f_i(u^\varepsilon)(x + \xi, t + h) - f_i(u^\varepsilon)(x, t)| dx dt \leq C(|\xi| + |h|) + r(\varepsilon), \quad (64)$$

where C and r have been updated. Denoting by

$$A_i(u) = \int_0^u (\sigma - b_i) f'_i(\sigma) d\sigma,$$

we obtain from (63) and (64)

$$\int_0^{T-h} \int_K |A_i(u^\varepsilon)(x + \xi, t + h) - A_i(u^\varepsilon)(x, t)| dx dt \leq C(|\xi| + |h|) + r(\varepsilon), \quad (65)$$

with a new update for C and r . Thus we deduce from Proposition 3.1 that, up to an extraction, $A_i(u^\varepsilon)$ converges almost everywhere toward $\overline{A}_i \in BV(K \times (0, T))$. It follows from (12) that A_i is an increasing function, and so we obtain the convergence almost everywhere in $K \times (0, T)$ of u^ε toward a measurable function v .

Since for all $\varepsilon > 0$, $0 \leq u^\varepsilon \leq 1$, there exists $u \in L^\infty(\mathbb{R} \times (0, T))$, $0 \leq u \leq 1$ such that u^ε converges to u in the $L^\infty(\mathbb{R} \times (0, T))$ - \star -weak sense, we thus have, up to an extraction,

$$u^\varepsilon \rightarrow u \text{ a.e. in } K \times (0, T). \quad (66)$$

Since (66) holds for any compact subset K of Ω_i , for $i = 1, 2$, we can claim that, up to an extraction,

$$u^\varepsilon \rightarrow u \text{ a.e. in } \mathbb{R} \times (0, T).$$

Moreover, since $A_i(u)$ belongs to $BV(\mathbb{R} \times (0, T))$, we can claim that $A_i(u)$ admits a strong trace on $\{x = 0\} \times (0, T)$ (see for instance [4]). Using once again the fact that A_i^{-1} is a continuous function, we can claim that u admits also a strong trace on each side of the interface. \square

3.2 convergence toward the entropy solution

In this section, it is proven that the limit value u for the family $(u^\varepsilon)_\varepsilon$ exhibited previously is the entropy solution described in Definition 1.2.

Proposition 3.3 *Let $u_0 \in BV(\mathbb{R})$, $0 \leq u_0 \leq 1$ a.e., and let $(u_0^\varepsilon)_\varepsilon$ a family of approximation of u_0 given by Lemma 2.1, and let $(u^\varepsilon)_\varepsilon$ be the induced sequence of bounded flux solution of (8)-(28)-(29). Then under Assumption (12)*

$$\lim_{\varepsilon \rightarrow 0} u^\varepsilon = u \text{ in } L_{loc}^p(\mathbb{R} \times [0, T]), \quad \forall p \in [1, \infty)$$

where u is the unique entropy solution to (10)-(11) associated to initial data u_0 .

Proof: Thanks to Proposition 3.2, we can suppose that there exists $u \in L^\infty(\mathbb{R} \times (0, T))$ such that, up to a subsequence,

$$u^\varepsilon \rightarrow u \text{ in } L_{loc}^1(\mathbb{R} \times (0, T)) \text{ as } \varepsilon \rightarrow 0, \quad (67)$$

then

$$f_i(u^\varepsilon) \rightarrow f_i(u) \text{ in } L_{loc}^1(\Omega_i \times (0, T)) \text{ as } \varepsilon \rightarrow 0.$$

Furthermore, thanks to Proposition 2.3, one has

$$\varepsilon \partial_x \varphi_i(u^\varepsilon) \rightarrow 0 \text{ in } L_{loc}^1(\Omega_i \times (0, T)) \text{ as } \varepsilon \rightarrow 0. \quad (68)$$

As a consequence, letting ε tend to 0 in (36) provides (recall that u_0^ε tends to u_0 in $L_{loc}^1(\mathbb{R})$) that u is a weak solution to (10)-(11), i.e. that it satisfies (19).

Since for $\varepsilon > 0$, $(\varepsilon \varphi_i)^{-1}$ is a continuous function, it follows from the work of Carrillo [21] that it fulfills the following entropy inequalities: $\forall \kappa \in [0, 1]$, $\forall \psi \in \mathcal{D}^+(\Omega_i \times [0, T])$

$$\begin{aligned} & \int_0^T \int_{\Omega_i} |u^\varepsilon - \kappa| \partial_t \psi \, dx dt + \int_{\Omega_i} |u_0^\varepsilon - \kappa| \psi(\cdot, 0) \, dx \\ & + \int_0^T \int_{\Omega_i} (F_i(u^\varepsilon, \kappa) - \varepsilon \partial_x |\varphi_i(u^\varepsilon) - \varphi_i(\kappa)|) \partial_x \psi \, dx dt \geq 0. \end{aligned} \quad (69)$$

The assertion (67) particularly yields that for $i = 1, 2$ and for all $\kappa \in [0, 1]$,

$$F_i(u^\varepsilon, \kappa) \rightarrow F_i(u, \kappa) \text{ in } L_{loc}^1(\mathbb{R} \times (0, T)) \text{ as } \varepsilon \rightarrow 0. \quad (70)$$

On the other hand, it follows from (68) that

$$\varepsilon \partial_x |\varphi_i(u^\varepsilon) - \varphi_i(\kappa)| \rightarrow 0 \text{ in } L_{loc}^1(\mathbb{R} \times (0, T)) \text{ as } \varepsilon \rightarrow 0. \quad (71)$$

Taking (67), (70) and (71) into account in (69) provides the inequality (20).

The last point remaining to check is that the interface entropy condition (18) holds. Let $(\kappa^\varepsilon)_\varepsilon$ be a family of steady states to $(\mathcal{P}^\varepsilon)$ converging in $L_{loc}^1(\mathbb{R})$ as ε tends to 0 to κ_{opt} defined in (16)-(17). Recall that such a family exists thanks to Proposition 2.9. Then for all fixed $\varepsilon > 0$, κ^ε is a steady bounded flux solution. Hence, it follows from (32) that

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}} |u^\varepsilon - \kappa^\varepsilon| \partial_t \psi \, dx dt + \int_{\mathbb{R}} |u_0^\varepsilon - \kappa^\varepsilon| \psi(\cdot, 0) \, dx dt \\ & + \int_0^T \sum_{i=1,2} \int_{\Omega_i} (F_i(u^\varepsilon, \kappa^\varepsilon) - \varepsilon \partial_x |\varphi_i(u^\varepsilon) - \varphi_i(\kappa^\varepsilon)|) \partial_x \psi \, dx dt \geq 0. \end{aligned} \quad (72)$$

It follows from Proposition 2.3 that

$$\varepsilon \partial_x |\varphi_i(u^\varepsilon) - \varphi_i(\kappa^\varepsilon)| \rightarrow 0 \text{ in } L^1_{loc}(\Omega_i \times (0, T)) \text{ as } \varepsilon \rightarrow 0.$$

Letting ε tend to 0 in (72) provides directly the fourth point (18) the fourth point in Definition 1.2. \square

We state now the main result, which is in fact the extension of Proposition 3.3 to a larger class of initial data.

Theorem 3.4 (main result) *Let $u_0 \in L^\infty(\mathbb{R})$, $0 \leq u_0 \leq 1$, and let $(u_0^\varepsilon)_\varepsilon$ a family of approximation of u_0 given by Lemma 2.1, and let $(u^\varepsilon)_\varepsilon$ be the induced sequence of bounded flux solution of $(\mathcal{P}^\varepsilon)$. Then under Assumption (12),*

$$\lim_{\varepsilon \rightarrow 0} u^\varepsilon = u \text{ in } L^p_{loc}(\mathbb{R} \times [0, T]), \quad \forall p \in [1, \infty)$$

where u is the unique entropy solution to (10)-(11) associated to initial data u_0 .

Proof: Let $u_0 \in L^\infty(\mathbb{R})$, $0 \leq u_0 \leq 1$, and let $\nu > 0$. There exists $u_{0,\nu}$ in $BV(\mathbb{R})$, $0 \leq u_{0,\nu} \leq 1$ such that for all $R > 0$,

$$\|u_{0,\nu} - u_0\|_{L^1(-R,R)} \leq C(R)\nu. \quad (73)$$

If one regularizes $u_{0,\nu}$ into $u_{0,\nu}^\varepsilon$ using Lemma 2.1, and if one denotes by u_ν^ε the associated unique bounded flux solution, we have seen that u_ν^ε converges almost everywhere to u_ν as ε tends to 0. As previously, we denote by u_0^ε the regularization of u_0 obtained via Lemma 2.1, and u^ε the unique associated bounded flux solution.

Let $R > 0$, then we have

$$\begin{aligned} \int_0^T \int_{-R}^R |u^\varepsilon - u| dx dt &\leq \int_0^T \int_{-R}^R |u^\varepsilon - u_0^\varepsilon| dx dt + \int_0^T \int_{-R}^R |u_0^\varepsilon - u_{0,\nu}^\varepsilon| dx dt \\ &\quad + \int_0^T \int_{-R}^R |u_{0,\nu}^\varepsilon - u_\nu^\varepsilon| dx dt \\ &\quad + \int_0^T \int_{-R}^R |u_\nu^\varepsilon - u_\nu| dx dt \\ &\quad + \int_0^T \int_{-R}^R |u_\nu - u| dx dt. \end{aligned} \quad (74)$$

The contraction principle stated in Theorem 1.3 yields

$$\int_0^T \int_{-R}^R |u_\nu(x, t) - u(x, t)| dx dt \leq T \int_{-R-MT}^{R+MT} |u_{0,\nu}(x) - u_0(x)| dx \quad (75)$$

where $M \geq \max_i \text{Lip}(f_i)$.

We denote by $\zeta(x, t) = \min(1, (R + 1 + M(T - t) - |x|)^+)$. One has $\zeta = 1$ on $(-R, R)$, $\zeta \geq 0$ and $\zeta \in L^1(\mathbb{R})$ thus

$$\int_0^T \int_{-R}^R |u^\varepsilon - u_\nu^\varepsilon| dx dt \leq \int_0^T \int_{\mathbb{R}} |u^\varepsilon - u_\nu^\varepsilon| \zeta dx dt. \quad (76)$$

It follows from Proposition 1.6 that for all $\psi \in W^{1,1}(\mathbb{R} \times (0, T))$ with $\psi \geq 0$ a.e. and $\psi(\cdot, T) = 0$, one has

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}} |u^\varepsilon(x, t) - u_\nu^\varepsilon(x, t)| \partial_t \psi(x, t) dx dt + \int_{\mathbb{R}} |u_0^\varepsilon(x) - u_{0,\nu}^\varepsilon(x)| \psi(x, 0) dx \\ &+ \int_0^T \sum_{i=1,2} \int_{\Omega_i} \text{sign}(u^\varepsilon(x, t) - u_\nu^\varepsilon(x, t)) (f_i(u^\varepsilon)(x, t) - f_i(u_\nu^\varepsilon)(x, t)) \partial_x \psi(x, t) dx dt \\ &- \int_0^T \sum_{i=1,2} \int_{\Omega_i} \varepsilon \partial_x |\varphi_i(u^\varepsilon)(x, t) - \varphi_i(u_\nu^\varepsilon)(x, t)| \partial_x \psi(x, t) dx dt \geq 0. \end{aligned} \quad (77)$$

We denote by

$$\begin{aligned} \Lambda_1(t) &= [-R - 1 - M(T - t), -R - M(T - t)], \\ \Lambda_2(t) &= [R + M(T - t), R + 1 + M(T - t)], \end{aligned}$$

$$\vartheta(x) = \begin{cases} 1 & \text{if } x < 0, \\ -1 & \text{if } x > 0, \end{cases}$$

so that

$$\partial_x \zeta(x, t) = \sum_{i=1,2} \vartheta(x) \chi_{\Lambda_i(t)}(x), \quad \partial_t \zeta(x, t) = -M \sum_{i=1,2} \chi_{\Lambda_i(t)}(x).$$

Taking $\psi(x, t) = (T - t)\zeta(x, t)$ in (77) yields :

$$\begin{aligned} & - \int_0^T \int_{\mathbb{R}} |u^\varepsilon(x, t) - u_\nu^\varepsilon(x, t)| \zeta(x, t) dx dt + T \int_{\mathbb{R}} |u_0^\varepsilon(x) - u_{0,\nu}^\varepsilon(x)| \zeta(x, 0) dx \\ & \quad - M \int_0^T (T - t) \sum_{i=1,2} \int_{\Lambda_i(t)} |u^\varepsilon(x, t) - u_\nu^\varepsilon(x, t)| dx dt \\ & + \int_0^T (T - t) \sum_{i=1,2} \int_{\Lambda_i(t)} \text{sign}(u^\varepsilon(x, t) - u_\nu^\varepsilon(x, t)) (f_i(u^\varepsilon)(x, t) - f_i(u_\nu^\varepsilon)(x, t)) dx dt \\ & \quad - \int_0^T (T - t) \sum_{i=1,2} \int_{\Lambda_i(t)} \varepsilon \vartheta(x) \partial_x |\varphi_i(u^\varepsilon)(x, t) - \varphi_i(u_\nu^\varepsilon)(x, t)| dx dt \geq 0. \end{aligned}$$

Since $M \geq \max_i \text{Lip}(f_i)$, one has

$$\begin{aligned} & \int_0^T (T - t) \sum_{i=1,2} \int_{\Lambda_i(t)} \text{sign}(u^\varepsilon(x, t) - u_\nu^\varepsilon(x, t)) (f_i(u^\varepsilon)(x, t) - f_i(u_\nu^\varepsilon)(x, t)) dx dt \\ & \leq M \int_0^T (T - t) \sum_{i=1,2} \int_{\Lambda_i(t)} |u^\varepsilon(x, t) - u_\nu^\varepsilon(x, t)| dx dt \end{aligned}$$

and thus

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}} |u^\varepsilon(x, t) - u_\nu^\varepsilon(x, t)| \zeta(x, t) dx dt \leq T \int_{\mathbb{R}} |u_0^\varepsilon(x) - u_{0,\nu}^\varepsilon(x)| \zeta(x, 0) dx \\ & \quad - \int_0^T (T - t) \sum_{i=1,2} \int_{\Lambda_i(t)} \varepsilon \vartheta(x) \partial_x |\varphi_i(u^\varepsilon)(x, t) - \varphi_i(u_\nu^\varepsilon)(x, t)| dx dt. \end{aligned} \quad (78)$$

We deduce, using (75) (76) and (78) in (74), that

$$\begin{aligned} & \int_0^T \int_{-R}^R |u^\varepsilon(x, t) - u(x, t)| dx dt \leq T \int_{\mathbb{R}} |u_0^\varepsilon(x) - u_{0,\nu}^\varepsilon(x)| \zeta(x, 0) dx \\ & \quad - \int_0^T (T - t) \sum_{i=1,2} \int_{\Lambda_i(t)} \varepsilon \vartheta(x) \partial_x |\varphi_i(u^\varepsilon)(x, t) - \varphi_i(u_\nu^\varepsilon)(x, t)| dx dt \\ & + T \int_{-R-MT}^{R+MT} |u_{0,\nu}(x) - u_0(x)| dx + \int_0^T \int_{-R}^R |u_\nu^\varepsilon(x, t) - u_\nu(x, t)| dx dt. \end{aligned} \quad (79)$$

We can now let ε tend to 0. Thanks to Proposition 2.3, we can claim that

$$\lim_{\varepsilon \rightarrow 0} \int_0^T (T - t) \sum_{i=1,2} \int_{\Lambda_i(t)} \varepsilon \vartheta(x) \partial_x |\varphi_i(u^\varepsilon)(x, t) - \varphi_i(u_\nu^\varepsilon)(x, t)| dx dt = 0.$$

We also deduce from Proposition 3.3 that

$$\lim_{\varepsilon \rightarrow 0} \int_0^T \int_{-R}^R |u_\nu^\varepsilon(x, t) - u_\nu(x, t)| dx dt = 0.$$

Since $\zeta(x, 0)$ is compactly supported, it follows from Lemma 2.1 and the dominated convergence theorem that

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}} |u_0^\varepsilon(x) - u_{0,\nu}^\varepsilon(x)| \zeta(x, 0) dx = \int_{\mathbb{R}} |u_0(x) - u_{0,\nu}(x)| \zeta(x, 0) dx.$$

Thus (79) becomes:

$$\begin{aligned} \limsup_{\varepsilon \rightarrow 0} \int_0^T \int_{-R}^R |u^\varepsilon(x, t) - u(x, t)| dx dt &\leq T \int_{-R-MT}^{R+MT} |u_{0,\nu}(x) - u_0(x)| dx \\ &+ \int_{\mathbb{R}} |u_0(x) - u_{0,\nu}(x)| \zeta(x, 0) dx. \end{aligned} \quad (80)$$

The inequality (80) holds for any $\nu > 0$, and letting ν tend to 0 leads to

$$\lim_{\varepsilon \rightarrow 0} \int_0^T \int_{-R}^R |u^\varepsilon(x, t) - u(x, t)| dx dt = 0,$$

then u^ε tends to u in $L^1_{loc}(\mathbb{R} \times [0, T])$ as $\varepsilon \rightarrow 0$. Since $0 \leq u^\varepsilon \leq 1$ a.e., Hölder inequality gives the convergence in $L^p_{loc}(\mathbb{R} \times [0, T])$ for all finite p . \square

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