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SKEW GROUP ALGEBRAS OF PATH ALGEBRAS AND PREPROJECTIVE ALGEBRAS

LAURENT DEMONET

ABSTRACT. We compute explicitly up to Morita-equivalence the skew group algebra of a finite group acting on the path algebra of a quiver and the skew group algebra of a finite group acting on a preprojective algebra. These results generalize previous results of Reiten and Riedtmann [RR] for a cyclic group acting on the path algebra of a quiver and of Reiten and Van den Bergh [RV, proposition 2.13] for a finite subgroup of $SL(\mathbb{C}X \oplus \mathbb{C}Y)$ acting on $\mathbb{C}[X, Y]$.

1. INTRODUCTION AND MAIN RESULTS

Let k be an algebraically closed field and G be a finite group such that the characteristic of k does not divide the cardinality of G . If Λ is a k -algebra and if G acts on the right on Λ , the action being denoted exponentially, the skew group algebra of Λ under the action of G is by definition the k -algebra whose underlying k -vector space is $\Lambda \otimes_k k[G]$ and whose multiplication is linearly generated by $(a \otimes g)(a' \otimes g') = aa'^{g^{-1}} \otimes gg'$ for all $a, a' \in \Lambda$ and $g, g' \in G$ (see [RR]). It will be denoted by ΛG . Identifying $k[G]$ and Λ with subalgebras of ΛG , an alternative definition is

$$\Lambda G = \langle \Lambda, k[G] \mid \forall (g, a) \in G \times \Lambda, g^{-1}ag = a^g \rangle_{k\text{-alg}}$$

Let now $Q = (I, A)$ be a quiver where I denotes the set of vertices and A the set of arrows. Consider an action of G on the path algebra kQ permuting the set of primitive idempotents $\{e_i \mid i \in I\}$ and stabilizing the vector space spanned by the arrows. Note that this is more general than an action coming from an action of G on Q since an arrow may be sent to a linear combination of arrows. We now define a new quiver Q_G . We first need some notation.

Let \tilde{I} be a set of representatives of the classes of I under the action of G . For $i \in I$, let G_i denote the subgroup of G stabilizing e_i , let $i_\circ \in \tilde{I}$ be the representative of the class of i and let $\kappa_i \in G$ be such that $i_\circ^{\kappa_i} = i$.

For $(i, j) \in \tilde{I}^2$, G acts on $O_i \times O_j$ where O_i and O_j are the orbits of i and j under the action of G . A set of representatives of the classes of this action will be denoted by F_{ij} .

For $i, j \in I$, define $M_{ij} \subset kQ$ to be the vector space spanned by the arrows from i to j . We regard M_{ij} as a right $k[G_i \cap G_j]$ -module by restricting the action of G .

The quiver Q_G has vertex set

$$I_G = \bigcup_{i \in \tilde{I}} \{i\} \times \text{irr}(G_i)$$

where $\text{irr}(G_i)$ is a set of representatives of isomorphism classes of irreducible representations of G_i . The set of arrows of Q_G from (i, ρ) to (j, σ) is a basis of

$$\bigoplus_{(i', j') \in F_{ij}} \text{Hom}_{\text{mod } k[G_{i'} \cap G_{j'}]} \left((\rho \cdot \kappa_{i'})_{G_{i'} \cap G_{j'}}, (\sigma \cdot \kappa_{j'})_{G_{i'} \cap G_{j'}} \otimes_k M_{i'j'} \right)$$

where the representation $\rho \cdot \kappa_{i'}$ of $G_{i'}$ is the same as ρ as a vector space, and $(\rho \cdot \kappa_{i'})_g = \rho_{\kappa_{i'} g \kappa_{i'}^{-1}}$ for $g \in G_{i'} = \kappa_{i'}^{-1} G_i \kappa_{i'}$. Table 1 gives two examples of quivers Q_G . A detailed example is also computed in section 2.

We can now state the two main results of this paper and two corollaries :

Theorem 1. *There is an equivalence of categories*

$$\text{mod } k(Q_G) \simeq \text{mod } (kQ) G.$$

Theorem 1 was proved by Reiten and Riedtmann in [RR, §2] for cyclic groups.

The following theorem deals with the case of preprojective algebras. The definition of the preprojective algebra Λ_Q of a quiver Q is recalled in section 3.2.

Theorem 2. *If G acts on $k\overline{Q}$, where \overline{Q} is the double quiver of Q , by permuting the primitive idempotents e_i and stabilizing the linear subspace of $k\overline{Q}$ spanned by the arrows, and if, for all $g \in G$, $r^g = r$ where r is the preprojective relation of this quiver, then $(\overline{Q})_G$ is of the form \overline{Q}' for some quiver Q' and $(\Lambda_Q)_G$ is Morita equivalent to $\Lambda_{Q'}$.*

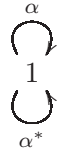
One can always extend an action on kQ to an action on $k\overline{Q}$ and this yields :

Corollary 3. *The action of G on a path algebra kQ permuting the primitive idempotents and stabilizing the linear subspace of $k\overline{Q}$ spanned by the arrows induces naturally an action of G on $k\overline{Q}$ and $(\overline{Q})_G$ is isomorphic to the double quiver of Q_G . Moreover, there is an equivalence of categories*

$$\text{mod } \Lambda_{Q_G} \simeq \text{mod } \Lambda_Q G.$$

Theorem 2 and corollary 3 will be used in [Dem] for constructing 2-Calabi-Yau categorifications of skew-symmetrizable cluster algebras. Another corollary, which is an easy consequence of the definition of the McKay graph, is :

Corollary 4. *Let Q be the quiver*



and G a finite subgroup of $\text{SL}(\mathbb{C}\alpha \oplus \mathbb{C}\alpha^*)$. Then

- (1) *There is an identification of Q_G with a double quiver \overline{Q}' such that the non oriented underlying graph of Q' is isomorphic to the affine Dynkin diagram corresponding to G through the McKay correspondence.*
- (2) *We have $kQ/(\alpha\alpha^* - \alpha^*\alpha) \simeq k[\alpha, \alpha^*]$ and there is an equivalence of categories*

$$\text{mod } (k[\alpha, \alpha^*]G) \simeq \text{mod } \Lambda_{Q'}.$$

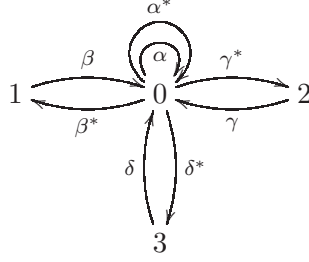
Corollary 4 was proved by a geometrical method in [RV, proof of proposition 2.13] (see also [CBH, theorem 0.1]).

TABLE 1. Examples of computations of Q_G

Q	G	Q_G
$ \begin{array}{c} 1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow n-1 \\ \searrow \qquad \qquad \qquad \nearrow \\ \qquad \qquad \qquad n \\ \nearrow \qquad \qquad \qquad \searrow \\ 1' \longrightarrow 2' \longrightarrow \cdots \longrightarrow (n-1)' \end{array} $	$\mathbb{Z}/2\mathbb{Z}$	$ \begin{array}{c} \qquad \qquad \qquad \qquad \qquad \qquad \nearrow n_+ \\ 1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow n-1 \\ \qquad \qquad \qquad \qquad \qquad \qquad \searrow n_- \end{array} $
$ \begin{array}{c} \alpha \\ \curvearrowright \\ 1 \\ \curvearrowleft \\ \beta \end{array} $	$G \subset \text{SL}(\mathbb{C}\alpha \oplus \mathbb{C}\beta)$ type A_n	

2. AN EXAMPLE

Suppose that $k = \mathbb{C}$ and that Q is the following quiver



Let also

$$G = \langle a, b \mid a^3 = b^2, b^4 = 1, aba = b \rangle$$

be the binary dihedral group of order 12. One lets G act on kQ by :

	e_0	e_1	e_2	e_3	α	α^*	β	β^*	γ	γ^*	δ	δ^*
a	e_0	e_2	e_3	e_1	$\zeta^{-1}\alpha$	$\zeta\alpha^*$	γ	γ^*	δ	δ^*	β	β^*
b	e_0	e_1	e_3	e_2	α^*	$-\alpha$	$-\beta$	$-\beta^*$	$-\delta$	$-\delta^*$	$-\gamma$	$-\gamma^*$

where ζ is a primitive sixth root of unity.

Using the notation of the introduction, one can choose $\tilde{I} = \{0, 1\}$, $\kappa_0 = \kappa_1 = 1$, $\kappa_2 = a$, $\kappa_3 = a^2$. One has $G_0 = G$, $G_1 = \langle b \rangle \simeq \mathbb{Z}/4\mathbb{Z}$, $G_2 = \langle ba \rangle \simeq \mathbb{Z}/4\mathbb{Z}$, $G_3 = \langle ab \rangle \simeq \mathbb{Z}/4\mathbb{Z}$. One can also choose $F_{0,0} = \{(0, 0)\}$, $F_{0,1} = \{(0, 1)\}$, $F_{1,0} = \{(1, 0)\}$ and $F_{1,1} = \{(1, 1), (1, 2), (2, 1)\}$.

The irreducible representations of $\mathbb{Z}/4\mathbb{Z}$ will be denoted by θ_α where $\alpha \in \{i, -1, -i, 1\}$ is the scalar action of a specified generator (b , ba or ab when $\mathbb{Z}/4\mathbb{Z}$ is realized as G_1 , G_2 or G_3). The group G has six irreducible representations : four of degree 1 of the form $a \mapsto \alpha^2$, $b \mapsto \alpha$ for each $\alpha \in \{i, -1, -i, 1\}$, which will be denoted by λ_α , and two of degree 2 :

$$\rho : a \mapsto \begin{pmatrix} \zeta^{-1} & 0 \\ 0 & \zeta \end{pmatrix} \quad b \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and

$$\sigma : a \mapsto \begin{pmatrix} \zeta^{-2} & 0 \\ 0 & \zeta^2 \end{pmatrix} \quad b \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

One checks easily that $\lambda_i \otimes \rho \simeq \sigma$, $\rho \otimes \rho \simeq \sigma \oplus \lambda_1 \oplus \lambda_{-1}$, $\rho \otimes \sigma \simeq \rho \oplus \lambda_i \oplus \lambda_{-i}$, $\sigma \otimes \sigma \simeq \sigma \oplus \lambda_1 \oplus \lambda_{-1}$. The other product formulas are deduced from these. One computes $M_{0,0} = \rho$, $M_{0,1} = M_{(1,0)} = \lambda_{-1}$ and $M_{(1,1)} = M_{(1,2)} = M_{(2,1)} = 0$. The vertices of Q_G are then 0_i , 0_{-1} , 0_{-i} , 0_1 , 0_ρ , 0_σ , 1_{-1} and 1_1 where we write $0_\alpha = (0, \lambda_\alpha)$ and $1_\alpha = (1, \theta_\alpha)$ for simplicity. One has

$$\mathrm{Hom}_G(\rho, \sigma \otimes \rho) \simeq \mathrm{Hom}_G(\rho, \rho \oplus \lambda_i \oplus \lambda_{-i}) \simeq \mathbb{C}$$

and therefore there is one arrow from 0_ρ to 0_σ ,

$$\mathrm{Hom}_G(\lambda_1, \sigma \otimes \rho) \simeq 0$$

and therefore there is no arrow from 0_1 to 0_σ ,

$$\mathrm{Hom}_G(\lambda_i, \sigma \otimes \rho) \simeq \mathbb{C}$$

and therefore there is one arrow from 0_i to 0_σ ,

$$\mathrm{Hom}_{\mathbb{Z}/4\mathbb{Z}}(\theta_i, \sigma|_{\mathbb{Z}/4\mathbb{Z}} \otimes \theta_{-1}) \simeq \mathrm{Hom}_{\mathbb{Z}/4\mathbb{Z}}(\theta_i, \theta_{-1} \oplus \theta_1) \simeq 0$$

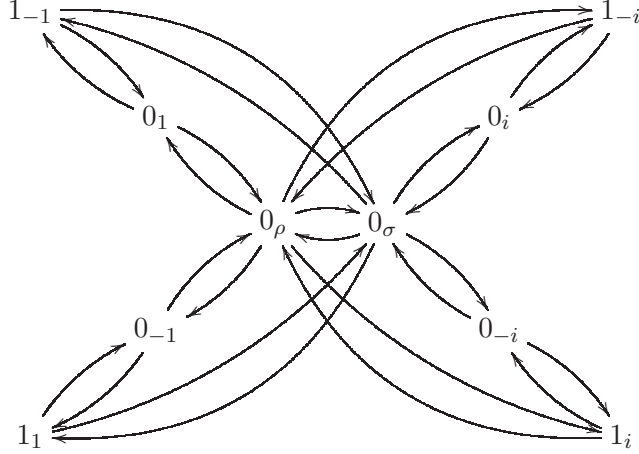
and therefore there is no arrow from 1_i to 0_σ ,

$$\mathrm{Hom}_{\mathbb{Z}/4\mathbb{Z}}(\theta_1, \sigma|_{\mathbb{Z}/4\mathbb{Z}} \otimes \theta_{-1}) \simeq \mathrm{Hom}_{\mathbb{Z}/4\mathbb{Z}}(\theta_1, \theta_{-1} \oplus \theta_1) \simeq \mathbb{C}$$

and therefore there is one arrow from 1_1 to 0_σ ,

$$\mathrm{Hom}_{\mathbb{Z}/4\mathbb{Z}}(\theta_1, \lambda_{-1}|_{\mathbb{Z}/4\mathbb{Z}} \otimes \theta_{-1}) \simeq \mathbb{C}$$

and therefore there is one arrow from 1_1 to 0_{-1} . All the other computations can be done in the same way. Finally, Q_G is the following quiver :



where one can remark that the full subgraph having vertices $\{0_i, 0_{-1}, 0_{-i}, 0_1, 0_\rho, 0_\sigma\}$ is the affine Dynkin diagram corresponding to G in the McKay correspondence, as expected. Hence $\text{mod}(\mathbb{C}Q)G \simeq \text{mod } \mathbb{C}Q_G$. Moreover, it is easy to check that the preprojective relation $\alpha\alpha^* - \alpha^*\alpha + \beta\beta^* - \beta^*\beta + \gamma\gamma^* - \gamma^*\gamma + \delta\delta^* - \delta^*\delta$ is stable under the action of G and therefore there is an equivalence of Morita between $\Lambda_{Q_1}G$ and Λ_{Q_2} where $\overline{Q}_1 = Q$, $\overline{Q}_2 = Q_G$, and $\Lambda_{Q_1}, \Lambda_{Q_2}$ are the preprojective algebras of Q_1 and Q_2 .

3. PROOFS OF THE MAIN PROPOSITIONS

3.1. Proof of theorem 1. We retain the notation of section 1. Thus Q is a quiver, G acts on kQ by stabilizing the set of primitive idempotents corresponding to vertices and the vector space spanned by the arrows, \tilde{I} is a fixed set of representatives of the G -orbits of I , etc.. Let R be the subalgebra of kQ generated by the primitive idempotents and $M \subset kQ$ be the linear subspace spanned by the arrows, seen as an R -bimodule. Define $T_0 = R$ and for every positive integer n , $T_n = T_{n-1} \otimes_R M$. Then, recall that the tensor algebra $T(R, M)$ is $\bigoplus_{i \geq 0} T_i$ endowed with the canonical product. It is clear that kQ is canonically isomorphic to $T(R, M)$ on which the action of G is graded.

As G stabilizes R and M , one can define the skew-group algebra RG which is a subalgebra of $(kQ)G$. Thus, $MG = M \otimes_k k[G]$ is a sub- RG -bimodule of $(kQ)G$ and one gets easily a canonical isomorphism $(kQ)G \simeq T(R, M)G \simeq T(RG, MG)$ which maps $(m_1 \otimes m_2 \otimes \dots \otimes m_n) \otimes g$ to $(m_1 \otimes 1) \otimes (m_2 \otimes 1) \otimes \dots \otimes (m_{n-1} \otimes 1) \otimes (m_n \otimes g) = (m_1 \otimes 1) \otimes (m_2 \otimes 1) \otimes \dots \otimes (m_{n-1} \otimes g) \otimes (m_n^g \otimes 1) = \dots = (m_1 \otimes 1) \otimes (m_2 \otimes g) \otimes \dots \otimes (m_{n-1}^g \otimes 1) \otimes (m_n^g \otimes 1) = (m_1 \otimes g) \otimes (m_2^g \otimes 1) \otimes \dots \otimes (m_{n-1}^g \otimes 1) \otimes (m_n^g \otimes 1)$.

Let now $e = \sum_{i \in \tilde{I}} e_i \in R \subset RG$ which is an idempotent. Then, according to [RR, proposition 1.6] and its proof,

$$\begin{aligned} \text{mod } RG &\simeq \text{mod } e(RG)e \\ N &\mapsto eN \\ (RG)e \otimes_{e(RG)e} N' &\leftarrow N' \end{aligned}$$

is a Morita equivalence between RG and $e(RG)e$. Hence, it is a classical and easy fact that there is an equivalence of categories

$$\begin{aligned} \text{mod}(kQ)G &\simeq \text{mod } T(RG, MG) \simeq \text{mod } T(e(RG)e, e(MG)e) \\ N &\mapsto eN \\ (RG)e \otimes_{e(RG)e} N' &\leftarrow N' \end{aligned}$$

using the isomorphism $MG \simeq (RG)e \otimes_{e(RG)e} e(MG)e \otimes_{e(RG)e} e(RG)$. One has $e(RG)e \simeq \prod_{i \in \tilde{I}} k[G_i]$ where, for each $i \in \tilde{I}$, G_i is the stabilizer of e_i . As G_i is semi-simple, one can fix, for each $i \in \tilde{I}$ and $\rho \in \text{irr}(G_i)$, $\tilde{e}_{i\rho}$ to be a primitive idempotent of $k[G_i]$ corresponding to ρ . Let $\tilde{e} = \sum_{i \in \tilde{I}} \sum_{\rho \in \text{irr}(G_i)} \tilde{e}_{i\rho}$ which satisfies $e\tilde{e}e = \tilde{e}$. Then we have a Morita equivalence between $e(RG)e$ and $\tilde{e}e(RG)e\tilde{e} = \tilde{e}(RG)\tilde{e}$, and, as before, between $T(e(RG)e, e(MG)e)$ and $T(\tilde{e}(RG)\tilde{e}, \tilde{e}(MG)\tilde{e})$. Moreover, using the notation I_G of the introduction, $\tilde{e}(RG)\tilde{e} \simeq \prod_{(i,\rho) \in I_G} k\tilde{e}_{i\rho}$ and therefore, it is enough to compute $\tilde{e}_{j\sigma}(MG)\tilde{e}_{i\rho} = \tilde{e}_{j\sigma}e_j(MG)e_i\tilde{e}_{i\rho}$ for each (i, ρ) and (j, σ) in I_G to finish the proof of theorem 1. Remark now that

$$e_j(MG)e_i = \sum_{(i',j') \in O_i \times O_j} G_j \kappa_{j'} M_{i'j'} \kappa_{i'}^{-1} G_i = \bigoplus_{(i',j') \in F_{ij}} G_j \kappa_{j'} M_{i'j'} \kappa_{i'}^{-1} G_i$$

and therefore, if one denotes $G_{i'j'} = G_{i'} \cap G_{j'}$ for every $i', j' \in I$,

$$\begin{aligned} \tilde{e}_{j\sigma}e_j(MG)e_i\tilde{e}_{i\rho} &\simeq \bigoplus_{(i',j') \in F_{ij}} \text{Hom}_k(k, \tilde{e}_{j\sigma} G_j \kappa_{j'} M_{i'j'} \kappa_{i'}^{-1} G_i \tilde{e}_{i\rho}) \\ &\simeq \bigoplus_{(i',j') \in F_{ij}} \text{Hom}_{k[G_i]}(\rho, \sigma \otimes_{k[G_j]} G_j \kappa_{j'} M_{i'j'} \kappa_{i'}^{-1} G_i) \\ &\simeq \bigoplus_{(i',j') \in F_{ij}} \text{Hom}_{k[G_{i'}]}(\rho \cdot \kappa_{i'}, (\sigma \cdot \kappa_{j'}) \otimes_{k[G_{j'}]} G_{j'} M_{i'j'} G_{i'}) \\ &\simeq \bigoplus_{(i',j') \in F_{ij}} \text{Hom}_{k[G_{i'}]}(\rho \cdot \kappa_{i'}, (\sigma \cdot \kappa_{j'}) \otimes_{k[G_{j'}]} k[G_{j'}] \\ &\quad \otimes_{k[G_{i'j'}]} G_{i'j'} M_{i'j'} G_{i'j'} \otimes_{k[G_{i'j'}]} k[G_{i'}]) \end{aligned}$$

and, because of the relations defining $(kQ)G$, the multiplication in $(kQ)G$ induces an isomorphism of $(G_{i'j'}, G_{i'j'})$ -bimodules $k[G_{i'j'}] \otimes_k M_{i'j'} \simeq G_{i'j'} M_{i'j'} G_{i'j'}$. Note that the action of $(G_{i'j'}, G_{i'j'})$ on $k[G_{i'j'}] \otimes_k M_{i'j'}$ is defined here by $g(v \otimes m)h = ghv \otimes h^{-1}mh = ghv \otimes m^h$ (recall that we use the multiplication notation for the product in $(kQ)G$ and the exponential notation for the action of G on kQ fixed at the beginning). Therefore

$$\begin{aligned} \tilde{e}_{j\sigma}e_j(MG)e_i\tilde{e}_{i\rho} &\simeq \bigoplus_{(i',j') \in F_{ij}} \text{Hom}_{k[G_{i'}]}(\rho \cdot \kappa_{i'}, (\sigma \cdot \kappa_{j'}) \\ &\quad \otimes_{k[G_{i'j'}]} (k[G_{i'j'}] \otimes_k M_{i'j'}) \otimes_{k[G_{i'j'}]} k[G_{i'}]) \\ &\simeq \bigoplus_{(i',j') \in F_{ij}} \text{Hom}_{k[G_{i'}]}(\rho \cdot \kappa_{i'}, ((\sigma \cdot \kappa_{j'})|_{G_{i'j'}} \otimes_k M_{i'j'}) \otimes_{k[G_{i'j'}]} k[G_{i'}]) \\ &\simeq \bigoplus_{(i',j') \in F_{ij}} \text{Hom}_{k[G_{i'j'}]}((\rho \cdot \kappa_{i'})|_{G_{i'j'}}, (\sigma \cdot \kappa_{j'})|_{G_{i'j'}} \otimes_k M_{i'j'}) \end{aligned}$$

where the representation $\rho \cdot \kappa_{i'}$ of $G_{i'}$ is the same as ρ as a vector space and, if $g \in G_{i'} = \kappa_{i'}^{-1} G_i \kappa_{i'}$, $(\rho \cdot \kappa_{i'})_g = \rho_{\kappa_{i'} g \kappa_{i'}^{-1}}$. It concludes the proof of theorem 1. Note that we go from the penultimate line to the last one by the classical adjunction between induction and restriction of representations.

3.2. Proof of theorem 2. We retain the notation of section 3.1. Define the R -bimodule $\overline{M} = M \oplus M^*$ (the R -bimodule structure on M^* is the natural one : $(afb)(m) = f(bma)$ for $a, b \in R$, $f \in M^*$ and $m \in M$). The tensor algebra $T(R, \overline{M})$ is the path algebra of the double quiver \overline{Q} of Q . The non-degenerate skew-symmetric bilinear form defined on \overline{M} by $\langle m + f, m' + f' \rangle = f'(m) - f(m')$ where $m, m' \in M$ and $f, f' \in M^*$ satisfies, for every $a, b \in R$ and $m, n \in \overline{M}$, $\langle amb, n \rangle = \langle m, bna \rangle$. If $\{x_i\}_{i \in S}$ is a k -basis of \overline{M} , then denote by $\{x_i^*\}_{i \in S}$ its (left) dual basis for the bilinear form $\langle -, - \rangle$ (that is the one satisfying $\langle x_i^*, x_j \rangle = \delta_{ij}$ for every

$i, j \in S$). Then the element $r = \sum_{i \in S} x_i \otimes x_i^* \in \overline{M} \otimes_R \overline{M}$ is independent of the choice of the basis. By definition, r is the preprojective relation corresponding to Q and the preprojective algebra Λ_Q of Q is $T(R, \overline{M})/(r)$. For more details about preprojective algebras, see for example [DR] or [Rin].

Suppose now that the group G acts on $k\overline{Q}$ by stabilizing the set of primitive idempotents corresponding to vertices and the k -subspace spanned by the arrows. Then it stabilizes r if and only if it stabilizes the bilinear form $\langle -, - \rangle$ (indeed, for $g \in G$, $r^g = r$ implies that $\{x_i^{*g}\}_{i \in S}$ is the left dual basis of $\{x_i^g\}_{i \in S}$ since for every $i \in S$, $(\text{Id}_{\overline{M}} \otimes \langle -, - \rangle)(r^g \otimes x_i^g) = (\text{Id}_{\overline{M}} \otimes \langle -, - \rangle)(r \otimes x_i^g) = x_i^g$). Extend now $\langle -, - \rangle$ to a bilinear form on $\overline{M}G$ by setting

$$\langle m \otimes g, n \otimes h \rangle = \begin{cases} \langle m, n^h \rangle & \text{if } gh = 1 \\ 0 & \text{else} \end{cases}$$

which is clearly skew-symmetric and non-degenerate. Moreover, for $a, b \in RG$ and $m, n \in \overline{M}G$, one has easily $\langle amb, n \rangle = \langle m, bna \rangle$. If $\{x_i\}_{i \in S}$ is a basis of \overline{M} and $\{x_i^*\}_{i \in S}$ is its left dual basis, then $\{x_i^{*g} \otimes g^{-1}\}_{(i,g) \in S \times G}$ is the left dual basis of the basis $\{x_i \otimes g\}_{(i,g) \in S \times G}$ of $\overline{M}G$. Hence, the preprojective relation r_G corresponding to $\langle -, - \rangle$ in $\overline{M}G \otimes_{RG} \overline{M}G$ is

$$\begin{aligned} r_G &= \sum_{(i,g) \in S \times G} (x_i \otimes g) \otimes (x_i^{*g} \otimes g^{-1}) = \sum_{(i,g) \in S \times G} (x_i \otimes 1) (1 \otimes g) \otimes (x_i^{*g} \otimes g^{-1}) \\ &= \sum_{(i,g) \in S \times G} (x_i \otimes 1) \otimes (1 \otimes g) (x_i^{*g} \otimes g^{-1}) = \sum_{(i,g) \in S \times G} (x_i \otimes 1) \otimes (x_i^* \otimes 1) = \#G \times r \end{aligned}$$

where the preprojective relation r of $T(R, \overline{M})$ is mapped by the canonical inclusion from $\overline{M} \otimes_R \overline{M}$ to $\overline{M}G \otimes_{RG} \overline{M}G$. As $\#G$ is invertible in k , one gets

$$(T(R, \overline{M})/(r)) G \simeq T(RG, \overline{M}G)/(r) = T(RG, \overline{M}G)/(r_G).$$

Moreover, for $\langle -, - \rangle$, $\tilde{e}(\overline{M}G)\tilde{e}$ and $(1 - \tilde{e})(\overline{M}G) + (\overline{M}G)(1 - \tilde{e})$ are orthogonal supplementary subspaces. Thus, $\langle -, - \rangle$ restricts to a skew-symmetric non-degenerate bilinear form on $\tilde{e}(\overline{M}G)\tilde{e}$ which satisfies, for every $a, b \in \tilde{e}(RG)\tilde{e}$ and $m, n \in \tilde{e}(\overline{M}G)\tilde{e}$, $\langle amb, n \rangle = \langle m, bna \rangle$. By taking a basis of $\overline{M}G$ which is the union of a basis of $\tilde{e}(\overline{M}G)\tilde{e}$ and a basis of $(1 - \tilde{e})(\overline{M}G) + (\overline{M}G)(1 - \tilde{e})$, it is clear that the equivalence of categories of section 3.1 restricts to an equivalence

$$\begin{aligned} \text{mod}((kQ)G/(r_G)) &\simeq \text{mod}(T(RG, \overline{M}G)/(r_G)) \simeq \text{mod}(T(\tilde{e}(RG)\tilde{e}, \tilde{e}(\overline{M}G)\tilde{e})/(r_{\tilde{e}})) \\ &\quad N \mapsto \tilde{e}N \\ (RG)\tilde{e} \otimes_{\tilde{e}(RG)\tilde{e}} N' &\leftarrow N' \end{aligned}$$

where $r_{\tilde{e}}$ is the preprojective relation in $T(\tilde{e}(RG)\tilde{e}, \tilde{e}(\overline{M}G)\tilde{e}) \simeq k(\overline{Q})_G \simeq k\overline{Q}'$. It completes the proof of theorem 2 (it is enough to take a maximal isotropic subspace of $\tilde{e}(\overline{M}G)\tilde{e}$ to find arrows of Q' which is of course non unique).

3.3. Proof of corollary 3. We retain the previous notation. If G acts on kQ by stabilizing the set of primitive idempotents and M , then its action can be extended to an action on $\overline{M} = M \oplus M^*$ using the contragredient representation on M^* . Then $\overline{M}G = MG \oplus M^*G$ as RG -bimodules. Moreover, MG is clearly maximal isotropic for the bilinear form $\langle -, - \rangle$ extended on $\overline{M}G$ as before. Hence $(\overline{Q})_G = \overline{Q}_G$ and it proves corollary 3.

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