Asymptotic behavior of two-phase flows in heterogeneous porous media for capillarity depending only on space. II. Non-classical shocks to model oil-trapping

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Asymptotic behavior of two-phase flows in heterogeneous porous media for capillarity depending only on space. II. Non-classical shocks to model oil-trapping

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Abstract

We consider a one-dimensional problem modeling two-phase flow in heterogeneous porous media made of two homogeneous subdomains, with discontinuous capillarity at the interface between them. We suppose that the capillary forces vanish inside the domains, but not on the interface. Under the assumption that the gravity forces and the capillary forces are oriented in opposite directions, we show that the limit, for vanishing diffusion, is not in general the optimal entropy solution of the hyperbolic scalar conservation law as in the first paper of the series [10]. A non-classical shock can occur at the interface, modeling oil-trapping.

key words. *scalar conservation laws with discontinuous flux, non-classical shock, two-phase flow, porous media, discontinuous capillarity*

AMS subject classification. *35L65, 35L67, 76S05*

1 Introduction

The models of two-phase flows provide good first approximations to predict the motions of oil in the subsoil. Although the theoretical knowledge concerning the question of the existence and the uniqueness of the solution to such models for homogeneous porous media [4, 15] and for media with regular enough variations [16] is quite complete, few results are available for discontinuous media, as for example media made of several rock types [3, 7, 9, 11, 13, 18].

One says that oil-trapping occurs when some oil can not pass through interfaces between different rocks. Such a phenomenon plays an important role in the basin modeling, to predict the position of eventual reservoirs where oil could be collected. As already explained in [7, 35], discontinuities of the capillary pressure field can induce the so-called oil-trapping phenomenon.

The effects of capillarity, which play a crucial role in oil-trapping, seem to play a less important role concerning the motion of oil in homogeneous porous media, and can sometime be neglected to provide the so-called Buckley-Leverett equation.

In this paper, we show that even if the dependence of the capillary pressure with respect to the oil-saturation of the fluid vanishes, the capillary pressure field still plays a crucial role to determine the saturation profile. In order to carry out this study, we restrict our frame to the one-dimensional case. We will strongly use some recent results [9, 11, 13] obtained on flows in heterogeneous media with discontinuous capillary forces.

We consider a one-dimensional porous medium, made of two different rocks, represented by $\Omega_1 = \mathbb{R}^\star_-$ and $\Omega_2 = \mathbb{R}^\star_+$. Let $\pi(u, x)$ be the capillary pressure, then it it is well known (see e.g. the

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introduction of the associate paper [10]) that, if both phases have different densities, the equation governing the two phase flow can be written

$$\partial_t u + \partial_x \Big(qc(u,x) + g(u,x)\,(1 - C\partial_x \pi(u,x)) \Big) = 0, \tag{1}$$

where u is the *oil saturation* of the fluid, q is the total flow rate, supposed to be a nonnegative constant, C is a constant depending on the buoyancy forces and

$$c(u,x) = c_i(u), \quad g(u,x) = g_i(u), \quad \text{and} \quad \pi(u,x) = \pi_i(u) \quad \text{if } x \in \Omega_i.$$

The functions c_i are supposed to be increasing and Lipschitz continuous with $c_i(0) = 0$ and $c_i(1) = 1$, while g_i are supposed to be Lipschitz continuous, strictly positive in $(0,1)$ satisfying $g_i(0) = g_i(1) = 0$ and π_i are increasing Lipschitz continuous functions.

Physical experiments suggest that the dependence of π_i with respect to u can be weak, at least for u far from 0 and 1. So we want to choose $\pi_1(u) = P_1$, and $\pi_2(u) = P_2$. The equation (1) turns formally to the scalar conservation law with discontinuous flux function

$$\partial_t u + \partial_x f(u,x) = 0, \tag{2}$$

where $f(u,x)$ (resp. $f_i(u)$) is equal to $qc(u,x) - g(u,x)$ (resp. $qc_i(u) - g_i(u)$).

Such conservation laws have been widely studied in the last years. For a large overview on this topic, we refer to the introduction of [8], or in a lesser extent to the associated paper [10]. In particular, it has been proven by Adimurthi, Mishra and Veerappa Gowda [2] that there might exist an infinite number of L^1-contraction semi-groups corresponding to the equation (2). Among them, in the case where the functions f_i have at most a single extremum in (0,1), we mention the so-called *optimal entropy solution* which corresponds to the unique entropy solution in the case of a continuous flux function $f_1 = f_2 = f$. We refer to [2] and to the first part of this communication [10] for a discussion on the so-called *optimal entropy condition*.

In the sequel of this paper, we suppose that

(H1) for $i \in \{1,2\}$, there exists a value $u_i^\star \in [0,1)$ such that $f_i(u_i^\star) = q$, f_i is increasing on $[0, u_i^\star]$ and $f_i(s) > q$ for all $s \in (u_i, 1)$.

We refer to Figure 1 for an illustration of the previous assumption. We denote by

$$\varphi_i(u) = C \int_0^u g_i(s)\, ds.$$

For technical reasons, we have to assume that

(H2) there exist $R > 0$, $\alpha > 0$ and $m \in (0,1)$ such that

$$f_1 \circ \varphi_1^{-1}(s) \geq q + R(\varphi_1(1) - s)^m \quad \text{if } s \in [\varphi_1(1) - \alpha, \varphi_1(1)]. \tag{3}$$

These assumptions are fulfilled by models widely used by the engineers, for which a classical choice of c_i, g_i is

$$c_i(u) = \frac{u^{a_i}}{u^{\alpha_i} + \frac{a}{b}(1-u)^{\beta_i}}, \qquad g_i(u) = K_i \frac{u^{\alpha_i}(1-u)^{\beta_i}}{bu^{a_i} + a(1-u)^{\beta_i}},$$

where $\alpha_i, \beta_i \geq 1$ and a, b are given constants.

The goal of this paper is to show that if the capillary forces at the level of the interface $\{x = 0\}$ are oriented in the opposite sense with respect to the gravity forces (in our case $P_1 < P_2$), then a *non classical* stationary shock can occur at the interface. It was shown by Kaasschieter [24] that if the capillary pressure field is continuous at the interface (corresponding to the case $P_1 = P_2$), then the good notion of solution is the one of *optimal entropy solution*, computed by Adimurthi, Jaffré and Veerappa Gowda using a Godunov-type scheme [1]. We have pointed out in [10] that if the

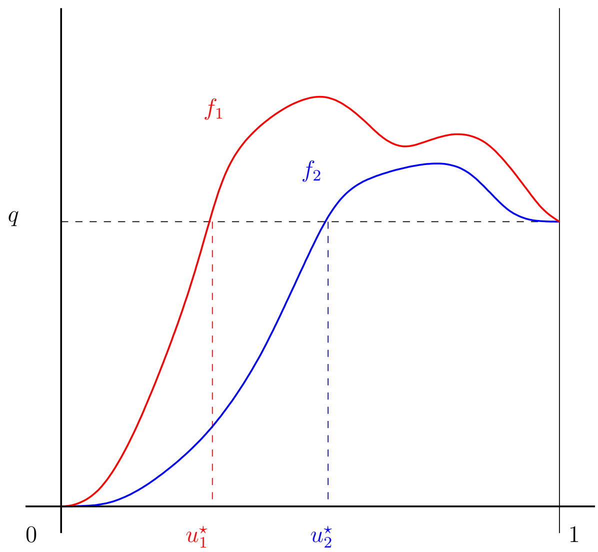

Figure 1: example of functions f_i satisfying Assumption **(H1)**. Note the we have note supposed, as it is done in [1, 8], that f_i has a single local extremum in $(0,1)$, but all the extrema have to be strictly greater than q.

capillary forces and the gravity forces are oriented in the same sense, the good notion of solution is also the one of *optimal entropy solution*. If the assumptions stated above are fulfilled, if $P_1 < P_2$ and if the initial data u_0 is large enough to ensure that both phases move in opposite directions, i.e.

$$u_i^\star \le u_0(x) \le 1 \quad \text{a.e. in } \Omega_i, \tag{4}$$

we will show that the limit is not the optimal entropy solution, but the entropy solution to the problem

$$\begin{cases} \partial_t u + \partial_x f_i(u) = 0, \\ u(x=0^-) = 1 \text{ and } u(x=0^+) = u_2^\star, \\ u(t=0) = u_0. \end{cases} \tag{$\mathcal{P}_{\rm lim}$}$$

In the sequel, we denote by a^+ (resp. a^-) the positive (resp. negative) part of a, i.e. $\max(0,a)$ (resp. $\max(0,-a)$), and for $i=1,2$, for $u,\kappa \in [0,1]$, one denotes by

$$\Phi_{i+}(u,\kappa) = \begin{cases} f_i(u) - f_i(\kappa) & \text{if } u \ge \kappa, \\ 0 & \text{otherwise,} \end{cases}$$

$$\Phi_{i-}(u,\kappa) = \begin{cases} f_i(\kappa) - f_i(u) & \text{if } u \le \kappa, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\Phi_i(u,\kappa) = \Phi_{i+}(u,\kappa) + \Phi_{i-}(u,\kappa) = f_i(\max(u,\kappa)) - f_i(\min(u,\kappa)).$$

We can now define the notion of solution to ($\mathcal{P}_{\rm lim}$), which is in fact an entropy in each subdomain Ω_i, with an internal boundary condition at the level of the interface.

Definition 1.1 (solution to ($\mathcal{P}_{\rm lim}$)) *Let $u_0 \in L^\infty(\mathbb{R})$, $u_i^\star \le u_0(x) \le 1$ a.e. in Ω_i, A function u is said to be a solution of ($\mathcal{P}_{\rm lim}$) if it belongs to $L^\infty(\mathbb{R}\times\mathbb{R}_+)$, $u_i^\star \le u \le 1$ a.e. in $\Omega_i \times (0,T)$, and for $i=1,2$, for all $\psi \in \mathcal{D}^+(\overline{\Omega}_i \times \mathbb{R}_+)$, for all $\kappa \in [0,1]$,*

$$\int_{\mathbb{R}_+}\int_{\Omega_i} (u(x,t)-\kappa)^\pm \partial_t\psi dxdt + \int_{\Omega_i} (u_0(x)-\kappa)^\pm \psi(x,0)dx$$
$$+ \int_{\mathbb{R}_+}\int_{\Omega_i} \Phi_{i\pm}(u(x,t),\kappa)\partial_x\psi(x,t)dxdt + M_{f_i}\int_{\mathbb{R}_+} (\overline{u}_i - \kappa)^\pm \psi(0,t)dt \ge 0, \tag{5}$$

where M_{f_i} is a Lipschitz constant of f_i, and $\overline{u}_1 = 1$, $\overline{u}_2 = u_2^\star$.

For a given u_0 in $L^\infty(\mathbb{R})$, there exists a unique solution u to ($\mathcal{P}_{\lim}$) in the sense of Definition 1.1, which is in fact made on an apposition of two entropy solutions in $\mathbb{R}_\pm \times \mathbb{R}_+$. We refer to [27, 28] and [38] for proofs of existence and uniqueness to solutions to the problem ($\mathcal{P}_{\lim}$). Moreover, thanks to [12], one can suppose that u belongs to $\mathcal{C}(\mathbb{R}_+; L^1_{loc}(\mathbb{R}))$.

Theorem 1.2 *Let $u_0 \in L^\infty(\mathbb{R})$ with $u_i^\star \le u_0 \le 1$ a.e. in Ω_i, then there exists a unique solution to ($\mathcal{P}_{\lim}$) in the sense of Definition 1.1. Furthermore, if v is another solution to ($\mathcal{P}_{\lim}$) corresponding to $v_0 \in L^\infty(\mathbb{R})$ with $u_i^\star \le v_0 \le 1$ a.e. in Ω_i, then for all $\mathbb{R} > 0$, for all $t \in \mathbb{R}_+$*

$$\int_{-R}^{R} (u(x,t) - v(x,t))^\pm dx \le \int_{-R-M_f t}^{R+M_f t} (u_0(x) - v_0(x))^\pm dx \tag{6}$$

where M_f is a Lipschitz constant of both f_i.

Assume now that both phases move in the same direction:

$$0 \le u_0(x) \le u_i^\star \quad \text{a.e. in } \Omega_i, \tag{7}$$

then it will be shown that the relevant solution u to the problem is the unique entropy solution defined below.

Definition 1.3 *A function u is said to be an entropy solution if it belongs to $L^\infty(\mathbb{R} \times \mathbb{R}_+)$, $0 \le u \le u_i^\star$ a.e. in $\Omega_i \times (0,T)$, and for $i = 1, 2$, for all $\psi \in \mathcal{D}^+(\mathbb{R} \times \mathbb{R}_+)$, for all $\kappa \in [0, 1]$,*

$$\begin{aligned}\int_{\mathbb{R}_+} \int_{\mathbb{R}} |u(x,t) - \kappa|\, \partial_t \psi dx dt + \int_{\mathbb{R}} |u_0(x) - \kappa|\, \psi(x,0) dx \\ + \int_{\mathbb{R}_+} \sum_{i \in \{1,2\}} \int_{\Omega_i} \Phi_i(u(x,t), \kappa) \partial_x \psi(x,t) dx dt + |f_2(\kappa) - f_1(\kappa)| \int_{\mathbb{R}_+} \psi(0,t) dt \ge 0.\end{aligned} \tag{8}$$

Thanks to Assumption **(H1)**, there exist no $\chi \in [0, \max u_i^\star]$ such that $f_1(\chi) = f_2(\chi)$, f_1 is decreasing and f_2 is increasing on $(\chi - \delta, \chi + \delta)$ for some $\delta > 0$. Then the notion of entropy solution described by (8) introduced by Towers [33, 34] is equivalent to the notion of *optimal entropy solution* introduced in [2] (see also [8]). We take advantage of this by using the very simple algebraic relation (8).

It has been proven that the entropy solution u exists and is unique for general flux functions f [6, Chapters 4 and 5]. In particular, the following comparison and L^1-contraction principle holds.

Theorem 1.4 *Let $u_0 \in L^\infty(\mathbb{R})$ with $0 \le u_0 \le u_i^\star$ a.e. in Ω_i, then there exists a unique entropy solution in the sense of Definition 1.3. Furthermore, if v is another entropy solution corresponding to $v_0 \in L^\infty(\mathbb{R})$ with $0 \le v_0 \le u_i^\star$ a.e. in Ω_i, then for all $\mathbb{R} > 0$, for all $t \in \mathbb{R}_+$*

$$\int_{-R}^{R} (u(x,t) - v(x,t))^\pm dx \le \int_{-R-M_f t}^{R+M_f t} (u_0(x) - v_0(x))^\pm dx \tag{9}$$

where M_f is a Lipschitz constant of both f_i.

1.1 non classical shock at the interface

As already mentioned, the optimal entropy solution can be seen as a extension to the case of discontinuous flux functions of the usual entropy solution [25] obtained for a regular flux function. We will now illustrate that it is not the case with the solution to ($\mathcal{P}_{\lim}$). Assume for the moment (it will be proved later) that in the case where $u_0(x) \in (u_i^\star, 1)$ a.e. in Ω_i, the corresponding solution u to ($\mathcal{P}_{\lim}$) admits $\overline{u}_i$ as strong trace on the interface. One has the following *Rankine-Hugoniot* relation

$$f_1(\overline{u}_1) = f_2(\overline{u}_2) = q,$$

then u is a weak solution to (2), i.e. it satisfies for all $\psi \in \mathcal{D}(\mathbb{R} \times \mathbb{R}_+)$:

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}} u \partial_t \psi \, dx dt + \int_{\mathbb{R}} u_0 \psi(\cdot, 0)\, dx + \int_{\mathbb{R}_+} \int_{\mathbb{R}} f(u, \cdot) \partial_x \psi \, dx dt = 0. \tag{10}$$

Firstly, suppose for the sake of simplicity that $f_1(u) = f_2(u) = f(u)$, and that $q = 0$, then $u_i^\star = 0$ for $i \in \{1, 2\}$. The function

$$u(x) = \begin{cases} 1 & \text{if } x < 0, \\ 0 & \text{if } x > 0 \end{cases}$$

is then a steady solution to ($\mathcal{P}_{\text{lim}}$) satisfying (5). However, since

$$\frac{f(1) - f(s)}{1 - s} < 0 \quad \text{for all } s \in (0, 1),$$

the discontinuity at $\{x = 0\}$ does not fulfill the usual *Oleinik entropy condition* (see e.g. [31]). This discontinuity is thus said to be a *non-classical shock*.

Suppose now that $f_1'(1) < 0$ and that $f_2'(u_2^\star) > 0$, then the pair $(1, u_2^\star)$ is a stationary *undercompressible shock-wave*, that are prohibited for optimal entropy solutions [2] as for classical entropy solutions in the case of regular flux functions.

Remark. *1.5 It has been pointed out in [2] that allowing a* connection (A, B)*, i.e. a stationary undercompressible wave between the left state A and the right state B at the interface lead to another L^1-contraction semi-group (see [2, 8, 21]), which is so-called entropy solution of type (A, B). However, we rather use the denomination* non-classical shock *for the connection between A and B since, as stressed above, the corresponding solution violates some fundamental properties of the classical entropy solutions.*

1.2 oil-trapping modeled by the non-classical shock

In this section, we assume that $q = 0$. Let u be the solution of the problem ($\mathcal{P}_{\text{lim}}$) corresponding to the initial data u_0. Assume that u admits strong traces on the interface. The flow-rate of oil going from Ω_1 to Ω_2 through the interface is given by

$$f_1(\overline{u}_1) = f_2(\overline{u}_2) = 0.$$

Thus the oil cannot overcome the interface from Ω_1 to Ω_2, thus if one supposes that u_0 belongs to $L^\infty(\mathbb{R})$, with $0 \leq u_0 \leq 1$ a.e., then the quantity of oil standing between $x = -R$ (R is an arbitrary positive number) and $x = 0$ can only grow.

Indeed, let $t_2 > t_1 \geq 0$, let $\zeta_n(x) = \min(1, n(x+R)^+, nx^-)$ and $\theta_m(t) = \min(1, m(t-t_1), m(t_2-t))$. Choosing $\psi(x, t) = \zeta_n(x)\theta_m(t)$ in (10) for $m, n \in \mathbb{N}$ yields, using the positivity of f_1

$$\int_{t_1}^{t_2} \left(\int_{-R}^{0} u(x, t)\zeta_n(x)dx \right) \partial_t\theta_m(t)dt + \int_{t_1}^{t_2} \theta_m(t) \left(\frac{1}{n} \int_{-1/n}^{0} f_1(u(x, t))dx \right) dt \leq 0.$$

Since u admits a strong trace on the interface,

$$\lim_{n\to\infty} \frac{1}{n} \int_{-1/n}^{0} f_1(u(x, t))dx = f_1(\overline{u}_1) = 0.$$

Then we obtain

$$\int_{t_1}^{t_2} \left(\int_{-R}^{0} u(x, t)dx \right) \partial_t\theta_m(t)dt \leq 0. \tag{11}$$

The solution u belong to $\mathcal{C}(\mathbb{R}_+; L^1(\mathbb{R}))$ thanks to [12], thus taking the limit as $m \to \infty$ in (11) provides

$$\int_{-R}^{0} u(x, t_1)\, dx \leq \int_{-R}^{0} u(x, t_2)\, dx.$$

Suppose now that $q \geq 0$. Thanks to what follows, we are able to solve the Riemann problem at the interface for any initial data

$$u_0(x) = \begin{cases} u_\ell & \text{if} \quad x < 0, \\ u_r & \text{if} \quad x > 0. \end{cases}$$

The study of the Riemann problem is carried out in Section 5, leading to the following result.

- If $u_\ell > u_1^\star$, then $u_1 = 1$ and $u_2 = u_2^\star$. We obtain the expected non-classical shock at the interface.
- If $u_\ell \le u_1^\star$, then $u_1 = u_\ell$ and u_2 is the unique value of $[0, u_2^\star]$ such that $f_2(u_2) = f_1(u_\ell)$.

Using Assumption **(H1)**, this particularly implies that in both cases, the flux at the interface is given by

$$f_1(u_1) = f_2(u_2) = G_1(u_\ell, 1) \tag{12}$$

where G_1 is the Godunov solver corresponding to the flux function f_1:

$$G_1(a,b) = \begin{cases} \min\limits_{s\in[a,b]} f_1(s) & \text{if} \quad a \le b, \\ \max\limits_{s\in[b,a]} f_1(s) & \text{if} \quad a > b. \end{cases}$$

This particularly yields that for any initial data $u_0 \in L^\infty(\mathbb{R})$ with $0 \le u_0 \le 1$, the restriction $u_{|\Omega_1}$ of the solution u to Ω_1 is the unique entropy solution to

$$\begin{cases} \partial_t u + \partial_x f_1(u) = 0 & \text{in } \Omega_1 \times \mathbb{R}_+, \\ u(\cdot, 0) = u_0 & \text{in } \Omega_1, \\ u(0, \cdot) = \gamma & \text{in } \mathbb{R}_+ \end{cases} \tag{13}$$

for $\gamma = 1$. Since the solution u to the problem (13) is a non-decreasing function of the prescribed trace γ on $\{x = 0\}$, we can claim as in [10] that

$$u_{|\Omega_1} = \sup_{\substack{\gamma\in L^\infty(\mathbb{R}_+)\\ 0\le\gamma\le 1}} \{ \, v \text{ solution to (13)} \, \}.$$

In particular, u is the unique weak solution (i.e. satisfying (10)) that is entropic in each subdomain and that minimizes the flux through the interface.

1.3 organization of the paper

We will introduce a family of approximate problems in Section 2, which takes into account the capillarity, with small dependance ε of the capillary pressure with respect to the saturation. We use the transmission conditions introduced in [9, 11, 13, 30] to connect the capillary pressure at the interface. For $\varepsilon > 0$, the problem $(\mathcal{P}^\varepsilon)$ admits a unique solution u^ε thanks to [11] and it is recalled that a comparison principle holds for the solutions of the approximate problem $(\mathcal{P}^\varepsilon)$. Particular sub- and super-solution are derived in order to show that if $u_0(x) \ge u_I^\star$ a.e. in Ω_i, then the limit u of the approximate solutions $(u^\varepsilon)_{\varepsilon>0}$ as ε tends to 0. An energy estimate is also derived.

In Section 3, letting ε tend to 0, since no strong pre-compactness can be derived on $(u^\varepsilon)_\varepsilon > 0$ in $L^1_{loc}(\mathbb{R} \times \mathbb{R}_+)$ from the available estimates, we use the notion of process solution [20], which is equivalent to the notion of measure valued solution introduced by DiPerna [17] (see also [27, 32]). The uniqueness of such a process solution allows us to claim that (u^ε) converges strongly in $L^1_{loc}((\mathbb{R} \times \mathbb{R}_+)$ towards the unique solution to $(\mathcal{P}_{\lim})$.

In Section 4, it is shown that if both phases move in the same direction, that is if $0 \le u_0 \le u_i^\star$ a.e. in Ω_i, then (u^ε) converges towards the unique entropy solution to the problem in the sense of Definition 1.3.

In Section 5, we complete the study of the Riemann problem at the interface.

2 The approximate problem

In this section, we take into account the effects of the capillarity, supposing that they are small. We will so build an approximate problem $(\mathcal{P}^\varepsilon)$, whose unknown u^ε will depend on a small parameter ε representing the dependance of the capillary pressure with respect to the saturation. We assume for the sake of simplicity that the capillary pressure in Ω_i is given by:

$$\pi_i^\varepsilon(u^\varepsilon) = P_i + \varepsilon u^\varepsilon. \tag{14}$$

It has been shown simultaneously in [9] and in [13] that a good way to connect the capillary pressures at the interface is to require

$$\tilde{\pi}_1^\varepsilon(u_1^\varepsilon) \cap \tilde{\pi}_2^\varepsilon(u_2^\varepsilon) \neq \emptyset, \tag{15}$$

where u_1^ε and u_2^ε are the traces of u^ε on the interface, and where $\tilde{\pi}_i^\varepsilon$ is the monotonous graph given by

$$\tilde{\pi}_i^\varepsilon(s) = \begin{cases} \pi_i^\varepsilon(s) & \text{if } s \in (0,1), \\ (-\infty, P_i] & \text{if } s = 0, \\ [P_i + \varepsilon, \infty) & \text{if } s = 1. \end{cases}$$

We suppose that the capillary force is oriented in the sense of decreasing x, i.e. $P_1 < P_2$ (the capillary force goes from the high capillary pressure to the low capillary pressure). Since ε is assumed to be a small parameter, we can suppose that $0 < \varepsilon < P_2 - P_1$, so that the relation (15) turns to

$$u_1^\varepsilon = 1 \text{ or } u_2^\varepsilon = 0. \tag{16}$$

The flux function in Ω_i is then given by:

$$F_i^\varepsilon(x,t) = f_i(u^\varepsilon)(x,t) - \varepsilon \partial_x \varphi_i(u^\varepsilon)(x,t).$$

Because of the conservation of mass, we require the continuity of the flux functions at the interface. Thus the approximate problem becomes

$$\begin{cases} \partial_t u^\varepsilon + \partial_x F_i^\varepsilon = 0, \\ u^\varepsilon(x = 0^-) = 1 \text{ or } u^\varepsilon(x = 0^+) = 0, \\ F_1^\varepsilon(0^-) = F_2^\varepsilon(0^+), \\ u(t = 0) = u_0. \end{cases} \tag{$\mathcal{P}^\varepsilon$}$$

We are not able to prove the uniqueness of a weak solution of ($\mathcal{P}_{\lim}$) if the flux F_i^ε "only" belongs to $L^2(\overline{\Omega}_i \times \mathbb{R}_+)$, and we will define the notion of prepared initial data, so that the flux belongs to $L^\infty(\Omega_i \times \mathbb{R}_+)$. In this latter case, the uniqueness holds.

2.1 bounded flux solutions

We define now the notion of bounded flux solution, that was introduced in this framework in [11, 13].

Definition 2.1 (bounded flux solution to ($\mathcal{P}^\varepsilon$)) *Let $u_0 \in L^\infty(\mathbb{R})$, $0 \le u_0 \le 1$, a function u^ε is said to be a bounded flux solution if*

1. $u^\varepsilon \in L^\infty(\mathbb{R} \times \mathbb{R}_+)$, $0 \le u \le 1$;
2. $\partial_x \varphi_i(u^\varepsilon) \in L^\infty(\Omega_i \times \mathbb{R}_+) \cap L^2_{loc}(\mathbb{R}_+; L^2(\Omega_i))$;
3. $u_1^\varepsilon(t)\,(1 - u_2^\varepsilon(t)) = 0$ *for almost all* $t \ge 0$, *where* u_i^ε *denotes the trace of* $u^\varepsilon_{|\Omega_i}$ *on* $\{x = 0\}$.
4. $\forall \psi \in \mathcal{D}(\mathbb{R} \times \mathbb{R}_+)$,

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}} u^\varepsilon(x,t) \partial_t \psi(x,t) dx dt + \int_{\mathbb{R}} u_0(x) \psi(x,0) dx + \int_{\mathbb{R}_+} \sum_{i \in \{1,2\}} \int_{\Omega_i} [f_i(u^\varepsilon) - \varepsilon \partial_x \varphi_i(u^\varepsilon)]\, \partial_x \psi(x,t) dx dt = 0. \tag{17}$$

Remark. *2.2 Such a bounded-flux u^ε solution belongs to $\mathcal{C}(\mathbb{R}_+; L^1_{\text{loc}}(\mathbb{R}))$, in the sense that there exists $\tilde{u^\varepsilon}$ in $\mathcal{C}(\mathbb{R}_+; L^1_{\text{loc}}(\mathbb{R}))$ such that $u^\varepsilon(t) = \tilde{u^\varepsilon}(t)$ for almost all $t \ge 0$ (see [12]). More precisely, all $t \ge 0$ is a Lebesgue point for u^ε. So, the slight abuse of notation consisting in considering $u^\varepsilon(t)$ for all $t \ge 0$ will not lead to any confusion.*

Proposition 2.3 *Let u and v be two bounded-flux solutions associated to initial data u_0, v_0, then for all $\psi \in \mathcal{D}^+(\mathbb{R}\times\mathbb{R}_+)$,*

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}} (u-v)^{\pm}\,\partial_t\psi dxdt + \int_{\mathbb{R}} (u_0-v_0)^{\pm}\,\psi(\cdot,0)dx$$
$$+\int_{\mathbb{R}_+}\sum_i\int_{\Omega_i}\left(\Phi_{i\pm}(u,v) - \varepsilon\partial_x\left(\varphi_i(u)-\varphi_i(v)\right)^{\pm}\right)\partial_x\psi dxdt \geq 0. \tag{18}$$

We state now a theorem which is a generalization in the case of unbounded domains of Theorem 3.1 and Theorem 4.1 stated in [11].

Theorem 2.4 (existence–uniqueness for bounded flux solutions) *Let $u_0 \in L^\infty(\mathbb{R})$ with $0 \leq u_0 \leq 1$ such that:*

- *there exists a function $\hat{u} \in L^\infty(\mathbb{R})$, with $0 \leq \hat{u} \leq 1$ a.e. in $\mathbb{R}$, satisfying $\partial_x\hat{u} \in L^1(\mathbb{R})\cap L^\infty(\mathbb{R})$ and such that $(u_0-\hat{u}) \in L^1(\mathbb{R})$*
- *$\partial_x\varphi_i(u_0) \in L^\infty(\Omega_i)$*
- *$\lim_{x\nearrow 0} u_0(x) = 1$ or $\lim_{x\searrow 0} u_0(x) = 0$.*

Then there exists a unique bounded flux solution u^ε to the problem ($\mathcal{P}^\varepsilon$) in the sense of Definition 2.1 satisfying $(u^\varepsilon - \hat{u}) \in L^1(\mathbb{R})$. Furthermore, if $u^\varepsilon, v^\varepsilon$ are two bounded flux solutions associated to initial data u_0, v_0 then

$$u_0(x) \geq v_0(x) \text{ a.e. in } \mathbb{R} \quad \Rightarrow u^\varepsilon(x,t) \geq v^\varepsilon(x,t) \text{ a.e. in } \mathbb{R} \text{ for all } t \geq 0. \tag{19}$$

Obviously, the existence of a bounded flux solution can not be extended to any initial data in $L^1(\mathbb{R})$. Indeed, the initial data u_0 has at least to involve bounded initial flux, i.e. $\partial_x\varphi_i(u_0) \in L^\infty(\mathbb{R})$. An additional natural assumption is needed to ensure the existence of such a bounded flux solution : the connection in the graphical sense of the capillary pressures at the interface.

If $(u_0 - \hat{u})$ and $(v_0 - \hat{u})$ belong to L^1 for the same $\hat{u}$, then (18) yields that the bounded flux solutions u^ε and v^ε corresponding to u_0 and v_0 satisfy the following contraction principle: $\forall t \in \mathbb{R}_+$,

$$\int_{\mathbb{R}} (u^\varepsilon(x,t) - v^\varepsilon(x,t))^{\pm} dx \leq \int_{\mathbb{R}} (u_0(x) - v_0(x))^{\pm} dx,$$

providing the uniqueness result stated in Theorem 2.4.

2.2 particular sub- and super-solutions

We will study particular steady states of the approximate problem ($\mathcal{P}^\varepsilon$). We will consider steady bounded flux solutions s^ε corresponding to a zero water flow rate, i.e.

$$f_i(s^\varepsilon) - \varepsilon\frac{\mathrm{d}}{\mathrm{d}x}\varphi_i(s^\varepsilon) = q \quad \text{ in } \Omega_i. \tag{20}$$

For $\varepsilon > 0$, there are infinitely many solutions s^ε of the equation (20). We will construct some particular solutions, that will permit us to show that the limit u as ε tends to 0 of bounded flux solutions u^ε corresponding to large initial data admits the expected strong traces on the interface $\{x = 0\}$.

We will introduce now particular solutions of the ordinary differential equation

$$y' = f_i \circ \varphi_i^{-1}(y) - q. \tag{21}$$

Lemma 2.5 *Let $\eta > 0$, there exists a solution y^η to (21) for $i = 1$ which is nondecreasing on $(-\infty, -0]$ equal to $\varphi_1(1)$ on $[-\eta, 0)$, satisfying $y^\eta(x) < \varphi_1(1)$ if $x < -\eta$ and $\lim_{x\to-\infty} y^\eta(x) = u_1^\star$.*

Proof: Consider the problem

$$\left\{\begin{array}{rcll} w'(x) & = & R(\varphi_i(1)-w(x))^m & \text{if} \quad x<-\eta, \\ w(-\eta) & = & \varphi_1(1), & \end{array}\right. \tag{22}$$

where R and m are constants given by Assumption **(H2)**. The function

$$w^\eta(x) = \varphi_i(1) - (R(1-m)(-x-\eta))^{\frac{1}{1-m}}$$

is a solution of (22). Because of **(H2)**, there exists a neighborhood $(-\eta-\delta,-\eta]$ of η such that w^η is a super-solution of the problem

$$\left\{\begin{array}{rcll} y'(x) & = & f_1\circ\varphi_1^{-1}(y)-q & \text{if} \quad x<-\eta, \\ y(-\eta) & = & \varphi_1(1). & \end{array}\right. \tag{23}$$

Then there exists y^η solution to (23) such that $y^\eta(x)=\varphi_1(1)$ if $x\in(-\eta,0)$ and

$$y^\eta(x)\le w^\eta(x) \qquad \text{on } (-\eta-\delta,-\eta].$$

In particular, y^η is not constant equal to 1. Thanks to **(H1)**, the function y^η is increasing on the set $\{\, x\in\Omega_1 \mid y^\eta(x)\in(\varphi_1(u_1^\star),\varphi_1(1))\,\}$. Assume that there exists $x_\star<-\eta$ such that $y^\eta(x_\star)=\varphi_1(u_1^\star)$, then one sets $y^\eta(x)=\varphi_1(u_1^\star)$ for all $x\in(-\infty,x_\star]$. If $y^\eta(x)>\varphi_1(u_1^\star)$ for all $x<0$, then y^η is increasing on $(-\infty,-\eta)$. Thus it admits a limit as x tends to $-\infty$, and it is clear that the only possible limit is $u_1^\star$. □

Lemma 2.6 *Let $\eta>0$, then there exists a solution z^η to (21) for $i=2$ which is nondecreasing on $\mathbb{R}$ satisfying $z^\eta(x)\le\varphi_2\left(\frac{1+u_2^\star}{2}\right)$ if $x\le\eta$, $z^\eta(x)\ge\varphi_2\left(\frac{1+u_2^\star}{2}\right)$ if $x\ge\eta$ and $\lim_{x\to\infty} z^\eta(x)=\varphi_2(1)$, $\lim_{x\to-\infty} z(x)=u_2^\star$.*

Proof: The problem

$$\left\{\begin{array}{rcll} z'(x) & = & f_2\circ\varphi_2^{-1}(z(x))-q & \text{if} \quad x\in\mathbb{R}, \\ z(\eta) & = & \varphi_2\left(\frac{1+u_2^\star}{2}\right). & \end{array}\right.$$

admits a (unique) solution in $\mathcal{C}^1(\mathbb{R},[0,1])$. Since $u_2^\star$ is a constant solution of (21) for $i=2$, then one has $z(x)\ge u_2^\star$ in $\mathbb{R}$. Thanks to **(H1)**, the function z is nondecreasing. This implies that it admits limits respectively in $-\infty$ and in $+\infty$. The only possible values for this limits are respectively $u_2^\star$ and 1. □

Proposition 2.7 *Let $\eta>0$, then there exists two families of steady bounded flux solutions $(\underline{s}^{\varepsilon,\eta})_{\varepsilon>0}$ and $(\overline{s}^{\varepsilon,\eta})_{\varepsilon>0}$ tending in $L^1_{loc}(\mathbb{R})$ as $\varepsilon\to0$ respectively towards*

$$\underline{s}^\eta : x\mapsto \left\{\begin{array}{lll} u_1^\star & \textit{if} & x<-\eta, \\ 1 & \textit{if} & x\in(-\eta,0), \\ u_2^\star & \textit{if} & x>0, \end{array}\right.$$

and

$$\overline{s}^\eta : x\mapsto \left\{\begin{array}{lll} 1 & \textit{if} & x<0, \\ u_2^\star & \textit{if} & x\in(0,\eta), \\ 1 & \textit{if} & x>\eta. \end{array}\right.$$

Proof: We set

$$\underline{s}^{\varepsilon,\eta}(x) = \left\{\begin{array}{lll} y^\eta\left(\frac{x+\eta}{\varepsilon}-\eta\right) & \text{if} & x<0, \\ u_2^\star & \text{if} & x>0, \end{array}\right. \tag{24}$$

and

$$\overline{s}^{\varepsilon,\eta}(x) = \left\{\begin{array}{lll} 1 & \text{if} & x<0, \\ z^\eta\left(\frac{x-\eta}{\varepsilon}+\eta\right) & \text{if} & x>0, \end{array}\right. \tag{25}$$

where the functions y^η and z^η have been defined in Lemmas 2.5 and 2.6. Since the functions $\varphi_i(\underline{s}^{\varepsilon,\eta})$ and $\varphi_i(\overline{s}^{\varepsilon,\eta})$ are monotone in Ω_i, there derivatives $\frac{\mathrm{d}}{\mathrm{d}x}\varphi_i(\underline{s}^{\varepsilon,\eta})$ and $\frac{\mathrm{d}}{\mathrm{d}x}\varphi_i(\overline{s}^{\varepsilon,\eta})$ belong to $L^1(\mathbb{R})$, and also to $L^\infty(\mathbb{R})$ because $\underline{s}^{\varepsilon,\eta}$ and $\overline{s}^{\varepsilon,\eta}$ are solutions to (20). Thus they belong to $L^2(\mathbb{R})$. Hence, for fixed ε, $\underline{s}^{\varepsilon,\eta}$ and $\overline{s}^{\varepsilon,\eta}$ are bounded flux solutions to the problem $(\mathcal{P}^\varepsilon)$. The convergence as $\varepsilon\to0$ towards the functions $\overline{s}^\eta$ and $\underline{s}^\eta$ is a direct consequence of Lemmas 2.5 and 2.6. □

2.3 a $L^2((0,T);H^1(\Omega_i))$ estimate

Our goal is now to derive an estimate which ensures that the effects of capillarity vanish almost everywhere in $\Omega_i \times \mathbb{R}_+$ as ε tends to 0.

Proposition 2.8 *Let $u_0 \in L^\infty(\mathbb{R})$ with $0 \le u_0 \le 1$ a.e. satisfying the assumptions of Theorem 2.4 and let u^ε be the corresponding bounded flux solution. Then for all $\varepsilon \in (0,1)$, for all $T > 0$, there exists C depending only on u_0, g_i, φ_i, T such that*

$$\sqrt{\varepsilon}\|\partial_x \varphi_i(u^\varepsilon)\|_{L^2(\Omega_i\times(0,T))} \le C. \tag{26}$$

This particularly ensures that

$$\varepsilon\|\partial_x \varphi_i(u^\varepsilon)\|_{L^2(\Omega_i\times(0,T))} \to 0 \qquad \text{as } \varepsilon \to 0. \tag{27}$$

The idea of the proof of Proposition 2.8 is formally to choose $(u^\varepsilon - \hat{u})\psi$ as test function in (17) for a function $x \mapsto \psi(x)$ compactly supported in Ω_i. Using the fact that the flux F_i^ε is uniformly bounded in $L^\infty(\Omega_i \times (0,T))$, we can let ψ tend towards χ_{Ω_i}, with $\chi_{\Omega_i}(x) = 1$ if $x \in \Omega_i$ and 0 otherwise, and the estimate (26) follows. To obtain (27), it suffices to multiply (26) by $\sqrt{\varepsilon}$. We refer to [10, Proposition 2.3] for a more details on the proof of Proposition 2.8.

2.4 approximation of the initial data

In order to ensure that the limit u of the approximate solutions u^ε as $\varepsilon \to 0$ admits the expected strong traces on the interface $\{x = 0\}$, we will perturb the initial data u_0.

Lemma 2.9 *Let $u_0 \in L^\infty(\mathbb{R})$ satisfying (4), then there exists $(u_0^{\varepsilon,\eta})_{\varepsilon,\eta}$ such that*

(a). $\underline{s}^{\varepsilon,\eta}(x) \le u_0^{\varepsilon,\eta}(x) \le \overline{s}^{\varepsilon,\eta}(x)$ a.e. in $\mathbb{R}$, where the functions $\underline{s}^{\varepsilon,\eta}$ and $\overline{s}^{\varepsilon,\eta}$ are defined in (24)-(25),

(b). $\varepsilon \left\|\partial_x \varphi_i(u_0^{\varepsilon,\eta})\right\|_{L^\infty(\Omega_i)} \le C$ where C depends neither on ε nor on η,

(c). $u_0^{\varepsilon,\eta} \to u_0$ in $L^1_{loc}(\mathbb{R})$ as $(\varepsilon,\eta) \to (0,0)$.

Proof: Let $(\rho_n)_{n\in\mathbb{N}^\star}$ be a sequence of mollifiers, then $\rho_n * u_0$ is a smooth function tending u_0 as $n \to \infty$. Then, for $\varepsilon > 0$, we choose $n \in \mathbb{N}^\star$ such that

$$\max\left\{n, \left\|\partial_x \varphi_i(u_0 * \rho_n)\right\|_{L^\infty(\Omega_i)}\right\} \ge \frac{1}{\varepsilon}, \tag{28}$$

and we define

$$u_0^{\varepsilon,\eta}(x) = \max\left\{\underline{s}^{\varepsilon,\eta}(x), \min\left\{\overline{s}^{\varepsilon,\eta}(x), u_0 * \rho_n(x)\right\}\right\}. \tag{29}$$

The point (a) is a direct consequence of (29). Letting $(\varepsilon,\eta) \to (0,0)$ in (29) yields

$$\lim_{(\varepsilon,\eta)\to(0,0)} u_0^{\varepsilon,\eta}(x) = \max\left\{u_i^\star, \min\left\{1, u_0(x)\right\}\right\}.$$

Since u_0 is supposed to satisfy (4), this provides

$$\lim_{(\varepsilon,\eta)\to(0,0)} u_0^{\varepsilon,\eta}(x) = u_0(x) \qquad \text{a.e. in } \mathbb{R}.$$

The point (c) follows. In order to establish (b), it suffices to note that there exist an open subset ω of $\mathbb{R}$ such that $u_0^{\varepsilon,\eta}(x)$ is equal to $u_0 * \rho_n(x)$ for $x \in \omega$, and such that $u_0^{\varepsilon,\eta}(x)$ is either equal to $\underline{s}^{\varepsilon,\eta}(x)$ or to $\overline{s}^{\varepsilon,\eta}(x)$ on $\omega^c = \mathbb{R}\setminus\omega$. It follows from (28) that

$$\varepsilon \left\|\partial_x \varphi_i(u_0^{\varepsilon,\eta})\right\|_{L^\infty(\Omega_i\cap\omega)} \le 1.$$

One has

$$f_i(u_0^{\varepsilon,\eta})(x) - \varepsilon\partial_x\varphi_i(u_0^{\varepsilon,\eta})(x) = q \qquad \text{a.e. in } \Omega_i \cap \omega^c,$$

thus

$$\varepsilon \left\|\partial_x \varphi_i(u_0^{\varepsilon,\eta})\right\|_{L^\infty(\Omega_i\cap\omega^c)} \le \|q - f_i\|_{L^\infty(u_i^\star,1)}.$$

This concludes the proof of Lemma 2.9. □

Definition 2.10 *A function u_0 is said to be a prepared initial data if it satisfies $(1 - u_0) \in L^1(\mathbb{R})$, $\partial_x \varphi_i(u_0) \in L^\infty(\Omega_i)$ and*

$$\underline{s}^{\varepsilon,\eta}(x) \leq u_0(x) \leq \overline{s}^{\varepsilon,\eta}(x) \text{ a.e. in } \mathbb{R} \tag{30}$$

for some $\varepsilon > 0, \eta > 0$.

Since the function $(\varepsilon, \eta) \mapsto \underline{s}^{\varepsilon,\eta}$ is decreasing with respect to both arguments and since the function $(\varepsilon, \eta) \mapsto \overline{s}^{\varepsilon,\eta}$ is increasing with respect to both arguments, if u_0 satisfies (30) for $\varepsilon = \varepsilon_0$ and $\eta = \eta_0$, then u_0 satisfies (30) for all (ε, η) such that $\varepsilon \leq \varepsilon_0$ and $\eta \leq \eta_0$. So the following Proposition is a direct consequence from (19).

Proposition 2.11 *Let u_0 be a prepared initial data satisfying (30) for $\varepsilon = \varepsilon_0$ and $\eta = \eta_0$, then for all $\varepsilon \leq \varepsilon_0$, for all $\eta \leq \eta_0$, the solution u^ε to $(\mathcal{P}^\varepsilon)$ satisfies*

$$\underline{s}^{\varepsilon,\eta}(x) \leq u^\varepsilon(x,t) \leq \overline{s}^{\varepsilon,\eta}(x) \quad \text{for a.e. } (x,t) \in \mathbb{R} \times \mathbb{R}_+.$$

3 Convergence

3.1 a compactness result

Since $(u^\varepsilon)_\varepsilon$ is uniformly bounded between 0 and 1, there exists $u \in L^\infty(\mathbb{R} \times (0,T))$ such that $u^\varepsilon \to u$ is the L^∞ weak-star sense. This is of course insufficient to pass in the limit in the nonlinear terms. Either greater estimates are needed, like for example a BV-estimate introduced in the work of Vol′pert [37] and in [10], or we have to use a weaker compactness result. This idea motivates the introduction of Young measures as in the papers of DiPerna [17] and Szepessy [32], or equivalently the notion of nonlinear weak star convergence, introduced in [19] and [20], which leads to the notion of process solution given in Definition 3.2.

Theorem 3.1 (Nonlinear weak star convergence) *Let $\mathcal{Q}$ be a Borelian subset of $\mathbb{R}^k$, and (u_n) be a bounded sequence in $L^\infty(\mathcal{Q})$. Then there exists $u \in L^\infty(\mathcal{Q} \times (0,1))$, such that up to a subsequence, u_n tends to u "in the non linear weak star sense" as $n \to \infty$, i.e.: $\forall g \in \mathcal{C}(\mathbb{R}, \mathbb{R})$,*

$$g(u_n) \to \int_0^1 g(u(\cdot, \alpha))d\alpha \text{ for the weak star topology of } L^\infty(\mathcal{Q}) \text{ as } n \to \infty.$$

We refer to [17] and [20] for the proof of Theorem 3.1.

3.2 convergence towards a process solution

Because of the lack of compactness, we have to introduce the notion of process solution, inspired from the notion of *measure valued solution* introduced by DiPerna [17].

Definition 3.2 (process solution to $(\mathcal{P}_{\lim})$) *A function $u \in L^\infty(\mathbb{R} \times \mathbb{R}_+ \times (0,1))$ is said to be a process solution to $(\mathcal{P}_{\lim})$ if $0 \leq u \leq 1$ and for $i = 1, 2$, $\forall \psi \in \mathcal{D}^+(\overline{\Omega}_i \times \mathbb{R}_+)$, $\forall \kappa \in [0,1]$,*

$$\int_{\mathbb{R}_+} \int_{\Omega_i} \int_0^1 (u(x,t,\alpha) - \kappa)^\pm \partial_t \psi(x,t) d\alpha dx dt + \int_{\Omega_i} (u_0(x) - \kappa)^\pm \psi(x,0) dx$$
$$+ \int_{\mathbb{R}_+} \int_{\Omega_i} \int_0^1 \Phi_{i\pm}(u(x,t,\alpha), \kappa) \partial_x \psi(x,t) d\alpha dx dt + M_{f_i} \int_{\mathbb{R}_+} (\overline{u}_i - \kappa)^\pm \psi(0,t) dt \geq 0,$$

where M_{f_i} is any Lipschitz constant of f_i, $\overline{u}_1 = 1$ and $\overline{u}_2 = u_2^\star$.

Lemma 3.3 *Let u_0 be a η-prepared initial data in the sense of Definition 2.10 for some $\eta > 0$, and let $(u^\varepsilon)_\varepsilon$ be the corresponding family of approximate solutions. Then*

$$u^\varepsilon(x,t) \to 1 \quad \text{for a.e. } (x,t) \in (-\eta, 0) \times \mathbb{R}_+, \tag{31}$$

$$u^\varepsilon(x,t) \to u_2^\star \quad \text{for a.e. } (x,t) \in (0, \eta) \times \mathbb{R}_+. \tag{32}$$

Proof: Firstly, since u_0 is a η-prepared initial data, there exists $\varepsilon_0 > 0$ such that

$$\underline{s}^{\varepsilon_0,\eta} \leq u_0 \leq \overline{s}^{\varepsilon_0,\eta}.$$

Then it follows from Proposition 2.11 that for all $\varepsilon \in (0, \varepsilon_0)$, for a.e. $(x,t) \in \mathbb{R} \times \mathbb{R}_+$

$$\underline{s}^{\varepsilon,\eta}(x) \leq u^\varepsilon(x,t) \leq \overline{s}^{\varepsilon,\eta}(x). \tag{33}$$

This particularly shows that for all $\varepsilon \in (0, \varepsilon_0)$, for a.e. $(x,t) \in (-\eta, 0) \times \mathbb{R}_+$,

$$u^\varepsilon(x,t) = 1,$$

thus (31) holds. The assertion (32) can be obtained by using Proposition 2.7 and the dominated convergence theorem. □

Proposition 3.4 (convergence towards a process solution) *Let u_0 be a prepared initial data in the sense of Definition 2.10, and let $(u^\varepsilon)_\varepsilon$ be the corresponding family of approximate solutions. Then, up to an extraction, u^ε converges in the nonlinear weak-star sense towards a process solution u to the problem $(\mathcal{P}_{\lim})$.*

Proof: Since u^ε is a weak solution of $(\mathcal{P}^\varepsilon)$, which is a non-fully degenerate parabolic problem, i.e. φ_i^{-1} is continuous, it follows from the work of Carrillo [14] that u^ε is an entropy weak solution, i.e.: $\forall \psi \in \mathcal{D}^+(\Omega_i \times \mathbb{R}_+)$, $\forall \kappa \in [0,1]$,

$$\int_{\mathbb{R}_+}\int_{\Omega_i} (u^\varepsilon(x,t) - \kappa)^\pm \partial_t \psi(x,t) dx dt + \int_{\Omega_i} (u_0(x) - \kappa)^\pm \psi(x,0) dx \\ + \int_{\mathbb{R}_+}\int_{\Omega_i} \left[\Phi_{i\pm}(u^\varepsilon(x,t),\kappa) - \varepsilon \partial_x (\varphi_i(u^\varepsilon)(x,t) - \varphi_i(\kappa))^\pm\right] \partial_x \psi(x,t) dx dt \geq 0.$$

This family of inequalities is only available for non-negative functions ψ compactly supported in Ω_i, and so vanishing on the interface $\{x = 0\}$. To overpass this difficulty, we use cut-off functions $\chi_{i,\delta}$.

Let $\delta > 0$, we denote by $\chi_{i,\delta}$ a smooth non-negative function, with $\chi_{i,\delta}(x) = 0$ if $x \notin \Omega_i$, and $\chi_{i,\delta}(x) = 1$ if $x \in \Omega_i$, $|x| \geq \delta$. Let $\psi \in \mathcal{D}^+(\overline{\Omega} \times \mathbb{R}_+)$, then $\psi\chi_{i,\delta} \in \mathcal{D}^+(\Omega_i \times \mathbb{R}_+)$ can be used as test function in (34). This yields

$$\int_{\mathbb{R}_+}\int_{\Omega_i} (u^\varepsilon - \kappa)^\pm \partial_t \psi \chi_{i,\delta} dx dt + \int_{\Omega_i} (u_0 - \kappa)^\pm \psi(\cdot,0)\chi_{i,\delta} dx \\ + \int_{\mathbb{R}_+}\int_{\Omega_i} \left[\Phi_{i\pm}(u^\varepsilon,\kappa) - \varepsilon\partial_x(\varphi_i(u^\varepsilon) - \varphi_i(\kappa))^\pm\right] \partial_x \psi \chi_{i,\delta} dx dt \\ + \int_{\mathbb{R}_+}\int_{\Omega_i} \left[\Phi_{i\pm}(u^\varepsilon,\kappa) - \varepsilon\partial_x(\varphi_i(u^\varepsilon) - \varphi_i(\kappa))^\pm\right] \psi \partial_x \chi_{i,\delta} dx dt \geq 0. \tag{34}$$

We can now let ε tend to 0. Thanks to Theorem 3.1, there exists $u \in L^\infty(\mathbb{R} \times \mathbb{R}_+ \times (0,1))$ such that

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}_+}\int_{\Omega_i} (u^\varepsilon(x,t) - \kappa)^\pm \partial_t \psi(x,t) \chi_{i,\delta}(x) dx dt = \\ \int_{\mathbb{R}_+}\int_{\Omega_i}\int_0^1 (u(x,t,\alpha) - \kappa)^\pm \partial_t \psi(x,t)\chi_{i,\delta}(x) d\alpha dx dt, \tag{35}$$

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}_+}\int_{\Omega_i} \Phi_{i\pm}(u^\varepsilon(x,t),\kappa) \partial_x \psi(x,t) \chi_{i,\delta}(x) dx dt = \\ \int_{\mathbb{R}_+}\int_{\Omega_i}\int_0^1 \Phi_{i\pm}(u(x,t,\alpha),\kappa) \partial_x \psi(x,t)\chi_{i,\delta}(x) d\alpha dx dt. \tag{36}$$

Thanks to Proposition 2.8, one has

$$\varepsilon \partial_x (\varphi_i(u^\varepsilon) - \varphi_i(\kappa))^\pm \text{ tends to } 0 \quad \text{a.e. in } \Omega_i \times (0,T) \text{ as } \varepsilon \to 0,$$

then

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}_+}\int_{\Omega_i}\varepsilon\partial_x(\varphi_i(u^\varepsilon)(x,t)-\varphi_i(\kappa))^{\pm}\partial_x\left(\psi(x,t)\chi_{i,\delta}(x)\right)dxdt=0. \tag{37}$$

Since u_0 is supposed to be a η-prepared initial data for some $\eta > 0$, we can claim thanks to Lemma 3.3 that $u^\varepsilon(x,t)$ converges almost everywhere on $(-\eta,\eta)\times\mathbb{R}_+$ towards $\overline{u}_i$ if $x\in\Omega_i$. Since for $\delta<\eta$ small enough, the support of $\partial_x\chi_{1,\delta}$ is included in the set where u^ε converges strongly, one has

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}_+}\int_{\Omega_i}\Phi_{i\pm}(u^\varepsilon(x,t),\kappa)\psi(x,t)\partial_x\chi_{i,\delta}(x)dxdt = \int_{\mathbb{R}_+}\int_{\Omega_i}\Phi_{i\pm}(\overline{u}_i,\kappa)\psi(x,t)\partial_x\chi_{i,\delta}(x)dxdt. \tag{38}$$

We let now δ tend to 0. Since $\chi_{i,\delta}(x)$ tends to 1 a.e. in Ω_i, (35) and (36) respectively provide

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\int_{\mathbb{R}_+}\int_{\Omega_i}(u^\varepsilon(x,t)-\kappa)^{\pm}\partial_t\psi(x,t)\chi_{i,\delta}(x)dxdt = \int_{\mathbb{R}_+}\int_{\Omega_i}\int_0^1(u(x,t,\alpha)-\kappa)^{\pm}\partial_t\psi(x,t)d\alpha dxdt, \tag{39}$$

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\int_{\mathbb{R}_+}\int_{\Omega_i}\Phi_{i\pm}(u^\varepsilon(x,t),\kappa)\partial_x\psi(x,t)\chi_{i,\delta}(x)dxdt = \int_{\mathbb{R}_+}\int_{\Omega_i}\int_0^1\Phi_{i\pm}(u(x,t,\alpha),\kappa)\partial_x\psi(x,t)d\alpha dxdt. \tag{40}$$

One has also

$$\lim_{\delta\to 0}\int_{\Omega_i}(u_0(x)-\kappa)^{\pm}\psi(x,0)\chi_{i,\delta}(x)dx = \int_{\Omega_i}(u_0(x)-\kappa)^{\pm}\psi(x,0)dx. \tag{41}$$

One has

$$|\Phi_{i\pm}(\overline{u}_i,\kappa)|\le M_{f_i}\,(\overline{u}_i-\kappa)^{\pm}$$

then

$$\left|\int_{\mathbb{R}_+}\int_{\Omega_i}\Phi_{i\pm}(\overline{u}_i,\kappa)\psi(x,t)\partial_x\chi_{i,\delta}(x)dxdt\right| \le M_{f_i}\,(\overline{u}_i-\kappa)^{\pm}\int_{\mathbb{R}_+}\int_{\Omega_i}\psi(x,t)\left|\partial_x\chi_{i,\delta}(x)\right|dxdt.$$

Since $|\partial_x\chi_{i,\delta}|$ tends to $\delta_{x=0}$ in the $\mathcal{M}(\mathbb{R})$-weak star sense where

$$\left\langle \delta_{x=0},\zeta\right\rangle_{\mathcal{M}(\mathbb{R}),\mathcal{C}_0(\mathbb{R})}=\zeta(0),$$

we obtain that

$$\liminf_{\delta\to 0}\lim_{\varepsilon\to 0}\int_{\mathbb{R}_+}\int_{\Omega_i}\Phi_{i\pm}(u^\varepsilon(x,t),\kappa)\psi(x,t)\partial_x\chi_{i,\delta}(x)dxdt \ge M_{f_i}\,(\overline{u}_i-\kappa)^{\pm}\int_{\mathbb{R}_+}\psi(0,t)dt. \tag{42}$$

Using (37),(39),(40),(41),(42) in (34) shows that u is a process solution in the sense of Definition 3.2. □

3.3 uniqueness of the (process) solution

It is clear that the notion of process solution is weaker than the one of solution given in Definition 1.1. We state here a theorem which claims the equivalence of the two notions, i.e. any process solution is a solution in the sense of Definition 1.1. Furthermore, such a solution is unique, and a L^1-contraction principle can be proven.

Theorem 3.5 (uniqueness of the (process) solutions) *There exists a unique process solution u to the problem ($\mathcal{P}_{\lim}$), and furthermore this solution does not depend on α, i.e. u is a solution to the problem ($\mathcal{P}_{\lim}$) in the sense of definition 1.1. Furthermore, if u_0, v_0 are two initial data in $L^\infty(\mathbb{R})$ satisfying (4) and let u and v be two solutions associated to those initial data, then for all $t \in [0,T)$,*

$$\int_{-R}^{R} (u(x,t) - v(x,t))^{\pm} dx \le \int_{-R-M_f t}^{R+M_f t} (u_0(x) - v_0(x))^{\pm} dx. \tag{43}$$

This theorem is a consequence of [38, Theorem 2]. Let $u(x,t,\alpha)$ and $v(x,t,\beta)$ be two process solutions corresponding to initial data u_0 and v_0. Classical Kato inequalities can be derived in each $\Omega_i \times \mathbb{R}_+$ by using the doubling variable technique: $\forall \psi \in \mathcal{D}^+(\Omega_i \times \mathbb{R}_+)$,

$$\begin{aligned}\int_{\mathbb{R}_+}\int_{\Omega_i}\int_0^1\int_0^1 &(u(x,t,\alpha) - v(x,t,\beta))^{\pm}\partial_t\psi(x,t)d\alpha d\beta dx dt\\ &+\int_{\Omega_i}(u_0(x)-v_0(x))^{\pm}\psi(x,0)dx\\ &+\int_{\mathbb{R}_+}\int_{\Omega_i}\int_0^1\int_0^1 \Phi_{i\pm}(u(x,t,\alpha),v(x,t,\beta))\partial_x\psi(x,t)d\alpha d\beta dx dt \ge 0.\end{aligned}$$

The treatment of the boundary condition at the interface is an adaptation to the case of process solution to the work of Otto summarized in [28] and detailed in [27] leading to (see [38, Lemma 2]): $\forall \psi \in \mathcal{D}^+(\overline{\Omega}_i \times \mathbb{R}_+)$,

$$\begin{aligned}\int_{\mathbb{R}_+}\int_{\Omega_i}\int_0^1\int_0^1 &(u(x,t,\alpha) - v(x,t,\beta))^{\pm}\partial_t\psi(x,t)d\alpha d\beta dx dt\\ &+\int_{\Omega_i}(u_0(x)-v_0(x))^{\pm}\psi(x,0)dx\\ &+\int_{\mathbb{R}_+}\int_{\Omega_i}\int_0^1\int_0^1 \Phi_{i\pm}(u(x,t,\alpha),v(x,t,\beta))\partial_x\psi(x,t)d\alpha d\beta dx dt \ge 0.\end{aligned} \tag{44}$$

Choosing

$$\psi_\varepsilon(x,s) = \begin{cases} 1 & \text{if} \quad |x| \le R + M_f s,\\ \dfrac{R+M_f s+\varepsilon-|x|}{\varepsilon} & \text{if} \quad R+M_f t \le |x| \le R+M_f s+\varepsilon\\ 0 & \text{if} \quad |x| \ge R+M_f s+\varepsilon \end{cases}$$

if $s \le t$ and $\psi_\varepsilon(x,s) = 0$ if $s > t$ as test function in (44) and letting ε tend to 0 provide the expected L^1-contraction principle (43).

Finally, if u and $\tilde{u}$ are two process solutions associated to the same initial data u_0, we obtain a L^1-contraction principle of the following form: for a.e. $t \in \mathbb{R}_+$,

$$\int_{\mathbb{R}}\int_0^1\int_0^1 (u(x,t,\alpha) - \tilde{u}(x,t,\beta))^{\pm} d\alpha d\beta dx \le 0,$$

thus $u(x,t,\alpha) = \tilde{u}(x,t,\beta)$ a.e. in $\mathbb{R}\times\mathbb{R}_+\times(0,1)\times(0,1)$. Hence u does not depend on the process variable α.

Theorem 3.6 *Let u_0 be a prepared initial data in the sense of Definition 2.10, and let u^ε be the corresponding solution to the approximate problem ($\mathcal{P}^\varepsilon$). Then u^ε converges to the unique solution u to ($\mathcal{P}_{\lim}$) associated to initial data u_0 in the $L^p((0,T);L^q(\mathbb{R}))$-sense, for all $p,q \in [1,\infty)$.*

Proof: We have seen in Proposition 3.4 that u^ε converges up to an extraction towards a process solution. The family $(u^\varepsilon)_\varepsilon$ admits so a unique adherence value, which is a solution thanks to Theorem 3.5, thus the whole family converges towards this unique limit u.

Let K denotes a compact subset of $\mathbb{R} \times [0, T]$, then one has

$$\iint_K (u^\varepsilon - u)^2 dxdt = \iint_K (u^\varepsilon)^2 \, dx - 2 \iint_K u^\varepsilon u dx + \iint_K u^2 dx.$$

Since u^ε converges in the nonlinear weak star sense towards u,

$$\lim_{\varepsilon \to 0} \iint_K (u^\varepsilon)^2 \, dx = \iint_K u^2 dx.$$

Moreover, u^ε converges in the L^∞ weak star topology towards u, then

$$\lim_{\varepsilon \to 0} \iint_K u^\varepsilon u dx = \iint_K u^2 dx.$$

Thus we obtain

$$\lim_{\varepsilon \to 0} \iint_K (u^\varepsilon - u)^2 dxdt = 0.$$

One concludes using the fact the $|u^\varepsilon - u| \leq 1$ for all $\varepsilon > 0$. □

3.4 initial data in $L^\infty(\mathbb{R})$

In this section, we extend the result of Theorem 3.6 to any initial data in $L^\infty(\mathbb{R})$ satisfying (4) thanks to density argument.

Theorem 3.7 *Let $u_0 \in L^\infty(\mathbb{R})$ satisfying (4), and let $(u_{0,n})_{n\in\mathbb{N}^\star}$ be a sequence of prepared initial data tending to u_0 in $L^1_{loc}(\mathbb{R})$. Then the sequence $(u_n)_n$ of solutions to ($\mathcal{P}_{\lim}$) corresponding to the sequence $(u_{0,n})$ of initial data converges in $\mathcal{C}(\mathbb{R}_+; L^1_{loc}(\mathbb{R}))$ towards the unique solution to ($\mathcal{P}_{\lim}$) corresponding to solution the initial data u_0.*

Proof: First, note that for all $u_0 \in L^\infty(\mathbb{R})$ satisfying (4), there exists a sequence $(u_{0,n})_{n\in\mathbb{N}^\star}$ of prepared initial data tending to u_0 in $L^1_{loc}(\mathbb{R})$ thanks to Lemma 2.9.

Thanks to (43), one has for $n, m \in \mathbb{N}^\star$, for all $t \in \mathbb{R}_+$

$$\int_{-R}^{R} (u_n(x,t) - u_m(x,t))^\pm dx \leq \int_{-R-M_f t}^{R+M_f t} (u_{0,n}(x) - u_{0,m}(x))^\pm dx,$$

then $(u_n)_n$ is a Cauchy sequence in $\mathcal{C}(\mathbb{R}_+; L^1_{loc}(\mathbb{R}))$. In particular, there exists u such that

$$\lim_{n\to\infty} u_n = u \quad \text{in } \mathcal{C}(\mathbb{R}_+; L^1_{loc}(\mathbb{R})).$$

It is then easy to check that u is the unique solution to ($\mathcal{P}_{\lim}$). □

4 Entropy solution for small initial data

In this section, we suppose that the initial data u_0 belongs to $L^1(\mathbb{R})$, and that

$$0 \leq u_0 \leq u_i^\star \quad \text{a.e. in } \Omega_i. \tag{45}$$

This initial data can be smoothed using following lemma whose proof is almost the same as the proof of Lemma 2.9.

Lemma 4.1 *There exists $(u_0^\varepsilon)_{\varepsilon>0} \subset L^1(\mathbb{R})$ such that*

- $\partial_x \varphi_i(u_0^\varepsilon) \in L^\infty(\Omega_i)$,

- $\operatorname{ess\,lim}_{x\nearrow 0} u_0^\varepsilon(x) = 1$,
- $\lim_{\varepsilon\to 0} u_0^\varepsilon = u_0$ *in* $L^1_{loc}(\mathbb{R})$.

For all $\varepsilon > 0$, there exists a unique bounded flux solution u^ε to $(\mathcal{P}^\varepsilon)$ corresponding to u_0^ε thanks to Theorem 2.4. The following theorem claims that as ε tends to 0, u^ε tends to the unique entropy solution in the sense of Definition 1.3.

Theorem 4.2 (convergence towards the entropy solution) *Let $u_0 \in L^\infty(\mathbb{R})$ satisfying (45) and let $(u_0^\varepsilon)_\varepsilon$ be a family of approximate initial data built in Lemma 4.1. Let u^ε be the bounded flux solution to $(\mathcal{P}^\varepsilon)$ corresponding to u_0^ε, then u^ε converges to u in $L^1_{loc}(\mathbb{R}\times\mathbb{R}_+)$ as ε tends to 0 where u is the unique entropy solution in the sense of Definition 1.3.*

Proof: Using the technics introduced in [10, Proposition 2.8], we can show that for all $\lambda \in [0, q]$ there exists a steady solution $\kappa_\lambda^\varepsilon$ to the problem $(\mathcal{P}^\varepsilon)$, corresponding to a constant flux

$$f_i(\kappa_\lambda^\varepsilon) - \varepsilon\partial\varphi_i(\kappa_\lambda^\varepsilon) = \lambda,$$

and such that this solution converges uniformly on each compact subset of $\mathbb{R}^\star$ as ε tends to 0 towards

$$\kappa_\lambda(x) = \min_\kappa \{f(\kappa, x) = \lambda\}.$$

Following the idea of Audusse and Perthame [5], we will now compare the limit u of u^ε as ε to 0 with the steady state κ_λ. Let $\lambda \in [0, q]$. Since u^ε and $\kappa_\lambda^\varepsilon$ are both bounded flux solutions, it follows from Proposition 2.3 that for all $\psi \in \mathcal{D}^+(\mathbb{R}\times\mathbb{R}_+)$,

$$\begin{aligned}\int_{\mathbb{R}_+}\int_{\mathbb{R}} (u^\varepsilon - \kappa_\lambda^\varepsilon)^\pm\, \partial_t\psi dx dt + \int_{\mathbb{R}} (u_0^\varepsilon - \kappa_\lambda^\varepsilon)^\pm\, \psi(\cdot, 0) dx \\ + \int_{\mathbb{R}_+}\sum_i \int_{\Omega_i} \left(\Phi_{i\pm}(u^\varepsilon, \kappa_\lambda^\varepsilon) - \varepsilon\partial_x\left(\varphi_i(u^\varepsilon) - \varphi_i(\kappa_\lambda^\varepsilon)\right)^+\right)\partial_x\psi dx dt \geq 0. \end{aligned} \tag{46}$$

Choosing $\lambda = q$ and $\psi(x,t) = (T-t)^+\xi(x)$ for some arbitrary $T > 0$ and some $\xi \in \mathcal{D}^+(\mathbb{R})$ yields

$$\int_0^T \int_\Omega (u^\varepsilon - \kappa_q^\varepsilon)^+ \xi dx dt \leq \int_0^T (T-t)\sum_{i=1,2}\int_{\Omega_i} \varepsilon\partial_x\left(\varphi_i(u^\varepsilon) - \varphi_i(\kappa_q^\varepsilon)\right)^+ \partial_x\xi dx dt. \tag{47}$$

Since u^ε is bounded between 0 and 1, it converges in the nonlinear weak star sense, thanks to Theorem 3.1 towards a function $u \in L^\infty(\mathbb{R}\times\mathbb{R}_+\times(0,1))$, with $0 \leq u \leq 1$ a.e.. Then (47) provides

$$u \leq \kappa_q = u_i^\star \quad \text{a.e. in } \Omega_i\times\mathbb{R}_+\times(0,1). \tag{48}$$

Let $\lambda \in [0, q]$, then taking the limit for $\varepsilon \to 0$ in (46) yields

$$\begin{aligned}\int_{\mathbb{R}_+}\int_{\mathbb{R}}\int_0^1 |u - \kappa_\lambda|\, \partial_t\psi d\alpha dx dt + \int_{\mathbb{R}} |u_0 - \kappa_\lambda|\, \psi(\cdot, 0) dx \\ + \int_{\mathbb{R}_+}\sum_i\int_{\Omega_i}\int_0^1 \Phi_i(u, \kappa_\lambda)\partial_x\psi d\alpha dx dt \geq 0.\end{aligned} \tag{49}$$

Suppose that $u_2^\star \geq u_1^\star$. Let $\kappa \in [0, u_2^\star]$, we denote by $\tilde\kappa = f_1^{-1}(f_2(\kappa)) \cap [0, u_1^\star]$. Then choosing $\lambda = f_2(\kappa)$ in (49), and letting ε tend to 0 gives: $\forall\kappa \in [0, u_2^\star]$, $\forall\psi \in \mathcal{D}^+(\mathbb{R}\times\mathbb{R}_+)$,

$$\begin{aligned}\int_0^T\int_{\Omega_1}\int_0^1 |u - \tilde\kappa|\partial_t\psi d\alpha dx dt + \int_{\Omega_1} |u_0 - \tilde\kappa|\psi(\cdot, 0) dx \\ + \int_{\mathbb{R}_+}\int_{\Omega_2}\int_0^1 |u - \kappa|\partial_t\psi d\alpha dx dt + \int_{\Omega_2} |u_0 - \kappa|\psi(\cdot, 0) dx \\ + \int_{\mathbb{R}_+}\int_0^1\left(\int_{\Omega_1}\Phi_1(u, \tilde\kappa)\partial_x\psi dx + \int_{\Omega_2}\Phi_2(u, \kappa)\partial_x\psi dx\right) d\alpha dt \geq 0.\end{aligned} \tag{50}$$

It follows from the work of Jose Carrillo [14] that the following entropy inequalities hold for test functions compactly supported in Ω_1: $\forall \kappa \in [0,1]$, $\forall \psi \in \mathcal{D}^+(\Omega_1 \times \mathbb{R}_+)$,

$$\int_{\mathbb{R}_+} \int_{\Omega_1} |u^\varepsilon - \kappa| \partial_t \psi dx dt + \int_{\Omega_1} |u_0^\varepsilon - \kappa| \psi(\cdot, 0) dx \\ + \int_{\mathbb{R}_+} \int_{\Omega_1} \left(\Phi_1(u^\varepsilon, \kappa) - \varepsilon \partial_x \left| \varphi_1(u^\varepsilon) - \varphi_1(\kappa) \right| \right) \partial_x \psi dx dt \geq 0. \tag{51}$$

Thus letting ε tend to 0 in (51) provides: $\forall \psi \in \mathcal{D}^+(\Omega_1 \times \mathbb{R}_+)$, $\forall \kappa \in [0,1]$,

$$\int_{\mathbb{R}_+} \int_{\Omega_1} \int_0^1 |u - \kappa| \partial_t \psi d\alpha dx dt + \int_{\Omega_1} |u_0 - \kappa| \psi(\cdot, 0) dx \\ + \int_{\mathbb{R}_+} \int_{\Omega_1} \int_0^1 \Phi_1(u, \kappa) \partial_x \psi d\alpha dx dt \geq 0. \tag{52}$$

Let $\delta > 0$, and let $\psi \in \mathcal{D}^+(\mathbb{R} \times \mathbb{R}_+)$, we define

$$\psi_{1,\delta}(x,t) = \psi(x,t) \chi_{1,\delta}(x), \qquad \psi_{2,\delta} = \psi - \psi_{1,\delta},$$

where $\chi_{1,\delta}$ is the cut-off function introduced in section 3.2. Then using $\psi_{1,\delta}$ as test function in (52) and $\psi_{2,\delta}$ in (50) leads to:

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}} \int_0^1 |u - \kappa| \partial_x \psi d\alpha dx dt + \int_{\mathbb{R}} |u_0 - \kappa| \psi(\cdot, 0) dx \\ + \int_{\mathbb{R}_+} \sum_i \int_{\Omega_i} \int_0^1 \Phi_i(u, \kappa) \partial_x \psi d\alpha dx dt \\ + \int_{\mathbb{R}_+} \int_{\Omega_1} \int_0^1 \left(\Phi_1(u, \kappa) - \Phi_1(u, \tilde{\kappa}) \right) \psi \partial_x \chi_{1,\delta} d\alpha dx dt \geq \mathcal{R}(\kappa, \psi, \delta), \tag{53}$$

where $\lim_{\delta \to 0} \mathcal{R}(\kappa, \psi, \delta) = 0$. Since f_1 is increasing on $[0, u_1^\star]$ and $f_1([u_1^\star, 1)) \subset [q, \infty)$, either $\kappa \leq u_1^\star$, or $f_1(\kappa) \geq f_1(u_1^\star)$. This ensures that

$$\Phi_1(u, \kappa) = |f_1(u) - f_1(\kappa)|, \qquad \forall u \in [0, u_1^\star], \quad \forall k \in [0, u_2^\star].$$

This yields

$$\begin{aligned} |\Phi_1(u, \kappa) - \Phi_1(u, \tilde{\kappa})| &= \big| |f_1(u) - f_1(\kappa)| - |f_1(u) - f_1(\tilde{\kappa})| \big| \\ &\leq |f_1(\kappa) - f_1(\tilde{\kappa})| = |f_1(\kappa) - f_2(\kappa)|. \end{aligned} \tag{54}$$

Taking the inequality (54) into account in (53), and letting $\delta \to 0$ provides:
$\forall \kappa \in [0, \|u\|_\infty]$, $\forall \psi \in \mathcal{D}^+(\mathbb{R} \times \mathbb{R}_+)$,

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}} \int_0^1 |u - \kappa| \partial_x \psi d\alpha dx dt + \int_{\mathbb{R}} |u_0 - \kappa| \psi(\cdot, 0) dx \\ + \int_{\mathbb{R}_+} \sum_i \int_{\Omega_i} \int_0^1 \Phi_i(u, \kappa) \partial_x \psi d\alpha dx dt + |f_1(\kappa) - f_2(\kappa)| \int_0^T \psi(0, \cdot) dt \geq 0.$$

Using the work of Florence Bachmann [6, Theorem 4.3], we can claim that u is the unique entropy solution to the problem. Particularly, u does not depend on α (introduced for the nonlinear weak star convergence). As proven in the proof of Theorem 3.6, this implies that u^ε converges in $L^1_{loc}(\mathbb{R} \times \mathbb{R}_+)$ towards u. □

5 Resolution of the Riemann problem

In this section, we complete the resolution of the Riemann problem at the interface $\{x = 0\}$, whose result has been given in section 1.2. Consider the initial data

$$u_0(x) = \begin{cases} u_\ell & \text{if} \quad x < 0, \\ u_r & \text{if} \quad x > 0. \end{cases}$$

We aim to determine the traces (u_1, u_2) at the interface of the solution $u(x,t)$ corresponding to u_0. This resolution has already been performed in the following cases.

(a). $u_1^\star < u_\ell \le 1$ and $u_2^\star \le u_r < 1$: it has been seen that $u_1 = 1$ and $u_2 = u_2^\star$.

(b). $0 \le u_\ell \le u_1^\star$ and $0 \le u_r \le u_2^\star$: Since u is the unique optimal entropy solution studied in [2, 24], then $u_1 = u_\ell$ and u_2 is the unique value in $[0, u_2^\star]$ such that $f_1(u_\ell) = f_2(u_2)$.

In the cases

(c). $u_1^\star < u_\ell \le 1$ and $u_r = 1$,

(d). $u_\ell = u_1^\star$ and $u_r = 1$,

it is possible to approach the solution u by bounded flux solutions u^ε that are constant equal to 1 in $\Omega_2 \times \mathbb{R}_+$. Then one obtains $u_1 = u_2 = 1$ for the case (c) and $u_1 = u_1^\star$ and $u_2 = 1$ for the case (d).

The last points we have to consider are

(e). $u_1^\star < u_\ell \le 1$ and $0 \le u_r < u_2^\star$,

(f). $0 \le u_\ell \le u_1^\star$ and $u_2^\star < u_r \le 1$.

To perform the study of the two last cases (e) and (f), we need the following lemmas that can be proved using similar arguments than those used in [11], particularly concerning the treatment of the boundary condition imposed on $\{x = 0\}$.

Lemma 5.1 *Let $u_r \in [0, u_2^\star)$. For all $\varepsilon > 0$, there exists a function v^ε solution to the problem*

$$\begin{cases} \partial_t v^\varepsilon + \partial_x\big(f_2(v^\varepsilon) - \varepsilon \partial_x \varphi_2(v^\varepsilon)\big) = 0 & \textit{if} \quad x > 0,\ t > 0, \\ f_2(v^\varepsilon) - \varepsilon \partial_x \varphi_2(v^\varepsilon) = f_2(u_2^\star) & \textit{if} \quad x = 0,\ t > 0, \\ v^\varepsilon = u_r & \textit{if} \quad x > 0,\ t = 0, \end{cases} \tag{55}$$

satisfying furthermore $u_r \le v^\varepsilon \le u_2^\star$ and $\partial_x \varphi_2(v^\varepsilon) \in L^\infty(\mathbb{R}_+ \times \mathbb{R}_+)$.

Lemma 5.2 *Let $u_\ell \in [0, u_1^\star]$, $u_r \in (u_2^\star, 1]$ and let u_2 be the unique value of $[0, u_2^\star]$ such that $f_2(u_2) = f_1(u_\ell)$. For all $\varepsilon > 0$ there exists a function w^ε solution to the problem*

$$\begin{cases} \partial_t w^\varepsilon + \partial_x\big(f_2(w^\varepsilon) - \varepsilon \partial_x \varphi_2(w^\varepsilon)\big) = 0 & \textit{if} \quad x > 0,\ t > 0, \\ f_2(w^\varepsilon) - \varepsilon \partial_x \varphi_2(w^\varepsilon) = f_2(u_2) = f_1(u_\ell) & \textit{if} \quad x = 0,\ t > 0, \\ w^\varepsilon = u_r & \textit{if} \quad x > 0,\ t = 0, \end{cases} \tag{56}$$

satisfying furthermore $u_2 \le w^\varepsilon \le u_r$ and $\partial_x \varphi_2(w^\varepsilon) \in L^\infty(\mathbb{R}_+ \times \mathbb{R}_+)$.

The case (e). Assume that $u_\ell > u_1^\star$ and $u_r < u_2^\star$. Let $(u_0^\eta)_\eta$ be a family of initial data such that $\partial_x \varphi_i(u_0^\eta) \in L^\infty(\Omega_i)$, $u_0^\eta(x) = 1$ for $x \in (-\eta, 0)$, $u_0^\eta(x) \in [u_\ell, 1]$ for a.e. $x \in \Omega_1$, $u_0^\eta(x) = u_r$ a.e. in Ω_2 and such that

$$\|u_0^\eta - u_\ell\|_{L^1(\Omega_1)} \le 2\eta.$$

Then thanks to Theorem 2.4, there exists a unique bounded flux solution $u^{\varepsilon,\eta}$ to the problem $(\mathcal{P}^\varepsilon)$ corresponding to the initial data u_0^η. It is easy to check that the solution defined in $\Omega_2 \times \mathbb{R}_+$ by the function v^ε introduced in Lemma 5.1 and coinciding in $\Omega_1 \times \mathbb{R}_+$ with the unique bounded flux solution corresponding to the initial data

$$\tilde{u}_0^\eta(x) = \begin{cases} u_0^\eta(x) & \text{if} \quad x < 0, \\ 1 & \text{if} \quad x > 0. \end{cases}$$

In particular, as ε tends to 0, it follows from arguments similar to those developed in the previous sections that $u^{\varepsilon,\eta}$ converges in $L^1_{loc}(\overline{\Omega}_i \times \mathbb{R}_+)$ towards the unique entropy solution to the problem problem

$$\begin{cases} \partial_t u^\eta + \partial_x f_1(u^\eta) = 0 & \text{if } x < 0,\ t > 0, \\ u^\eta = 1 & \text{if } x = 0,\ t > 0, \\ u^\eta = u_0^\eta & \text{if } x < 0,\ t = 0. \end{cases} \tag{57}$$

$$\begin{cases} \partial_t u + \partial_x f_2(u) = 0 & \text{if } x > 0,\ t > 0, \\ u = u_2^\star & \text{if } x = 0,\ t > 0, \\ u = u_r & \text{if } x > 0,\ t = 0. \end{cases} \tag{58}$$

Note that the trace condition on the interface $\{x = 0\}$ in (58) is fulfilled in a strong sense since $u_r \le u(x,t) \le u_2^\star$ a.e. in $\Omega_2 \times \mathbb{R}_+$ and f_2 is increasing on $[u_r, u_2^\star]$.

The solution to (57) depends continuously on the initial data in L^1_{loc}. Hence, letting η tend to 0 in (57) provides that the limit u of u^η is the unique entropy solution to the problem

$$\begin{cases} \partial_t u + \partial_x f_1(u) = 0 & \text{if } x < 0,\ t > 0, \\ u = 1 & \text{if } x = 0,\ t > 0, \\ u = u_\ell & \text{if } x < 0,\ t = 0. \end{cases}$$

Note that since $u_1^\star < u_\ell \le u \le 1$ and $\min_{s\in[u,1]} f_1(s) = f_1(1) = q$, the trace prescribed on the interface $\{x = 0\}$ is fulfilled in a strong sense. This particularly yields that in the case (e), the solution to the Riemann problem is given by

$$u_1 = 1, \qquad u_2 = u_2^\star.$$

The case (f). Following the technique used in [10] and in Section 4, there exists a unique function u_ℓ^ε solution to the problem:

$$\begin{cases} f_1(u_\ell^\varepsilon) - \varepsilon \dfrac{\mathrm{d}}{\mathrm{d}x}\varphi_1(u_\ell^\varepsilon) = f_1(u_\ell) & \text{if} \quad x < 0, \\ u_\ell^\varepsilon(0) = 1 & \text{if} \quad x = 0. \end{cases}$$

Let u^ε be the function defined by

$$u^\varepsilon(x,t) = \begin{cases} u_\ell^\varepsilon(x) & \text{if} \quad x < 0,\ t \ge 0, \\ w^\varepsilon(x,t) & \text{if} \quad x > 0,\ t \ge 0, \end{cases}$$

where w^ε is the function introduced in Lemma 5.2. Then u^ε is a bounded flux solution to the problem ($\mathcal{P}^\varepsilon$) in the sense of Definition 2.1.

One has

$$u_\ell^\varepsilon \to u_\ell \quad \text{in } L^1_{loc}(\Omega_1) \quad \text{as } \varepsilon \to 0,$$

and

$$w^\varepsilon \to w \quad \text{in } L^1_{loc}(\Omega_2 \times \mathbb{R}_+) \quad \text{as } \varepsilon \to 0$$

where w is the unique solution to

$$\begin{cases} \partial_t w + \partial_x f_2(w) = 0 & \text{if } x > 0,\ t > 0, \\ w = u_2 = f_2^{-1} \circ f_1(u_\ell) & \text{if } x = 0,\ t > 0, \\ w = u_r & \text{if } x > 0,\ t = 0. \end{cases}$$

Since $w(x,t) \in [u_2, u_r]$ a.e. in $\Omega_2 \times \mathbb{R}_+$ and since $\min_{s\in[u_2,w]} f_2(s) = f_2(u_2) = f_1(u_\ell)$, the trace $w = u_2$ is satisfied in a strong sense on $\{x = 0\}$. This yields that the solution to the Riemann problem in the case (f) is given by

$$u_1 = u_\ell, \qquad u_2 = f_2^{-1} \circ f_1(u_\ell).$$

6 Conclusion

The model presented here shows that for two-phase flows in heterogeneous porous media with negligible dependance of the capillary pressure with respect to the saturation, the good notion of solution is not always the entropy solution presented for example in [1, 6], and particular care as to be taken with respect to the orientation of the gravity forces. Indeed, some non classical shock can appear at the discontinuities of the capillary pressure field, leading to the phenomenon of oil trapping. We stress the fact that the non classical shocks appearing in our case have a different origin, and a different behavior of those suggested in the recent paper [36] (see also [26]). Indeed, in this latter paper, this lack of entropy was caused by the introduction of the dynamical capillary pressure [22, 23, 29], i.e. the capillary pressure is supposed to depend also on $\partial_t u$. In our problem, the lack of entropy comes only from the discontinuity of the porous medium.

In order to conclude this paper, we just want to stress that this model of piecewise constant capillary pressure curves can not lead to some interesting phenomenon. Indeed, if the capillary pressure functions π_i are such that $\pi_1((0,1)) \cap \pi_2((0,1)) \neq \emptyset$, it appears in [11, Section 6] (see also [7]) that some oil can overpass the boundary, and that only a finite quantity of oil can be definitely trapped. Moreover, this quantity is determined only by the capillary pressure curves and the difference between the volume mass of both phases, and does not depend on u_0. The model presented here, with total flow-rate q equal to zero, do not allow this phenomenon, and all the oil present in Ω_1 at the initial time remains trapped in Ω_1 for all $t \geq 0$.

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