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EXAMPLES OF H-HYPERSURFACES IN $\mathbb{H}^n \times \mathbb{R}$ AND GEOMETRIC APPLICATIONS

P. Bérard **R. Sa Earp**

Dedicated to Professor Manfredo do Carmo on the occasion of his 80th birthday

Abstract

In this paper we describe all rotation H-hypersurfaces in $\mathbb{H}^n \times \mathbb{R}$ and use them as barriers to prove existence and characterization of certain vertical H-graphs and to give symmetry and uniqueness results for compact H-hypersurfaces whose boundary is one or two parallel submanifolds in slices. We also describe examples of translation H-hypersurfaces in $\mathbb{H}^n \times \mathbb{R}$. For $n > 2$ we obtain a complete embedded translation hypersurface generated by a compact, simple, strictly convex curve. When $0 < H < \frac{n-1}{n}$ we obtain a complete non-entire vertical graph over the non-mean convex domain bounded by an equidistant hypersurface taking infinite boundary value data and infinite asymptotic boundary value data.

1 Introduction

Rotation and translation surfaces with constant mean curvature in $\mathbb{H}^2 \times \mathbb{R}$ have been studied in details in [8, 7, 9] together with applications. We have studied rotation and translation minimal hypersurfaces with applications in [2].

In this paper, we consider constant non-zero mean curvature hypersurfaces in $\mathbb{H}^n \times \mathbb{R}$.

We consider rotation H-hypersurfaces in Section 2.1. For $H > \frac{n-1}{n}$, we find the constant mean curvature sphere-like hypersurfaces obtained in [4] and the Delaunay-like hypersurfaces obtained in [6]. When $0 < H \leq \frac{n-1}{n}$, we obtain complete simply-connected hypersurfaces $\mathcal{S}_H$ which are entire vertical graphs above $\mathbb{H}^n$, as well as some complete embedded or complete immersed cylinders which are bi-graphs (Theorems 2.1 and 2.2). When $H = \frac{n-1}{n}$, the asymptotic behaviour of the height function of these hypersurfaces is exponential, and it only depends on the dimension when $n \geq 3$. In Section 3, we give geometric applications using the simply-connected rotation H-hypersurfaces $\mathcal{S}_H$ ($0 < H \leq \frac{n-1}{n}$) mentioned above as barriers. We give existence and characterization of vertical H-graphs ($0 < H \leq \frac{n-1}{n}$) over appropriate bounded domains (Proposition 3.2) as well as symmetry and uniqueness results for compact hypersurfaces whose boundary is one or two parallel submanifolds in slices (Theorems 3.3 and 3.4). These results generalize the 2-dimensional results obtained previously in [5].

We treat translation H-hypersurfaces in Section 2.3 (Theorem 2.4). When $n \geq 3$ and $H = \frac{n-1}{n}$, we in particular find a complete embedded hypersurface generated by a compact, simple, strictly convex curve.

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Key words: Constant mean curvature, Rotation and Translation hypersurfaces, Vertical graphs, Dirichlet problem.

When $0 < H < \frac{n-1}{n}$, we obtain a complete non-entire vertical graph over the non-mean convex domain bounded by an equidistant hypersurface Γ. This graph takes infinite boundary value data on Γ and it has infinite asymptotic boundary value data.

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2 Examples of H-hypersurfaces in $\mathbb{H}^n \times \mathbb{R}$

We consider the ball model for the hyperbolic space $\mathbb{H}^n$,

$$\mathbb{B} := \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1^2 + \cdots + x_n^2 < 1\},$$

with the hyperbolic metric $g_{\mathbb{B}}$,

$$g_{\mathbb{B}} := 4\big(1 - (x_1^2 + \cdots + x_n^2)\big)^{-2}\big(dx_1^2 + \cdots + dx_n^2\big),$$

and the product metric

$$\widehat{g} = g_{\mathbb{B}} + dt^2$$

on $\mathbb{H}^n \times \mathbb{R}$.

2.1 Rotation H-hypersurfaces in $\mathbb{H}^n \times \mathbb{R}$

The mean curvature equation for rotation hypersurfaces,

$$nH(\rho)\sinh^{n-1}(\rho) = \partial_\rho\Big(\sinh^{n-1}(\rho)\dot{\lambda}(\rho)(1 + \dot{\lambda}^2(\rho))^{-1/2}\Big)$$

can be established using the flux formula, see Appendix A. We consider rotation hypersurfaces about $\{0\} \times \mathbb{R}$, where ρ denotes the hyperbolic distance to the axis and the mean curvature is taken with respect to the unit normal pointing upwards.

Minimal rotation hypersurfaces in $\mathbb{H}^n \times \mathbb{R}$ have been studied in [8] in dimension 2 and in [2] in higher dimensions. In this Section we consider the case in which H is a non-zero constant. We may assume that H is positive.

Integrating the above differential equation, we obtain the equation for the generating curves of rotation H-hypersurfaces in $\mathbb{H}^n \times \mathbb{R}$,

$$\dot{\lambda}(\rho)\big(1+\dot{\lambda}^2(\rho)\big)^{-1/2}\sinh^{n-1}(\rho) = nH\int_0^\rho \sinh^{n-1}(t)\,dt + d \tag{2.1}$$

for $H > 0$ and for some constant d.

This equation has been studied in [5, 8] in dimension 2 (with a different constant d).

Notations. For later purposes we introduce some notations.

• For $m \geq 0$, we define the function $I_m(t)$ by

$$I_m(t) := \int_0^t \sinh^m(r)\,dr. \tag{2.2}$$

• For $H > 0$ and $d \in \mathbb{R}$, we define the functions,

$$\begin{cases} M_{H,d}(t) & := & \sinh^{n-1}(t) - nHI_{n-1}(t) - d, \\ P_{H,d}(t) & := & \sinh^{n-1}(t) + nHI_{n-1}(t) + d, \\ Q_{H,d}(t) & := & \big[nHI_{n-1}(t)+d\big]\big[M_{H,d}(t)\,P_{H,d}(t)\big]^{-1/2}, \\ & & \text{when the square root exists.} \end{cases} \tag{2.3}$$

We see from (2.1) that $\dot{\lambda}(t)$ has the sign of $nHI_{n-1}(t) + d$. It follows that λ is given, up to an additive constant, by

$$\lambda_{H,d}(\rho) = \int_{\rho_0}^\rho \frac{nHI_{n-1}(t)+d}{\sqrt{\sinh^{2n-2}(t) - \big(nHI_{n-1}(t)+d\big)^2}}\,dt$$

or, with the above notations,

$$\lambda_{H,d}(\rho) = \int_{\rho_0}^\rho \frac{nHI_{n-1}(t)+d}{\sqrt{M_{H,d}(t)P_{H,d}(t)}}\,dt = \int_{\rho_0}^\rho Q_{H,d}(t)\,dt \tag{2.4}$$

where the integration interval $[\rho_0, \rho]$ is contained in the interval in which the square-root exists. The existence and behaviour of the function $\lambda_{H,d}$ depend on the signs of the functions $nHI_{n-1}(t) + d$, $M_{H,d}(t)$ and $P_{H,d}(t)$.

Up to vertical translations, the rotation hypersurfaces about the axis $\{0\} \times \mathbb{R}$, with constant mean curvature $H > 0$ with respect to the unit normal pointing upwards, can be classified according to the sign of $H - \frac{n-1}{n}$ and to the sign of d. We state three theorems depending on the value of H.

Theorem 2.1 (Rotation H-hypersurfaces with $H = \frac{n-1}{n}$)

1. *When $d = 0$, the hypersurface $\mathcal{S}_{\frac{n-1}{n}}$ is a simply-connected entire vertical graph above $\mathbb{H}^n \times \{0\}$, tangent to the slice at 0, generated by a strictly convex curve. The height function $\lambda(\rho)$ on $\mathcal{S}_{\frac{n-1}{n}}$ grows exponentially.*

2. *When $d > 0$, the hypersurface $\mathcal{C}_{\frac{n-1}{n}}$ is a complete embedded cylinder, symmetric with respect to the slice $\mathbb{H}^n \times \{0\}$. The parts $\mathcal{C}^{\pm}_{\frac{n-1}{n}} := \mathcal{C}_{\frac{n-1}{n}} \cap \mathbb{H}^n \times \mathbb{R}_{\pm}$ are vertical graphs above the exterior of a ball $B(0, a)$, for some constant $a > 0$ depending on d. The height function $\lambda(\rho)$ on $\mathcal{C}^{\pm}_{\frac{n-1}{n}}$ grows exponentially. When $n = 2$, the solution exists when $0 < d < 1$ only.*

3. *When $d < 0$, the hypersurface $\mathcal{D}_{\frac{n-1}{n}}$ is complete and symmetric with respect to the slice $\mathbb{H}^n \times \{0\}$. It has self-intersections along a sphere in $\mathbb{H}^n \times \{0\}$. The parts $\mathcal{D}^{\pm}_{\frac{n-1}{n}} := \mathcal{D}_{\frac{n-1}{n}} \cap \mathbb{H}^n \times \mathbb{R}_{\pm}$ are vertical graphs above the exterior of a ball $B(0, a)$, for some constant $a > 0$ depending on d. The height function $\lambda(\rho)$ on $\mathcal{D}^{\pm}_{\frac{n-1}{n}}$ grows exponentially.*

The asymptotic behaviour of the height function when ρ tends to infinity is as follows.

$$\begin{cases} \text{For } \ n = 2, \ \lambda(\rho) \sim \frac{e^{\rho/2}}{\sqrt{1-d}}. \\ \text{For } \ n = 3, \ \lambda(\rho) \sim \frac{1}{2\sqrt{2}} \int^{\rho} \frac{e^t}{\sqrt{t}}\, dt. \\ \text{For } \ n \geq 4, \ \lambda(\rho) \sim a(n) e^{b(n)t}, \quad \text{for some positive constants } \ a(n), b(n). \end{cases}$$

The generating curves are obtained by symmetries from the curves (=) (standing for $H = \frac{n-1}{n}$) which appear in Figures 1-3.

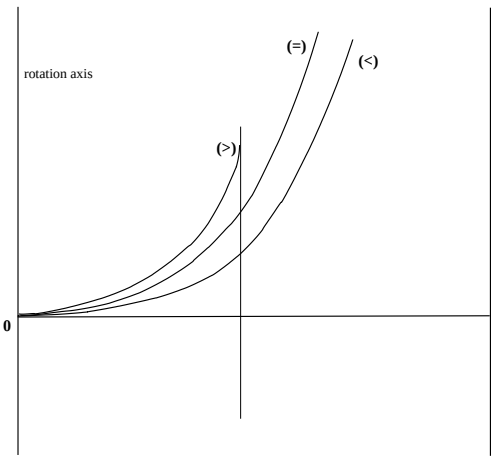

Figure 1: Case $d = 0$

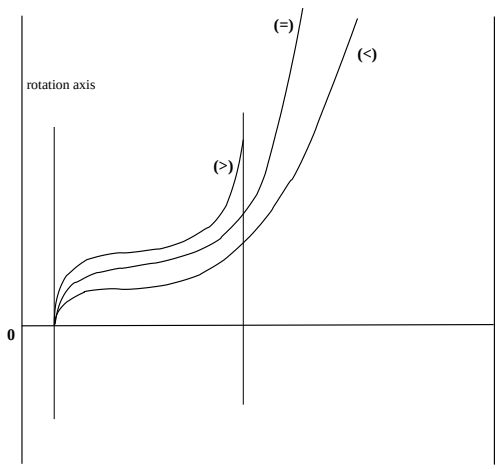

Figure 2: Case $d > 0$

Remark. When $n = 2$ the asymptotic growth depends on the value of the integration contant d.

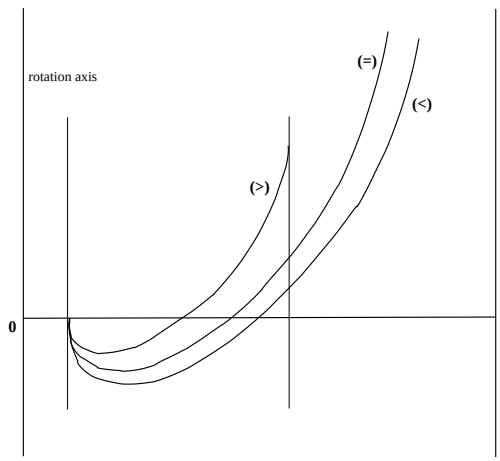

Figure 3: Case $d < 0$

Theorem 2.2 (Rotation H-hypersurfaces with $0 < H < \frac{n-1}{n}$)

1. *When $d = 0$, the hypersurface $\mathcal{S}_H$ is a simply-connected entire vertical graph above $\mathbb{H}^n \times \{0\}$, tangent to the slice at 0, generated by a strictly convex curve. The height function $\lambda(\rho)$ on $\mathcal{S}_H$ grows linearly.*

2. *When $d > 0$, the hypersurface $\mathcal{C}_H$ is a complete embedded cylinder, symmetric with respect to the slice $\mathbb{H}^n \times \{0\}$. The parts $\mathcal{C}_H^{\pm} := \mathcal{C}_H \cap \mathbb{H}^n \times \mathbb{R}_{\pm}$ are vertical graphs above the exterior of a ball $B(0, a)$, for some constant $a > 0$ depending on H and d. The height function $\lambda(\rho)$ on $\mathcal{C}_H^{\pm}$ grows linearly.*

3. *When $d < 0$, the hypersurface $\mathcal{D}_H$ is complete and symmetric with respect to the slice $\mathbb{H}^n \times \{0\}$. It has self-intersections along a sphere in $\mathbb{H}^n \times \{0\}$. The parts $\mathcal{D}_H^{\pm} := \mathcal{D}_H \cap \mathbb{H}^n \times \mathbb{R}_{\pm}$ are vertical graphs above the exterior of a ball $B(0, a)$, for some constant $a > 0$ depending on H and d. The height function $\lambda(\rho)$ on $\mathcal{D}_H^{\pm}$ grows linearly.*

The asymptotic behaviour of the height function when ρ tends to infinity is given by

$$\lambda(\rho) \sim \frac{\frac{nH}{n-1}}{\sqrt{1 - (\frac{nH}{n-1})^2}}\, \rho.$$

The generating curves are obtained by symmetries from the curves $(<)$ (standing for $H < \frac{n-1}{n}$) which appear in Figures 1-3.

Theorem 2.3 (Rotation H-hypersurfaces with $H > \frac{n-1}{n}$)

1. *When $d = 0$, the hypersurface $\mathcal{K}_H$ is compact and diffeomorphic to an n-dimensional sphere. It is generated by a compact, simple, strictly convex curve.*

2. *When $d > 0$, the hypersurface $\mathcal{U}_H$ is complete, embedded and periodic in the $\mathbb{R}$-direction. It looks like an unduloid and is contained in a domain of the form $B(0,b) \setminus B(0,a) \times \mathbb{R}$, for some constants $0 < a < b$, depending on H and d.*

3. *When $d < 0$, the hypersurface $\mathcal{N}_H$ is complete and periodic in the $\mathbb{R}$-direction. It has self-intersections, looks like a nodoid and is contained in a domain of the form $B(0,b) \setminus B(0,a) \times \mathbb{R}$, for some constants $0 < a < b$ depending on H and d.*

The generating curves are obtained by symmetries from the curves (>) (standing for $H > \frac{n-1}{n}$) which appear in Figures 1-3.

Remarks

1. Constant mean curvature rotation hypersurfaces with $H > \frac{n-1}{n}$ were obtained in [4] and [6].

2. The hypersurfaces $\mathcal{S}_H$ and the upper (lower) halves of the cylinders $\mathcal{C}_H$ in Theorems 2.1 and 2.2 are stable (as vertical graphs).

2.2 Proofs of Theorem 2.1 - 2.3

The proofs follow from an analysis of the asymptotic behaviour of $I_m(t)$ (Formula (2.2)) when t goes to infinity and from an analysis of the signs of the functions $nHI_{n-1}(t) + d$, $M_{H,d}(t)$ and $P_{H,d}(t)$ (Formulas (2.3)), using the tables which appear below.

When $d = 0$, using (2.1) one can show that $\ddot{\lambda} > 0$ and conclude that the generating curve is strictly convex. When $d \leq 0$, the formula for $\ddot{\lambda}$ also shows that the curvature extends continuously at the vertical points.

Proof of Theorem 2.1

Assume $H = \frac{n-1}{n}$.

When $d = 0$, the functions $M_{H,0}$ and $P_{H,0}$ are non-negative and vanish at $t = 0$. Near 0 we have $Q_{H,0}(t) \sim Ht$ and hence $\lambda_{H,0}(\rho) = \int_0^\rho Q_{H,0}(t)\,dt \sim \frac{H}{2}\rho^2$.

When $d > 0$, the function $Q_{H,d}$ exists on an interval $]a_{H,d}, \infty[$ for some constant $a_{H,d} > 0$ and the integral $\int_{a_{H,d}}^\rho Q_{H,d}(t)\,dt$ converges at $a_{H,d}$.

When $d < 0$, the function $Q_{H,d}$ exists on an interval $]\alpha_{H,d}, \infty[$ for some constant $\alpha_{H,d} > 0$, changes sign from negative to positive, the integral $\int_{\alpha_{H,d}}^{\rho} Q_{H,d}(t)\, dt$ converges at $\alpha_{H,d}$ and the curve has a vertical tangent at this point. The generating curve can be extended by symmetry to a complete curve with one self-intersection.

Using the recurrence relations for the functions $I_m(t)$ one can determine their asymptotic behaviour at infinity and deduce the precise exponential growth of the height function $\lambda(\rho)$.

□

Proof of Theorem 2.2

Assume $0 < H < \frac{n-1}{n}$.

When $d = 0$, the functions $M_{H,0}$ and $P_{H,0}$ are non-negative and vanish at $t = 0$. Near 0 we have $Q_{H,0}(t) \sim Ht$ and hence $\lambda_{H,0}(\rho) = \int_0^{\rho} Q_{H,0}(t)\, dt \sim \frac{H}{2}\rho^2$.

When $d > 0$, the function $Q_{H,d}$ exists on an interval $]a_{H,d}, \infty[$ for some constant $a_{H,d} > 0$ and the integral $\int_{a_{H,d}}^{\rho} Q_{H,d}(t)\, dt$ converges at $a_{H,d}$.

When $d < 0$, the function $Q_{H,d}$ changes sign from negative to positive, exists on an interval $]\alpha_{H,d}, \infty[$ for some constant $\alpha_{H,d} > 0$, the integral $\int_{\alpha_{H,d}}^{\rho} Q_{H,d}(t)\, dt$ converges at $\alpha_{H,d}$ and the generating curve has a vertical tangent at this point. The generating curve can be extended by symmetry to a complete curve with one self-intersection.

Using the recurrence relations for the functions $I_m(t)$ one can determine their asymptotic behaviour at infinity and deduce the precise linear growth of the height function $\lambda(\rho)$.

□

Proof of Theorem 2.3

Assume $H > \frac{n-1}{n}$.

When $d = 0$, $Q_{H,0}(t)$ exists on some interval $]0, a_{H,0}[$ for some positive $a_{H,0}$ and the integral $\lambda_{H,0}(\rho) = \int_0^{\rho} Q_{H,0}(t)\, dt$ converges at 0 and at $a_{H,0}$. The generating curve has a horizontal tangent at 0 and a vertical tangent at a_H. It can be extended by symmetries to a closed embedded convex curve.

When $d > 0$, the function $Q_{H,d}(t)$ exists on an interval $]b_{H,d}, c_{H,d}[$ for some constants $0 < b_{H,d} < c_{H,d}$ and the integral converges at the limits of this interval. The generating curve at these points is vertical. It can be extended by symmetry to a complete embedded periodic curve (unduloid).

When $d < 0$, the function $Q_{H,d}(t)$ exists on an interval $]\beta_{H,d}, \gamma_{H,d}[$ for some constants $0 < \beta_{H,d} < \gamma_{H,d}$, changes sign from negative to positive and the integral converges at the limits of this interval. The generating curve at these points is vertical. The generating curve can extended by symmetries to a complete periodic curve with self-intersections (nodoid).

□

Remark. We note that the integrand $Q_{H,d}(t)$ in (2.4) is an increasing function of H for t and d fixed. This fact provides the relative positions of the curves $\lambda_{H,d}(\rho)$ when ρ and d are fixed. The curve corresponding to $H > \frac{n-1}{n}$ is above the curve corresponding to $H = \frac{n-1}{n}$ which is above the curve corresponding to $H < \frac{n-1}{n}$. See Figures 1 to 3.

The above sketches of proof can be completed using the details below.

• We have the following relations for the functions I_m,

$$\begin{cases} m=0 & I_0(t) & = & t, \\ m=1 & I_1(t) & = & \cosh(t)-1, \\ m=2 & 2I_2(t) & = & \sinh(t)\cosh(t)-t, \\ m=3 & 3I_3(t) & = & \sinh^2(t)\cosh(t)-2(\cosh(t)-1), \\ m\geq 2 & mI_m(t) & = & \sinh^{m-1}(t)\cosh(t)-(m-1)I_{m-2}(t). \end{cases} \tag{2.5}$$

For $m \geq 5$, the asymptotic behavior of $I_m(t)$ near infinity is given by,

$$\begin{cases} mI_m(t) & = & \sinh^{m-3}(t)\cosh(t)\big(\sinh^2(t)-\frac{m-1}{m-2}\big)+O(e^{(m-4)t}), \\ mI_m(t) & = & \sinh^{m-1}(t)\cosh(t)\big(1+O(e^{-2t})\big). \end{cases} \tag{2.6}$$

The same holds for $m = 4$ with remainder term $O(t)$ in the first relation.

• The derivative of $P_{H,d}$ is positive for t positive. The behaviour of the function $P_{H,d}(t)$ is summarized in the following table.

$n \geq 2$	$0 < H$		
t	0		∞
$\partial_t P_{H,d}$		$+$	
$P_{H,d}(t)$	d	$\nearrow$	∞

• The derivative of $M_{H,d}$ is given by $\partial_t M_{H,d}(t) = (n-1)\sinh^{n-1}(t)\big(\coth(t)-\frac{nH}{n-1}\big)$. For $H > \frac{n-1}{n}$, we denote by C_H the number such that $\coth(C_H) = \frac{nH}{n-1}$. The behaviour of the function $M_{H,d}(t)$ is summarized in the following tables.

$n=2$	$0<H\leq\frac{1}{2}$					$H>\frac{1}{2}$				
t	0				∞	0		C_H		∞
$\partial_t M_{H,d}$		$+$					$+$	0	$-$	
$M_{H,d}(t)$	$-d$		$\nearrow$	$\begin{cases}\infty, & H<\frac{1}{2}\\ 1-d, & H=\frac{1}{2}\end{cases}$		$-d$	$\nearrow$	$f_H(d)$	$\searrow$	$-\infty$

$n\geq 3$	$0<H\leq\frac{n-1}{n}$			$H>\frac{n-1}{n}$				
t	0		∞	0		C_H		∞
$\partial_t M_{H,d}$		$+$			$+$	0	$-$	
$M_{H,d}(t)$	$-d$	$\nearrow$	∞	$-d$	$\nearrow$	$f_H(d)$	$\searrow$	$-\infty$

where $f_H(d) := M_{H,d}(C_H) = \sinh^{n-1}(C_H) - nHI_{n-1}(C_H) - d$.

The signs and zeroes of the functions $M_{H,d}(t)$ and $P_{H,d}(t)$ when $d \neq 0$ are summarized in the following charts, together with the existence domain of the function $Q_{H,d}$.

When $d > 0$, we have

$n=2$	$0<H<\frac{1}{2}$, $H=\frac{1}{2}$,		$0<d$ $0<d<1$,		
$n\geq 3$		$0<H\leq\frac{n-1}{n}$ $0<d$			
t	0		$a_{H,d}$		∞
$M_{H,d}$		$-$	0	$+$	
$P_{H,d}$		$+$	$+$	$+$	
$Q_{H,d}$		$\nexists$	$+\infty$	$\exists$	

$n\geq 2$	$H>\frac{n-1}{n}$ $0<d<D_H$					
t	0	$b_{H,d}$	C_H	$c_{H,d}$		∞
$M_{H,d}$	$-$	0	$+$	0	$-$	
$P_{H,d}$	$+$		$+$		$+$	
$Q_{H,d}$	$\nexists$	$+\infty$	$\exists$	$+\infty$	$\nexists$	

where $D_H := \sinh^{n-1}(C_H) - nHI_{n-1}(C_H)$.

When $d < 0$, we have the following tables.

$n \geq 2$	$0 < H \leq \frac{n-1}{n}$ $d < 0$			
t	0		$\alpha_{H,d}$		∞
$M_{H,d}$		$+$		$+$	
$P_{H,d}$		$-$	0	$+$	
$Q_{H,d}$		$\nexists$	$-\infty$	$\exists$	

Note that the function $Q_{H,d}$ changes sign from negative to positive when t goes from $\alpha_{H,d}$ to infinity.

$n \geq 2$	$H > \frac{n-1}{n}$ $d < 0$					
t	0		$\gamma_{H,d}$		$\beta_{H,d}$		∞
$M_{H,d}$		$+$	$+$	$+$	0	$-$	
$P_{H,d}$		$-$	0	$+$	$+$	$+$	
$Q_{H,d}$		$\nexists$	$-\infty$	$\exists$	$+\infty$	$\nexists$	

Note that the function $Q_{H,d}$ changes sign from negative to positive when t goes from $\gamma_{H,d}$ to $\beta_{H,d}$.

2.3 Translation invariant H-hypersurfaces in $\mathbb{H}^n \times \mathbb{R}$

2.3.1 Translation hypersurfaces

• **Definitions and Notations.** We consider γ a geodesic through 0 in $\mathbb{H}^n$ and the totally geodesic vertical plane $\mathbb{V} = \gamma \times \mathbb{R} = \{(\gamma(\rho), t) \mid (\rho, t) \in \mathbb{R} \times \mathbb{R}\}$ where ρ is the signed hyperbolic distance to 0 on γ.

Take $\mathbb{P}$ a totally geodesic hyperplane in $\mathbb{H}^n$, orthogonal to γ at 0. We consider the hyperbolic translations with respect to the geodesics δ through 0 in $\mathbb{P}$. We shall refer to these translations as translations with respect to $\mathbb{P}$. These isometries of $\mathbb{H}^n$ extend "slice-wise" to isometries of $\mathbb{H}^n \times \mathbb{R}$.

In the vertical plane $\mathbb{V}$, we consider the curve $c(\rho) := \big(\tanh(\rho/2), \mu(\rho)\big)$.

In $\mathbb{H}^n \times \{\mu(\rho)\}$, we translate the point $c(\rho)$ by the translations with respect to $\mathbb{P} \times \{\mu(\rho)\}$ and we get the equidistant hypersurface $\mathbb{P}_\rho$ passing through $c(\rho)$, at distance ρ from $\mathbb{P} \times \{\mu(\rho)\}$. The curve c then generates a *translation hypersurface* $M = \cup_\rho \mathbb{P}_\rho$ in $\mathbb{H}^n \times \mathbb{R}$.

• **Principal curvatures.** The principal directions of curvature of M are the tangent to the curve c in $\mathbb{V}$ and the directions tangent to $\mathbb{P}_\rho$. The corresponding principal curvatures with respect to the unit normal pointing upwards are given by

$$\begin{cases} k_{\mathbb{V}} &= \ddot{\mu}(\rho)\big(1+\dot{\mu}^2(\rho)\big)^{-3/2}, \\ k_{\mathbb{P}} &= \dot{\mu}(\rho)\big(1+\dot{\mu}^2(\rho)\big)^{-1/2}\tanh(\rho). \end{cases}$$

The first equality comes from the fact that $\mathbb{V}$ is totally geodesic and flat. The second equality follows from the fact that $\mathbb{P}_\rho$ is totally umbilic and at distance ρ from $\mathbb{P} \times \{\mu(\rho)\}$ in $\mathbb{H}^n \times \{\mu(\rho)\}$.

• **Mean curvature.** The mean curvature of the translation hypersurface M associated with μ is given by

$$nH(\rho)\cosh^{n-1}(\rho) = \partial_\rho\Big(\cosh^{n-1}(\rho)\dot{\mu}(\rho)\big(1+\dot{\mu}^2(\rho)\big)^{-1/2}\Big). \tag{2.7}$$

2.3.2 Constant mean curvature translation hypersurfaces

We may assume that $H \geq 0$. The generating curves of translation hypersurfaces with constant mean curvature H are given by the differential equation

$$\dot{\mu}(\rho)\big(1+\dot{\mu}^2(\rho)\big)^{-1/2}\cosh^{n-1}(\rho) = nH\int_0^\rho \cosh^{n-1}(t)\,dt + d \tag{2.8}$$

for some integration constant d.

Minimal translation hypersurfaces have been studied in [7, 9] in dimension 2 and in [2] in higher dimensions. Constant mean curvature ($H \neq 0$) translation hypersurfaces have been treated in [7] in dimension 2. The purpose of the present section is to investigate the higher dimensional translation H-hypersurfaces.

Notations. For later purposes, we introduce some notations.

• For $m \geq 0$, we define the functions

$$J_m(r) := \int_0^r \cosh^m(t)\,dt. \tag{2.9}$$

• For $H > 0$ and $d \in \mathbb{R}$, we introduce the functions,

$$\begin{cases} R_{H,d}(t) &= \cosh^{n-1}(t) - nHJ_{n-1}(t) - d, \\ S_{H,d}(t) &= \cosh^{n-1}(t) + nHJ_{n-1}(t) + d, \\ T_{H,d}(t) &= \big[nHJ_{n-1}(t)+d\big]\big[R_{H,d(t)}S_{H,d}(t)\big]^{-1/2}. \end{cases} \tag{2.10}$$

We note from (2.8) that $\dot{\mu}(t)$ has the sign of $nHJ_{n-1}(t)+d$. It follows that μ is given (up to an additive contant) by

$$\mu_{H,d}(\rho)=\int_{\rho_0}^{\rho}\big[nHJ_{n-1}(t)+d\big]\big[\cosh^{2n-2}(t)-\big(nHJ_{n-1}(t)+d\big)^2\big]^{-1/2}\,dt$$

or, using the above notations,

$$\mu_{H,d}(\rho)=\int_{\rho_0}^{\rho}\big[nHJ_{n-1}(t)+d\big]\big[R_{H,d(t)}\,S_{H,d}(t)\big]^{-1/2}\,dt=\int_{\rho_0}^{\rho}T_{H,d}(t)\,dt\,, \tag{2.11}$$

where the integration interval $[\rho_0,\rho]$ is contained in the interval in which the square root exists. The existence and behaviour of the function $\mu_{H,d}$ depend on the signs of the functions $nHJ_{n-1}(t)+d$, $R_{H,d}(t)$ and $S_{H,d}(t)$.

For $H=\frac{n-1}{n}$, we give a complete description of the corresponding translation H-hypersurfaces. For $0<H<\frac{n-1}{n}$, we prove the existence of a complete non-entire H-graph with infinite boundary data and infinite asymptotic behaviour. The other cases can be treated similarly using the tables below.

Theorem 2.4 (Translation H-hypersurfaces, with $n\geq 3$ and $H=\frac{n-1}{n}$)

1. *When $d=0$, $\mathcal{T}_0$ is a complete embedded smooth hypersurface generated by a compact, simple, strictly convex curve. The hypersurface is symmetric with respect to a horizontal hyperplane and the parts above and below this hyperplane are vertical graphs. The hypersurface also admits a vertical symmetry. The asymptotic boundary of $\mathcal{T}_0$ is topologically a cylinder.*
2. *When $0<d<1$, the hypersurface $\mathcal{T}_d$ is similar to $\mathcal{T}_0$ except that it is not smooth.*
3. *When $d\leq -1$, $\mathcal{T}_d$ is a smooth complete immersed hypersurface with self-intersections and horizontal symmetries. The asymptotic boundary of $\mathcal{T}_d$ is topologically a cylinder.*
4. *When $-1<d<0$, the hypersurface $\mathcal{T}_d$ looks like $\mathcal{T}_{-1}$ except that it is not smooth.*

Remark. When $d\geq 1$, the differential equation (2.8) does not have solutions.

Theorem 2.5 (Complete H-graph with infinite boundary data)

There exists a complete translation hypersurface $\mathcal{T}_H$, with $0<H<\frac{n-1}{n}$, such that

1. *$\mathcal{T}_H$ is a complete monotone vertical H-graph over the non mean convex side of an equidistant hypersurface $\Gamma\subset\mathbb{H}^n$ with mean curvature $\frac{nH}{n-1}$,*
2. *$\mathcal{T}_H$ takes infinite boundary value data on Γ and infinite asymptotic boundary data.*

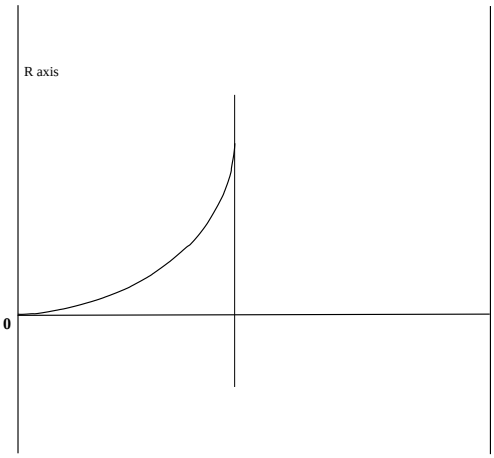

Figure 4: $n \geq 3, H = \frac{n-1}{n}, d = 0$

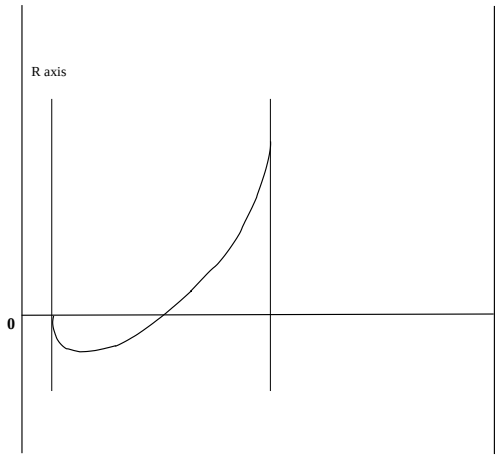

Figure 5: $n \geq 3, H = \frac{n-1}{n}, d < -1$

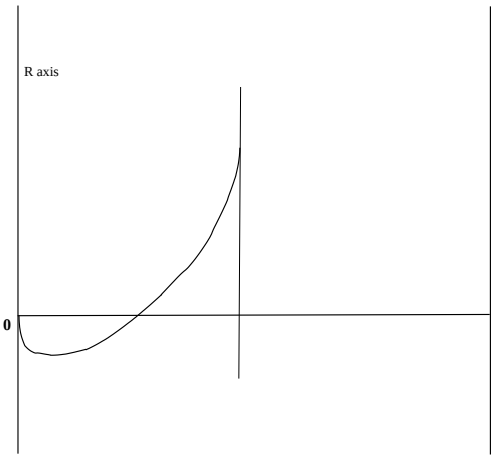

Figure 6: $n \geq 3, H = \frac{n-1}{n}, d = -1$

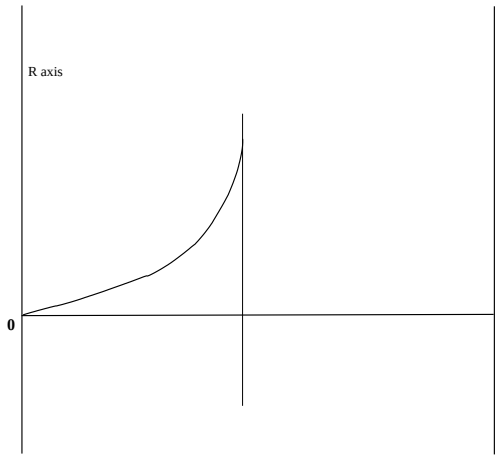

Figure 7: $n \geq 3, H = \frac{n-1}{n}, 0 < d < 1$

2.4 Proof of Theorem 2.4

The proof of Theorem 2.4 follows from an analysis of the asymptotic behaviour of the functions $J_m(t)$ (Formula (2.9)) when t goes to infinity and from an analysis of the signs of the functions $R_{H,d}$ and $S_{H,d}$ (Formulas (2.10)) depending on the signs of $H - \frac{n-1}{n}$ and d.

• We have the relations

$$\left\{ \begin{array}{lcl} J_0(t) & = & t, \\ J_1(t) & = & \sinh(t), \\ 2J_2(t) & = & \sinh(t)\cosh(t) + t, \\ 3J_3(t) & = & \sinh(t)\cosh^2(t) + 2J_1(t), \\ mJ_m(t) & = & \sinh(t)\cosh^{m-1}(t) + (m-1)J_{m-2}(t), \quad \text{for} \quad m \geq 3. \end{array} \right. \tag{2.12}$$

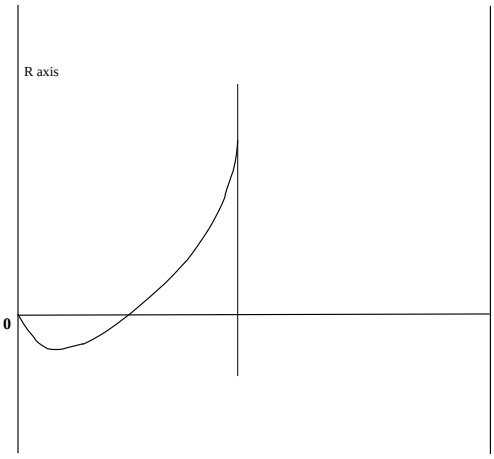

Figure 8: $n \geq 3, H = \frac{n-1}{n}, -1 < d < 0$

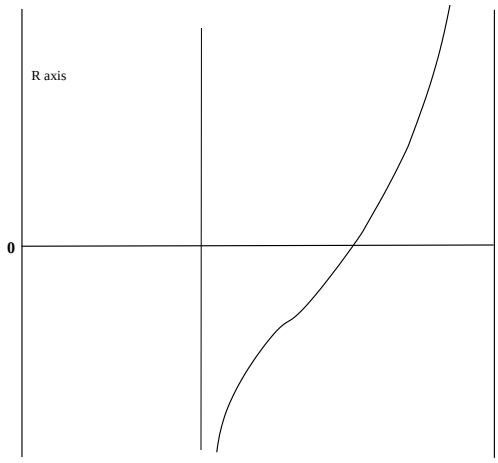

Figure 9: $n \geq 2, H < \frac{n-1}{n}$

These relations give us the asymptotic behaviour of the functions $J_m(t)$ when t tends to infinity. In particular,

$$mJ_m(t) = \sinh(t)\cosh^{m-1}(t) + \frac{m-1}{m-2}\sinh(t)\cosh^{m-3}(t) + O(e^{(m-4)t}), \quad \text{for} \quad m \geq 5$$

with the remainder term replaced by $O(t)$ when $m = 4$.

• **The function** $S_{H,d}(t)$

For all $H > 0$, the function $S_{H,d}$ increases from $1 + d$ to $+\infty$. Its behaviour is summarized in the following table.

	Case $0 < H$		$d \geq -1$	
t	0			$+\infty$
$S_{H,d}(t)$	$1 + d \geq 0$		$\nearrow$	$+\infty$

	Case $0 < H$		$d < -1$		
t	0		$\alpha_{H,d}$		$+\infty$
$S_{H,d}(t)$	$1 + d < 0$	$\nearrow$	0	$\nearrow$	$+\infty$

(2.13)

• **The function** $R_{H,d}(t)$

The derivative of $R_{H,d}(t)$ is given by $\partial_t R_{H,d}(t) = (n-1)\cosh^{n-1}(t)[\tanh(t) - \frac{nH}{n-1}]$. For $0 < H < \frac{n-1}{n}$, let t_H be the value such that $\tanh(t_H) = \frac{nH}{n-1}$.

When $H \neq \frac{n-1}{n}$,

$$R_{H,d}(t) \sim \frac{1}{2}(1 - \frac{nH}{n-1})\cosh^{n-2}(t)e^t \quad \text{near} \quad t = +\infty. \tag{2.14}$$

When $H = \frac{n-1}{n}$ and when t tends to $+\infty$, $R_{H,d}(t)$ tends to $-\infty$ for $n \geq 3$ and to $-d$ for $n = 2$.

The behaviour of the function $R_{H,d}(t)$ is summarized in the following table.

	Case	$0 < H < \frac{n-1}{n}$		
t	0	t_H		$+\infty$
$R_{H,d}(t)$	$1-d$ $\searrow$	$R_{H,d}(t_H)$	$\nearrow$	$+\infty$

	Case	$H = \frac{n-1}{n}$	
t	0		$+\infty$
$R_{H,d}(t)$	$1-d$	$\searrow$	$\begin{cases} -\infty, & n \geq 3 \\ -d, & n = 2 \end{cases}$

	Case	$H > \frac{n-1}{n}$	
t	0		$+\infty$
$R_{H,d}(t)$	$1-d$	$\searrow$	$-\infty$

(2.15)

Proof of Theorem 2.4, continued

We now investigate the behaviour of the solution μ to Equation (2.8) when $n \geq 3$ and $H = \frac{n-1}{n}$ (for $n = 2$, see [7]).

According to Table (2.13), the function $S_{H,d}$ increases from $1 + d$ to $+\infty$ and we have to consider two cases, *(i)* $d \geq -1$, in which case $S_{H,d}$ is always non-negative and *(ii)* $d < -1$, in which case $S_{H,d}$ has one zero $\alpha_{H,d}$ such that

$$\cosh^{n-1}(\alpha_{H,d}) + nHJ_{n-1}(\alpha_{H,d}) + d = 0.$$

According to Table (2.15), the function $R_{H,d}$ decreases from $1 - d$ to $\begin{cases} -\infty, & n \geq 3 \\ -d, & n = 2 \end{cases}$, depending on the value of n. It follows that we have two cases, *(i)* $d \geq 1$, in which case the function $R_{H,d}$ is always non-positive and *(ii)* $d < 1$, in which case it has one zero $c_{H,d}$ for $n \geq 3$. When it exists, the zero $c_{H,d}$ satisfies

$$\cosh^{n-1}(c_{H,d}) - nHJ_{n-1}(c_{H,d}) - d = 0.$$

Looking at the equations defining $\alpha_{H,d}$ and $c_{H,d}$ we see that $\alpha_{H,d} < c_{H,d}$ when they both exist.

The behaviour of the function μ is described in the following tables, see also Figures 4 to 8.

Case 1	$H = \frac{n-1}{n}$		$d < -1$		$n \geq 3$	
t	0	$\alpha_{H,d}$		$c_{H,d}$		$+\infty$
$R_{H,d}$	$+$	$+$	$+$	0	$-$	
$S_{H,d}$	$-$	0	$+$	$+$	$+$	
$T_{H,d}$	$\nexists$	$-\infty$	$\exists$	$+\infty$	$\nexists$	

(2.16)

The function μ is given by

$$\mu(\rho) = \int_{\rho_0}^{\rho} T_{H,d}(t)\, dt$$

for $\rho_0, \rho \in [\alpha_{H,d}, c_{H,d}]$ and the integral exists at both limits. Note that the integrand is negative near the lower limit while it is positive near the upper limit.

When $d = 0$, using (2.8) one can show that $\ddot{\mu} > 0$ and conclude that the generating curve is strictly convex. The formula for $\ddot{\mu}$ also shows that the curvature extends continuously at the vertical points.

The generating curve can be extended by symmetry and periodicity to give rise to a complete immersed hypersurface with self-intersections.

Case 2	$H = \frac{n-1}{n}$	$-1 \leq d < 1$	$n \geq 3$	
t	0		$c_{H,d}$	$+\infty$
$R_{H,d}$		$+$	0	$-$
$S_{H,d}$		$+$	$+$	$+$
$T_{H,d}$		$\exists$	$+\infty$	$\nexists$

(2.17)

The function μ is given by

$$\mu(\rho) = \int_{0}^{\rho} T_{H,d}(t)\, dt$$

for $\rho_0, \rho \in [0, c_{H,d}]$ and the integral exists at both ends. Note that the integrand has the sign of d near 0, with $\dot{\mu}(0) = d/\sqrt{1-d^2}$; it is positive near the upper bound with $\dot{\mu}(c_{H,d}) = +\infty$.

When $d = -1$, the original curve has a vertical tangent at 0. It can be extended by symmetry and periodicity to give rise to a complete immersed hypersurface with self-intersections.

When $d = 0$, the curve has a horizontal tangent and is strictly convex (use (2.8)). It can be extended by symmetry as a topological circle and gives rise to a complete embedded surface.

When $d \geq 1$, Equation (2.8) has no solution.

2.5 Proof of Theorem 2.5

Given n and H, such that $0 < H < \frac{n-1}{n}$, consider the function $R_{H,d}(t)$ and choose d_H such that $R_{H,d_H}(t_H) = 0$, where t_H is defined by $\tanh(t_H) = \frac{nH}{n-1}$, *i.e.* $d_H := \cosh^{n-1}(t_H) - nHJ_{n-1}(t_H)$.

It follows that $R_{H,d_H}(t) > 0$ for $t > t_H$ and hence the quantity $nHJ_{n-1}(t) + d_H$ does not change sign for $t > t_H$ and the same is true for $T_{H,d_H}(t)$.

Taking (2.10) into account, we choose $\rho_0 > t_H$ and define the generating curve by Formula (2.11).

We conclude that $\mu(\rho)$ is well-defined and strictly increasing for $\rho > t_H$. Moreover, $\mu(\rho)$ goes to $-\infty$, if $\rho \to t_H^+$. Notice that the mean curvature of the equidistant hypersurface at distance t_H to $\mathbb{P}$ is $\tanh(t_H) = \frac{nH}{n-1}$, by the choice of t_H.

Now recall that if $0 < H < \frac{n-1}{n}$, then $R_{H,d}(t) \sim \frac{1}{2}(1 - \frac{nH}{n-1})\cosh^{n-2}(t)e^t$, as $t \to \infty$. From this it follows that $T_{H,d}(t) = O(1)$, as $t \to \infty$. Thus $\mu(\rho) \to +\infty$, if $\rho \to \infty$. □

Remark. The situation when $n = 2$ is similar although the generating curves are defined on infinite intervals (see Figures 10 to 14). The corresponding surfaces have height functions tending to infinity when ρ tends to infinity. In particular, the surface $\mathcal{T}_0$ is a complete smooth entire graph above $\mathbb{H}^2$.

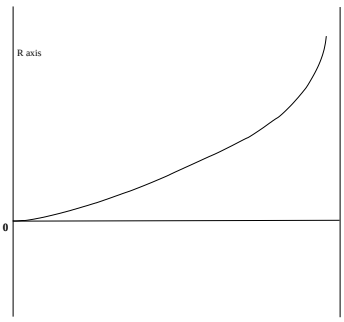

Figure 10: $n = 2, H = \frac{1}{2}$, $d = 0$

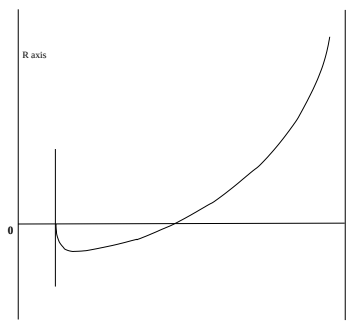

Figure 11: $n = 2, H = \frac{1}{2}$, $d < -1$

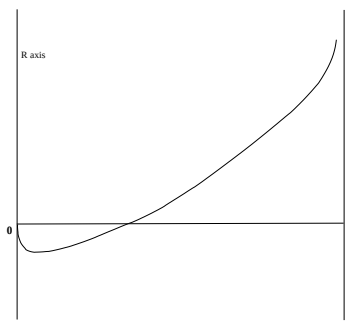

Figure 12: $n = 2, H = \frac{1}{2}$, $d = -1$

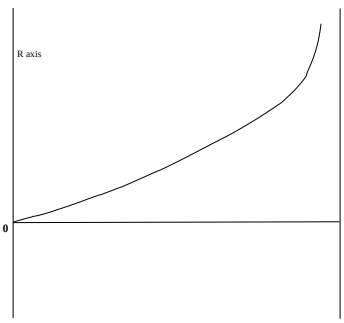

Figure 13: $n = 2, H = \frac{1}{2}$, $0 < d < 1$

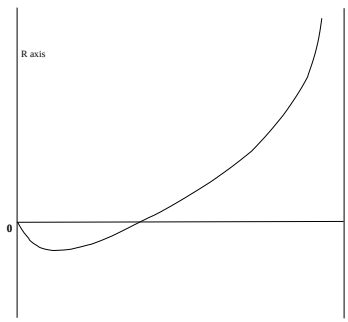

Figure 14: $n = 2, H = \frac{1}{2}$, $-1 < d < 0$

3 Applications, embedded minimal hypersurfaces with boundary contained in a slice

In this section we give some results in which we use the H-hypersurfaces constructed in Section 2 as barriers.

Recall from Section 2 that for $0 < H \leq \frac{n-1}{n}$ and for $d = 0$, there exist simply-connected rotation H-hypersurfaces $\mathcal{S}_H$ which are entire vertical graphs going to infinity at infinity. The unit normal of $\mathcal{S}_H$ points upwards. We call $\check{\mathcal{S}}_H$ the symmetric of $\mathcal{S}_H$ with respect to the slice $\mathbb{H}^n \times \{0\}$. Its unit normal points downwards. We call $\mathcal{C}(\mathcal{S}_H)$ the mean convex side of $\mathcal{S}_H$ (*i.e.* the connected component of the complement of $\mathcal{S}_H$ into which the unit normal points). We consider the set $\mathcal{R}$ of hypersurfaces obtained from $\mathcal{S}_H$ and $\check{S}_H$ by vertical or horizontal translations in $\mathbb{H}^n \times \mathbb{R}$. We denote by $\mathcal{C}(\mathcal{S})$ the mean convex side of a hypersurface $\mathcal{S} \in \mathcal{R}$.

The following Proposition generalizes to higher dimensions the *convex hull lemma* given in [5], Lemma 2.1.

Proposition 3.1 (Convex hull lemma) *Given K a compact subset in $\mathbb{H}^n \times \mathbb{R}$, let $\mathcal{F}_K^H$ denote the subset of domains B in $\mathbb{H}^n \times \mathbb{R}$ which contain K and such that $B = \mathcal{C}(\mathcal{S})$ for some $\mathcal{S} \in \mathcal{R}$. Let M be a compact connected immersed hypersurface in $\mathbb{H}^n \times \mathbb{R}$ with mean curvature H.*

1. *If H is a constant in $]0, \frac{n-1}{n}]$, then $M \subset \mathcal{F}_{\partial M}^H$.*
2. *If $0 < H(x) \leq \frac{n-1}{n}$ for all $x \in M$, then $M \subset \mathcal{F}_{\partial M}^{(n-1)/n}$.*

Proof. Because M is compact, taking into account the asymptotic behaviour of $\mathcal{S}_H$ (see Theorems 2.1-2.2), there exists some vertical translation τ such that $M \subset \mathcal{C}(\tau(\mathcal{S}_H))$ so that the set of hypersurfaces in $\mathcal{R}$ such that $M \subset \mathcal{C}(\mathcal{S})$ is non empty. Take any $\mathcal{S} \in \mathcal{R}$ such that $M \subset \mathcal{C}(\mathcal{S})$ and translate $\mathcal{S}$ horizontally along some geodesic until it touches M at some point p. We claim that p

cannot be an interior point. Indeed, assume that p is an interior point and let p_0 denote the projection of p onto $\mathbb{H}^n$. Both hypersurfaces $\mathcal{S}$ and M would be vertical graphs near p_0, corresponding respectively to functions u, v such that $u(p_0) = v(p_0)$ and $u \leq v$ in a neighborhood of p_0. By the maximum principle, this would imply that $M = \mathcal{S}$ a contradiction. The Proposition follows.

□

In the applications below, we consider a hypersurface Γ in $\mathbb{H}^n$ with the following properties.

$$\begin{cases} \Gamma & \text{is smooth, compact, connected, embedded,} \\ \Gamma = \partial\Omega, & \Omega \text{ a bounded domain in } \mathbb{H}^n, \\ \Gamma & \text{has all its principal curvatures } > 1, \end{cases} \tag{3.18}$$

where the principal curvatures are taken with respect to the unit normal to $\partial\Omega$ pointing inwards.

Given a hypersurface Γ satisfying Properties (3.18), there exists some radius R such that for any point p, the ball $B_{p,R} \subset \mathbb{H}^n$ with radius R is tangent to p at Γ and $\Gamma \subset B_{p,R}$. We denote by

$$\mathcal{S}_{p,+} \quad \text{and} \quad \mathcal{S}_{p,-} \tag{3.19}$$

the two hypersurfaces in $\mathcal{R}$ passing through the sphere $\partial B_{p,R}$ and symmetric with respect to the slice $\mathbb{H}^n \times \{0\}$.

We first prove an existence result for a Dirichlet problem.

Proposition 3.2 *Let $\Omega \subset \mathbb{H}^n \times \{0\}$ be a bounded domain with smooth boundary Γ satisfying (3.18). Then, for any $H, 0 < H \leq \frac{n-1}{n}$, there exists a vertical graph M_Γ over Ω in $\mathbb{H}^n \times \mathbb{R}$, with constant mean curvature H with respect to the upward pointing normal. This means that there exists a function $u : \Omega \to \mathbb{R}$, smooth up to the boundary, such that $u|_\Gamma = 0$, and whose graph $\{(x, u(x)) \mid x \in \Omega\}$ has constant mean curvature H with respect to the unit normal pointing upwards.*

Remark. The graph M_Γ having positive mean curvature with respect to the upward pointing normal, must lie below the slice $\mathbb{H}^n \times \{0\}$. The symmetric $\check{M}_\Gamma$ with respect to the slice lies above the slice and has positive mean curvature with respect to the normal pointing downwards.

Proof of Proposition 3.2

• We first consider the case $H = \frac{n-1}{n}$.

By our assumption on Γ, using the hypersurfaces (3.19) and the Convex hull lemma, Proposition 3.1, any solution to our Dirichlet problem must be contained in $\mathcal{C}(\mathcal{S}_{p,-}) \cap \mathcal{C}(\mathcal{S}_{p,+})$. This provides a priori height estimates and boundary gradient estimates on the solution.

We could use [10] and classical elliptic theory [3], to get existence for our Dirichlet problem when $H = \frac{n-1}{n}$. We shall instead apply [10] directly. Indeed, in our case, the mean curvature H_Γ of Γ satisfies $H_\Gamma > 1 = H\frac{n}{n-1}$, and the Ricci curvature of $\mathbb{H}^n$ satisfies $\mathrm{Ric} = -(n-1) \geq -\frac{n^2}{n-1}H^2$. Theorem 1.4 in [10] states that under theses assumptions there exists a vertical graph over Ω with boundary Γ and constant mean curvature $H = \frac{n-1}{n}$.

• We now consider the case $0 < H \leq \frac{n-1}{n}$.

We use the graphs constructed previously as barriers to obtain a priori height estimates and apply the interior and global gradient estimates of [10] to conclude.

We consider the Dirichlet problem (P_t) for $0 \leq t \leq 1$,

$$\begin{cases} \mathrm{div}\big(\dfrac{\nabla u}{W}\big) & = & t\,(n-1) & \text{in } \Omega \\ u & = & 0 & \text{on } \Gamma \end{cases}$$

where $u \in C^2(\Omega)$ is the height function, ∇u its gradient and $W = (1 + |\nabla u|^2)^{1/2}$, and where the gradient and the divergence are taken with respect to the metric on $\mathbb{H}^n$. This is the equation for *vertical* H-graphs in $\mathbb{H}^n \times \mathbb{R}$. It is elliptic of divergence type.

By the first step, we have obtained the solution u_1 for the Dirichlet problem (P_1). The solution for (P_0) is the trivial solution $u_0 = 0$. By the maximum principle, using the fact that vertical translations are positive isometries for the product metric, and the existence of the solutions u_1 and u_0, we have that any $C^1(\overline{\Omega})$ solution u_t of the Dirichlet problem (P_t) stays above u_1 and below u_0. This yields a priori height and boundary gradient estimates, independently of t and u_t. Global gradient estimates follow Theorem 1.1 and Theorem 3.1 in [10]. We have therefore $C^1(\overline{\Omega})$ a priori estimates independently of t and u_t. The existence of the solution u_t for $0 < t < 1$ now follows from classical elliptic theory, see [3] or Theorem A.7 in [1].

This completes the proof of Proposition 3.2. □

We now generalize to higher dimensions results obtained in [5].

Theorem 3.3 *Let M be an embedded compact connected H-hypersurface in $\mathbb{H}^n \times \mathbb{R}$, with $0 < H \leq \frac{n-1}{n}$. Assume that the boundary Γ is an $(n-1)$-submanifold in $\mathbb{H}^n \times \{0\}$ satisfying (3.18).*

1. *The hypersurface M is either the graph M_Γ given by Proposition 3.2 or its symmetric $\check{M}_\Gamma$.*
2. *Assume furthermore that Γ is symmetric with respect to some hyperbolic hyperplane P in $\mathbb{H}^n \times \{0\}$ and that each connected component of $\Gamma \setminus P$ is a graph above P. Then M is symmetric with respect to the vertical hyperplane $P \times \mathbb{R}$ and each connected component of $M \setminus P \times \mathbb{R}$ is a horizontal graph. In particular, if Γ is an $(n-1)$-sphere, the hypersurface M is part of the rotation surface given by Theorem 2.1.*

Proof of Theorem 3.3.

Let Ω be the bounded domain such that $\Gamma = \partial\Omega$ and let $\mathcal{C} = \overline{\Omega} \times \mathbb{R}$ be the vertical cylinder above $\overline{\Omega}$. We claim that $M \subset \mathcal{C}$ and that $M \cap \mathcal{C} = \Gamma$. Indeed, at each $p \in \Gamma$, we have the hypersurfaces $\mathcal{S}_{p,+}$ and $\mathcal{S}_{p,-}$ given by (3.19). It follows from the Convex hull lemma, Proposition 3.1, that M is in the convex hull of such hypersurfaces and hence that $M \subset \mathcal{C}$ and $M \cap \mathcal{C} = \Gamma$.

By Proposition 3.2, we have two vertical graphs above Ω, $M_+ \subset \mathbb{H}^n \times \mathbb{R}_+$ with constant mean curvature H with respect to the normal pointing downwards and $M_- \subset \mathbb{H}^n \times \mathbb{R}_-$ with constant mean curvature H with respect to the normal pointing upwards.

We claim that M is a vertical graph contained either in $\mathbb{H}^n \times \mathbb{R}_+$ or in $\mathbb{H}^n \times \mathbb{R}_-$. If not, making reflexions with respect to slices $\mathbb{H}^n \times \{t\}$ starting from t_+ the highest height on M we would obtain a contradiction by the maximum principle. If M were not contained in one of the half-spaces, we would have highest and lowest interior points at which the normal would point downwards, *resp.* upwards by the maximum principle.

We claim that $M = M_+$ or $M = M_-$. Assume that $M \subset \mathbb{H}^n \times \mathbb{R}_+$ (the proof is similar if M is contained in the lower half-space). Translating M_+ vertically upwards very far and then coming down, we see that $\tau(M_+)$ cannot touch M before the boundaries coincide (maximum principle). It follows that M must be below M_+. Doing the same thing with M, we see that M must be above M_+. It follows finally that $M = M_+$.
Assume now that Γ is symmetric with respect to a hyperbolic hyperplane P and assume that each connected component of $\Gamma \setminus P$ is a horizontal graph. We can then use Alexandrov Reflection Principle in vertical hyperplanes $P_t \times \mathbb{R}$ in ambient space, obtained by applying horizontal translations along geodesics orthogonal to P, to the vertical hyperplane $P \times \mathbb{R}$ of symmetry of Γ, and conclude that M is symmetric with respect to $P \times \mathbb{R}$. Moreover, Alexandrov Reflection Principle ensures that each connected component of $M \setminus P \times \mathbb{R}$ is a horizontal graph.

When Γ is an $(n-1)$-sphere, we can apply the preceding result to prove that M is rotationally symmetric.

□

Recall from [2] that the height of the family of minimal catenoids in $\mathbb{H}^n \times \mathbb{R}$ is $\frac{\pi}{n-1}$.

Theorem 3.4 *Let Γ satisfy (3.18). Consider two copies of Γ in different slices $\Gamma_+ = \Gamma \times \{a\}$ and $\Gamma_- = \Gamma \times \{-a\}$ for some $a > 0$. Let M be a compact connected embedded H-hypersurface such that $\partial M = \Gamma_+ \cup \Gamma_-$, with $0 < H \leq \frac{n-1}{n}$. Assume that $2a \geq \frac{\pi}{n-1}$.*

1. *Assume that Γ is symmetric with respect to a hyperbolic hyperplane P and that each connected component of $\Gamma \setminus P$ is a graph above P. Then M is symmetric with respect to the vertical hyperplane $P \times \mathbb{R}$ and each connected component of $M \setminus P \times \mathbb{R}$ is a horizontal graph.*

2. *Assume that Γ is an $(n-1)$-sphere. Then M is part of the complete embedded rotation hypersurface given by Theorem 2.1 and 2.2 and containing Γ. It follows that M is symmetric with respect to the slice $\mathbb{H}^n \times \{0\}$ and the parts of M above and below the slice of symmetry are vertical graphs.*

Proof of Theorem 3.4.

Let $\Omega_+ = \Omega \times \{a\}$ and $\Omega_- = \Omega \times \{-a\}$. By the Convex hull Lemma, Proposition 3.1, using the hypersurfaces given by (3.19) we have that $M \cap \overline{\mathrm{ext}(\Omega_+)} = \Gamma_+$ and $M \cap \overline{\mathrm{ext}(\Omega_-)} = \Gamma_-$.

We claim that $M \cap (\overline{\Omega} \times \mathbb{R}) = \Gamma_+ \cap \Gamma_-$. Let $M_{\Gamma,a}$ be the graph above $\overline{\Omega_+}$ contained in $\mathbb{H}^n \times [a, \infty[$ and $M_{\Gamma,-a}$ be the graph below $\overline{\Omega_-}$ contained in $\mathbb{H}^n \times]-\infty, a]$, given by Theorem 3.3.

Consider $\widetilde{M} = M_{\Gamma,a} \cap M \cap M_{\Gamma,-a}$ oriented by the mean curvature vector of M by continuity. Take the family of (minimal) catenoids symmetric with respect to $\mathbb{H}^n \times \{0\}$ with rotation axis some $\{\bullet\} \times \mathbb{R}$. Coming from infinity with such catenoids, using the assumption that $2a \geq \frac{\pi}{n-1}$ and the fact that the catenoids have height $< \frac{\pi}{n-1}$, we see that one catenoid will eventually touch $\widetilde{M}$ at some interior point in M. This implies that the normal to M at this point is the same as the normal to the catenoid at the same point (maximum principle) and hence that the normal to M points inside $\widetilde{M}$.

Assume that $M \cap (\Omega \times \{a\}) \neq \emptyset$ (*resp.* that $M \cap (\Omega \times \{-a\}) \neq \emptyset$). Then at the highest point of M the normal would be pointing upwards (*resp.* downwards) and we would get a contradiction with the maximum principle by considering the horizontal slice (a minimal hypersurface) at this point.

Finally, $M \cap (\overline{\Omega} \times \mathbb{R}) = \Gamma_+ \cap \Gamma_-$ and the normal to M points inside $M \cup \Omega_+ \cup \Omega_-$.

To conclude, we use Alexandrov Reflection Principle in vertical hyperplanes $P_t \times \mathbb{R}$ in ambient space, obtained by applying horizontal translations along the horizontal geodesic orthogonal to P, to the hyperplane $P \times \mathbb{R}$ of symmetry of Γ. We conclude that M is symmetric about $P \times \mathbb{R}$ and that each connected component of $M \setminus P \times \mathbb{R}$ is a horizontal graph. This complete the proof of the first statement in the theorem.

If Γ is spherical then M is a rotation hypersurface. As the mean curvature vector points into the region of ambient space that contains the axes, by the geometric classification of the rotation H-hypersurfaces with constant mean curvature $H \leq (n-1)/n$ given by Theorems 2.1 and 2.2, it follows that M is part of a complete embedded rotation hypersurface $\overline{M}$. It follows that $\overline{M}$ has a slice of symmetry at $\mathbb{H}^n \times \{0\}$ and each connected component of $\overline{M}$ above and below $t = 0$ is a complete vertical graph over the exterior of a round ball in $t = 0$.

□

A Vertical flux formula in $\mathbb{H}^n \times \mathbb{R}$

Let $f : M \looparrowright \widehat{M} := \mathbb{H}^n \times \mathbb{R}$ be an isometric immersion. Let h denote the function $h : \widehat{M} \to \mathbb{R}$, such that $h(x,t) = t$ and let $h_M = h|_M$ be the restriction of the function h to the hypersurface M, *i.e.* the height function of M. We let $\widehat{g}$ be the (product) metric on $\widehat{M}$ and Δ_M be the (non-positive) Laplacian on M, for the induced metric $g := f^*\widehat{g}$.

Proposition A.1 *With the above notations we have*

$$\Delta_M h_M = n\widehat{g}(\overrightarrow{H}, \partial_t)$$

where $\overrightarrow{H}$ is the (normalized) mean curvature vector of the immersion and ∂_t the vertical vector-field along $\mathbb{R}$.

Remark. When f admits a unit normal field N_M the above formula boils down to $\Delta_M h_M = nHv_M$ where H is the (normalized) mean curvature in the direction N_M and v_M the vertical component of N_M, $v_M := \widehat{g}(N_M, \partial_t)$.

Proof. Take a local orthonormal frame $\{E_i\}_{i=1}^n$ for M near a point $m \in M$ and extend it locally in a neighborhood of m in $\widehat{M}$. Then

$$\begin{aligned}
\Delta_M h_M &= \sum_{i=1}^{n} \big\{ \big(E_i \cdot (E_i \cdot h_M)\big) - (D_{E_i} E_i) \cdot h_M \big\} \\
&= \sum_{i=1}^{n} \big\{ E_i \cdot \big(dh_M(E_i)\big) - dh_M(D_{E_i} E_i) \big\} \\
&= \sum_{i=1}^{n} \big\{ E_i \cdot \big(dh(E_i)\big) - dh(D_{E_i} E_i) \big\} \\
&= \sum_{i=1}^{n} \big\{ (\widehat{D}_{E_i} dh)(E_i) + dh(\widehat{D}_{E_i} E_i) - dh(D_{E_i} E_i) \big\}.
\end{aligned}$$

In the product space $\widehat{M} = \mathbb{H}^n \times \mathbb{R}$, we have $\widehat{D}_E dh = 0$ for all $E \in \mathcal{X}(\widehat{M})$. It follows that

$$\Delta_M h_M = \sum_{i=1}^{n} dh(\widehat{D}_{E_i} E_i - D_{E_i} E_i) = \sum_{i=1}^{n} dh(A(E_i, E_i))$$

where A is the second fundamental form of the immersion. Finally,

$$\Delta_M h_M = dh(\mathrm{Tr}(A)) = n dh(\overrightarrow{H})$$

which is the formula in the Theorem.

□

Corollary A.2 *Let Ω be a compact domain on M with unit inner normal $\nu_{\partial\Omega}$ to $\partial\Omega$ in Ω. Then*

$$\int_\Omega \Delta_M h_M \, d\mu_M = -\int_{\partial\Omega} dh_M(\nu_{\partial\Omega}) \, d\sigma_{\partial\Omega} = n \int_\Omega \widehat{g}(\overrightarrow{H}, \partial_t) \, d\mu_M.$$

Proof. Divergence Theorem.

□

Applications to rotation H-hypersurfaces

Let us consider a rotation hypersurface M given by the parametrization

$$X(\rho, \xi) = \big(\tanh(\rho/2)\xi, \lambda(\rho)\big)$$

with $\rho > 0$ and $\xi \in S^{n-1}$ and choose the unit normal pointing upwards.

Consider the domain

$$\Omega(\rho_0, \rho) := X([\rho_0, \rho] \times S^{n-1}) \subset M.$$

We have

$$X_\rho(\rho,\xi)\big(\frac{\xi}{2\cosh^2(\rho/2)}, \dot\lambda(\rho)\big),$$

$$v_M(\rho,\xi) = (1+\dot\lambda^2(\rho))^{-1/2},$$

$$d\mu_M = (1+\dot\lambda^2(\rho))^{1/2}\sinh^{n-1}(\rho)\,d\rho\,d\mu_S,$$

$$\nu_{\partial\Omega(\rho_0,\rho)(X(\rho,\xi))} = (1+\dot\lambda^2(\rho))^{-1/2}X_\rho(\rho,\xi),$$

$$d\sigma_{X(\{\rho\}\times S^{n-1})} = \sinh^{n-1}(\rho)\,d\mu_S.$$

The above Corollary applied to $\Omega(\rho_0,\rho)$ gives

$$-\mathrm{Vol}(S^{n-1})\sinh^{n-1}(t)\dot\lambda(t)(1+\dot\lambda^2(t))^{-1/2}\Big|_{\rho_0}^{\rho} = -n\mathrm{Vol}(S^{n-1})\int_{\rho_0}^{\rho} H(t)\sinh^{n-1}(t)\,dt.$$

Looking for rotation surfaces with *constant mean curvature* H we find

$$\sinh^{n-1}(\rho)\dot\lambda(\rho)(1+\dot\lambda^2(\rho))^{-1/2} = nH\int_{\rho_0}^{\rho}\sinh^{n-1}(t)\,dt + F(\rho_0)$$

where the constant $F(\rho_0) := \sinh^{n-1}(\rho_0)\dot\lambda(\rho_0)(1+\dot\lambda^2(\rho_0))^{-1/2}$ is the flux through $X(\{\rho_0\}\times S^{n-1})$.

References

[1] Barbosa, J. L.; Sa Earp, R., *Prescribed mean curvature hypersurfaces in $\mathbb{H}^{n+1}$ with planar boundary, II.* Séminaire de théorie spectrale et géométrie, Grenoble, 16 (1998), 43–79.

[2] Bérard, P.; Sa Earp, R., *Minimal hypersurfaces in $\mathbb{H}^n \times \mathbb{R}$, total curvature and index*, (2008) (arXiv:0808.3838).

[3] Gilbarg, D.; S. Trudinger, N., *Elliptic partial differential equations of second order.* Springer-Verlag, Berlin, second edition, (1983).

[4] Hsiang, W.-T.; Hsiang, W.-Y., *On the uniqueness of isoperimetric solutions and imbedded soap bubbles in noncompact symmetric spaces*, I. Invent. Math., 98(1) (1989), 39–58.

[5] Nelli, B.; Sa Earp, R.; Santos, W.; Toubiana, E., *Uniqueness of H-surfaces in* $\mathbb{H}^2\times\mathbb{R}$, $|H|\leq 1/2$, *with boundary one or two parallel horizontal circles*, Ann. Global Anal. Geom., 33 (2008), 307–321.

[6] Pedrosa, R. H.L.; Ritoré, M., *Isoperimetric domains in the Riemannian product of a circle with a simply connected space form and applications to free boundary problems*, Indiana Univ. Math. J., 48(4) (1999), 1357–1394.

[7] Sa Earp, R., *Parabolic and hyperbolic screw motion surfaces in* $\mathbb{H}^2\times\mathbb{R}$, Journal of the Australian Math. Soc., 85 (2008), 113–143 (DOI: 10.1017/S1446788708000013).

[8] Sa Earp, R.; Toubiana, E., *Screw motion surfaces in* $\mathbb{H}^2\times\mathbb{R}$ *and* $\mathbb{S}^2\times\mathbb{R}$, Illinois J. Math., 49(4) (2005), 1323–1362.

[9] Sa Earp, R.; Toubiana, R., *An asymptotic theorem for minimal surfaces and existence results for minimal graphs in* $\mathbb{H}^2\times\mathbb{R}$, Math. Annalen, 342 (2008), 309–331 (DOI 10.1007/s00208-008-0237-0).

[10] Spruck, J., *Interior gradient estimates and existence theorems for constant mean curvature graphs in* $M\times\mathbb{R}$, Preprint, www.math.jhu.edu/∼js/grad2.pdf.

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