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# Asymptotic behavior of two-phase flows in heterogeneous porous media for capillarity depending only on space.

## II. Non-classical shocks to model oil-trapping

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### Abstract

We consider a one-dimensional problem modeling two-phase flow in heterogeneous porous media made of two homogeneous subdomains, with discontinuous capillarity at the interface between them. We suppose that the capillary forces vanish inside the domains, but not on the interface. Under the assumption that the gravity forces and the capillary forces are oriented in opposite directions, we show that the limit, for vanishing diffusion, is not the entropy solution of the hyperbolic scalar conservation law. A non-classical shock occurs on the interface, modeling oil-trapping.

**key words.** *scalar conservation laws with discontinuous flux, non-classical shock, two-phase flow, porous media, discontinuous capillarity*

**AMS subject classification.** *35L60, 35L67, 76S05*

## 1 Introduction

The models of two-phase flows are a good first approximation to predict the motions of oil in the subsoil. Although the theoretical knowledge concerning the question of the existence and the uniqueness of the solution to such models for homogeneous porous media [3, 15] and for media with regular enough variations [16] is quite complete, few results are available for discontinuous media, as for example media made of several rock types [2, 8, 7, 10, 13, 18].

One says that oil-trapping occurs when some oil can not pass through interfaces between different rocks. Such a phenomenon plays an important role in the basin modeling, to predict the position of eventual reservoirs where oil could be collected. As already explained in [7, 30], discontinuities of the capillary pressure field can induce the so-called oil-trapping phenomenon.

The effects of capillarity, which play a crucial role in oil-trapping, seem to play a less important role concerning the motion of oil in homogeneous porous media, and can sometime be neglected to provide the so-called Buckley-Leverett equation.

In this paper, we show that even if the dependence of the capillary pressure with respect to the oil-saturation of the fluid vanishes, the capillary pressure field still plays a crucial role to determine the saturation profile. In order to carry out this study, we restrict our frame to the one-dimensional case. We will strongly use some recent results [10, 13] obtained on flows in heterogeneous media with discontinuous capillary forces.

Let  $\Omega_1 = \mathbb{R}_-$ ,  $\Omega_2 = \mathbb{R}_+$ , and let  $\pi_i$ ,  $i = 1, 2$  be two derivable increasing functions on  $[0, 1]$  called capillary pressure functions, where  $\pi_i^{-1}$  is furthermore assume to be locally Lipschitz continuous on  $(0, 1)$ . We suppose that the convection is only involved by the gravity forces, which work in the sense of increasing  $x$ .

Let  $\pi(u, x)$  be the capillary pressure, then it has been shown in the previous chapters (but see also [7, 18]) that the equation governing the two phase flow can be written

$$\partial_t u + \partial_x \left( g(u, x) [1 - C(x)] \partial_x \pi(u, x) \right) = 0, \quad (1)$$

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where

$$g(u, x) = g_i(u), \quad C(x) = C_i, \quad \text{and} \quad \pi(u, x) = \pi_i(u) \quad \text{if } x \in \Omega_i.$$

Physical experiments let think that the dependence of  $\pi_i$  with respect to  $u$  can be weak, at least for  $u$  far from 0 and 1. So we want to choose  $\pi_1(u) = P_1$ , and  $\pi_2(u) = P_2$ . The equation (1) becomes

$$\partial_t u + \partial_x g(u, x) = 0.$$

We suppose that for  $i = 1, 2$ ,  $g_i$  fulfills the following assumptions.

**Assumptions 1.1** For  $i = 1, 2$ ,

(H1)  $g_i$  is a Lipschitz continuous function,

(H2)  $g_i(0) = g_i(1) = 0$ , and  $g_i(s) > 0$  if  $s \in (0, 1)$ ,

We denote by  $\varphi_i$  ( $i = 1, 2$ ) the strictly increasing continuous function defined by  $\varphi_i(s) = C_i \int_0^s g_i(a) da$ . We furthermore assume that

(H3) there exist  $R > 0$ ,  $\alpha > 0$  and  $m \in (0, 1)$  such that

$$g_i \circ \varphi_i^{-1}(s) \geq R s^m \quad \text{if } s \in [0, \alpha], \quad (2)$$

$$g_i \circ \varphi_i^{-1}(s) \geq R(\varphi_i(1) - s)^m \quad \text{if } s \in [\varphi_i(1) - \alpha, \varphi_i(1)]. \quad (3)$$

These assumptions are fulfilled by the models used by the engineers, for which a classical choice of  $g_i$  is

$$g_i(u) = K_i \frac{u^{\alpha_i} (1-u)^{\beta_i}}{u^{\alpha_i} + C(1-u)^{\beta_i}},$$

where  $\alpha_i, \beta_i \geq 1$ .

The goal of this chapter is to show that if the capillary forces at the level of the interface  $\{x = 0\}$  are oriented in the inverse sense with respect to the gravity forces (in our case  $P_1 < P_2$ ), then a non classical stationary shock occurs at the interface. It has been recently shown in [9, 11] that if the capillary forces and the gravity forces are oriented in the same sense, the good notion of solution is the one of entropy solution [5]. If the assumptions stated above are fulfilled, and if  $P_1 < P_2$ , we will show that the limit is not the entropy solution, but a solution to the problem

$$\begin{cases} \partial_t u + \partial_x g_i(u) = 0, \\ u(x = 0^-) = 1 \text{ and } u(x = 0^+) = 0, \\ u(t = 0) = u_0. \end{cases} \quad (\mathcal{P}_{\text{lim}})$$

In the sequel, we denote by  $a^+$  (resp.  $a^-$ ) the positive (resp. negative) part of  $a$ , i.e.  $\max(0, a)$  (resp.  $\max(0, -a)$ ), and for  $i = 1, 2$ , for  $u, \kappa \in [0, 1]$ , one denotes by

$$G_{i+}(u, \kappa) = \begin{cases} g_i(u) - g_i(\kappa) & \text{if } u \geq \kappa, \\ 0 & \text{otherwise,} \end{cases}$$

$$G_{i-}(u, \kappa) = \begin{cases} g_i(\kappa) - g_i(u) & \text{if } u \leq \kappa, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$G_i(u, \kappa) = G_{i+}(u, \kappa) + G_{i-}(u, \kappa) = g_i(\max(u, \kappa)) - g_i(\min(u, \kappa)).$$

We can now define the notion of solution to  $(\mathcal{P}_{\text{lim}})$ , which is in fact an entropy in each subdomain  $\Omega_i$ , with an internal boundary condition at the level of the interface.

**Definition 1.1 (solution to  $(\mathcal{P}_{\text{lim}})$ )** Let  $u_0 \in L^\infty(\mathbb{R})$ ,  $0 \leq u_0 \leq 1$ , A function  $u$  is said to be a solution of  $(\mathcal{P}_{\text{lim}})$  if it belongs to  $L^\infty(\mathbb{R} \times (0, T))$ ,  $0 \leq u \leq 1$ , and for  $i = 1, 2$ , for all  $\psi \in \mathcal{D}^+(\overline{\Omega}_i \times [0, T])$ , for all  $\kappa \in [0, 1]$ ,

$$\begin{aligned} & \int_0^T \int_{\Omega_i} (u(x, t) - \kappa)^\pm \partial_t \psi dx dt + \int_{\Omega_i} (u_0(x) - \kappa)^\pm \psi(x, 0) dx \\ & + \int_0^T \int_{\Omega_i} G_{i\pm}(u(x, t), \kappa) \partial_x \psi(x, t) dx dt + M_{g_i} \int_0^T (\bar{u}_i - \kappa)^\pm \psi(0, t) dt \geq 0, \end{aligned} \quad (4)$$

where  $M_{g_i}$  is a Lipschitz constant of  $g_i$ , and  $\bar{u}_1 = 1$ ,  $\bar{u}_2 = 0$ .

For a given  $u_0$  in  $L^\infty(\mathbb{R})$ ,  $0 \leq u_0 \leq 1$  there exists a unique solution  $u$  to  $(\mathcal{P}_{\text{lim}})$  in the sense of definition 1.1, which is in fact made on an apposition of two entropy solutions in  $\mathbb{R}_\pm \times (0, T)$ . The assumptions on  $g_i$  will ensure us that the conditions at the interface  $\lim_{x \nearrow 0} u(x, t) = 1$  and  $\lim_{x \searrow 0} u(x, t) = 0$  are fulfilled for almost every  $t \in (0, T)$ .

## 1.1 non classical shock at the interface

The study of entropy solutions of hyperbolic scalar conservation laws with discontinuous flux functions has been performed recently by Christian Klingenberg and Nils H. Risebro [22], John D. Towers [28, 29], Nicolas Seguin and Julien Vovelle [26], and in the work of Florence Bachmann [4, 5, 6]. They introduced the entropy formulation for this type of problems.  $u \in L^\infty(\mathbb{R} \times (0, T))$  is said to be an entropy solution if  $0 \leq u \leq 1$  and  $\forall \kappa \in [0, 1], \forall \psi \in \mathcal{D}^+(\mathbb{R} \times (0, T))$ ,

$$\int_0^T \int_{\mathbb{R}} |u(x, t) - \kappa| \partial_t \psi(x, t) dx dt + \int_{\mathbb{R}} |u_0(x) - \kappa| \psi(x, 0) dx + \int_0^T \sum_{i=1,2} \int_{\Omega_i} G_i(u, \kappa) \partial_x \psi(x, t) dx dt + |g_1(\kappa) - g_2(\kappa)| \int_0^T \psi(0, t) dt \geq 0. \quad (5)$$

Suppose that an entropy solution  $u$  admits strong traces  $u_i$  on the interface  $\{x = 0\}$ , as it is for example the case if  $g_i$ ,  $i = 1, 2$  are genuinely non linear as in [4], using a result of Alexis Vasseur [32]. Then choosing a test function  $\psi_\eta(x, t) \geq 0$  such that  $\psi_\eta(0, t) = 1$ , and  $\psi(x, t) = 0$  if  $|x| \geq \eta$ , and letting  $\eta$  tend to 0 leads to

$$G_1(u_1, \kappa) - G_2(u_2, \kappa) + |g_1(\kappa) - g_2(\kappa)| \geq 0. \quad (6)$$

We will show that the solution to  $(\mathcal{P}_{\text{lim}})$  is not an entropy solution, and particularly that the conditions at the interface involved by the formulation (4) do not fulfill inequality (6).

Suppose for the moment that the traces  $u|_{\Omega_i}(0) = \bar{u}_i$  of the solution to  $(\mathcal{P}_{\text{lim}})$  are fulfilled in a strong sense as it will be the case in the sequel. Since  $g_1(\bar{u}_1) = g_2(\bar{u}_2)$ , the Rankine-Hugoniot condition is fulfilled at the interface, and it is then easy to check that a  $u$  solution to  $(\mathcal{P}_{\text{lim}})$  is a weak solution, i.e.:  $\forall \psi \in \mathcal{D}(\mathbb{R} \times [0, T))$ ,

$$\int_0^T \int_{\mathbb{R}} u(x, t) \partial_t \psi(x, t) dx dt + \int_0^T u_0(x) \psi(x, 0) dx - \int_0^T \sum_{i=1,2} \int_{\Omega_i} g_i(u)(x, t) \partial_x \psi(x, t) dx dt = 0. \quad (7)$$

It follows also from (4), taking test functions compactly supported in  $\Omega_i \times [0, T)$  that  $u$  is entropic in  $\Omega_i$ , and the lack of entropy can only come from the discontinuity between  $\bar{u}_1$  and  $\bar{u}_2$  at the interface. Since  $\bar{u}_1 = 1$  and  $\bar{u}_2 = 0$ , we have for all  $\kappa \in [0, 1]$ ,  $G_1(\bar{u}_1, \kappa) = -g_1(\kappa)$ , and  $G_2(\bar{u}_2, \kappa) = g_2(\kappa)$ . Using the fact that  $g_i(\kappa) > 0$  if  $\kappa \in (0, 1)$ , this implies that

$$G_1(\bar{u}_1, \kappa) - G_2(\bar{u}_2, \kappa) + |g_1(\kappa) - g_2(\kappa)| < 0, \quad \forall \kappa \in (0, 1). \quad (8)$$

The inequality (8) ensures that the solution to  $(\mathcal{P}_{\text{lim}})$  given by  $u(x) = 1$  if  $x < 0$  and  $u(x) = 0$  if  $x > 0$  is not an entropy solution, and so the stationary shock at  $\{x = 0\}$  is not classical, i.e. the discontinuity does not satisfy entropy conditions.

## 1.2 oil-trapping modeled by the non-classical shock

Let  $u$  be a solution of the problem  $(\mathcal{P}_{\text{lim}})$ , admitting strong traces on the interface. The flow-rate of oil going from  $\Omega_1$  to  $\Omega_2$  through the interface is given by

$$g_1(\bar{u}_1) = g_2(\bar{u}_2) = 0.$$

Thus the oil cannot overcome the interface from  $\Omega_1$  to  $\Omega_2$ , thus if one supposes that  $u_0$  belongs to  $L^\infty(\mathbb{R})$ , with  $0 \leq u_0 \leq 1$  a.e., then the quantity of oil standing between  $x = -R$  ( $R$  is an arbitrary positive number) and  $x = 0$  can only grow.

Indeed, let  $t_2 > t_1 \geq 0$ , let  $\zeta_n(x) = \min(1, n(x+R)^+, nx^-)$  and  $\theta_m(t) = \min(1, m(t-t_1), m(t_2-t))$ . Choosing  $\psi(x, t) = \zeta_n(x)\theta_m(t)$  in (7) for  $m, n \in \mathbb{N}$  yields, using the positivity of  $g_1$

$$\int_{t_1}^{t_2} \left( \int_{-R}^0 u(x, t) \zeta_n(x) dx \right) \partial_t \theta_m(t) dt + \int_{t_1}^{t_2} \theta_m(t) \left( \frac{1}{n} \int_{-1/n}^0 g_1(u(x, t)) dx \right) dt \leq 0.$$

Since  $u$  admits a strong trace on the interface,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \int_{-1/n}^0 g_1(u(x, t)) dx = g_1(\bar{u}_1) = 0.$$

Then we obtain

$$\int_{t_1}^{t_2} \left( \int_{-R}^0 u(x, t) dx \right) \partial_t \theta_m(t) dt \leq 0. \quad (9)$$

The solution  $u$  belong to  $\mathcal{C}([0, T]; L^1(\mathbb{R}))$  thanks to [12], thus taking the limit as  $m \rightarrow \infty$  in (9) provides

$$\int_{-R}^0 u(t_1) dt \leq \int_{-R}^0 u(t_2) dt.$$

### 1.3 organization of the paper

We will introduce a family of approximate problems ( $\mathcal{P}^\varepsilon$ ) in Section 2, which takes into account the capillarity, with small dependance  $\varepsilon$  of the capillary pressure with respect to the saturation  $u^\varepsilon$ . We use the transmission conditions introduced in [10, 13] to connect the capillary pressure at the interface. Then we show that if the initial data is prepared, i.e. if  $u_0(x) = \bar{u}_i$  for  $x \in \Omega_i$ ,  $|x| < \eta$  for some  $\eta > 0$ , then, if the dependance  $\varepsilon$  of the capillary pressure with respect to the saturation  $u^\varepsilon$  is small enough, one has  $u^\varepsilon(x, t) = \bar{u}_i$  for  $x \in \Omega_i$ ,  $|x| < \eta/2$ . This ensures the existence of strong traces on the interface for the limit  $u$  of  $u^\varepsilon$  as  $\varepsilon \rightarrow 0$ . We also derive from a uniform bound on the flux a  $L^2((0, T); H^1(\Omega_i))$ -estimate, with ensures that the diffusive effects due to the capillary pressure vanish when we let  $\varepsilon$  tend to 0.

In Section 3, we let this parameter  $\varepsilon$  tend to 0. The only estimate not depending on  $\varepsilon$  we have on the approximate solution is a  $L^\infty(\Omega_i \times (0, T))$  estimate, i.e.  $0 \leq u^\varepsilon \leq 1$ . This ensures that there exists  $u \in L^\infty(\mathbb{R} \times (0, T))$  such that  $u^\varepsilon$  tends to  $u$  for the  $L^\infty(\mathbb{R} \times (0, T))$  weak star topology. This is of course not sufficient to pass to the limit in the nonlinearities. To avoid this difficulty, we use the notion of process solution [19], which is equivalent to the notion of measure valued solutions of Ronald DiPerna [17]. We show that the approximate solution  $u^\varepsilon$  tends to a process solution  $u$  as  $\varepsilon$  tends to 0. We use then the uniqueness of the process solution to claim that  $u^\varepsilon$  tends almost everywhere to the unique solution to ( $\mathcal{P}_{\text{lim}}$ ).

Finally, in section 4, we point out that in the case a global convection term is added to the problem, the limit  $u^\varepsilon$  can either be an entropy solution, or a non-classical shock can occur. No complete answer is given in this more complex case.

## 2 The approximate problem

In this section, we take into account the effects of the capillarity, supposing that they are small. We will so build an approximate problem ( $\mathcal{P}^\varepsilon$ ), whose unknown  $u^\varepsilon$  will depend on a small parameter  $\varepsilon$  representing the dependance of the capillary pressure with respect to the saturation. We assume that the capillary pressure in  $\Omega_i$  is given by:

$$\pi_i^\varepsilon(u^\varepsilon) = P_i + \varepsilon u^\varepsilon. \quad (10)$$

It has been shown in previous papers [10, 13] that a good way to connect the capillary pressures at the interface is to require

$$\tilde{\pi}_1^\varepsilon(u_1^\varepsilon) \cap \tilde{\pi}_2^\varepsilon(u_2^\varepsilon) \neq \emptyset, \quad (11)$$

where  $u_1^\varepsilon$  and  $u_2^\varepsilon$  are the traces of  $u^\varepsilon$  on the interface, and where  $\tilde{\pi}_i^\varepsilon$  is the monotonous graph given by

$$\tilde{\pi}_i^\varepsilon(s) = \begin{cases} \pi_i^\varepsilon(s) & \text{if } s \in (0, 1), \\ (-\infty, P_i] & \text{if } s = 0, \\ [P_i + \varepsilon, \infty) & \text{if } s = 1. \end{cases}$$

We suppose that the capillary forces are oriented in the sense of decreasing  $x$ , i.e.  $P_1 < P_2$  (the capillary forces go from the high pressure to the low pressure). Since  $\varepsilon$  is assumed to be a small parameter, we can suppose that  $0 < \varepsilon < P_2 - P_1$ , so that the relation (11) becomes

$$u_1^\varepsilon = 1 \text{ or } u_2^\varepsilon = 0. \quad (12)$$

The flux function in  $\Omega_i$  is then given by:

$$F_i^\varepsilon(x, t) = g_i(u^\varepsilon)(x, t) - \varepsilon \partial_x \varphi_i(u^\varepsilon)(x, t),$$

where  $\varphi_i(s) = C_i \int_0^s g_i(a) da$ . Because of the conservation of mass, we require the connection of the flux functions at the interface. Thus the approximate problem becomes

$$\begin{cases} \partial_t u^\varepsilon + \partial_x F_i^\varepsilon = 0, \\ u^\varepsilon(x = 0^-) = 1 \text{ or } u^\varepsilon(x = 0^+) = 0, \\ F_1^\varepsilon(0^-) = F_2^\varepsilon(0^+), \\ u(t = 0) = u_0. \end{cases} \quad (\mathcal{P}^\varepsilon)$$

We are not able to prove the uniqueness of a weak solution of  $(\mathcal{P}_{\text{lim}})$  if the flux  $F_i^\varepsilon$  "only" belongs to  $L^2(\overline{\Omega}_i \times (0, T))$ , and we will define the notion of prepared initial data, so that the flux belongs to  $L^\infty(\Omega_i \times (0, T))$ . In this latter case, the uniqueness holds.

## 2.1 prepared initial data

The following lemma states that one can approach the initial data  $u_0$  by a function  $u_{0,\eta}$  for which the expected non-entropic discontinuity already occurs at the interface.

**Lemma 2.1 (prepared initial data)** *Let  $u_0 \in L^1(\mathbb{R})$ ,  $0 \leq u_0 \leq 1$ , then there exists  $u_{0,\eta}$  such that*

- i)  $u_{0,\eta} \in C^\infty(\mathbb{R}^*)$ ,  $0 \leq u_{0,\eta} \leq 1$ ,
- ii)  $u_{0,\eta}(x) = 1$  on  $(-\eta, 0)$ , and  $u_{0,\eta}(x) = 0$  on  $(0, \eta)$ ,
- iii)  $\lim_{\eta \rightarrow 0} u_{0,\eta} = u_0(x)$  in  $L^1(\mathbb{R})$ .

A function  $u_{0,\eta}$  fulfilling i) and ii) is said to be a  $\eta$ -prepared initial data. An initial data is said to be prepared if it is  $\eta$ -prepared for some  $\eta > 0$ .

The proof of this lemma, which is close to the one of [9, Lemma 2.1], is left to the reader.

## 2.2 bounded flux solutions

We define now the notion of bounded flux solution, that was introduced in this framework in [10, 13].

**Definition 2.1 (bounded flux solution to  $(\mathcal{P}^\varepsilon)$ )** *Let  $u_0 \in L^\infty(\mathbb{R})$ ,  $0 \leq u_0 \leq 1$ , a function  $u^\varepsilon$  is said to be a bounded flux solution if*

1.  $u^\varepsilon \in L^\infty(\mathbb{R} \times (0, T))$ ,  $0 \leq u \leq 1$ ;
2.  $\partial_x \varphi_i(u^\varepsilon) \in L^\infty(\mathbb{R} \times (0, T))$ ;
3.  $u_1^\varepsilon(t)(1 - u_2^\varepsilon(t)) = 0$  for almost all  $t \in (0, T)$ , where  $u_i^\varepsilon$  denotes the trace of  $u^\varepsilon$  on  $\{x = 0\}$ .
4.  $\forall \psi \in \mathcal{D}(\mathbb{R} \times [0, T])$ ,

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}} u^\varepsilon(x, t) \partial_t \psi(x, t) dx dt + \int_{\mathbb{R}} u_0(x) \psi(x, 0) dx \\ & + \int_0^T \int_{\mathbb{R}} [g_i(u^\varepsilon) - \varepsilon \partial_x \varphi_i(u^\varepsilon)] \partial_x \psi(x, t) dx dt = 0. \end{aligned} \quad (13)$$

**Remark.** Such a bounded-flux  $u^\varepsilon$  solution belongs to  $\mathcal{C}([0, T]; L^1_{\text{loc}}(\mathbb{R}))$ , in the sense that there exists  $\tilde{u}^\varepsilon$  in  $\mathcal{C}([0, T]; L^1_{\text{loc}}(\mathbb{R}))$  such that  $u^\varepsilon(t) = \tilde{u}^\varepsilon(t)$  for almost all  $t \in [0, T]$  (see [12]). More precisely, all  $t \in [0, T]$  is a Lebesgue point for  $u^\varepsilon$ . So, the slight abuse of notation consisting in considering  $u^\varepsilon(t)$  for all  $t \in [0, T]$  will not lead to any confusion.

**Proposition 2.2** Let  $u$  and  $v$  be two bounded-flux solutions associated to initial data  $u_0, v_0$ , then for all  $\psi \in \mathcal{D}^+(\mathbb{R} \times [0, T])$ ,

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}} (u - v)^\pm \partial_t \psi dx dt + \int_{\mathbb{R}} (u_0 - v_0)^\pm \psi(\cdot, 0) dx \\ & + \int_0^T \sum_i \int_{\Omega_i} \left( G_{i\pm}(u, v) - \partial_x (\varphi_i(u) - \varphi_i(v))^+ \right) \partial_x \psi dx dt \geq 0. \end{aligned} \quad (14)$$

We state now a theorem which is a generalization in the case of unbounded domains of Theorem 3.1 and Theorem 4.1 stated in [10].

**Theorem 2.3 (existence–uniqueness for bounded flux solutions)** Let  $u_0$  be a prepared initial data, then there exists a bounded flux solution  $u^\varepsilon$  to the problem  $(\mathcal{P}^\varepsilon)$  in the sense of definition 2.1. Furthermore, if  $u^\varepsilon, v^\varepsilon$  are two bounded flux solutions associated to initial data  $u_0, v_0$ , then for all  $t \in [0, T]$ ,

$$\int_{\mathbb{R}} (u^\varepsilon(t) - v^\varepsilon(t))^\pm dx \leq \int_{\mathbb{R}} (u_0 - v_0)^\pm dx. \quad (15)$$

Obviously, the existence of a bounded flux solution can not be extended to any initial data in  $L^1(\mathbb{R})$ . Indeed, the initial data  $u_0$  has at least to involve bounded initial flux, i.e.  $\partial_x \varphi_i(u_0) \in L^\infty(\mathbb{R})$ . An additional natural assumption is needed to ensure the existence of such a bounded flux solution : the connection in the graphical sense of the capillary pressures at the interface.

It is also interesting to remark that the inequalities (15) ensure the uniqueness of the bounded-flux solution as soon as the initial data belongs to  $L^1(\mathbb{R})$ . More precisely, if  $u_0$  and  $v_0$  are two prepared initial data belonging to  $L^1(\mathbb{R})$ , the following comparison-contraction principle holds:

$$\int_{\mathbb{R}} (u^\varepsilon(t) - v^\varepsilon(t))^\pm dx \leq \int_{\mathbb{R}} (u_0 - v_0)^\pm dx, \quad \forall t \in [0, T].$$

### 2.3 particular sub- and super-solutions

Thanks to Lemma 2.1, we can approach in some sense any initial data  $u_0 \in L^\infty$  by a prepared initial data  $u_{0,\eta}$  which is smooth in subdomains  $\mathbb{R}_-$  and  $\mathbb{R}_+$ , and which is constant on a small interval on each side of the interface. For the sake of simplicity, we will remove the  $\eta$  in the notation, and so we suppose for the moment that  $u_0$  is a prepared initial data. Let  $u^\varepsilon$  be the unique solution to the approximate problem  $(\mathcal{P}^\varepsilon)$ . We compare  $u^\varepsilon$  and particular sub- and super-solutions, and so we deduce in proposition 2.5 that for  $\varepsilon$  small enough,  $u^\varepsilon$  is constant on a small interval on each side of the interface.

We will introduce now particular solutions of the ordinary differential equation

$$y'_i = g_i \circ \varphi_i^{-1}(y_i). \quad (16)$$

**Lemma 2.4** There exists a solution  $y_i$  to (16) and  $C(g_i, \varphi_i) > 0$  such that  $y_i(x) = 0$  if  $x \leq 0$  and  $y_i(x) = \varphi_i(1)$  if  $x \geq C(g_i, \varphi_i)$ .

#### Proof

The ordinary differential equation  $w' = R w^m$  with initial condition  $w(0) = 0$  admits multiple solutions, and  $w(x) = (R(1-m)x)^{\frac{1}{1-m}}$  if  $x > 0$  and  $w(x) = 0$  if  $x \leq 0$  is a particular solution. It follows from inequality (2) that  $w$  is a subsolution to (16) on a neighborhood of  $\{x = 0\}$ , so there exists a solution  $y_i$  to (16) such that  $y_i \geq w$  and  $y_i = 0$  on  $\mathbb{R}_-$  (see e.g. [21]). This ensures the existence of  $\eta_0 > 0$  such that  $y_i(\eta_0) = \alpha$ .

We consider now the initial value problem  $z' = R(\varphi_i(1) - z)^m$ , with  $z(0) = \varphi_i(1)$ . It admits of

course also multiple solutions, since  $z(x) = 1$  if  $x \geq 0$  and  $z(x) = \varphi_i(1) - (R(1-m)(-x))^{\frac{1}{1-m}}$  if  $x \leq 0$  is a particular solution. Then, thanks to (3), there exists  $\tilde{y}_i$  solution to (16) with  $\tilde{y}_i = \varphi_i(1)$  if  $x \geq 0$  and  $\tilde{y}_i \leq z$  if  $x \leq 0$ . This ensures the existence of  $\eta_1 < 0$  such that  $\tilde{y}_i(\eta_1) = \varphi_i(1) - \alpha$ . Since  $g_i \circ \varphi_i^{-1}$  is Lipschitz continuous on  $[\alpha, 1 - \alpha]$ , and  $g_i \circ \varphi_i^{-1}(s) \geq \beta > 0$  if  $s \in [\alpha, 1 - \alpha]$ , then there exists a unique increasing solution  $y$  to (16) with  $y(\eta_0) = \alpha$ . This solution is greater than

$$\hat{y}_i(x) = \min((\varphi_i(1) - \alpha), \max(\alpha, \alpha \exp(\beta x - \eta_0))),$$

thus in particular, there exists  $\gamma > 0$  such that  $y_i(\gamma) = (\varphi_i(1) - \alpha)$ .

The function  $y_i$  can be extended by  $y_i(x) = \tilde{y}_i(x - \gamma + \eta_1)$  if  $x \geq \gamma$ , and so this function reaches the value  $\varphi_i(1)$  in finite  $x$ .  $\square$

**Proposition 2.5** *Let  $u_0$  be a  $\eta$ -prepared initial data, then there exists  $\varepsilon_0$ , such that for all  $\varepsilon \in (0, \varepsilon_0)$ ,  $u^\varepsilon(x) = 1$  on  $(-\eta/2, 0)$  and  $u^\varepsilon(x) = 0$  on  $(0, \eta/2)$ .*

**Proof**

Let  $C = C(g_i, \phi_i)$  be a positive real value chosen as in lemma 2.4, i.e. chosen such that a solution  $y_i$  to (16) needs less space than  $C$  to pass from the state 0 to the state  $\varphi_i(1)$ . We define  $\varepsilon_0 = \frac{\eta}{2C}$ , and let  $\varepsilon \in (0, \varepsilon_0)$

$$\underline{w}(x) = \begin{cases} \varphi_1^{-1} \left( y_1 \left( \frac{x + \eta}{\varepsilon} \right) \right) & \text{if } x < 0, \\ 0 & \text{if } x > 0, \end{cases} \quad (17)$$

$$\overline{w}(x) = \begin{cases} 1 & \text{if } x < 0, \\ \varphi_2 \left( y_2 \left( \frac{x - \eta/2}{\varepsilon} \right) \right) & \text{if } x > 0. \end{cases} \quad (18)$$

It follows from lemma 2.4 that  $\underline{w}(x) = 0$  if  $x \notin (-\eta, 0)$ ,  $\overline{w}(x) = 1$  if  $x \notin (0, \eta)$ , and that  $\underline{w}(x) = 1$  if  $x \in (-\eta/2, 0)$ ,  $\overline{w}(x) = 1$  if  $x \in (0, \eta/2)$ .

This functions have been built so that  $\underline{w}$  is a sub-solution and  $\overline{w}$  is a super-solution to the problem  $(\mathcal{P}^\varepsilon)$ . The comparison principle ensures that for every  $t \in [0, T]$ ,

$$\underline{w} \leq u^\varepsilon(\cdot, t) \leq \overline{w}.$$

This particularly ensures that  $u^\varepsilon(x, \cdot) = 1$  for a.e.  $x \in (-\eta/2, 0)$  and  $u^\varepsilon(x, \cdot) = 0$  for a.e.  $x \in (0, \eta/2)$ .  $\square$

## 2.4 a $L^2((0, T); H^1(\Omega_i))$ estimate

Our goal is now to derive an estimate which ensures that the effects of capillarity vanish almost everywhere in  $\Omega_i \times (0, T)$  as  $\varepsilon$  tends to 0.

**Proposition 2.6** *Let  $u_0 \in L^1(\mathbb{R})$  be a prepared initial data, and  $u^\varepsilon$  the bounded flux solution associated to  $u_0$ . Then there exists  $C$  depending only on  $u_0, g_i, \varphi_i, T$  such that*

$$\sqrt{\varepsilon} \|\varphi_i(u^\varepsilon)\|_{L^2((0, T); H^1(\Omega_i))} \leq C. \quad (19)$$

*This particularly ensures that*

$$\varepsilon \|\partial_x \varphi_i(u^\varepsilon)\|_{L^2((0, T); H^1(\Omega_i))} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0. \quad (20)$$

The idea of the proof of proposition 2.6 is formally to choose  $u^\varepsilon \psi$  as test function in (13) for a function  $x \mapsto \psi(x)$  compactly supported in  $\Omega_i$ . Using the fact that the flux  $F_i^\varepsilon$  is uniformly bounded in  $L^\infty(\Omega_i \times (0, T))$ , we can let  $\psi$  tend towards  $\chi_{\Omega_i}$ , with  $\chi_{\Omega_i}(x) = 1$  if  $x \in \Omega_i$  and 0 otherwise, and the estimate (19) follows. To obtain (20), it suffices to multiply (19) by  $\sqrt{\varepsilon}$ . We refer to [9, Proposition 2.2] for a rigorous proof of Proposition 2.6.

**Remark.** *In [9, Proposition 2.2], the corresponding estimate was local, i.e. a  $L^2((0, T); H_{\text{loc}}^1(\overline{\Omega}_i))$  estimate. But in our frame, since  $u_0$  is supposed to belong to  $L^1(\mathbb{R}) \subset L^2(\mathbb{R})$ , the estimate can be extended without any difficulty to the whole  $\Omega_i$ .*



### 3 Convergence

#### 3.1 a compactness result

Since  $(u^\varepsilon)_\varepsilon$  is uniformly bounded between 0 and 1, there exists  $u \in L^\infty(\mathbb{R} \times (0, T))$  such that  $u^\varepsilon \rightarrow u$  is the  $L^\infty$  weak-star sense. This is of course insufficient to pass in the limit in the nonlinear terms. Either greater estimates are needed, like for example a  $BV$ -estimate introduced in the work of Vol'pert [33] and in [9], or we have to use a weaker compactness result. This idea motivates the introduction of Young measures as in the papers of DiPerna [17] and Szepessy [27], or equivalently the notion of nonlinear weak star convergence, introduced in [19] and [20], which will lead to the notion of process solution given in definition 3.1.

**Theorem 3.1 (Nonlinear weak star convergence)** *Let  $\mathcal{Q}$  be a Borelian subset of  $\mathbb{R}^k$ , and  $(u_n)$  be a bounded sequence in  $L^\infty(\mathcal{Q})$ . Then there exists  $u \in L^\infty(\mathcal{Q} \times (0, 1))$ , such that up to a subsequence,  $u_n$  tends to  $u$  "in the non linear weak star sense" as  $n \rightarrow \infty$ , i.e.:  $\forall g \in \mathcal{C}(\mathbb{R}, \mathbb{R})$ ,*

$$g(u_n) \rightarrow \int_0^1 g(u(\cdot, \alpha)) d\alpha \text{ for the weak star topology of } L^\infty(\mathcal{Q}) \text{ as } n \rightarrow \infty.$$

We refer to [17] and [19] for the proof of Theorem 3.1.

#### 3.2 convergence towards a process solution

Because of the lack of compactness, we have to introduce the notion of process solution, inspired from the notion of *measure valued solution* introduced by DiPerna [17].

**Definition 3.1 (process solution to  $(\mathcal{P}_{\text{lim}})$ )** *A function  $u \in L^\infty(\mathbb{R} \times (0, T) \times (0, 1))$  is said to be a process solution to  $(\mathcal{P}_{\text{lim}})$  if  $0 \leq u \leq 1$  and for  $i = 1, 2$ ,  $\forall \psi \in \mathcal{D}^+(\overline{\Omega}_i \times (0, T))$ ,  $\forall \kappa \in [0, 1]$ ,*

$$\begin{aligned} & \int_0^T \int_{\Omega_i} \int_0^1 (u(x, t, \alpha) - \kappa)^\pm \partial_t \psi(x, t) d\alpha dx dt + \int_{\Omega_i} (u_0(x) - \kappa)^\pm \psi(x, 0) dx \\ & + \int_0^T \int_{\Omega_i} \int_0^1 G_{i\pm}(u(x, t, \alpha), \kappa) \partial_x \psi(x, t) d\alpha dx dt + M_{g_i} \int_0^T (\bar{u}_i - \kappa)^\pm \psi(0, t) dt \geq 0, \end{aligned}$$

where  $M_{g_i}$  is any Lipschitz constant of  $g_i$ ,  $\bar{u}_1 = 1$  and  $\bar{u}_2 = 0$ .

**Proposition 3.2 (convergence towards a process solution)** *Let  $u_0$  be a  $\eta$ -prepared initial data, and let  $(u^\varepsilon)_\varepsilon$  be the corresponding family of approximate solutions. Then, up to an extraction,  $u^\varepsilon$  converges in the nonlinear weak-star sense towards a process solution  $u$  to the problem  $(\mathcal{P}_{\text{lim}})$ .*

#### Proof

Since  $u^\varepsilon$  is a weak solution of  $(\mathcal{P}^\varepsilon)$ , which is a non-fully degenerate parabolic problem, i.e.  $\varphi_i^{-1}$  is continuous, it follows from the work of Carrillo [14] that  $u^\varepsilon$  is an entropy weak solution, i.e.:  $\forall \psi \in \mathcal{D}^+(\Omega_i \times [0, T])$ ,  $\forall \kappa \in [0, 1]$ ,

$$\begin{aligned} & \int_0^T \int_{\Omega_i} (u^\varepsilon(x, t) - \kappa)^\pm \partial_t \psi(x, t) dx dt + \int_{\Omega_i} (u_0(x) - \kappa)^\pm \psi(x, 0) dx \\ & + \int_0^T \int_{\Omega_i} [G_{i\pm}(u^\varepsilon(x, t), \kappa) - \varepsilon \partial_x (\varphi_i(u^\varepsilon)(x, t) - \varphi_i(\kappa))^\pm] \partial_x \psi(x, t) dx dt \geq 0. \end{aligned}$$

This family of inequalities is only available for non-negative functions  $\psi$  compactly supported in  $\Omega_i$ , and so vanishing on the interface  $\{x = 0\}$ . To overpass this difficulty, we use cut-off functions  $\chi_{i,\delta}$ .

Let  $\delta > 0$ , we denote by  $\chi_{i,\delta}$  a smooth non-negative function, with  $\chi_{i,\delta}(x) = 0$  if  $x \notin \Omega_i$ , and  $\chi_{i,\delta}(x) = 1$  if  $x \in \Omega_i$ ,  $|x| \geq \delta$ . Let  $\psi \in \mathcal{D}^+(\overline{\Omega} \times [0, T])$ , then  $\psi \chi_{i,\delta} \in \mathcal{D}^+(\Omega_i \times [0, T])$  can be used

as test function in (21). This yields

$$\begin{aligned} & \int_0^T \int_{\Omega_i} (u^\varepsilon(x, t) - \kappa)^\pm \partial_t \psi(x, t) \chi_{i, \delta}(x) dx dt + \int_{\Omega_i} (u_0(x) - \kappa)^\pm \psi(x, 0) \chi_{i, \delta}(x) dx \\ & + \int_0^T \int_{\Omega_i} [G_{i\pm}(u^\varepsilon(x, t), \kappa) - \varepsilon \partial_x(\varphi_i(u^\varepsilon)(x, t) - \varphi_i(\kappa))^\pm] \partial_x \psi(x, t) \chi_{i, \delta}(x) dx dt \\ & + \int_0^T \int_{\Omega_i} [G_{i\pm}(u^\varepsilon(x, t), \kappa) - \varepsilon \partial_x(\varphi_i(u^\varepsilon)(x, t) - \varphi_i(\kappa))^\pm] \psi(x, t) \partial_x \chi_{i, \delta}(x) dx dt \geq 0. \end{aligned}$$

Since  $u_0$  is supposed to be a  $\eta$ -prepared initial data, we can claim thanks to proposition 2.5 that, if  $\varepsilon$  is small enough,  $u$  is constant equal to  $\bar{u}_i$  on a neighborhood of  $\{x = 0\}$  in  $\Omega_i$ . It follows that for  $\delta$  small enough, the support of  $\partial_x \chi_{i, \delta}$  is included in the set where  $u^\varepsilon = \bar{u}_i$ , and so the last term in presious inequality becomes can be rewritten in a very simple way.

$$\begin{aligned} & \int_0^T \int_{\Omega_i} (u^\varepsilon(x, t) - \kappa)^\pm \partial_t \psi(x, t) \chi_{i, \delta}(x) dx dt + \int_{\Omega_i} (u_0(x) - \kappa)^\pm \psi(x, 0) \chi_{i, \delta}(x) dx \\ & + \int_0^T \int_{\Omega_i} [G_{i\pm}(u^\varepsilon(x, t), \kappa) - \varepsilon \partial_x(\varphi_i(u^\varepsilon)(x, t) - \varphi_i(\kappa))^\pm] \partial_x \psi(x, t) \chi_{i, \delta}(x) dx dt \\ & + \int_0^T \int_{\Omega_i} G_{i\pm}(\bar{u}_i, \kappa) \psi(x, t) \partial_x \chi_{i, \delta}(x) dx dt \geq 0. \end{aligned} \quad (21)$$

We denote by  $n(i)$  the inward normal to  $\Omega_i$ , i.e.  $n(1) = -1$  and  $n(2) = +1$ . Then letting  $\delta$  tend to 0 in (21) yields:  $\forall \psi \in \mathcal{D}^+(\bar{\Omega}_i \times [0, T])$ ,  $\forall \kappa \in [0, 1]$ ,

$$\begin{aligned} & \int_0^T \int_{\Omega_i} (u^\varepsilon(x, t) - \kappa)^\pm \partial_t \psi(x, t) dx dt + \int_{\Omega_i} (u_0(x) - \kappa)^\pm \psi(x, 0) dx \\ & + \int_0^T \int_{\Omega_i} [G_{i\pm}(u^\varepsilon(x, t), \kappa) - \varepsilon \partial_x(\varphi_i(u^\varepsilon)(x, t) - \varphi_i(\kappa))^\pm] \partial_x \psi(x, t) dx dt \\ & + n(i) \int_0^T G_{i\pm}(\bar{u}_i, \kappa) \psi(0, t) dt \geq 0. \end{aligned} \quad (22)$$

Let  $M_{g_i}$  be a Lipschitz constant of  $g_i$ , then

$$|G_{i\pm}(\bar{u}_i, \kappa)| \leq M_{g_i} (\bar{u}_i - \kappa)^\pm,$$

and it follows from (22) that

$$\begin{aligned} & \int_0^T \int_{\Omega_i} (u^\varepsilon(x, t) - \kappa)^\pm \partial_t \psi(x, t) dx dt + \int_{\Omega_i} (u_0(x) - \kappa)^\pm \psi(x, 0) dx \\ & + \int_0^T \int_{\Omega_i} [G_{i\pm}(u^\varepsilon(x, t), \kappa) - \varepsilon \partial_x(\varphi_i(u^\varepsilon)(x, t) - \varphi_i(\kappa))^\pm] \partial_x \psi(x, t) dx dt \\ & + M_{g_i} \int_0^T \int_{\Omega_i} (\bar{u}_i - \kappa)^\pm \psi(0, t) dt \geq 0. \end{aligned} \quad (23)$$

We can now let  $\varepsilon$  tend to 0. We deduce from Proposition 2.6 that, up to an extraction, for all  $\kappa \in [0, 1]$ ,

$$\varepsilon \partial_x(\varphi_i(u^\varepsilon) - \varphi_i(\kappa))^\pm \text{ tends to } 0 \quad \text{a.e. in } \Omega_i \times (0, T) \text{ as } \varepsilon \rightarrow 0, \quad (24)$$

and using  $0 \leq u^\varepsilon \leq 1$ , Theorem 3.1 ensures the existence of  $u \in L^\infty(\mathbb{R} \times (0, T) \times (0, 1))$  such that

$$(u^\varepsilon - \kappa)^\pm \rightarrow \int_0^1 (u(\cdot, \cdot, \alpha) - \kappa)^\pm d\alpha \quad \text{as } \varepsilon \rightarrow 0, \quad (25)$$

$$G_{i\pm}(u^\varepsilon, \kappa) \rightarrow \int_0^1 G_{i\pm}(u(\cdot, \cdot, \alpha), \kappa) d\alpha \quad \text{as } \varepsilon \rightarrow 0. \quad (26)$$

Letting  $\varepsilon$  tend to 0 in (23), using (24), (25) and (26) yields:  $\forall \psi \in \mathcal{D}^+(\bar{\Omega}_i \times (0, T)), \forall \kappa \in [0, 1]$ ,

$$\begin{aligned} & \int_0^T \int_{\Omega_i} \int_0^1 (u(x, t, \alpha) - \kappa)^\pm \partial_t \psi(x, t) d\alpha dx dt + \int_{\Omega_i} (u_0(x) - \kappa)^\pm \psi(x, 0) dx \\ & + \int_0^T \int_{\Omega_i} \int_0^1 G_{i\pm}(u(x, t, \alpha), \kappa) \partial_x \psi(x, t) d\alpha dx dt + M_{g_i} \int_0^T \int_{\Omega_i} (\bar{u}_i - \kappa)^\pm \psi(0, t) dt \geq 0. \end{aligned}$$

Thus  $u$  is a process solution in the sense of Definition 3.1.  $\square$

### 3.3 uniqueness of the (process) solution

It is clear that the notion of process solution is weaker than the one of solution given in Definition 1.1. We state here a theorem which claims the equivalence of the two notions, i.e. any process solution is a solution in the sense of Definition 1.1. Furthermore, such a solution is unique, and a  $L^1$ -contraction principle can be proven.

**Theorem 3.3 (uniqueness of the (process) solutions)** *There exists a unique process solution  $u$  to the problem  $(\mathcal{P}_{\text{lim}})$ , and furthermore this solution does not depend on  $\alpha$ , i.e.  $u$  is a solution to the problem  $(\mathcal{P}_{\text{lim}})$  in the sense of definition 1.1. Furthermore, if  $u_0, v_0$  are two initial data in  $L^1(\mathbb{R})$ , and let  $u$  and  $v$  be two solutions associated to those initial data, then for almost every  $t \in [0, T]$ ,*

$$\int_{\mathbb{R}} (u(x, t) - v(x, t))^\pm dx \leq \int_{\mathbb{R}} (u_0(x) - v_0(x))^\pm dx. \quad (27)$$

The proof of this theorem can be done using a doubling variable method. The presence of the process variable  $\alpha$  does not lead to any difficulties all along the proof. At the end, if  $u$  and  $\tilde{u}$  are two process solutions associated to the same initial data  $u_0$ , we obtain a  $L^1$ -contraction principle of the following form: for a.e.  $t \in [0, T]$ ,

$$\int_{\mathbb{R}} \int_0^1 \int_0^1 (u(x, t, \alpha) - \tilde{u}(x, t, \beta))^\pm d\alpha d\beta dx \leq 0,$$

and thus  $u = \tilde{u}$ , and  $u$  does not depend on  $\alpha$ . In the doubling variable method, the treatment of the boundary conditions has been performed by Felix Otto in his PhD Thesis, summarized in [24], and explained in [23]. We refer to this later reference and to [34] for a complete proof of theorem 3.3.

**Proposition 3.4** *Let  $u_0$  be a prepared initial data, and let  $u^\varepsilon$  be the corresponding solution to the approximate problem  $(\mathcal{P}^\varepsilon)$ . Then  $u^\varepsilon$  converges to the unique solution  $u$  associated to initial data  $u_0$  in the  $L^p((0, T); L^q(\mathbb{R}))$ -sense, for all  $p, q \in [1, \infty)$ .*

We have seen in proposition 3.2 that  $u^\varepsilon$  converges up to extraction to a process solution. The family  $(u^\varepsilon)_\varepsilon$  admits so a unique adherence value, which is a solution thanks to Theorem 3.3, thus all the family converges towards this unique limit  $u$ .

Let  $K$  denotes a compact subset of  $\mathbb{R} \times [0, T]$ , then one has

$$\iint_K (u^\varepsilon - u)^2 dx dt = \iint_K (u^\varepsilon)^2 dx - 2 \iint_K u^\varepsilon u dx + \iint_K u^2 dx.$$

Since  $u^\varepsilon$  converges in the nonlinear weak star sense towards  $u$ ,

$$\lim_{\varepsilon \rightarrow 0} \iint_K (u^\varepsilon)^2 dx = \iint_K u^2 dx.$$

Moreover,  $u^\varepsilon$  converges in the  $L^\infty$  weak star topology towards  $u$ , then

$$\lim_{\varepsilon \rightarrow 0} \iint_K u^\varepsilon u dx = \iint_K u^2 dx.$$

Thus we obtain

$$\lim_{\varepsilon \rightarrow 0} \iint_K (u^\varepsilon - u)^2 dx dt = 0.$$

One concludes using the fact the  $|u^\varepsilon - u| \leq 1$  for all  $\varepsilon > 0$ .

### 3.4 initial data $u_0$ in $L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$

The notion of bounded flux solution is too restrictive to provide existence for any  $u_0$  in  $L^\infty(\mathbb{R})$ , but the existence–uniqueness frame can be extended to general initial data using density arguments.

Let  $u_0 \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$ , there exists thanks to Lemma 2.1 a sequence  $(u_{0,\eta})_\eta$  of prepared initial data tending to  $u_0$ . Let  $(u_\eta^\varepsilon)_\eta$  be the associated family of solutions to the problem  $(\mathcal{P}^\varepsilon)$ . Thanks to (15), one has for almost every  $t \in (0, T)$ , for all  $\eta, \delta > 0$ ,

$$\int_{\mathbb{R}} (u_\eta^\varepsilon(x, t) - u_\delta^\varepsilon(x, t))^\pm dx \leq \int_{\mathbb{R}} (u_{0,\eta} - u_{0,\delta})^\pm dx.$$

We can thus build a Cauchy sequence in  $\mathcal{C}([0, T]; L^1(\mathbb{R}))$ , and thus there exists a unique  $u^\varepsilon$  limit of  $u_\eta^\varepsilon$  for  $\eta \rightarrow 0$ , and

$$\int_{\mathbb{R}} (u_\eta^\varepsilon(x, t) - u^\varepsilon(x, t))^\pm dx \leq \int_{\mathbb{R}} (u_{0,\eta} - u_0)^\pm dx. \quad (28)$$

This  $u^\varepsilon$ , obtained as limit of approximation, and so-called SOLA to the problem, fulfills the problem  $(\mathcal{P}^\varepsilon)$  in a weaker sense, as it is explained in [10, 13]. Moreover, the function  $u^\varepsilon$  is the limit of the finite volume approximation introduced in [10].

Let  $u$  (resp.  $u_\eta$ ) be the solution to  $(\mathcal{P}_{\text{lim}})$  associated to  $u_0$  (resp.  $u_{0,\eta}$ ). We now aim to show that  $u^\varepsilon$  tends to  $u$  as  $\varepsilon$  tends to 0. For a.e.  $t \in (0, T)$ ,

$$\begin{aligned} \int_{\mathbb{R}} |u(x, t) - u^\varepsilon(x, t)| dx &\leq \int_{\mathbb{R}} |u(x, t) - u_\eta(x, t)| dx \\ &\quad + \int_{\mathbb{R}} |u_\eta(x, t) - u_\eta^\varepsilon(x, t)| dx \\ &\quad + \int_{\mathbb{R}} |u_\eta^\varepsilon(x, t) - u^\varepsilon(x, t)| dx. \end{aligned} \quad (29)$$

Using (27) and (28) in (29) yields

$$\begin{aligned} \int_{\mathbb{R}} |u(x, t) - u^\varepsilon(x, t)| dx &\leq \int_{\mathbb{R}} |u_\eta(x, t) - u_\eta^\varepsilon(x, t)| dx \\ &\quad + 2 \int_{\mathbb{R}} |u_{0,\eta}(x, t) - u_0(x, t)| dx. \end{aligned} \quad (30)$$

Letting  $\varepsilon$  tend to 0 in (30), it follows from Proposition 3.4 that

$$\limsup_{\varepsilon \rightarrow 0} \int_{\mathbb{R}} |u(x, t) - u^\varepsilon(x, t)| dx \leq 2 \int_{\mathbb{R}} |u_{0,\eta}(x, t) - u_0(x, t)| dx. \quad (31)$$

Since (31) holds for any  $\eta$ , we can let  $\eta$  tend to 0 and we obtain

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}} |u(x, t) - u^\varepsilon(x, t)| dx = 0,$$

and we have proven the following theorem.

**Theorem 3.5** *Let  $u_0 \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$ ,  $0 \leq u_0 \leq 1$ , and let  $u^\varepsilon$  be the unique SOLA corresponding to  $u_0$ . Then  $u^\varepsilon$  tends almost everywhere in  $\mathbb{R} \times (0, T)$  towards the unique solution  $u$  to  $(\mathcal{P}_{\text{lim}})$  as  $\varepsilon$  tends to 0.*

## 4 The problem with global convection

It is natural to consider the case where the contribution of the global convection is not equal to zero. We consider now the equation

$$\partial_t u + \partial_x (qc_i(u) + g_i(u)) = 0,$$

where  $c_i$  is an increasing Lipschitz continuous function satisfying  $c_i(0) = 0$  and  $c_1(1) = 1$ . Denoting by  $f_i(s) = qc_i(s) + g_i(s)$ , the approximate problem  $(\mathcal{P}^\varepsilon)$  turns to

$$\begin{cases} \partial_t u^\varepsilon + \partial_x F_i^\varepsilon = 0, \\ F_i^\varepsilon = f_i(u^\varepsilon) - \varepsilon \partial_x \varphi_i(u^\varepsilon), \\ u^\varepsilon(x = 0^-) = 1 \text{ or } u^\varepsilon(x = 0^+) = 0, \\ F_1^\varepsilon(0^-) = F_2^\varepsilon(0^+), \\ u(t = 0) = u_0. \end{cases} \quad (\mathcal{P}_{\text{conv}}^\varepsilon)$$

The construction of  $f_i$  ensures the relations

$$f_i(0) = 0, \quad f_i(1) = q, \quad i = 1, 2,$$

where  $q$  is the total flow-rate, which is supposed to be constant with respect to time. We also assume, without loss of generality, that  $q > 0$ . We assume furthermore that

$$f_i^{-1}([q, \infty)) \text{ is an interval } [u_i^*, 1], \quad (32)$$

$$f_i \text{ is strictly increasing on } [0, u_i^*]. \quad (33)$$

The results of existence/uniqueness for bounded flux solutions still hold for the problem with

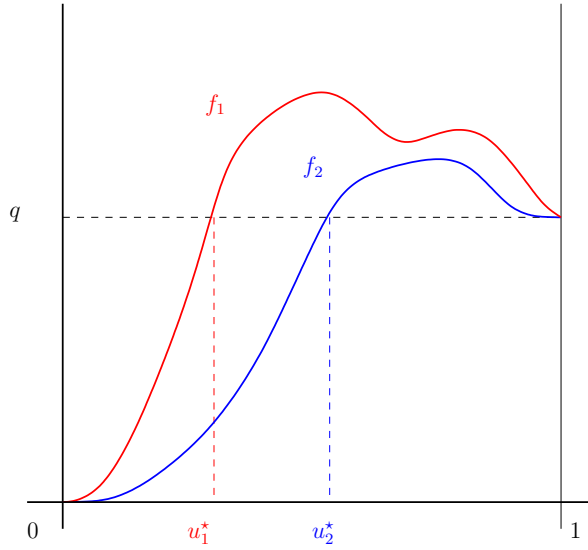


Figure 1: the functions  $f_i$  ( $i = 1, 2$ ).

global convection. More precisely, the following theorem, corresponding to Theorem 2.3 for global convection, holds.

**Theorem 4.1** *Let  $\varepsilon > 0$ , and let  $u_0 \in L^1(\mathbb{R})$  (resp. s.t.  $(1 - u_0) \in L^1(\mathbb{R})$ ) be a smooth function, then there exists a unique bounded flux solution  $u^\varepsilon \in L^1(\mathbb{R} \times (0, T))$  (resp. s.t.  $(1 - u^\varepsilon) \in L^1(\mathbb{R} \times (0, T))$ ), i.e. fulfilling*

1.  $0 \leq u^\varepsilon \leq 1$  a.e. in  $\mathbb{R} \times (0, T)$ ;
2.  $\partial_x \varphi_i(u^\varepsilon) \in L^\infty(\mathbb{R} \times (0, T))$ ;
3.  $\forall \psi \in \mathcal{D}(\mathbb{R} \times [0, T))$ ,

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}} u^\varepsilon \partial_t \psi dx dt + \int_{\mathbb{R}} u_0 \psi(\cdot, 0) dx \\ & + \int_0^T \sum_i \int_{\Omega_i} (f_i(u^\varepsilon) - \varepsilon \partial_x \varphi_i(u^\varepsilon)) \partial_x \psi dx dt = 0. \end{aligned}$$

We will now show that if  $q > 0$ ,  $u^\varepsilon$  can either tend to an entropy solution or toward a weak solution involving a non-classical shock at the interface  $\{x = 0\}$  as  $\varepsilon$  tends to 0. The following study is partial, but we are not able to give a general characterization of the limit  $u$  of  $u^\varepsilon$  for any initial data  $u_0$ .

## 4.1 entropy solution for small initial data

In this section, we first suppose that the initial data  $u_0$  belongs to  $L^1(\mathbb{R})$ , and that

$$0 \leq u_0 \leq u_i^* \quad \text{a.e. in } \Omega_i. \quad (34)$$

**Theorem 4.2 (convergence towards the entropy solution)** *Assume that (32), (33) and (34) hold. Let  $u^\varepsilon$  be the SOLA to the approximate problem  $(\mathcal{P}^\varepsilon)$ , then  $u^\varepsilon$  converges to  $u$  in  $L^p((0, T); L^q(\mathbb{R}))$  as  $\varepsilon$  tends to 0, for all  $p, q \in [1, \infty)$ , where  $u$  is the unique entropy solution, i.e. the unique function fulfilling (5).*

### Proof

The question of the uniqueness of the entropy solution for hyperbolic scalar conservation laws with discontinuous flux functions has been widely studied during the last years, leading to numerous papers. We quote Florence Bachmann's PhD, and particularly [5, Chapter 4] for a complete answer.

In such a case, using the technics introduced in [9, Proposition 2.8], we can show that for all  $\lambda \in [0, q]$  there exists a steady solution  $\kappa_\lambda^\varepsilon$  to the problem  $(\mathcal{P}^\varepsilon)$ , corresponding to a constant flux

$$f_i(\kappa_\lambda^\varepsilon) - \varepsilon \partial \varphi_i(\kappa_\lambda^\varepsilon) = \lambda,$$

and such that this solution converges uniformly on each compact subset of  $\mathbb{R}^*$  towards

$$\kappa_\lambda(x) = \min_{\kappa} \{f(\kappa, x) = \lambda\}.$$

Let  $\lambda \in [0, q]$ . Since  $u^\varepsilon$  and  $\kappa_\lambda^\varepsilon$  are both SOLAs, it follows from Proposition 2.2 that for all  $\psi \in \mathcal{D}^+(\mathbb{R} \times [0, T))$ ,

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}} (u^\varepsilon - \kappa_\lambda^\varepsilon)^\pm \partial_t \psi dx dt + \int_{\mathbb{R}} (u_0 - \kappa_\lambda^\varepsilon)^\pm \psi(\cdot, 0) dx \\ & + \int_0^T \sum_i \int_{\Omega_i} \left( G_{i\pm}(u^\varepsilon, \kappa_\lambda^\varepsilon) - \varepsilon \partial_x (\varphi_i(u^\varepsilon) - \varphi_i(\kappa_\lambda^\varepsilon))^+ \right) \partial_x \psi dx dt \geq 0, \end{aligned} \quad (35)$$

where  $G_{i\pm}, G_i$  have been updated, i.e. for all  $s, \kappa \in [0, 1]$ ,

$$G_{i\pm}(s, \kappa) = \text{sign}_\pm(s - \kappa) (f_i(s) - f_i(\kappa)), \quad G_i(s, \kappa) = \text{sign}(s - \kappa) (f_i(s) - f_i(\kappa)).$$

Choosing  $\lambda = q$  and  $\psi(x, t) = (T - t)\xi(x, t)$  for  $\xi \in \mathcal{D}^+(\mathbb{R})$  yields

$$\int_0^T \int_{\Omega} (u^\varepsilon - \kappa_q^\varepsilon)^+ \xi dx dt \leq \int_0^T (T - t) \sum_{i=1,2} \int_{\Omega_i} \varepsilon \partial_x (\varphi_i(u^\varepsilon) - \varphi_i(\kappa_q^\varepsilon))^+ \partial_x \xi dx dt. \quad (36)$$

Since  $u^\varepsilon$  is bounded between 0 and 1, it converges in the nonlinear weak star sense, thanks to Theorem 3.1 towards a function  $u \in L^\infty(\mathbb{R} \times (0, T) \times (0, 1))$ , with  $0 \leq u \leq 1$  a.e.. Then (36) provides

$$u \leq \kappa_q = u_i^* \quad \text{a.e. in } \Omega_i \times (0, T) \times (0, 1). \quad (37)$$

Let  $\lambda \in [0, q]$ , then taking the limit for  $\varepsilon \rightarrow 0$  in (35) yields

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}} \int_0^1 |u - \kappa_\lambda| \partial_t \psi d\alpha dx dt + \int_{\mathbb{R}} |u_0 - \kappa_\lambda| \psi(\cdot, 0) dx \\ & + \int_0^T \sum_i \int_{\Omega_i} \int_0^1 G_i(u, \kappa_\lambda) \partial_x \psi d\alpha dx dt \geq 0. \end{aligned} \quad (38)$$

Suppose that  $u_1^* \leq u_2^*$ . Let  $\kappa \in [0, u_2^*]$ , we denote by  $\tilde{\kappa} = f_1^{-1}(f_2(\kappa)) \cap [0, u_1^*]$ . Then choosing  $\lambda = f_2(\kappa)$  in (38), and letting  $\varepsilon$  tend to 0 gives:  $\forall \kappa \in [0, u_2^*], \forall \psi \in \mathcal{D}^+(\mathbb{R} \times [0, T))$ ,

$$\begin{aligned} & \int_0^T \int_{\Omega_1} \int_0^1 |u - \kappa| \partial_t \psi d\alpha dx dt + \int_{\Omega_1} |u_0 - \kappa| \psi(\cdot, 0) dx \\ & + \int_0^T \int_{\Omega_2} \int_0^1 |u - \tilde{\kappa}| \partial_t \psi d\alpha dx dt + \int_{\Omega_2} |u_0 - \tilde{\kappa}| \psi(\cdot, 0) dx \\ & + \int_0^T \int_0^1 \left( \int_{\Omega_1} G_1(u, \kappa) \partial_x \psi dx + \int_{\Omega_2} G_2(u, \tilde{\kappa}) \partial_x \psi dx \right) d\alpha dt \geq 0. \end{aligned} \quad (39)$$

It follows from the work of Jose Carrillo [14] that the following entropy inequalities hold for test functions compactly supported in  $\Omega_1$ :  $\forall \kappa \in [0, 1], \forall \psi \in \mathcal{D}^+(\Omega_1 \times [0, T])$ ,

$$\begin{aligned} & \int_0^T \int_{\Omega_1} |u^\varepsilon - \kappa| \partial_t \psi dx dt + \int_{\Omega_1} |u_0 - \kappa| \psi(\cdot, 0) dx \\ & + \int_0^T \int_{\Omega_1} (G_1(u^\varepsilon, \kappa) - \varepsilon \partial_x |\varphi_1(u^\varepsilon) - \varphi_1(\kappa)|) \partial_x \psi dx dt \geq 0. \end{aligned}$$

Thus letting  $\varepsilon$  tend to 0 provides:  $\forall \psi \in \mathcal{D}^+(\Omega_1 \times [0, T]), \forall \kappa \in [0, 1]$ ,

$$\begin{aligned} & \int_0^T \int_{\Omega_1} \int_0^1 |u - \kappa| \partial_t \psi d\alpha dx dt + \int_{\Omega_1} |u_0 - \kappa| \psi(\cdot, 0) dx \\ & + \int_0^T \int_{\Omega_1} \int_0^1 G_1(u, \kappa) \partial_x \psi d\alpha dx dt \geq 0. \end{aligned} \quad (40)$$

Let  $\delta > 0$ , and let  $\psi \in \mathcal{D}^+(\mathbb{R} \times [0, T])$ , we define

$$\psi_{1,\delta}(x, t) = \psi(x, t) \chi_{1,\delta}(x), \quad \psi_{2,\delta} = \psi - \psi_{1,\delta},$$

where  $\chi_{1,\delta}$  is the cut-off function introduced in section 3.2. Then using  $\psi_{1,\delta}$  as test function in (40) and  $\psi_{2,\delta}$  in (39) leads to:

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}} \int_0^1 |u - \kappa| \partial_x \psi d\alpha dx dt + \int_{\mathbb{R}} |u_0 - \kappa| \psi(\cdot, 0) dx \\ & + \int_0^T \sum_i \int_{\Omega_i} \int_0^1 G_i(u, \kappa) \partial_x \psi d\alpha dx dt \\ & + \int_0^T \int_{\Omega_1} \int_0^1 (G_1(u, \kappa) - G_1(u, \tilde{\kappa})) \psi \partial_x \chi_{1,\delta} d\alpha dx dt \geq \mathcal{R}(\kappa, \psi, \delta), \end{aligned} \quad (41)$$

where  $\lim_{\delta \rightarrow 0} \mathcal{R}(\kappa, \psi, \delta) = 0$ . Since  $f_1$  is increasing on  $[0, u_1^*]$  and  $f_1([u_1^*, 1]) \subset [q, \infty)$ , either  $\kappa \leq u_1^*$ , or  $f_1(\kappa) \geq g_1(f_1^*)$ . This ensures that

$$G_1(u, \kappa) = |f_1(u) - f_1(\kappa)|, \quad \forall \kappa \in [0, u_2^*].$$

This yields

$$\begin{aligned} |G_1(u, \kappa) - G_1(u, \tilde{\kappa})| &= ||f_1(u) - f_1(\kappa)| - |f_1(u) - f_1(\tilde{\kappa})|| \\ &\leq |f_1(\kappa) - f_1(\tilde{\kappa})| = |f_1(\kappa) - f_2(\kappa)|. \end{aligned} \quad (42)$$

Taking the inequality (42) into account in (41), and letting  $\delta \rightarrow 0$  provides:  $\forall \kappa \in [0, \|u\|_\infty], \forall \psi \in \mathcal{D}^+(\mathbb{R} \times [0, T])$ ,

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}} \int_0^1 |u - \kappa| \partial_x \psi d\alpha dx dt + \int_{\mathbb{R}} |u_0 - \kappa| \psi(\cdot, 0) dx \\ & + \int_0^T \sum_i \int_{\Omega_i} \int_0^1 G_i(u, \kappa) \partial_x \psi d\alpha dx dt + |f_1(\kappa) - f_2(\kappa)| \int_0^T \psi(0, \cdot) dt \geq 0. \end{aligned}$$

Using the work of Florence Bachmann [5, Theorem 4.3], we can claim that  $u$  is the unique entropy solution to the problem. Particularly,  $u$  does not depend on  $\alpha$  (introduced for the nonlinear weak star convergence). As proven in section 3.3, this implies that  $u^\varepsilon$  converges in  $L^1(\mathbb{R} \times (0, T))$  towards  $u$ .  $\square$

## 4.2 non-classical shock for large initial data

Changing  $u^\varepsilon$  by  $(1 - u^\varepsilon)$  in  $(\mathcal{P}^\varepsilon)$  does not change the nature of the problem. So we can extend the notion of bounded flux solution to any initial data  $u_0 \in L^\infty(\mathbb{R}), 0 \leq u_0 \leq 1$  a.e., such that  $(1 - u_0)$  belongs to  $L^1(\mathbb{R})$ .

**Theorem 4.3** Assume that  $u_0 \in L^\infty(\mathbb{R})$ , such that  $(u_0)|_{\Omega_i} \geq u_i^*$ , and such that there exists a corresponding bounded-flux solution ( $u^\varepsilon$ ) for all  $\varepsilon > 0$ . Suppose furthermore that

$$\exists \eta > 0 \text{ s.t. } u_0(x) = 1 \text{ if } -\eta \leq x \leq 0, \text{ and } u_0(x) = u_2^* \text{ if } 0 \leq x \leq \eta.$$

Then  $u^\varepsilon$  tends to  $u$  in  $L^p((0, T); L^r_{\text{loc}}(\mathbb{R}))$  for all  $p, r \in [1, \infty)$ , where  $u|_{\Omega_i}$  is the unique entropy solution to

$$\begin{cases} \partial_t u + \partial_x f_i(u) = 0 & \text{on } \Omega_i \times (0, T), \\ u(\cdot, 0) = u_0 & \text{on } \Omega_i, \\ u(0, \cdot) = \begin{cases} 1 & \text{if } i = 1, \\ u_2^* & \text{if } i = 2. \end{cases} & \text{on } (0, T). \end{cases} \quad (43)$$

**Proof**

Using similar technics as in section 2.3, we can show that, for  $\varepsilon$  small enough, there exists a steady solution  $\underline{u}^\varepsilon$  to the problem  $(\mathcal{P}^\varepsilon)$ , corresponding to the equation

$$f_i(\underline{u}^\varepsilon) - \varepsilon \partial_x \varphi_i(\underline{u}^\varepsilon) = q \quad \text{in } \Omega_i,$$

and satisfying

$$\begin{aligned} \lim_{x \rightarrow -\infty} \underline{u}^\varepsilon(x) &= u_1^*, \\ \underline{u}^\varepsilon(x) &= 1 \text{ if } x \in \left(-\frac{\eta}{2}, 0\right), \\ \underline{u}^\varepsilon(x) &= u_2^* \text{ if } x \in \Omega_2. \end{aligned} \quad (44)$$

Then  $\underline{u}^\varepsilon$  is a subsolution with respect to  $u^\varepsilon$ .

Similarly, for  $\varepsilon$  small enough, there exists a steady solution  $\bar{u}^\varepsilon$  to the problem  $(\mathcal{P}^\varepsilon)$ , corresponding to the equation

$$f_i(\bar{u}^\varepsilon) - \varepsilon \partial_x \varphi_i(\bar{u}^\varepsilon) = q \quad \text{in } \Omega_i,$$

and satisfying

$$\begin{aligned} \bar{u}^\varepsilon(x) &= 1 \text{ if } x \notin (0, \eta), \\ \bar{u}^\varepsilon(x) &\leq \frac{u_2^* + 1}{2} < 1 \text{ if } x \in \left(0, \frac{\eta}{2}\right). \end{aligned} \quad (46)$$

Then for a.e.  $(x, t) \in \mathbb{R} \times (0, T)$ , one has

$$\underline{u}^\varepsilon(x) \leq u^\varepsilon(x, t) \leq \bar{u}^\varepsilon(x). \quad (47)$$

As previously, the following convergence properties hold:

- there exists  $u \in L^\infty(\mathbb{R} \times (0, T) \times (0, 1))$  such that  $u^\varepsilon$  tends to  $u$  in the nonlinear weak star sense as  $\varepsilon$  tends to 0;
- $\varepsilon \partial_x \varphi_i(u^\varepsilon)$  tends to 0 in  $L^1(\Omega_i \times (0, T))$  as  $\varepsilon$  tends to 0.

Thanks to (44),  $u$  admits a strong trace equal to 1 on  $\{x = 0\} \times (0, T) \times (0, 1)$ . This implies that  $u$  is the unique solution of the following family of inequalities:  $\forall \kappa \in [0, 1], \forall \psi \in \mathcal{D}^+(\bar{\Omega}_1 \times [0, T])$ ,

$$\begin{aligned} & \int_0^T \int_{\Omega_1} \int_0^1 (u - \kappa)^\pm \partial_t \psi d\alpha dx dt + \int_{\Omega_1} (u_0 - \kappa)^\pm \psi(\cdot, 0) dx \\ & + \int_0^T \int_{\Omega_1} \int_0^1 G_{1\pm}(u, \kappa) d\alpha dx dt + M_{f_1} (1 - \kappa)^\pm \int_0^T \psi(0, \cdot) dt \geq 0, \end{aligned}$$

where  $M_{f_1}$  is an arbitrary Lipschitz constant of  $f_1$ . Such a process solution  $(x, t, \alpha) \mapsto u(x, t, \alpha)$  is unique (see [34]), and does not depend on the process variable  $\alpha$ .

Let us now turn to the behavior of the solution in  $\Omega_2$ . Let  $\theta \in \mathcal{D}([0, T])$ , and  $\xi_\delta(x) = \left(1 - \frac{|x|}{\delta}\right)^+$  for some  $\delta > 0$ . Choosing  $\psi(x, t) = \theta(t)\xi_\delta(x)$  in (13) provides

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}} u^\varepsilon \partial_t \theta \xi_\delta dx dt + \int_{\mathbb{R}} u_0 \theta(0) \xi_\delta dx \\ & + \int_0^T \sum_i \int_{\Omega_i} (f_i(u^\varepsilon) - \varepsilon \partial_x \varphi_i(u^\varepsilon)) \theta \partial_x \xi_\delta dx dt = 0. \end{aligned}$$



Choosing  $\delta \leq \eta_2$ , and  $\varepsilon$  small enough to ensure that (44) and (46) hold leads to

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}} u^\varepsilon \partial_t \theta \xi_\delta dx dt + \int_{\mathbb{R}} u_0 \theta(0) \xi_\delta dx \\ & + \int_0^T \theta \left( q - \frac{1}{\delta} \int_0^\delta (f_2(u^\varepsilon) - \varepsilon \partial_x \varphi_2(u^\varepsilon)) dx \right) dt = 0. \end{aligned}$$

We let  $\varepsilon$  tend to 0 and we obtain

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}} \int_0^1 u \partial_t \theta \xi_\delta d\alpha dx dt + \int_{\mathbb{R}} u_0 \theta(0) \xi_\delta dx \\ & + \int_0^T \theta \left( q - \frac{1}{\delta} \int_0^\delta \int_0^1 f_2(u) d\alpha dx \right) dt = 0. \end{aligned}$$

Letting now  $\theta$  tend to  $\chi_{[0,T)}$  and  $\delta$  tend to 0 provides

$$\lim_{\delta \rightarrow 0} \int_0^1 \int_0^T \left( \frac{1}{\delta} \int_0^\delta (f_2(u(x, t, \alpha)) - f_2(u_2^*)) dx \right) dt d\alpha = 0. \quad (48)$$

Thanks to (45) and (46), the function  $u$  is essentially bounded between  $u_2^*$  and  $\frac{1+u_2^*}{2}$  on  $(0, \frac{\eta}{2}) \times (0, T) \times (0, 1)$ . Since one has supposed that

$$f_2(s) > f_2(u_2^*) \quad \text{if } s \in (u_2^*, 1), \quad (49)$$

then (48) implies that

$$\lim_{\delta \rightarrow 0} \int_0^1 \int_0^T \left( \frac{1}{\delta} \int_0^\delta |f_2(u(x, t, \alpha)) - f_2(u_2^*)| dx \right) dt d\alpha = 0. \quad (50)$$

Particularly, for all  $\beta > 0$ ,

$$\text{meas} \left\{ (x, t, \alpha) \mid 0 \leq x \leq \delta \text{ and } f_2(u(x, t, \alpha)) \geq f_2(u_2^*) + \beta \right\} = o(\delta).$$

Furthermore, using the fact that  $f(s) > f(u_2^*)$  if  $s \in (u_2^*, \frac{1+u_2^*}{2})$ , there exists  $\gamma_0 > 0$  such that for all  $\gamma \in (0, \gamma_0)$ , there exists  $\beta > 0$  such that

$$\left\{ (x, t, \alpha) \mid u(x, t, \alpha) \geq u_2^* + \gamma \right\} \subset \left\{ (x, t, \alpha) \mid f_2(u(x, t, \alpha)) \geq f_2(u_2^*) + \beta \right\}.$$

Then for all  $\gamma \in (0, \gamma_0)$ ,

$$\text{meas} \left\{ (x, t, \alpha) \mid 0 \leq x \leq \delta \text{ and } u(x, t, \alpha) \geq u_2^* + \gamma \right\} = o(\delta).$$

It follows from (50) that

$$\lim_{\delta \rightarrow 0} \int_0^1 \int_0^T \left( \frac{1}{\delta} \int_0^\delta |u(x, t, \alpha) - u_2^*| dx \right) dt d\alpha = 0,$$

and  $u$  admits a strong trace on  $\{x = 0\} \times (0, T) \times (0, 1)$  which depends neither on  $t$  nor on  $\alpha$ . This implies that  $u|_{\Omega_2}$  is the unique solution to the following family of inequalities:  $\forall \psi \in \mathcal{D}^+(\overline{\Omega_2} \times [0, T])$ ,  $\forall \kappa \in [0, 1]$ ,

$$\begin{aligned} & \int_0^T \int_{\Omega_2} \int_0^1 (u - \kappa)^\pm \partial_t \psi d\alpha dx dt + \int_{\Omega_2} (u_0 - \kappa)^\pm \psi(\cdot, 0) dx \\ & + \int_0^T \int_{\Omega_2} \int_0^1 G_{2\pm}(u, \kappa) d\alpha dx dt + M_{f_2} (1 - \kappa)^\pm \int_0^T \psi(0, \cdot) dt \geq 0, \end{aligned}$$

where  $M_{f_2}$  is an arbitrary Lipschitz constant of  $f_2$ . Then, the uniqueness result stated in [34] ensures that  $u|_{\Omega_2}$  is unique and does not depend on  $\alpha$ . The strong convergence of  $u^\varepsilon$  to  $u$  can be obtained as in section 3.3.  $\square$

## 5 conclusion

The model presented here shows that for two-phase flows in heterogeneous porous media with negligible dependence of the capillary pressure with respect to the saturation, the good notion of solution is not always the entropy solution presented for example in [1, 5], and particular care has to be taken with respect to the orientation of the gravity forces. Indeed, some non classical shock can appear at the discontinuities of the capillary pressure field, leading to the phenomenon of oil trapping. We stress the fact that the non classical shocks appearing in our case have a different origin, and a different behavior of those suggested in the recent paper [31]. Indeed, in this latter paper, this lack of entropy was caused by the introduction of the dynamical capillary pressure [25], i.e. the capillary pressure is supposed to depend also on  $\partial_t u$ . In our problem, the lack of entropy comes only from the discontinuity of the porous medium.

In order to conclude this paper, we just want to stress that this model of piecewise constant capillary pressure curves can not lead to some interesting phenomenon. Indeed, if the capillary pressure functions  $\pi_i$  are such that  $\pi_1(0, 1) \cap \pi_2((0, 1)) \neq \emptyset$ , it appears in [10, Section 6] that some oil can overpass the boundary, and that only a finite quantity of oil can be definitely trapped. Moreover, this quantity is determined only by the capillary pressure curves and the difference between the volume mass of both phases, and does not depend on  $u_0$ . The model presented here, with total flow-rate  $q$  equal to zero, do not allow this phenomenon, and all the oil present in  $\Omega_1$  at the initial time remains trapped in  $\Omega_1$  for all  $t \geq 0$ .

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