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Spanning Trees of Bounded Degree Graphs

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► **To cite this version:**

| John Michael Robson. Spanning Trees of Bounded Degree Graphs. 2008. hal-00360110

HAL Id: hal-00360110

<https://hal.science/hal-00360110>

Preprint submitted on 10 Feb 2009

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Appendix

The following conditions are tested to check that the multipliers for cuts with both parts having more than one vertex are subject to the same lower bounds as those where one part is a singleton. In each case the format is the same. Writing d for the degree bound, c for the cut value, i and p for the number of independent and dependent pairs:

$(c - \text{maxu} - \text{maxv}) + f[\text{maxu}, d] + f[\text{maxv}, d] + (2 * i + p)(f[2, d] - 2) \geq f[c, d]$
followed by the numerical values of the two sides.

Case of a cut of value 4, with maxu= 2 and maxv= 2

There are 2 dependent pairs incident at u1 or v1 (1 at u1 and 1 at v1)

For u and v greedy partition is (2, 2);

2 independent and 2 dependent pairs (excluding pairs incident at u1 or v1)

Degree bound 5: $0+f[5,2]+f[5,2]+(2*2+2)(f[5,2]-2) \geq f[5,4]$ (7.200000>6.912000)

Degree bound 6: $0+f[6,2]+f[6,2]+(2*2+2)(f[6,2]-2) \geq f[6,4]$ (6.666664>6.351852)

Degree bound 7: $0+f[7,2]+f[7,2]+(2*2+2)(f[7,2]-2) \geq f[7,4]$ (6.285712>5.970845)

Degree bound 8: $0+f[8,2]+f[8,2]+(2*2+2)(f[8,2]-2) \geq f[8,4]$ (6.000000>5.695313)

Degree bound 9: $0+f[9,2]+f[9,2]+(2*2+2)(f[9,2]-2) \geq f[9,4]$ (5.777776>5.486968)

Degree bound 10: $0+f[10,2]+f[10,2]+(2*2+2)(f[10,2]-2) \geq f[10,4]$ (5.600000>5.324000)

Degree bound 11: $0+f[11,2]+f[11,2]+(2*2+2)(f[11,2]-2) \geq f[11,4]$ (5.454544>5.193088)

Case of a cut of value 5, with maxu= 2 and maxv= 2

There are 2 dependent pairs incident at u1 or v1 (1 at u1 and 1 at v1)

For u and v greedy partition is (2, 2, 1);

6 independent and 2 dependent pairs (excluding pairs incident at u1 or v1)

Degree bound 6: $1+f[6,2]+f[6,2]+(2*6+2)(f[6,2]-2) \geq f[6,5]$ (10.333328>9.263117)

Degree bound 7: $1+f[7,2]+f[7,2]+(2*6+2)(f[7,2]-2) \geq f[7,5]$ (9.571424>8.529779)

Degree bound 8: $1+f[8,2]+f[8,2]+(2*6+2)(f[8,2]-2) \geq f[8,5]$ (9.000000>8.009033)

Degree bound 9: $1+f[9,2]+f[9,2]+(2*6+2)(f[9,2]-2) \geq f[9,5]$ (8.555552>7.620790)

Degree bound 10: $1+f[10,2]+f[10,2]+(2*6+2)(f[10,2]-2) \geq f[10,5]$ (8.200000>7.320500)

Degree bound 11: $1+f[11,2]+f[11,2]+(2*6+2)(f[11,2]-2) \geq f[11,5]$ (7.909088>7.081484)

Case of a cut of value 5, with maxu= 3 and maxv= 2

There are 4 dependent pairs incident at u1 or v1 (3 at u1 and 1 at v1)

For u greedy partition is (3, 2); for v greedy partition is (2, 2, 1);

4 independent and 2 dependent pairs (excluding pairs incident at u1 or v1)

Degree bound 6: $0+f[6,3]+f[6,2]+(2*4+2)(f[6,2]-2) \geq f[6,5]$ (9.749996>9.263117)

Degree bound 7: $0+f[7,3]+f[7,2]+(2*4+2)(f[7,2]-2) \geq f[7,5]$ (9.061221>8.529779)

Degree bound 8: $0+f[8,3]+f[8,2]+(2*4+2)(f[8,2]-2) \geq f[8,5]$ (8.546875>8.009033)

Degree bound 9: $0+f[9,3]+f[9,2]+(2*4+2)(f[9,2]-2) \geq f[9,5]$ (8.148146>7.620790)

Degree bound 10: $0+f[10,3]+f[10,2]+(2*4+2)(f[10,2]-2) \geq f[10,5]$ (7.830000>7.320500)

Degree bound 11: $0+f[11,3]+f[11,2]+(2*4+2)(f[11,2]-2) \geq f[11,5]$ (7.570246>7.081484)

Case of a cut of value 6, with maxu= 2 and maxv= 2

There are 2 dependent pairs incident at u1 or v1 (1 at u1 and 1 at v1)

For u and v greedy partition is (2, 2, 2);

9 independent and 4 dependent pairs (excluding pairs incident at u_1 or v_1)
Degree bound 7: $2f[7,2]+f[7,2]+(2*9+4)(f[7,2]-2) \geq f[7,6]$ (12.857136>11.697983)
Degree bound 8: $2f[8,2]+f[8,2]+(2*9+4)(f[8,2]-2) \geq f[8,6]$ (12.000000>10.812195)
Degree bound 9: $2f[9,2]+f[9,2]+(2*9+4)(f[9,2]-2) \geq f[9,6]$ (11.333328>10.161053)
Degree bound 10: $2f[10,2]+f[10,2]+(2*9+4)(f[10,2]-2) \geq f[10,6]$ (10.800000>9.663060)
Degree bound 11: $2f[11,2]+f[11,2]+(2*9+4)(f[11,2]-2) \geq f[11,6]$ (10.363632>9.270306)

Case of a cut of value 6, with $\max_u = 3$ and $\max_v = 2$

There are 4 dependent pairs incident at u_1 or v_1 (3 at u_1 and 1 at v_1)
For u greedy partition is (3, 3); for v greedy partition is (2, 2, 2);
6 independent and 5 dependent pairs (excluding pairs incident at u_1 or v_1)
Degree bound 7: $1f[7,3]+f[7,2]+(2*6+5)(f[7,2]-2) \geq f[7,6]$ (12.061219>11.697983)
Degree bound 8: $1f[8,3]+f[8,2]+(2*6+5)(f[8,2]-2) \geq f[8,6]$ (11.296875>10.812195)
Degree bound 9: $1f[9,3]+f[9,2]+(2*6+5)(f[9,2]-2) \geq f[9,6]$ (10.703700>10.161053)
Degree bound 10: $1f[10,3]+f[10,2]+(2*6+5)(f[10,2]-2) \geq f[10,6]$ (10.230000>9.663060)
Degree bound 11: $1f[11,3]+f[11,2]+(2*6+5)(f[11,2]-2) \geq f[11,6]$ (9.842972>9.270306)

Case of a cut of value 6, with $\max_u = 3$ and $\max_v = 3$

There are 6 dependent pairs incident at u_1 or v_1 (3 at u_1 and 3 at v_1)
For u and v greedy partition is (3, 2, 1);
7 independent and 2 dependent pairs (excluding pairs incident at u_1 or v_1)
Degree bound 7: $0f[7,3]+f[7,3]+(2*7+2)(f[7,2]-2) \geq f[7,6]$ (12.408158>11.697983)
Degree bound 8: $0f[8,3]+f[8,3]+(2*7+2)(f[8,2]-2) \geq f[8,6]$ (11.593750>10.812195)
Degree bound 9: $0f[9,3]+f[9,3]+(2*7+2)(f[9,2]-2) \geq f[9,6]$ (10.962960>10.161053)
Degree bound 10: $0f[10,3]+f[10,3]+(2*7+2)(f[10,2]-2) \geq f[10,6]$ (10.460000>9.663060)
Degree bound 11: $0f[11,3]+f[11,3]+(2*7+2)(f[11,2]-2) \geq f[11,6]$ (10.049584>9.270306)

Case of a cut of value 6, with $\max_u = 4$ and $\max_v = 2$

There are 7 dependent pairs incident at u_1 or v_1 (6 at u_1 and 1 at v_1)
For u greedy partition is (4, 2); for v greedy partition is (2, 2, 1, 1);
6 independent and 2 dependent pairs (excluding pairs incident at u_1 or v_1)
Degree bound 7: $0f[7,4]+f[7,2]+(2*6+2)(f[7,2]-2) \geq f[7,6]$ (12.256555>11.697983)
Degree bound 8: $0f[8,4]+f[8,2]+(2*6+2)(f[8,2]-2) \geq f[8,6]$ (11.445313>10.812195)
Degree bound 9: $0f[9,4]+f[9,2]+(2*6+2)(f[9,2]-2) \geq f[9,6]$ (10.820298>10.161053)
Degree bound 10: $0f[10,4]+f[10,2]+(2*6+2)(f[10,2]-2) \geq f[10,6]$ (10.324000>9.663060)
Degree bound 11: $0f[11,4]+f[11,2]+(2*6+2)(f[11,2]-2) \geq f[11,6]$ (9.920358>9.270306)

Case of a cut of value 7, with $\max_u = 2$ and $\max_v = 2$

There are 2 dependent pairs incident at u_1 or v_1 (1 at u_1 and 1 at v_1)
For u and v greedy partition is (2, 2, 2, 1);
15 independent and 4 dependent pairs (excluding pairs incident at u_1 or v_1)
Degree bound 8: $3f[8,2]+f[8,2]+(2*15+4)(f[8,2]-2) \geq f[8,7]$ (16.000000>14.191006)
Degree bound 9: $3f[9,2]+f[9,2]+(2*15+4)(f[9,2]-2) \geq f[9,7]$ (14.999992>13.171735)
Degree bound 10: $3f[10,2]+f[10,2]+(2*15+4)(f[10,2]-2) \geq f[10,7]$ (14.200000>12.400927)
Degree bound 11: $3f[11,2]+f[11,2]+(2*15+4)(f[11,2]-2) \geq f[11,7]$ (13.545448>11.798571)

Case of a cut of value 7, with $\max_u = 3$ and $\max_v = 2$
 There are 4 dependent pairs incident at u_1 or v_1 (3 at u_1 and 1 at v_1)
 For u greedy partition is (3, 3, 1); for v greedy partition is (2, 2, 2, 1);
 12 independent and 5 dependent pairs (excluding pairs incident at u_1 or v_1)
 Degree bound 8: $2f[8,3]+f[8,2]+(2*12+5)(f[8,2]-2) \geq f[8,7]$ (15.296875>14.191006)
 Degree bound 9: $2f[9,3]+f[9,2]+(2*12+5)(f[9,2]-2) \geq f[9,7]$ (14.370364>13.171735)
 Degree bound 10: $2f[10,3]+f[10,2]+(2*12+5)(f[10,2]-2) \geq f[10,7]$ (13.630000>12.400927)
 Degree bound 11: $2f[11,3]+f[11,2]+(2*12+5)(f[11,2]-2) \geq f[11,7]$ (13.024788>11.798571)

Case of a cut of value 7, with $\max_u = 3$ and $\max_v = 3$
 There are 6 dependent pairs incident at u_1 or v_1 (3 at u_1 and 3 at v_1)
 For u and v greedy partition is (3, 3, 1);
 9 independent and 6 dependent pairs (excluding pairs incident at u_1 or v_1)
 Degree bound 8: $1f[8,3]+f[8,3]+(2*9+6)(f[8,2]-2) \geq f[8,7]$ (14.593750>14.191006)
 Degree bound 9: $1f[9,3]+f[9,3]+(2*9+6)(f[9,2]-2) \geq f[9,7]$ (13.740736>13.171735)
 Degree bound 10: $1f[10,3]+f[10,3]+(2*9+6)(f[10,2]-2) \geq f[10,7]$ (13.060000>12.400927)
 Degree bound 11: $1f[11,3]+f[11,3]+(2*9+6)(f[11,2]-2) \geq f[11,7]$ (12.504128>11.798571)

Case of a cut of value 7, with $\max_u = 4$ and $\max_v = 2$
 There are 7 dependent pairs incident at u_1 or v_1 (6 at u_1 and 1 at v_1)
 For u greedy partition is (4, 3); for v greedy partition is (2, 2, 2, 1);
 9 independent and 5 dependent pairs (excluding pairs incident at u_1 or v_1)
 Degree bound 8: $1f[8,4]+f[8,2]+(2*9+5)(f[8,2]-2) \geq f[8,7]$ (14.695313>14.191006)
 Degree bound 9: $1f[9,4]+f[9,2]+(2*9+5)(f[9,2]-2) \geq f[9,7]$ (13.820296>13.171735)
 Degree bound 10: $1f[10,4]+f[10,2]+(2*9+5)(f[10,2]-2) \geq f[10,7]$ (13.124000>12.400927)
 Degree bound 11: $1f[11,4]+f[11,2]+(2*9+5)(f[11,2]-2) \geq f[11,7]$ (12.556720>11.798571)

Case of a cut of value 7, with $\max_u = 4$ and $\max_v = 3$
 There are 9 dependent pairs incident at u_1 or v_1 (6 at u_1 and 3 at v_1)
 For u greedy partition is (4, 2, 1); for v greedy partition is (3, 2, 1, 1);
 10 independent and 2 dependent pairs (excluding pairs incident at u_1 or v_1)
 Degree bound 8: $0f[8,4]+f[8,3]+(2*10+2)(f[8,2]-2) \geq f[8,7]$ (14.992188>14.191006)
 Degree bound 9: $0f[9,4]+f[9,3]+(2*10+2)(f[9,2]-2) \geq f[9,7]$ (14.079556>13.171735)
 Degree bound 10: $0f[10,4]+f[10,3]+(2*10+2)(f[10,2]-2) \geq f[10,7]$ (13.354000>12.400927)
 Degree bound 11: $0f[11,4]+f[11,3]+(2*10+2)(f[11,2]-2) \geq f[11,7]$ (12.763332>11.798571)

Case of a cut of value 7, with $\max_u = 5$ and $\max_v = 2$
 There are 11 dependent pairs incident at u_1 or v_1 (10 at u_1 and 1 at v_1)
 For u greedy partition is (5, 2); for v greedy partition is (2, 2, 1, 1, 1);
 8 independent and 2 dependent pairs (excluding pairs incident at u_1 or v_1)
 Degree bound 8: $0f[8,5]+f[8,2]+(2*8+2)(f[8,2]-2) \geq f[8,7]$ (14.759033>14.191006)
 Degree bound 9: $0f[9,5]+f[9,2]+(2*8+2)(f[9,2]-2) \geq f[9,7]$ (13.843008>13.171735)
 Degree bound 10: $0f[10,5]+f[10,2]+(2*8+2)(f[10,2]-2) \geq f[10,7]$ (13.120500>12.400927)
 Degree bound 11: $0f[11,5]+f[11,2]+(2*8+2)(f[11,2]-2) \geq f[11,7]$ (12.536026>11.798571)

Case of a cut of value 8, with $\max_u = 2$ and $\max_v = 2$

There are 2 dependent pairs incident at u_1 or v_1 (1 at u_1 and 1 at v_1)
 For u and v greedy partition is (2, 2, 2, 2);
 20 independent and 6 dependent pairs (excluding pairs incident at u_1 or v_1)
 Degree bound 9: $4+f[9,2]+f[9,2]+(2*20+6)(f[9,2]-2) \geq f[9,8]$ (18.666656>16.726013)
 Degree bound 10: $4+f[10,2]+f[10,2]+(2*20+6)(f[10,2]-2) \geq f[10,8]$ (17.600000>15.589737)
 Degree bound 11: $4+f[11,2]+f[11,2]+(2*20+6)(f[11,2]-2) \geq f[11,8]$ (16.727264>14.709907)

Case of a cut of value 8, with $\max_u = 3$ and $\max_v = 2$
 There are 4 dependent pairs incident at u_1 or v_1 (3 at u_1 and 1 at v_1)
 For u greedy partition is (3, 3, 2); for v greedy partition is (2, 2, 2, 2);
 17 independent and 7 dependent pairs (excluding pairs incident at u_1 or v_1)
 Degree bound 9: $3+f[9,3]+f[9,2]+(2*17+7)(f[9,2]-2) \geq f[9,8]$ (18.037028>16.726013)
 Degree bound 10: $3+f[10,3]+f[10,2]+(2*17+7)(f[10,2]-2) \geq f[10,8]$ (17.030000>15.589737)
 Degree bound 11: $3+f[11,3]+f[11,2]+(2*17+7)(f[11,2]-2) \geq f[11,8]$ (16.206604>14.709907)

Case of a cut of value 8, with $\max_u = 3$ and $\max_v = 3$
 There are 6 dependent pairs incident at u_1 or v_1 (3 at u_1 and 3 at v_1)
 For u and v greedy partition is (3, 3, 2);
 14 independent and 8 dependent pairs (excluding pairs incident at u_1 or v_1)
 Degree bound 9: $2+f[9,3]+f[9,3]+(2*14+8)(f[9,2]-2) \geq f[9,8]$ (17.407400>16.726013)
 Degree bound 10: $2+f[10,3]+f[10,3]+(2*14+8)(f[10,2]-2) \geq f[10,8]$ (16.460000>15.589737)
 Degree bound 11: $2+f[11,3]+f[11,3]+(2*14+8)(f[11,2]-2) \geq f[11,8]$ (15.685944>14.709907)

Case of a cut of value 8, with $\max_u = 4$ and $\max_v = 2$
 There are 7 dependent pairs incident at u_1 or v_1 (6 at u_1 and 1 at v_1)
 For u greedy partition is (4, 4); for v greedy partition is (2, 2, 2, 2);
 12 independent and 9 dependent pairs (excluding pairs incident at u_1 or v_1)
 Degree bound 9: $2+f[9,4]+f[9,2]+(2*12+9)(f[9,2]-2) \geq f[9,8]$ (17.042516>16.726013)
 Degree bound 10: $2+f[10,4]+f[10,2]+(2*12+9)(f[10,2]-2) \geq f[10,8]$ (16.124000>15.589737)
 Degree bound 11: $2+f[11,4]+f[11,2]+(2*12+9)(f[11,2]-2) \geq f[11,8]$ (15.374900>14.709907)

Case of a cut of value 8, with $\max_u = 4$ and $\max_v = 3$
 There are 9 dependent pairs incident at u_1 or v_1 (6 at u_1 and 3 at v_1)
 For u greedy partition is (4, 3, 1); for v greedy partition is (3, 3, 1, 1);
 13 independent and 6 dependent pairs (excluding pairs incident at u_1 or v_1)
 Degree bound 9: $1+f[9,4]+f[9,3]+(2*13+6)(f[9,2]-2) \geq f[9,8]$ (17.301776>16.726013)
 Degree bound 10: $1+f[10,4]+f[10,3]+(2*13+6)(f[10,2]-2) \geq f[10,8]$ (16.354000>15.589737)
 Degree bound 11: $1+f[11,4]+f[11,3]+(2*13+6)(f[11,2]-2) \geq f[11,8]$ (15.581512>14.709907)

Case of a cut of value 8, with $\max_u = 4$ and $\max_v = 4$
 There are 12 dependent pairs incident at u_1 or v_1 (6 at u_1 and 6 at v_1)
 For u and v greedy partition is (4, 2, 1, 1);
 14 independent and 2 dependent pairs (excluding pairs incident at u_1 or v_1)
 Degree bound 9: $0+f[9,4]+f[9,4]+(2*14+2)(f[9,2]-2) \geq f[9,8]$ (17.640596>16.726013)
 Degree bound 10: $0+f[10,4]+f[10,4]+(2*14+2)(f[10,2]-2) \geq f[10,8]$ (16.648000>15.589737)
 Degree bound 11: $0+f[11,4]+f[11,4]+(2*14+2)(f[11,2]-2) \geq f[11,8]$ (15.840716>14.709907)

Case of a cut of value 8, with $\max_u = 5$ and $\max_v = 2$

There are 11 dependent pairs incident at u_1 or v_1 (10 at u_1 and 1 at v_1)

For u greedy partition is (5, 3); for v greedy partition is (2, 2, 2, 1, 1);

12 independent and 5 dependent pairs (excluding pairs incident at u_1 or v_1)

Degree bound 9: $1+f[9,5]+f[9,2]+(2*12+5)(f[9,2]-2) \geq f[9,8]$ (17.287450>16.726013)

Degree bound 10: $1+f[10,5]+f[10,2]+(2*12+5)(f[10,2]-2) \geq f[10,8]$ (16.320500>15.589737)

Degree bound 11: $1+f[11,5]+f[11,2]+(2*12+5)(f[11,2]-2) \geq f[11,8]$ (15.536024>14.709907)

Case of a cut of value 8, with $\max_u = 5$ and $\max_v = 3$

There are 13 dependent pairs incident at u_1 or v_1 (10 at u_1 and 3 at v_1)

For u greedy partition is (5, 2, 1); for v greedy partition is (3, 2, 1, 1, 1);

13 independent and 2 dependent pairs (excluding pairs incident at u_1 or v_1)

Degree bound 9: $0+f[9,5]+f[9,3]+(2*13+2)(f[9,2]-2) \geq f[9,8]$ (17.546710>16.726013)

Degree bound 10: $0+f[10,5]+f[10,3]+(2*13+2)(f[10,2]-2) \geq f[10,8]$ (16.550500>15.589737)

Degree bound 11: $0+f[11,5]+f[11,3]+(2*13+2)(f[11,2]-2) \geq f[11,8]$ (15.742636>14.709907)

Case of a cut of value 8, with $\max_u = 6$ and $\max_v = 2$

There are 16 dependent pairs incident at u_1 or v_1 (15 at u_1 and 1 at v_1)

For u greedy partition is (6, 2); for v greedy partition is (2, 2, 1, 1, 1, 1);

10 independent and 2 dependent pairs (excluding pairs incident at u_1 or v_1)

Degree bound 9: $0+f[9,6]+f[9,2]+(2*10+2)(f[9,2]-2) \geq f[9,8]$ (17.272159>16.726013)

Degree bound 10: $0+f[10,6]+f[10,2]+(2*10+2)(f[10,2]-2) \geq f[10,8]$ (16.263060>15.589737)

Degree bound 11: $0+f[11,6]+f[11,2]+(2*10+2)(f[11,2]-2) \geq f[11,8]$ (15.452120>14.709907)

Case of a cut of value 9, with $\max_u = 2$ and $\max_v = 2$

There are 2 dependent pairs incident at u_1 or v_1 (1 at u_1 and 1 at v_1)

For u and v greedy partition is (2, 2, 2, 2, 1);

28 independent and 6 dependent pairs (excluding pairs incident at u_1 or v_1)

Degree bound 10: $5+f[10,2]+f[10,2]+(2*28+6)(f[10,2]-2) \geq f[10,9]$ (21.800000>19.292299)

Degree bound 11: $5+f[11,2]+f[11,2]+(2*28+6)(f[11,2]-2) \geq f[11,9]$ (20.636352>18.053067)

Case of a cut of value 9, with $\max_u = 3$ and $\max_v = 2$

There are 4 dependent pairs incident at u_1 or v_1 (3 at u_1 and 1 at v_1)

For u greedy partition is (3, 3, 3); for v greedy partition is (2, 2, 2, 2, 1);

23 independent and 9 dependent pairs (excluding pairs incident at u_1 or v_1)

Degree bound 10: $4+f[10,3]+f[10,2]+(2*23+9)(f[10,2]-2) \geq f[10,9]$ (20.830000>19.292299)

Degree bound 11: $4+f[11,3]+f[11,2]+(2*23+9)(f[11,2]-2) \geq f[11,9]$ (19.752056>18.053067)

Case of a cut of value 9, with $\max_u = 3$ and $\max_v = 3$

There are 6 dependent pairs incident at u_1 or v_1 (3 at u_1 and 3 at v_1)

For u and v greedy partition is (3, 3, 3);

18 independent and 12 dependent pairs (excluding pairs incident at u_1 or v_1)

Degree bound 10: $3+f[10,3]+f[10,3]+(2*18+12)(f[10,2]-2) \geq f[10,9]$ (19.860000>19.292299)

Degree bound 11: $3+f[11,3]+f[11,3]+(2*18+12)(f[11,2]-2) \geq f[11,9]$ (18.867760>18.053067)

Case of a cut of value 9, with $\max_u = 4$ and $\max_v = 2$
 There are 7 dependent pairs incident at u_1 or v_1 (6 at u_1 and 1 at v_1)
 For u greedy partition is (4, 4, 1); for v greedy partition is (2, 2, 2, 2, 1);
 20 independent and 9 dependent pairs (excluding pairs incident at u_1 or v_1)
 Degree bound 10: $3f[10,4]+f[10,2]+(2*20+9)(f[10,2]-2) \geq f[10,9]$ (20.324000>19.292299)
 Degree bound 11: $3f[11,4]+f[11,2]+(2*20+9)(f[11,2]-2) \geq f[11,9]$ (19.283988>18.053067)

Case of a cut of value 9, with $\max_u = 4$ and $\max_v = 3$
 There are 9 dependent pairs incident at u_1 or v_1 (6 at u_1 and 3 at v_1)
 For u greedy partition is (4, 4, 1); for v greedy partition is (3, 3, 2, 1);
 17 independent and 10 dependent pairs (excluding pairs incident at u_1 or v_1)
 Degree bound 10: $2f[10,4]+f[10,3]+(2*17+10)(f[10,2]-2) \geq f[10,9]$ (19.754000>19.292299)
 Degree bound 11: $2f[11,4]+f[11,3]+(2*17+10)(f[11,2]-2) \geq f[11,9]$ (18.763328>18.053067)

Case of a cut of value 9, with $\max_u = 4$ and $\max_v = 4$
 There are 12 dependent pairs incident at u_1 or v_1 (6 at u_1 and 6 at v_1)
 For u and v greedy partition is (4, 3, 1, 1);
 18 independent and 6 dependent pairs (excluding pairs incident at u_1 or v_1)
 Degree bound 10: $1f[10,4]+f[10,4]+(2*18+6)(f[10,2]-2) \geq f[10,9]$ (20.048000>19.292299)
 Degree bound 11: $1f[11,4]+f[11,4]+(2*18+6)(f[11,2]-2) \geq f[11,9]$ (19.022532>18.053067)

Case of a cut of value 9, with $\max_u = 5$ and $\max_v = 2$
 There are 11 dependent pairs incident at u_1 or v_1 (10 at u_1 and 1 at v_1)
 For u greedy partition is (5, 4); for v greedy partition is (2, 2, 2, 2, 1);
 16 independent and 9 dependent pairs (excluding pairs incident at u_1 or v_1)
 Degree bound 10: $2f[10,5]+f[10,2]+(2*16+9)(f[10,2]-2) \geq f[10,9]$ (19.720500>19.292299)
 Degree bound 11: $2f[11,5]+f[11,2]+(2*16+9)(f[11,2]-2) \geq f[11,9]$ (18.717840>18.053067)

Case of a cut of value 9, with $\max_u = 5$ and $\max_v = 3$
 There are 13 dependent pairs incident at u_1 or v_1 (10 at u_1 and 3 at v_1)
 For u greedy partition is (5, 3, 1); for v greedy partition is (3, 3, 1, 1, 1);
 17 independent and 6 dependent pairs (excluding pairs incident at u_1 or v_1)
 Degree bound 10: $1f[10,5]+f[10,3]+(2*17+6)(f[10,2]-2) \geq f[10,9]$ (19.950500>19.292299)
 Degree bound 11: $1f[11,5]+f[11,3]+(2*17+6)(f[11,2]-2) \geq f[11,9]$ (18.924452>18.053067)

Case of a cut of value 9, with $\max_u = 5$ and $\max_v = 4$
 There are 16 dependent pairs incident at u_1 or v_1 (10 at u_1 and 6 at v_1)
 For u greedy partition is (5, 2, 1, 1); for v greedy partition is (4, 2, 1, 1, 1);
 18 independent and 2 dependent pairs (excluding pairs incident at u_1 or v_1)
 Degree bound 10: $0f[10,5]+f[10,4]+(2*18+2)(f[10,2]-2) \geq f[10,9]$ (20.244500>19.292299)
 Degree bound 11: $0f[11,5]+f[11,4]+(2*18+2)(f[11,2]-2) \geq f[11,9]$ (19.183656>18.053067)

Case of a cut of value 9, with $\max_u = 6$ and $\max_v = 2$
 There are 16 dependent pairs incident at u_1 or v_1 (15 at u_1 and 1 at v_1)
 For u greedy partition is (6, 3); for v greedy partition is (2, 2, 2, 1, 1, 1);
 15 independent and 5 dependent pairs (excluding pairs incident at u_1 or v_1)

Degree bound 10: $1+f[10,6]+f[10,2]+(2*15+5)(f[10,2]-2) \geq f[10,9]$ (19.863060>19.292299)
Degree bound 11: $1+f[11,6]+f[11,2]+(2*15+5)(f[11,2]-2) \geq f[11,9]$ (18.815754>18.053067)

Case of a cut of value 9, with $\max_u=6$ and $\max_v=3$
There are 18 dependent pairs incident at u_1 or v_1 (15 at u_1 and 3 at v_1)
For u greedy partition is (6, 2, 1); for v greedy partition is (3, 2, 1, 1, 1, 1);
16 independent and 2 dependent pairs (excluding pairs incident at u_1 or v_1)
Degree bound 10: $0+f[10,6]+f[10,3]+(2*16+2)(f[10,2]-2) \geq f[10,9]$ (20.093060>19.292299)
Degree bound 11: $0+f[11,6]+f[11,3]+(2*16+2)(f[11,2]-2) \geq f[11,9]$ (19.022366>18.053067)

Case of a cut of value 9, with $\max_u=7$ and $\max_v=2$
There are 22 dependent pairs incident at u_1 or v_1 (21 at u_1 and 1 at v_1)
For u greedy partition is (7, 2); for v greedy partition is (2, 2, 1, 1, 1, 1, 1);
12 independent and 2 dependent pairs (excluding pairs incident at u_1 or v_1)
Degree bound 10: $0+f[10,7]+f[10,2]+(2*12+2)(f[10,2]-2) \geq f[10,9]$ (19.800927>19.292299)
Degree bound 11: $0+f[11,7]+f[11,2]+(2*12+2)(f[11,2]-2) \geq f[11,9]$ (18.707657>18.053067)

Case of a cut of value 10, with $\max_u=2$ and $\max_v=2$
There are 2 dependent pairs incident at u_1 or v_1 (1 at u_1 and 1 at v_1)
For u and v greedy partition is (2, 2, 2, 2, 2);
35 independent and 8 dependent pairs (excluding pairs incident at u_1 or v_1)
Degree bound 11: $6+f[11,2]+f[11,2]+(2*35+8)(f[11,2]-2) \geq f[11,10]$ (24.545440>21.882506)

Case of a cut of value 10, with $\max_u=3$ and $\max_v=2$
There are 4 dependent pairs incident at u_1 or v_1 (3 at u_1 and 1 at v_1)
For u greedy partition is (3, 3, 3, 1); for v greedy partition is (2, 2, 2, 2, 2);
31 independent and 10 dependent pairs (excluding pairs incident at u_1 or v_1)
Degree bound 11: $5+f[11,3]+f[11,2]+(2*31+10)(f[11,2]-2) \geq f[11,10]$ (23.842962>21.882506)

Case of a cut of value 10, with $\max_u=3$ and $\max_v=3$
There are 6 dependent pairs incident at u_1 or v_1 (3 at u_1 and 3 at v_1)
For u and v greedy partition is (3, 3, 3, 1);
27 independent and 12 dependent pairs (excluding pairs incident at u_1 or v_1)
Degree bound 11: $4+f[11,3]+f[11,3]+(2*27+12)(f[11,2]-2) \geq f[11,10]$ (23.140484>21.882506)

Case of a cut of value 10, with $\max_u=4$ and $\max_v=2$
There are 7 dependent pairs incident at u_1 or v_1 (6 at u_1 and 1 at v_1)
For u greedy partition is (4, 4, 2); for v greedy partition is (2, 2, 2, 2, 2);
27 independent and 11 dependent pairs (excluding pairs incident at u_1 or v_1)
Degree bound 11: $4+f[11,4]+f[11,2]+(2*27+11)(f[11,2]-2) \geq f[11,10]$ (23.193076>21.882506)

Case of a cut of value 10, with $\max_u=4$ and $\max_v=3$
There are 9 dependent pairs incident at u_1 or v_1 (6 at u_1 and 3 at v_1)
For u greedy partition is (4, 4, 2); for v greedy partition is (3, 3, 3, 1);
23 independent and 13 dependent pairs (excluding pairs incident at u_1 or v_1)
Degree bound 11: $3+f[11,4]+f[11,3]+(2*23+13)(f[11,2]-2) \geq f[11,10]$ (22.490598>21.882506)

Case of a cut of value 10, with $\max_u = 4$ and $\max_v = 4$
There are 12 dependent pairs incident at u_1 or v_1 (6 at u_1 and 6 at v_1)
For u and v greedy partition is (4, 4, 1, 1);
21 independent and 12 dependent pairs (excluding pairs incident at u_1 or v_1)
Degree bound 11: $2f[11,4]+f[11,4]+(2*21+12)(f[11,2]-2) \geq f[11,10]$ (22.204348>21.882506)

Case of a cut of value 10, with $\max_u = 5$ and $\max_v = 2$
There are 11 dependent pairs incident at u_1 or v_1 (10 at u_1 and 1 at v_1)
For u greedy partition is (5, 5); for v greedy partition is (2, 2, 2, 2, 2);
20 independent and 14 dependent pairs (excluding pairs incident at u_1 or v_1)
Degree bound 11: $3f[11,5]+f[11,2]+(2*20+14)(f[11,2]-2) \geq f[11,10]$ (22.081474>21.882506)

Case of a cut of value 10, with $\max_u = 5$ and $\max_v = 3$
There are 13 dependent pairs incident at u_1 or v_1 (10 at u_1 and 3 at v_1)
For u greedy partition is (5, 4, 1); for v greedy partition is (3, 3, 2, 1, 1);
22 independent and 10 dependent pairs (excluding pairs incident at u_1 or v_1)
Degree bound 11: $2f[11,5]+f[11,3]+(2*22+10)(f[11,2]-2) \geq f[11,10]$ (22.469904>21.882506)

Case of a cut of value 10, with $\max_u = 5$ and $\max_v = 4$
There are 16 dependent pairs incident at u_1 or v_1 (10 at u_1 and 6 at v_1)
For u greedy partition is (5, 3, 1, 1); for v greedy partition is (4, 3, 1, 1, 1);
23 independent and 6 dependent pairs (excluding pairs incident at u_1 or v_1)
Degree bound 11: $1f[11,5]+f[11,4]+(2*23+6)(f[11,2]-2) \geq f[11,10]$ (22.729108>21.882506)

Case of a cut of value 10, with $\max_u = 5$ and $\max_v = 5$
There are 20 dependent pairs incident at u_1 or v_1 (10 at u_1 and 10 at v_1)
For u and v greedy partition is (5, 2, 1, 1, 1);
23 independent and 2 dependent pairs (excluding pairs incident at u_1 or v_1)
Degree bound 11: $0f[11,5]+f[11,5]+(2*23+2)(f[11,2]-2) \geq f[11,10]$ (22.890232>21.882506)

Case of a cut of value 10, with $\max_u = 6$ and $\max_v = 2$
There are 16 dependent pairs incident at u_1 or v_1 (15 at u_1 and 1 at v_1)
For u greedy partition is (6, 4); for v greedy partition is (2, 2, 2, 2, 1, 1);
20 independent and 9 dependent pairs (excluding pairs incident at u_1 or v_1)
Degree bound 11: $2f[11,6]+f[11,2]+(2*20+9)(f[11,2]-2) \geq f[11,10]$ (22.361206>21.882506)

Case of a cut of value 10, with $\max_u = 6$ and $\max_v = 3$
There are 18 dependent pairs incident at u_1 or v_1 (15 at u_1 and 3 at v_1)
For u greedy partition is (6, 3, 1); for v greedy partition is (3, 3, 1, 1, 1, 1);
21 independent and 6 dependent pairs (excluding pairs incident at u_1 or v_1)
Degree bound 11: $1f[11,6]+f[11,3]+(2*21+6)(f[11,2]-2) \geq f[11,10]$ (22.567818>21.882506)

Case of a cut of value 10, with $\max_u = 6$ and $\max_v = 4$
There are 21 dependent pairs incident at u_1 or v_1 (15 at u_1 and 6 at v_1)
For u greedy partition is (6, 2, 1, 1); for v greedy partition is (4, 2, 1, 1, 1, 1);

22 independent and 2 dependent pairs (excluding pairs incident at u1 or v1)
Degree bound 11: $0+f[11,6]+f[11,4]+(2*22+2)(f[11,2]-2) \geq f[11,10]$ (22.827022>21.882506)

Case of a cut of value 10, with maxu= 7 and maxv= 2
There are 22 dependent pairs incident at u1 or v1 (21 at u1 and 1 at v1)
For u greedy partition is (7, 3); for v greedy partition is (2, 2, 2, 1, 1, 1, 1);
18 independent and 5 dependent pairs (excluding pairs incident at u1 or v1)
Degree bound 11: $1+f[11,7]+f[11,2]+(2*18+5)(f[11,2]-2) \geq f[11,10]$ (22.434927>21.882506)

Case of a cut of value 10, with maxu= 7 and maxv= 3
There are 24 dependent pairs incident at u1 or v1 (21 at u1 and 3 at v1)
For u greedy partition is (7, 2, 1); for v greedy partition is (3, 2, 1, 1, 1, 1, 1);
19 independent and 2 dependent pairs (excluding pairs incident at u1 or v1)
Degree bound 11: $0+f[11,7]+f[11,3]+(2*19+2)(f[11,2]-2) \geq f[11,10]$ (22.641539>21.882506)

Case of a cut of value 10, with maxu= 8 and maxv= 2
There are 29 dependent pairs incident at u1 or v1 (28 at u1 and 1 at v1)
For u greedy partition is (8, 2); for v greedy partition is (2, 2, 1, 1, 1, 1, 1, 1);
14 independent and 2 dependent pairs (excluding pairs incident at u1 or v1)
Degree bound 11: $0+f[11,8]+f[11,2]+(2*14+2)(f[11,2]-2) \geq f[11,10]$ (22.346265>21.882506)