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Laurent Demonet. Skew group algebras of path algebras and preprojective algebras. 2009. hal-00359394v1

**HAL Id: hal-00359394**

**<https://hal.science/hal-00359394v1>**

Preprint submitted on 6 Feb 2009 (v1), last revised 3 Dec 2009 (v2)

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# SKEW GROUP ALGEBRAS OF PATH ALGEBRAS AND PREPROJECTIVE ALGEBRAS

LAURENT DEMONET

ABSTRACT. We compute explicitly up to Morita-equivalence the skew group algebra of a finite group acting on the path algebra of a quiver and the skew group algebra of a finite group acting on a preprojective algebra. These results generalize previous results of Reiten and Riedtmann [RR85] for a cyclic group acting on the path algebra of a quiver and of Reiten and Van den Bergh [RV89, proposition 2.13] for a finite subgroup of  $SL(\mathbb{C}X \oplus \mathbb{C}Y)$  acting on  $\mathbb{C}[X, Y]$ .

## 1. INTRODUCTION AND MAIN RESULTS

Let  $k$  be an algebraically closed field and  $G$  be a finite group such that the characteristic of  $k$  does not divide the cardinality of  $G$ . If  $\Lambda$  is a  $k$ -algebra and if  $G$  acts on  $\Lambda$ , the action being denoted exponentially, the skew group algebra of  $\Lambda$  under the action of  $G$  is by definition the  $k$ -algebra whose underlying  $k$ -vector space is  $k[G] \otimes_k \Lambda$  and whose multiplication is linearly generated by  $(g \otimes a)(g' \otimes a') = gg' \otimes a^{g'^{-1}} a'$  for all  $g, g' \in G$  and  $a, a' \in \Lambda$  (see [RR85]). It will be denoted by  $\Lambda G$ . Identifying  $k[G]$  and  $\Lambda$  with subalgebras of  $\Lambda G$ , an alternative definition is

$$\Lambda G = \langle \Lambda, k[G] \mid \forall (g, a) \in G \times \Lambda, gag^{-1} = a^g \rangle_{k\text{-alg}}$$

Let now  $Q = (I, A)$  be a quiver where  $I$  denotes the set of vertices and  $A$  the set of arrows. Consider an action of  $G$  on the path algebra  $kQ$  permuting the set of primitive idempotents  $\{e_i \mid i \in I\}$ . Note that this is more general than an action coming from an action of  $G$  on  $Q$  since an arrow may be sent to a linear combination of arrows. We now define a new quiver  $Q_G$ . We first need some notation.

Let  $\tilde{I}$  be a set of representatives of the classes of  $I$  under the action of  $G$ . For  $i \in I$ , let  $G_i$  denote the subgroup of  $G$  stabilizing  $e_i$ , let  $i_o \in \tilde{I}$  be the representative of the class of  $i$  and let  $\kappa_i \in G$  be such that  $\kappa_i i_o = i$ .

For  $(i, j) \in \tilde{I}^2$ ,  $G$  acts on  $O_i \times O_j$  where  $O_i$  and  $O_j$  are the orbits of  $i$  and  $j$  under the action of  $G$ . A set of representatives of the classes of this action will be denoted by  $F_{ij}$ .

For  $i, j \in I$ , define  $A_{ij} = e_j (\text{rad}(kQ) / \text{rad}(kQ)^2) e_i$  where  $\text{rad}(kQ)$  is the Jacobson radical of  $kQ$ . We regard  $A_{ij}$  as a left  $k[G_i \cap G_j]$ -module by restricting the action of  $G$ .

The quiver  $Q_G$  has vertex set

$$I_G = \bigcup_{i \in \tilde{I}} \{i\} \times \text{irr}(G_i)$$

where  $\text{irr}(G_i)$  is a set of representatives of isomorphism classes of irreducible representations of  $G_i$ . The set of arrows of  $Q_G$  from  $(i, \rho)$  to  $(j, \sigma)$  is a basis of

$$\bigoplus_{(i', j') \in F_{ij}} \text{Hom}_{\text{mod } k[G_{i'} \cap G_{j'}]}((\kappa_{i'} \cdot \rho)_{G_{i'} \cap G_{j'}} \otimes A_{i'j'}, (\kappa_{j'} \cdot \sigma)_{G_{i'} \cap G_{j'}})^*$$

where the representation  $\kappa_{i'} \cdot \rho$  of  $G_{i'}$  is the same as  $\rho$  as a vector space, and  $(\kappa_{i'} \cdot \rho)_g = \rho_{\kappa_{i'}^{-1} g \kappa_{i'}}$  for  $g \in G_{i'} = \kappa_{i'} G_i \kappa_{i'}^{-1}$ . Table 1 gives two examples of quivers  $Q_G$ . A detailed example is also computed in section 2.

We can now state the two main results of this paper and two corollaries :

**Theorem 1.** *There is an equivalence of categories*

$$\text{mod } k(Q_G) \simeq \text{mod } (kQ) G.$$

Theorem 1 was proved by Reiten and Riedtmann in [RR85, §2] for cyclic groups. Its proof is split into two propositions (15 and 19). It relies on the category of representations of a group species which is introduced in section 3.

The following theorem deals with the case of preprojective algebras. The definition of the preprojective algebra  $\Lambda_Q$  of a quiver  $Q$  is recalled in section 3.

**Theorem 2.** *If  $G$  acts on  $k\overline{Q}$ , where  $\overline{Q}$  is the double quiver of  $Q$ , by permuting the primitive idempotents  $e_i$ , and if for all  $g \in G$ ,  $r^g = r$  where  $r$  is the preprojective relation of this quiver, then  $(\overline{Q})_G$  is of the form  $\overline{Q}'$  for some quiver  $Q'$  and  $(\Lambda_Q)G$  is Morita equivalent to  $\Lambda_{Q'}$ .*

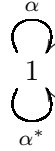
The proof of theorem 2 is split into two propositions (17 and 20) and uses lemma 21. One can always extend an action on  $kQ$  to an action on  $k\overline{Q}$  and this yields :

**Corollary 3.** *The action of  $G$  on a path algebra  $kQ$  permuting the primitive idempotents induces naturally an action of  $G$  on  $k\overline{Q}$  and  $(\overline{Q})_G$  is isomorphic to the double quiver of  $Q_G$ . Moreover, there is an equivalence of categories*

$$\text{mod } \Lambda_{Q_G} \simeq \text{mod } \Lambda_Q G.$$

Theorem 2 and corollary 3 will be used in [Dem] for constructing 2-Calabi-Yau categorifications of skew-symmetrizable cluster algebras. Another corollary related to the McKay correspondence is :

**Corollary 4.** *Let  $Q$  be the quiver*



and  $G$  a finite subgroup of  $\text{SL}(\mathbb{C}\alpha \oplus \mathbb{C}\alpha^*)$ . Then

- (1) *There is an identification of  $Q_G$  with a double quiver  $\overline{(Q')}$  such that the non oriented underlying graph of  $Q'$  is isomorphic to the affine Dynkin diagram corresponding to  $G$  through the McKay correspondence.*
- (2) *We have  $kQ/(\alpha\alpha^* - \alpha^*\alpha) \simeq k[\alpha, \alpha^*]$  and there is an equivalence of categories*

$$\text{mod } (k[\alpha, \alpha^*]G) \simeq \text{mod } \Lambda_{Q'}.$$

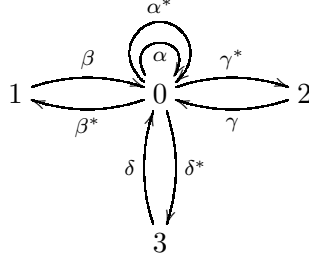
Corollary 4 was proved by a geometrical method in [RV89, proof of proposition 2.13] (see also [CBH98, theorem 0.1]).

TABLE 1. Examples of computations of  $Q_G$

$Q$	$G$	$Q_G$
$  \begin{array}{c}  1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow n-1 \\  \searrow \qquad \qquad \qquad \nearrow \\  \qquad \qquad \qquad n \\  \nearrow \qquad \qquad \qquad \searrow \\  1' \longrightarrow 2' \longrightarrow \cdots \longrightarrow (n-1)'  \end{array}  $	$\mathbb{Z}/2\mathbb{Z}$	$  \begin{array}{c}  \qquad \qquad \qquad \qquad \qquad \nearrow n_+ \\  1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow n-1 \\  \qquad \qquad \qquad \qquad \qquad \searrow n_-  \end{array}  $
$  \begin{array}{c}  \alpha \\  \circlearrowleft \\  1 \\  \circlearrowright \\  \beta  \end{array}  $	$  \begin{array}{c}  G \subset \\  \text{SL}(\mathbb{C}\alpha \oplus \mathbb{C}\beta) \\  \text{type } A_n  \end{array}  $	

## 2. AN EXAMPLE

Suppose that  $k = \mathbb{C}$  and that  $Q$  is the following quiver



Let also

$$G = \langle a, b \mid a^3 = b^2, b^4 = 1, aba = b \rangle$$

be the binary dihedral group of order 12. One lets  $G$  act on  $kQ$  by :

	$e_0$	$e_1$	$e_2$	$e_3$	$\alpha$	$\alpha^*$	$\beta$	$\beta^*$	$\gamma$	$\gamma^*$	$\delta$	$\delta^*$
$a$	$e_0$	$e_2$	$e_3$	$e_1$	$\zeta^{-1}\alpha$	$\zeta\alpha^*$	$\gamma$	$\gamma^*$	$\delta$	$\delta^*$	$\beta$	$\beta^*$
$b$	$e_0$	$e_1$	$e_3$	$e_2$	$\alpha^*$	$-\alpha$	$-\beta$	$-\beta^*$	$-\delta$	$-\delta^*$	$-\gamma$	$-\gamma^*$

where  $\zeta$  is a primitive sixth root of unity.

Using the notation of the introduction, one can choose  $\tilde{I} = \{0, 1\}$ ,  $\kappa_0 = \kappa_1 = 1$ ,  $\kappa_2 = a$ ,  $\kappa_3 = a^2$ . One has  $G_0 = G$ ,  $G_1 = \langle b \rangle \simeq \mathbb{Z}/4\mathbb{Z}$ ,  $G_2 = \langle ba \rangle \simeq \mathbb{Z}/4\mathbb{Z}$ ,  $G_3 = \langle ab \rangle \simeq \mathbb{Z}/4\mathbb{Z}$ . One can also choose  $F_{0,0} = \{(0, 0)\}$ ,  $F_{0,1} = \{(0, 1)\}$ ,  $F_{1,0} = \{(1, 0)\}$  and  $F_{1,1} = \{(1, 1), (1, 2), (2, 1)\}$ .

The irreducible representations of  $\mathbb{Z}/4\mathbb{Z}$  will be denoted by  $\theta_\alpha$  where  $\alpha \in \{i, -1, -i, 1\}$  is the scalar action of a specified generator ( $b$ ,  $ba$  or  $ab$  when  $\mathbb{Z}/4\mathbb{Z}$  is realized as  $G_1$ ,  $G_2$  or  $G_3$ ). The group  $G$  has six irreducible representations : four of degree 1 of the form  $a \mapsto \alpha^2$ ,  $b \mapsto \alpha$  for each  $\alpha \in \{i, -1, -i, 1\}$ , which will be denoted by  $\lambda_\alpha$ , and two of degree 2 :

$$\rho : a \mapsto \begin{pmatrix} \zeta^{-1} & 0 \\ 0 & \zeta \end{pmatrix} \quad b \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and

$$\sigma : a \mapsto \begin{pmatrix} \zeta^{-2} & 0 \\ 0 & \zeta^2 \end{pmatrix} \quad b \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

One checks easily that  $\lambda_i \otimes \rho \simeq \sigma$ ,  $\rho \otimes \rho \simeq \sigma \oplus \lambda_1 \oplus \lambda_{-1}$ ,  $\rho \otimes \sigma \simeq \rho \oplus \lambda_i \oplus \lambda_{-i}$ ,  $\sigma \otimes \sigma \simeq \sigma \oplus \lambda_1 \oplus \lambda_{-1}$ . The other product formulas are deduced from these. One computes  $A_{0,0} = \rho$ ,  $A_{0,1} = A_{(1,0)} = \lambda_{-1}$  and  $A_{(1,1)} = A_{(1,2)} = A_{(2,1)} = 0$ . The vertices of  $Q_G$  are then  $0_i$ ,  $0_{-1}$ ,  $0_{-i}$ ,  $0_1$ ,  $0_\rho$ ,  $0_\sigma$ ,  $1_{-1}$  and  $1_1$  where we write  $0_\alpha = (0, \lambda_\alpha)$  and  $1_\alpha = (1, \theta_\alpha)$  for simplicity. One has

$$\text{Hom}_G(\rho \otimes \rho, \sigma) \simeq \text{Hom}_G(\sigma \oplus \lambda_1 \oplus \lambda_{-1}, \sigma) \simeq \mathbb{C}$$

and therefore there is one arrow from  $0_\rho$  to  $0_\sigma$ ,

$$\text{Hom}_G(\lambda_1 \otimes \rho, \sigma) \simeq 0$$

and therefore there is no arrow from  $0_1$  to  $0_\sigma$ ,

$$\text{Hom}_G(\lambda_i \otimes \rho, \sigma) \simeq \mathbb{C}$$

and therefore there is one arrow from  $0_i$  to  $0_\sigma$ ,

$$\text{Hom}_{\mathbb{Z}/4\mathbb{Z}}(\theta_i \otimes \theta_{-1}, \sigma|_{\mathbb{Z}/4\mathbb{Z}}) \simeq \text{Hom}_{\mathbb{Z}/4\mathbb{Z}}(\theta_{-i}, \theta_1 \oplus \theta_{-1}) \simeq 0$$

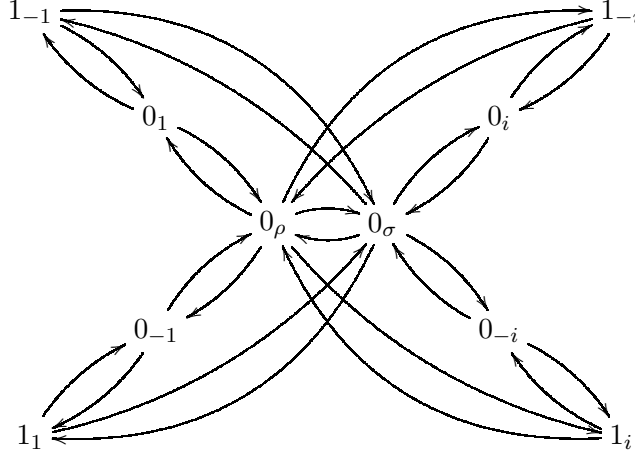
and therefore there is no arrow from  $1_i$  to  $0_\sigma$ ,

$$\text{Hom}_{\mathbb{Z}/4\mathbb{Z}}(\theta_1 \otimes \theta_{-1}, \sigma|_{\mathbb{Z}/4\mathbb{Z}}) \simeq \text{Hom}_{\mathbb{Z}/4\mathbb{Z}}(\theta_{-1}, \theta_1 \oplus \theta_{-1}) \simeq \mathbb{C}$$

and therefore there is one arrow from  $1_1$  to  $0_\sigma$ ,

$$\text{Hom}_{\mathbb{Z}/4\mathbb{Z}}(\theta_1 \otimes \theta_{-1}, \lambda_1|_{\mathbb{Z}/4\mathbb{Z}}) \simeq \mathbb{C}$$

and therefore there is one arrow from  $1_1$  to  $0_{-1}$ . All the other computations can be done in the same way. Finally,  $Q_G$  is the following quiver :



where one can remark that the full subgraph having vertices  $\{0_i, 0_{-1}, 0_{-i}, 0_1, 0_\rho, 0_\sigma\}$  is the affine Dynkin diagram corresponding to  $G$  in the McKay correspondence, as expected. Hence  $\text{mod}(\mathbb{C}Q)G \simeq \text{mod } \mathbb{C}Q_G$ . Moreover, it is easy to check that the preprojective relation  $\alpha\alpha^* - \alpha^*\alpha + \beta\beta^* - \beta^*\beta + \gamma\gamma^* - \gamma^*\gamma + \delta\delta^* - \delta^*\delta$  is stable under the action of  $G$  and therefore there is an equivalence of Morita between  $\Lambda_{Q_1}G$  and  $\Lambda_{Q_2}$  where  $\overline{Q}_1 = Q$ ,  $\overline{Q}_2 = Q_G$ , and  $\Lambda_{Q_1}, \Lambda_{Q_2}$  are the preprojective algebras of  $Q_1$  and  $Q_2$ .

### 3. REPRESENTATIONS OF GROUP SPECIES

Let  $k$  be a field.

**Definition 5.** A *group species* is a triple  $(I, (G_i)_{i \in I}, (A_{ij})_{(i,j) \in I^2})$  where  $I$  is a set, for each  $i \in I$ ,  $G_i$  is a group and for each  $(i, j) \in I^2$ ,  $A_{ij}$  is a  $(k[G_j], k[G_i])$ -bimodule (the first acting on the left and the second on the right).

Fix now such a group species  $Q = (I, (G_i)_{i \in I}, (A_{ij})_{(i,j) \in I^2})$

**Definition 6.** A *representation of  $Q$*  is a pair  $((V_i)_{i \in I}, (x_{ij})_{(i,j) \in I^2})$  where for each  $i \in I$ ,  $V_i$  is a  $k$ -representation of  $G_i$  and for each  $(i, j) \in I^2$ ,

$$x_{ij} \in \text{Hom}_{G_j}(A_{ij} \otimes_{G_i} V_i, V_j).$$

**Definition 7.** Let  $((V_i)_{i \in I}, (x_{ij})_{(i,j) \in I^2})$  and  $((V'_i)_{i \in I}, (x'_{ij})_{(i,j) \in I^2})$  be two representations of  $Q$ . A *morphism* from the first one to the second one is a family  $(f_i)_{i \in I} \in \prod_{i \in I} \text{Hom}_{G_i}(V_i, V'_i)$  such that for each  $(i, j) \in I^2$  the following diagram commute :

$$\begin{array}{ccc} A_{ij} \otimes_{G_i} V_i & \xrightarrow{x_{ij}} & V_j \\ \text{Id}_{A_{ij}} \otimes f_i \downarrow & & f_j \downarrow \\ A_{ij} \otimes_{G_i} V'_i & \xrightarrow{x'_{ij}} & V'_j \end{array}$$

*Remarks 8.*

- The previous definitions give rise to an abelian category.
- If for each  $i \in I$ ,  $G_i$  is the trivial group, we get back the classical definition of a quiver (up to the choice of a basis of each  $A_{ij}$ ) and of the category of representations of a quiver.
- If for each  $i \in I$ ,  $k[G_i]$  is replaced by a division algebra, we obtain the usual definition of a species (see for example [DR76]).

**Definition 9.** If  $V$  is a  $(k[H_1], k[H_2])$ -bimodule and  $V'$  is a  $(k[H_2], k[H_1])$ -bimodule then a *non degenerate pairing* between  $V$  and  $V'$  is a bilinear map  $\langle, \rangle$  from  $V \times V'$  to  $k$  such that for every  $v \in V \setminus \{0\}$  and  $v' \in V' \setminus \{0\}$ ,  $\langle v, - \rangle$  and  $\langle -, v' \rangle$  do not vanish and such that for every  $(h_1, h_2) \in H_1 \times H_2$  and  $(v, v') \in V \times V'$ ,  $\langle h_1 v h_2, v' \rangle = \langle v, h_2 v' h_1 \rangle$ . It induces an isomorphism

between  $V'$  and the dual bimodule of  $V$  (that is the natural structure of  $(k[H_2], k[H_1])$ -bimodule on  $V^*$ ).

**Definition 10.** A *double group species* is a triple  $(Q, (\langle, \rangle_{ij})_{(i,j) \in I^2}, (\varepsilon_{ij})_{(i,j) \in I^2})$  where  $Q = (I, (G_i)_{i \in I}, (A_{ij})_{(i,j) \in I^2})$  is a group species and for each  $(i, j) \in I^2$ ,  $\langle, \rangle_{ij}$  is a non degenerate pairing between  $A_{ij}$  and  $A_{ji}$  satisfying for every  $(a, a') \in A_{ij} \times A_{ji}$   $\langle a, a' \rangle_{ij} = \langle a', a \rangle_{ji}$ , and  $\varepsilon_{ij}$  is an automorphism of  $A_{ij}$  such that  $\varepsilon_{ij} = -{}^t \varepsilon_{ji}$ . As there is no possibility of confusion,  $\langle, \rangle_{ij}$  will be denoted by  $\langle, \rangle$ .

**Definition 11.** The *double group species of a group species*  $Q = (I, (G_i)_{i \in I}, (A_{ij})_{(i,j) \in I^2})$  is the group species  $\bar{Q} = (I, (G_i)_{i \in I}, (\bar{A}_{ij})_{(i,j) \in I^2})$  where for each  $(i, j) \in I^2$ ,

$$\bar{A}_{ij} = A_{ij} \oplus A_{ji}^*.$$

There is a canonical identification between  $\bar{A}_{ij}$  and  $\bar{A}_{ji}^*$  which leads to a non degenerate pairing between  $\bar{A}_{ij}$  and  $\bar{A}_{ji}$ . One can define  $\varepsilon_{ij}$  as  $\varepsilon_{ij} = \text{Id}_{A_{ij}} \oplus -\text{Id}_{A_{ji}^*}$ .

Fix now a double group species  $(Q, \langle, \rangle, \varepsilon)$ . For every pair  $(i, j)$  of vertices of  $Q$ , there is an isomorphism  $\text{Hom}_{G_j}(A_{ij} \otimes_{G_i} V, W) \simeq \text{Hom}_{G_i}(V, A_{ji} \otimes_{G_j} W)$  which is functorial in  $V \in \text{mod } k[G_i]$  and  $W \in \text{mod } k[G_j]$ . More precisely, choose a basis  $\mathcal{B}$  of  $A_{ij}$  and for every  $a \in \mathcal{B}$ , denote by  $a^*$  the element of  $A_{ji}$  corresponding to  $a$  in the dual basis of  $\mathcal{B}$ . Then, to each  $f \in \text{Hom}_{G_j}(A_{ij} \otimes_{G_i} V, W)$ , one can associate  $\tilde{f} \in \text{Hom}_{G_i}(V, A_{ji} \otimes_{G_j} W)$  such that for every  $v \in V$ ,

$$\tilde{f}(v) = \sum_{a \in \mathcal{B}} a^* \otimes f(a \otimes v)$$

which does not depend on the choice of  $\mathcal{B}$ . Conversely, to every  $\tilde{f} \in \text{Hom}_{G_i}(V, A_{ji} \otimes_{G_j} W)$ , one can associate  $f \in \text{Hom}_{G_j}(A_{ij} \otimes_{G_i} V, W)$  such that, for every  $a \otimes v \in A_{ij} \otimes_{G_i} V$ ,

$$f(a \otimes v) = \frac{1}{\#G} \varphi(a \otimes \tilde{f}(v))$$

where  $\varphi$  is defined by

$$\begin{aligned} \varphi : A_{ij} \otimes_{G_i} A_{ji} \otimes_{G_j} W &\rightarrow W \\ a' \otimes a'' \otimes w &\mapsto \sum_{g \in G} \langle a', a'' g^{-1} \rangle gw. \end{aligned}$$

It is easy to see that  $f \mapsto \tilde{f}$  and  $\tilde{f} \mapsto f$  are reciprocal one of each other.

**Definition 12.** A representation  $((V_i)_{i \in I}, (x_{ij})_{(i,j) \in I^2})$  of the double group species  $Q$  is said to satisfy the *preprojective relations* if for all  $i \in I$ ,

$$\sum_{j \in I} x_{ji} \circ (\varepsilon_{ji} \otimes \text{Id}_{V_j}) \circ \tilde{x}_{ij} = 0.$$

The full subcategory of the category of representations of  $Q$  consisting of these objects will be called the category of representations of  $Q$  satisfying the preprojective relations.

**Lemma 13.** *Up to an isomorphism of categories, the previous definition does not depend on the choice of  $\varepsilon$  or  $\langle, \rangle$ .*

*Proof.* Let  $\varepsilon'$  satisfy the same hypothesis as  $\varepsilon$ . With the hypothesis on  $\varepsilon$  and  $\varepsilon'$ , it is easy to choose, for every  $(i, j) \in I^2$  an automorphism  $\varphi_{ij}$  of  $A_{ij}$  such that  $\varphi_{ji} = \varepsilon'_{ji} {}^t \varphi_{ij}^{-1} \varepsilon_{ji}^{-1}$ . Then one obtains an automorphism of the category of representations of  $Q$  :

$$((V_i)_{i \in I}, (x_{ij})_{(i,j) \in I^2}) \mapsto ((V_i)_{i \in I}, (x_{ij} \circ (\varphi_{ij} \otimes \text{Id}_{V_i}))_{(i,j) \in I^2})$$

which sends the representations of  $Q$  satisfying the preprojective relations of type  $\varepsilon'$  to the representations of  $Q$  satisfying the preprojective relations of type  $\varepsilon$ .

Suppose now that  $\langle, \rangle'$  is another pairing satisfying the hypothesis. For every  $(i, j) \in I^2$ , for every basis  $\mathcal{B}$  of  $A_{ij}$  and every  $x \in \mathcal{B}$  denote by  $d_{\mathcal{B}}(x)$  (resp.  $d'_{\mathcal{B}}(x)$ ) the corresponding element in

the dual basis of  $\mathcal{B}$  for the pairing  $\langle, \rangle$  (resp.  $\langle, \rangle'$ ). Let  $\varepsilon'_{ij}$  be the automorphism of  $A_{ij}$  satisfying  $\varepsilon'_{ij}(d'_{\mathcal{B}}(x)) = \varepsilon_{ij}(d_{\mathcal{B}}(x))$  for every  $x \in \mathcal{B}$ . It does not depend on the basis  $\mathcal{B}$  (it is easy to see that the definition is stable under transvections and dilations of  $A_{ij}$ ). One has

$$\begin{aligned}\varepsilon_{ij} = -{}^t\varepsilon_{ji} &\Leftrightarrow \forall x \in \mathcal{B}, \varepsilon_{ij} \left( d_{\varepsilon_{ji}(d_{\mathcal{B}}(\mathcal{B}))}(\varepsilon_{ji}(d_{\mathcal{B}}(x))) \right) = -x \\ \varepsilon'_{ij} = -{}^t\varepsilon'_{ji} &\Leftrightarrow \forall x \in \mathcal{B}, \varepsilon'_{ij} \left( d'_{\varepsilon'_{ji}(d'_{\mathcal{B}}(\mathcal{B}))}(\varepsilon'_{ji}(d'_{\mathcal{B}}(x))) \right) = -x\end{aligned}$$

and therefore, using the definition of  $\varepsilon'_{ij}$ ,  $\varepsilon_{ij} = -{}^t\varepsilon_{ji} \Leftrightarrow \varepsilon'_{ij} = -{}^t\varepsilon'_{ji}$ . It is easy to see that the preprojective relations for  $\langle, \rangle$  and  $\varepsilon$  are the same than those for  $\langle, \rangle'$  and  $\varepsilon'$ , which ends the proof.  $\square$

- Remarks 14.*
- Because of the previous lemma, we will not specify  $\varepsilon$  or  $\langle, \rangle$  in general.
  - If  $G$  is the trivial group and  $\overline{Q}$  is the double quiver of some quiver  $Q$ , we obtain the classical definition of the representations of the preprojective algebra of a quiver.
  - If, for  $i \in I$ ,  $k[G_i]$  is replaced by a division algebra, one gets back the definition of the preprojective algebra of a species introduced in [DR80] (for more details, see also [Rin98, §6]).

#### 4. PROOFS OF THE MAIN PROPOSITIONS

We retain the notation of section 1. Thus  $Q$  is a quiver,  $G$  acts on  $kQ$  by stabilizing primitive idempotents,  $\tilde{I}$  is a fixed set of representatives of the  $G$ -orbits of  $I$ , etc.. For  $(i, j) \in \tilde{I}^2$ , define

$$\tilde{A}_{ij} = \bigoplus_{(i', j') \in F_{ij}} G_j \kappa_{j'}^{-1} A_{i'j'} \kappa_{i'} G_i \subset (kQ)G$$

which is a  $(k[G_j], k[G_i])$ -bimodule. Remark that, for every  $(i', j') \in I^2$  and  $(i'', j'')$  belonging to the  $G$ -orbit of  $(i', j')$ , one has

$$G_j \kappa_{j'}^{-1} A_{i'j'} \kappa_{i'} G_i = G_j \kappa_{j''}^{-1} A_{i''j''} \kappa_{i''} G_i$$

and therefore, the previous decomposition does not depend on the choice of  $F_{ij}$ . This fact will be used in the computations of proposition 17.

Now let  $\tilde{Q}_G$  be the group species defined by  $\tilde{Q}_G = \left( \tilde{I}, (G_i)_{i \in \tilde{I}}, (\tilde{A}_{ij})_{(i,j) \in \tilde{I}^2} \right)$ .

**Proposition 15.** *The category of representations of  $\tilde{Q}_G$  is equivalent to  $\text{mod}(kQ)G$ .*

*Proof.* Let  $V \in \text{mod}(kQ)G$ . For each  $i \in \tilde{I}$ , let  $V_i = e_i V$  which is a representation of  $G_i$  because for  $g \in G_i$ ,  $g$  and  $e_i$  commute in  $(kQ)G$ . For each  $(i, j) \in \tilde{I}^2$ , let

$$\begin{aligned}x_{ij} : \tilde{A}_{ij} \otimes_{G_i} V_i &\rightarrow V_j \\ a \otimes v &\mapsto av\end{aligned}$$

which is clearly a morphism of representations.

Hence, one gets a representation

$$\Phi(V) = \left( (V_i)_{i \in \tilde{I}}, (x_{ij})_{(i,j) \in \tilde{I}^2} \right)$$

of  $\tilde{Q}_G$ . If  $f \in \text{Hom}_{(kQ)G}(V, W)$ , it is easy that it induces a morphism  $\Phi(f)$  from  $\Phi(V)$  to  $\Phi(W)$  by taking the restrictions to the  $(V_i)_{i \in \tilde{I}}$ . So  $\Phi$  is a functor.

Let  $f \in \text{Hom}_{(kQ)G}(V, W)$  and suppose that  $\Phi(f) = 0$ . If  $i \in I$ , for every  $v \in e_i V$ ,

$$f(v) = \kappa_i f(\kappa_i^{-1} v) = \kappa_i \Phi(f)(\kappa_i^{-1} v) = 0.$$

In other terms,  $\Phi$  is faithful. Let now

$$f' : \Phi(V) = \left( (V_i)_{i \in \tilde{I}}, (x_{ij})_{(i,j) \in \tilde{I}^2} \right) \rightarrow \Phi(W) = \left( (W_i)_{i \in \tilde{I}}, (y_{ij})_{(i,j) \in \tilde{I}^2} \right)$$

be a morphism. For each  $i \in I$ , define  $f_i : e_i V \rightarrow e_i W$  by  $f_i(v) = \kappa_i f'_{i_o}(\kappa_i^{-1}v)$  and  $f : V \rightarrow W$  by  $f = \bigoplus_{i \in I} f_i$ . Now, for each  $g \in G$ ,  $i \in I$  and  $v \in V_i$ , one have

$$f(gv) = f_{gi}(gv) = \kappa_{gi} f'_{i_o}(\kappa_{gi}^{-1}gv) = \kappa_{gi} f'_{i_o}(\kappa_{gi}^{-1}g\kappa_i \kappa_i^{-1}v) = \kappa_{gi} \kappa_{gi}^{-1} g \kappa_i f'_{i_o}(\kappa_i^{-1}v) = g f_i(v) = g f(v)$$

the fourth equality being true because  $\kappa_{gi}^{-1}g\kappa_i \in G_{i_o}$ . Let  $(i, j) \in I^2$ ,  $a \in A_{ij}$  and  $v \in V$ . Let  $(i', j') \in F_{i_o j_o}$  and  $g \in G$  such that  $(i, j) = g(i', j')$ .

$$f(av) = f_j(av) = \kappa_j f'_{j_o}(\kappa_j^{-1}av) = \kappa_j f'_{j_o}(\kappa_j^{-1}g(g^{-1}ag)g^{-1}v) = g \kappa_{j'} f'_{j_o}(\kappa_{j'}^{-1}(g^{-1}ag)g^{-1}v)$$

as  $\kappa_{j'}^{-1}g\kappa_{j'} \in G_{j_o}$ . Hence

$$\begin{aligned} f(av) &= g \kappa_{j'} f'_{j_o}(x_{i_o j_o}(\kappa_{j'}^{-1}g^{-1}ag\kappa_{i'} \otimes \kappa_{i'}^{-1}g^{-1}v)) = g \kappa_{j'} y_{i_o j_o}(\kappa_{j'}^{-1}g^{-1}ag\kappa_{i'} \otimes f'_{j_o}(\kappa_{i'}^{-1}g^{-1}v)) \\ &= g \kappa_{j'}(\kappa_{j'}^{-1}g^{-1}ag\kappa_{i'} f'_{i_o}(\kappa_{i'}^{-1}g^{-1}v)) = ag\kappa_{i'} f'_{i_o}(\kappa_{i'}^{-1}g^{-1}v) \end{aligned}$$

and because  $\kappa_{i'}^{-1}g^{-1}\kappa_i \in G_{i_o}$ , one have

$$f(av) = a \kappa_i f'_{i_o}(\kappa_i^{-1}v) = a f_i(v) = a f(v)$$

so that  $f$  is a morphism in  $\text{mod}(kQ)G$ . Moreover, one has  $\Phi(f) = f'$  and then  $\Phi$  is full.

Let now  $\left( (V'_i)_{i \in \tilde{I}}, (x_{ij})_{(i,j) \in \tilde{I}^2} \right)$  be a representation of  $\tilde{Q}_G$ . For each  $i \in I$ , fix a vector space  $V_i$  and an isomorphism  $\iota_i : V'_{i_o} \xrightarrow{\sim} V_i$ . Let  $V = \bigoplus_{i \in I} V_i$ . For  $g \in G$ ,  $\kappa_{gi}^{-1}g\kappa_i \in G_{i_o}$  and then, for every  $v_i \in V_i$ , one can define

$$gv_i = \iota_{gi}(\kappa_{gi}^{-1}g\kappa_i \iota_i^{-1}(v_i))$$

which leads clearly to an action of  $G$  on  $V$ . For  $i \in I$ ,  $e_i \in (kQ)G$  acts on  $V$  by

$$e_i \sum_{i' \in I} v_{i'} = v_i.$$

Let  $(i, j) \in I^2$ . For  $a \in A_{ij}$ ,  $a$  acts on  $V$  by

$$a \sum_{\ell \in I} v_\ell = \iota_j \left( x_{i_o j_o} \left( \kappa_j^{-1} a \kappa_i \otimes \iota_i^{-1}(v_i) \right) \right)$$

which clearly defines an action of  $kQ$  on  $V$  (to see that  $\kappa_j^{-1}a\kappa_i \in \tilde{A}_{i_o j_o}$ , it is enough to choose  $(i', j') \in F_{i_o j_o}$  and  $g \in G$  such that  $g(i', j') = (i, j)$ ). Now, keeping the same notations, if  $h \in G$ ,

$$\begin{aligned} h \left( a \left( h^{-1} \sum_{\ell \in I} v_\ell \right) \right) &= h \left( a \sum_{\ell \in I} \iota_{h^{-1}\ell}(\kappa_{h^{-1}\ell}^{-1} h^{-1} \kappa_\ell \iota_\ell^{-1}(v_\ell)) \right) \\ &= h \iota_j \left( x_{i_o j_o} \left( \kappa_j^{-1} a \kappa_i \otimes \iota_i^{-1}(\iota_i(\kappa_i^{-1} h^{-1} \kappa_{hi} \iota_{hi}^{-1}(v_{hi}))) \right) \right) \\ &= \iota_{hj} \left( \kappa_{hj}^{-1} h \kappa_j x_{i_o j_o} \left( \kappa_j^{-1} a \kappa_i \otimes \kappa_i^{-1} h^{-1} \kappa_{hi} \iota_{hi}^{-1}(v_{hi}) \right) \right) \\ &= \iota_{hj} \left( x_{i_o j_o} \left( \kappa_{hj}^{-1} h \kappa_j \kappa_j^{-1} a \kappa_i \otimes \kappa_i^{-1} h^{-1} \kappa_{hi} \iota_{hi}^{-1}(v_{hi}) \right) \right) \end{aligned}$$

because  $\kappa_{hj}^{-1}h\kappa_j \in G_{j_o}$ . Hence

$$h \left( a \left( h^{-1} \sum_{\ell \in I} v_\ell \right) \right) = \iota_{hj} \left( x_{i_o j_o} \left( \kappa_{hj}^{-1} h a \kappa_i \kappa_i^{-1} h^{-1} \kappa_{hi} \otimes \iota_{hi}^{-1}(v_{hi}) \right) \right)$$

because  $\kappa_i^{-1}h^{-1}\kappa_{hi} \in G_{i_o}$ . Finally,

$$h \left( a \left( h^{-1} \sum_{\ell \in I} v_\ell \right) \right) = \iota_{hj} \left( x_{i_o j_o} \left( \kappa_{hj}^{-1} (h a h^{-1}) \kappa_{hi} \otimes \iota_{hi}^{-1}(v_{hi}) \right) \right) = (h a h^{-1}) \sum_{\ell \in I} v_\ell$$



and therefore an action of  $(kQ)G$  is defined on  $V$ . Moreover, it is easy to see that

$$\begin{aligned}\Phi(V) &\rightarrow \left( (V'_i)_{i \in \tilde{I}}, (x_{ij})_{(i,j) \in \tilde{I}^2} \right) \\ (v_i)_{i \in \tilde{I}} &\mapsto (\kappa_i v_i^{-1}(v_i))_{i \in \tilde{I}}\end{aligned}$$

is an isomorphism and therefore,  $\Phi$  is essentially surjective. As  $\Phi$  is fully faithful and essentially surjective,  $\Phi$  is an equivalence of categories.  $\square$

Suppose now that  $Q$  is a double quiver. We regard it as a double group species with trivial group at each vertex. Thus we have a pairing  $\langle, \rangle$  and an  $\varepsilon$ . Moreover, the group  $G$  still acts on  $kQ$ , and we will assume that  $\langle, \rangle$  and  $\varepsilon$  satisfy

$$(*) \quad \begin{aligned} \forall (i, j) \in I^2, \forall (a_1, a_2) \in A_{ij} \times A_{ji}, \forall g \in G, \\ \langle ga_1 g^{-1}, ga_2 g^{-1} \rangle_{gi, gj} = \langle a_1, a_2 \rangle_{ij} \text{ and } \varepsilon_{gi, gj}(ga_1 g^{-1}) = g \varepsilon_{ij}(a_1) g^{-1}. \end{aligned}$$

For each  $(i, j) \in \tilde{I}$ ,  $(i', j'), (i'', j'') \in F_{ij}$ ,  $a_1 \in A_{i'j'}$ ,  $a_2 \in A_{j''i''}$ ,  $(g_1, g_2) \in G_i$  and  $(h_1, h_2) \in G_j$ , let

$$\left\langle h_1 \kappa_{j'}^{-1} a_1 \kappa_{i'} g_1, g_2 \kappa_{i''}^{-1} a_2 \kappa_{j''} h_2 \right\rangle'_{ij} = \left\langle \kappa_{j''} h_2 h_1 \kappa_{j'}^{-1} a_1 \kappa_{i'} g_1 g_2 \kappa_{i''}^{-1}, a_2 \right\rangle$$

if  $(i'', j'') = (i', j')$  and  $\kappa_{j''} h_2 h_1 \kappa_{j'}^{-1} \kappa_{i'} g_1 g_2 \kappa_{i''}^{-1} = 1$  and

$$\left\langle h_1 \kappa_{j'}^{-1} a_1 \kappa_{i'} g_1, g_2 \kappa_{i''}^{-1} a_2 \kappa_{j''} h_2 \right\rangle'_{ij} = 0$$

otherwise. Define also  $\varepsilon'_{ij}$  by

$$\varepsilon'_{ij}(h_1 \kappa_{j'}^{-1} a_1 \kappa_{i'} g_1) = \frac{1}{\#G_i \#G_j} h_1 \kappa_{j'}^{-1} \varepsilon_{ij}(a_1) \kappa_{i'} g_1.$$

**Lemma 16.** *The bracket  $\langle, \rangle'$  and  $\varepsilon'$  endow  $\tilde{Q}_G$  with the structure of a double group species.*

*Proof.* First of all, if one chooses  $\bar{i}', \bar{j}', \bar{h}_1, \bar{a}_1, \bar{g}_1$  such that  $\bar{h}_1 \kappa_{\bar{j}'}^{-1} \bar{a}_1 \kappa_{\bar{i}'} \bar{g}_1 = h_1 \kappa_{j'}^{-1} a_1 \kappa_{i'} g_1$ , we get that  $\bar{a}_1 = \kappa_{\bar{j}'} \bar{h}_1^{-1} h_1 \kappa_{j'}^{-1} a_1 \kappa_{i'} g_1 \bar{g}_1^{-1} \kappa_{\bar{i}'}^{-1}$  and therefore, because of the presentation of  $(kQ)G$ , one has  $\kappa_{\bar{j}'} \bar{h}_1^{-1} h_1 \kappa_{j'}^{-1} \kappa_{i'} g_1 \bar{g}_1^{-1} \kappa_{\bar{i}'}^{-1} = 1$ . In other terms,  $\bar{h}_1 \kappa_{\bar{j}'}^{-1} \kappa_{\bar{i}'} \bar{g}_1 = h_1 \kappa_{j'}^{-1} \kappa_{i'} g_1$  and then  $\kappa_{j''} h_2 \bar{h}_1 \kappa_{\bar{j}'}^{-1} \kappa_{\bar{i}'} \bar{g}_1 g_2 \kappa_{i''}^{-1} = 1$  if and only if  $\kappa_{j''} h_2 h_1 \kappa_{j'}^{-1} \kappa_{i'} g_1 g_2 \kappa_{i''}^{-1} = 1$ . On the other hand,  $\kappa_{j'} h_1^{-1} \bar{h}_1 \kappa_{\bar{j}'}^{-1} = \kappa_{i'} g_1 \bar{g}_1^{-1} \kappa_{\bar{i}'}^{-1}$  which is an element that sends  $(\bar{i}', \bar{j}')$  to  $(i', j')$ . By definition of  $F_{ij}$ , one gets that  $(\bar{i}', \bar{j}') = (i', j')$  and then  $\langle, \rangle'$  is well defined. By the same reasoning,  $\varepsilon'$  is well defined.

It is now clear that  $\langle, \rangle'_{ij}$  is a non degenerate pairing and that for every  $(a, a') \in \tilde{A}_{ij} \times \tilde{A}_{ji}$ ,  $\langle a, a' \rangle'_{ij} = \langle a', a \rangle'_{ji}$ . Keeping the same notation as before, one gets

$$\begin{aligned} \#G_i \#G_j \left\langle {}^t \varepsilon'_{ji}(h_1 \kappa_{j'}^{-1} a_1 \kappa_{i'} g_1), g_2 \kappa_{i''}^{-1} a_2 \kappa_{j''} h_2 \right\rangle'_{ij} &= \left\langle h_1 \kappa_{j'}^{-1} a_1 \kappa_{i'} g_1, g_2 \kappa_{i''}^{-1} \varepsilon_{ji}(a_2) \kappa_{j''} h_2 \right\rangle'_{ij} \\ &= \begin{cases} \left\langle \kappa_{j''} h_2 h_1 \kappa_{j'}^{-1} a_1 \kappa_{i'} g_1 g_2 \kappa_{i''}^{-1}, \varepsilon_{ji}(a_2) \right\rangle & \text{if } (i'', j'') = (i', j') \text{ and } \kappa_{j''} h_2 h_1 \kappa_{j'}^{-1} \kappa_{i'} g_1 g_2 \kappa_{i''}^{-1} = 1 \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} \left\langle -\varepsilon_{ij}(\kappa_{j''} h_2 h_1 \kappa_{j'}^{-1} a_1 \kappa_{i'} g_1 g_2 \kappa_{i''}^{-1}), a_2 \right\rangle & \text{if } (i'', j'') = (i', j') \text{ and } \kappa_{j''} h_2 h_1 \kappa_{j'}^{-1} \kappa_{i'} g_1 g_2 \kappa_{i''}^{-1} = 1 \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} \left\langle -\kappa_{j''} h_2 h_1 \kappa_{j'}^{-1} \varepsilon_{ij}(a_1) \kappa_{i'} g_1 g_2 \kappa_{i''}^{-1}, a_2 \right\rangle & \text{if } (i'', j'') = (i', j') \text{ and } \kappa_{j''} h_2 h_1 \kappa_{j'}^{-1} \kappa_{i'} g_1 g_2 \kappa_{i''}^{-1} = 1 \\ 0 & \text{otherwise} \end{cases} \\ &= \#G_i \#G_j \left\langle -\varepsilon'_{ij}(h_1 \kappa_{j'}^{-1} a_1 \kappa_{i'} g_1), g_2 \kappa_{i''}^{-1} a_2 \kappa_{j''} h_2 \right\rangle'_{ij} \end{aligned}$$

which shows that  ${}^t \varepsilon'_{ji} = -\varepsilon'_{ij}$ .  $\square$

**Proposition 17.** *The category of representations of  $(kQ)G$  satisfying the preprojective relations is equivalent to the category of representations of  $\tilde{Q}_G$  satisfying the preprojective relations.*

*Proof.* For every  $(i, j) \in I^2$ , denote by  $\mathcal{B}_{ij}$  a basis of  $A_{ij}$ . For each  $(g, h), (g', h') \in G_{i_o} \times G_{j_o}$ , say that  $(g, h) \sim (g', h')$  if  $h\kappa_j^{-1}\kappa_i g = h'\kappa_j^{-1}\kappa_i g'$  and then fix a set  $G_{ij}$  of representatives of the equivalence classes for this relation. Then, for  $(i, j) \in \tilde{I}^2$ ,

$$\tilde{\mathcal{B}}_{ij} = \bigcup_{(i', j') \in F_{ij}} \bigcup_{(g, h) \in G_{i'j'}} h\kappa_{j'}^{-1}\mathcal{B}_{i'j'}\kappa_{i'}g$$

is a basis of  $\tilde{A}_{ij}$ . Indeed, it is an easy consequence of the definition of  $\tilde{A}_{ij}$  and of the relations in  $(kQ)G$ . Moreover, if  $h\kappa_{j'}^{-1}a\kappa_{i'}g \in \tilde{\mathcal{B}}_{ij}$  then the corresponding element of  $\tilde{A}_{ji}$  in the dual basis of  $\tilde{\mathcal{B}}_{ij}$  for  $\langle \cdot, \cdot \rangle'$  is  $(h\kappa_{j'}^{-1}a\kappa_{i'}g)^* = g^{-1}\kappa_{i'}^{-1}a^*\kappa_{j'}h^{-1}$  where  $a^*$  is the element of  $A_{j'i'}$  corresponding to  $a$  in the dual basis of  $\mathcal{B}_{i'j'}$  for  $\langle \cdot, \cdot \rangle$  (this is an easy computation). We prove now that the map

$$\begin{aligned} G_{i'j'} &\rightarrow I \\ (g, h) &\mapsto g^{-1}\kappa_{i'}^{-1}j' \end{aligned}$$

has image  $I_{i'j'} = \{j'' \in I \mid \exists g \in G, (i, j'') = g(i', j')\}$  and that each element of  $I_{i'j'}$  has exactly  $\#G_j$  preimages. First of all, it is clear that the image is a subset of  $I_{i'j'}$ . Let now  $j'' \in I_{i'j'}$ . There exists  $h_o \in G$  such that  $(i, j'') = h_o(i', j')$ . The equality  $i = h_o i'$  is equivalent to the fact that  $h_o = g_o^{-1}\kappa_{i'}^{-1}$  for some  $g_o \in G_i$ . The aim is now to count the  $(g, h) \in G_{i'j'}$  such that  $g^{-1}\kappa_{i'}^{-1}j' = g_o^{-1}\kappa_{i'}^{-1}j'$  or in other terms  $\kappa_{i'}g g_o^{-1}\kappa_{i'}^{-1} \in G_{j'}$ . Finally, as  $G_i = \kappa_{i'}^{-1}G_{i'}\kappa_{i'}g_o$ , one has to compute

$$\#((\kappa_{i'}^{-1}(G_{i'} \cap G_{j'})\kappa_{i'}g_o \times G_j) \cap G_{i'j'}).$$

If  $(g, h) \in \kappa_{i'}^{-1}(G_{i'} \cap G_{j'})\kappa_{i'}g_o \times G_j$ , its equivalence class for  $\sim$  is

$$\begin{aligned} &\left\{ \left( g', h\kappa_{j'}^{-1}\kappa_{i'}g g'^{-1}\kappa_{i'}^{-1}\kappa_{j'} \right) \in G_i \times G_j \right\} \\ &= \left\{ \left( g', h\kappa_{j'}^{-1}\kappa_{i'}g g'^{-1}\kappa_{i'}^{-1}\kappa_{j'} \right) \in \kappa_{i'}^{-1}(G_{i'} \cap G_{j'})\kappa_{i'}g_o \times G_j \right\} \end{aligned}$$

which is a subset of  $\kappa_{i'}^{-1}(G_{i'} \cap G_{j'})\kappa_{i'}g_o \times G_j$  of cardinality  $\#(G_{i'} \cap G_{j'})$ . One deduces easily that

$$\#((\kappa_{i'}^{-1}(G_{i'} \cap G_{j'})\kappa_{i'}g_o \times G_j) \cap G_{i'j'}) = \#G_j.$$

One also has

$$I = \coprod_{j \in \tilde{I}} \coprod_{(i', j') \in F_{ij}} I_{i'j'}.$$

Now, it is immediate that  $\text{mod } \Lambda_Q G$  can be seen as the category of representations of  $(kQ)G$  satisfying the preprojective relations, in the obvious way. Let  $V = ((V_i)_{i \in I}, (x_{ij})_{(i,j) \in I^2}) \in \text{mod}(kQ)G$ . Then, using the same notations as in the proposition 15,

$$\Phi(V) = \left( (V_i)_{i \in \tilde{I}}, (y_{ij})_{(i,j) \in \tilde{I}^2} \right)$$

satisfies the preprojective relations if and only if, for all  $i \in \tilde{I}$  and  $v \in V_i$ ,

$$\sum_{j \in \tilde{I}} y_{ji}((\varepsilon'_{ji} \otimes \text{Id}_{V_j})(\tilde{y}_{ij}(v))) = 0.$$

It is equivalent to :

$$\begin{aligned}
0 &= \sum_{j \in \tilde{I}} y_{ji} \left( (\varepsilon'_{ji} \otimes \text{Id}_{V_j}) \left( \sum_{a \in \tilde{\mathcal{B}}_{ij}} a^* \otimes y_{ij}(a \otimes v) \right) \right) \\
&= \sum_{j \in \tilde{I}} y_{ji} \left( \sum_{a \in \tilde{\mathcal{B}}_{ij}} \varepsilon'_{ji}(a^*) \otimes av \right) = \sum_{j \in \tilde{I}} \sum_{a \in \tilde{\mathcal{B}}_{ij}} \varepsilon'_{ji}(a^*) av \\
&= \sum_{j \in \tilde{I}} \sum_{(i', j') \in F_{ij}} \sum_{(g, h) \in G_{i'j'}} \sum_{a \in \mathcal{B}_{i'j'}} \varepsilon'_{ji}((h\kappa_{j'}^{-1}a\kappa_{i'}g)^*) h\kappa_{j'}^{-1}a\kappa_{i'}gv \\
&= \sum_{j \in \tilde{I}} \sum_{(i', j') \in F_{ij}} \sum_{(g, h) \in G_{i'j'}} \sum_{a \in \mathcal{B}_{i'j'}} \varepsilon'_{ji}(g^{-1}\kappa_{i'}^{-1}a^*\kappa_{j'}h^{-1}) h\kappa_{j'}^{-1}a\kappa_{i'}gv \\
&= \sum_{j \in \tilde{I}} \sum_{(i', j') \in F_{ij}} \sum_{(g, h) \in G_{i'j'}} \sum_{a \in \mathcal{B}_{i'j'}} \frac{1}{\#G_{i'}\#G_{j'}} g^{-1}\kappa_{i'}^{-1}\varepsilon_{j'i'}(a^*)a\kappa_{i'}gv \\
&= \frac{1}{\#G_i} \sum_{j \in \tilde{I}} \frac{1}{\#G_j} \sum_{(i', j') \in F_{ij}} \sum_{(g, h) \in G_{i'j'}} \sum_{a \in \mathcal{B}_{i'j'}} \varepsilon_{g^{-1}\kappa_{i'}^{-1}j', g^{-1}\kappa_{i'}^{-1}i'}(g^{-1}\kappa_{i'}^{-1}a^*\kappa_{i'}g) g^{-1}\kappa_{i'}^{-1}a\kappa_{i'}gv \\
&= \frac{1}{\#G_i} \sum_{j \in \tilde{I}} \frac{1}{\#G_j} \sum_{(i', j') \in F_{ij}} \sum_{(g, h) \in G_{i'j'}} \sum_{a \in \mathcal{B}_{i'j'}} \varepsilon_{g^{-1}\kappa_{i'}^{-1}j', i}((g^{-1}\kappa_{i'}^{-1}a\kappa_{i'}g)^*) g^{-1}\kappa_{i'}^{-1}a\kappa_{i'}gv \\
&= \frac{1}{\#G_i} \sum_{j \in \tilde{I}} \frac{1}{\#G_j} \sum_{(i', j') \in F_{ij}} \sum_{(g, h) \in G_{i'j'}} \sum_{a \in g^{-1}\kappa_{i'}^{-1}\mathcal{B}_{i'j'}\kappa_{i'}g} \varepsilon_{g^{-1}\kappa_{i'}^{-1}j', i}(a^*)av \\
&= \frac{1}{\#G_i} \sum_{j \in \tilde{I}} \frac{1}{\#G_j} \sum_{(i', j') \in F_{ij}} \sum_{(g, h) \in G_{i'j'}} \sum_{a \in \mathcal{B}_{i, g^{-1}\kappa_{i'}^{-1}j'}} \varepsilon_{g^{-1}\kappa_{i'}^{-1}j', i}(a^*)av \\
&= \frac{1}{\#G_i} \sum_{j \in \tilde{I}} \frac{1}{\#G_j} \sum_{(i', j') \in F_{ij}} \sum_{(g, h) \in G_{i'j'}} x_{g^{-1}\kappa_{i'}^{-1}j', i} \left( \left( \varepsilon_{g^{-1}\kappa_{i'}^{-1}j', i} \otimes \text{Id}_{V_{g^{-1}\kappa_{i'}^{-1}j'}} \right) (\tilde{x}_{i, g^{-1}\kappa_{i'}^{-1}j'}(v)) \right) \\
&= \frac{1}{\#G_i} \sum_{j \in \tilde{I}} x_{j, i} \left( (\varepsilon_{j, i} \otimes \text{Id}_{V_j}) (\tilde{x}_{i, j}(v)) \right)
\end{aligned}$$

which is the preprojective relation for  $V$  at the vertex  $i$ . Moreover, all preprojective relations of  $Q$  are conjugates by elements of  $G$  to some of these relations for  $i \in \tilde{I}$ .  $\square$

We suppose now that the characteristic of  $k$  does not divide the cardinality of  $G$  and that the simple representations of  $G$  on  $k$  are absolutely indecomposable, in the sense that for every simple representation  $\rho$  of  $G$ ,  $\rho \otimes_k \bar{k}$  is simple over the algebraic closure  $\bar{k}$  of  $k$ , or equivalently  $\text{End}_{k[G]}(\rho) \simeq k$ . In particular, this is the case when  $k = \bar{k}$ . As in section 1, define

$$I_G = \bigcup_{i \in \tilde{I}} \{i\} \times \text{irr}(G_i)$$

where  $\text{irr}(G_i)$  is a set of representatives of isomorphism classes of irreducible representations of  $G_i$ . For every  $(i, \rho), (j, \sigma) \in I_G$ , define

$$A'_{(i, \rho)(j, \sigma)} = \text{Hom}_{G_j}(\tilde{A}_{ij} \otimes_{G_i} \rho, \sigma)^*$$

Let now  $Q_G$  be the quiver  $(I_G, (A'_{ij})_{(i, j) \in I_G^2})$ .

*Remark 18.* One has another way to compute the arrows :

$$A'_{(i, \rho)(j, \sigma)} \simeq \bigoplus_{(i', j') \in F_{ij}} \text{Hom}_{G_{i'} \cap G_{j'}} \left( (\kappa_{i'} \cdot \rho)|_{G_{i'} \cap G_{j'}} \otimes_k A_{i'j'}, (\kappa_{j'} \cdot \sigma)|_{G_{i'} \cap G_{j'}} \right)^*$$

where the representation  $\kappa_{i'} \cdot \rho$  of  $G_{i'}$  is the same as  $\rho$  as a vector space and where, if  $g \in G_{i'} = \kappa_{i'} G_i \kappa_{i'}^{-1}$ ,  $(\kappa_{i'} \cdot \rho)_g = \rho_{\kappa_{i'}^{-1} g \kappa_{i'}}$ .

**Proposition 19.** *There is an equivalence of categories between  $\text{mod } kQ_G$  and the category of representations of  $\tilde{Q}_G$ .*

*Proof.* For each  $(i, \rho), (j, \sigma) \in I_G$  and for every pair of vector spaces  $(V, W)$ , there is a linear isomorphism :

$$\varphi_{(i,\rho),(j,\sigma)} : \text{Hom}_{\mathbb{C}} \left( \text{Hom}_{G_j} \left( \tilde{A}_{ij} \otimes_{G_i} \rho, \sigma \right)^* \otimes_k V, W \right) \rightarrow \text{Hom}_{G_j} \left( \tilde{A}_{ij} \otimes_{G_i} \rho \otimes_k V, \sigma \otimes_k W \right)$$

which is bifunctorial in  $(V, W)$  (contravariantly in  $V$  and covariantly in  $W$ ). It is defined by

$$\varphi_{(i,\rho),(j,\sigma)}(f)(a \otimes t \otimes v) = \sum_{\alpha \in \mathcal{B}} \alpha(a \otimes t) \otimes f(\alpha^* \otimes v)$$

which does not depend on the choice of a basis  $\mathcal{B}$  of  $\text{Hom}_{G_j} \left( \tilde{A}_{ij} \otimes_{G_i} \rho, \sigma \right)$ . Its inverse is defined by

$$\varphi_{(i,\rho),(j,\sigma)}^{-1}(g)(\alpha \otimes v) = \sum_{w \in \mathcal{B}} \alpha \left( (\text{Id}_{\sigma} \otimes w^*) \circ g \circ (\text{Id}_{\tilde{A}_{ij} \otimes \rho} \otimes v) \right) \otimes w$$

which does not depend on the choice of a basis  $\mathcal{B}$  of  $W$ .

Let  $\left( (V_i)_{i \in I_G}, (x_{ij})_{(i,j) \in I_G^2} \right)$  be a representation of  $Q_G$ . For each  $i \in \tilde{I}$ , let

$$V'_i = \bigoplus_{\rho \in \text{irr}(G_i)} \rho \otimes_k V_{(i,\rho)}.$$

We regard it as a representation of  $G_i$  and for each  $(i, j) \in \tilde{I}$ , we set

$$x'_{ij} = \left( \varphi_{(i,\rho),(j,\sigma)}(x_{(i,\rho),(j,\sigma)}) \right)_{(\sigma,\rho) \in \text{irr}(G_j) \times \text{irr}(G_i)}$$

with the usual matrix notation. Denote  $\Psi \left( (V_i)_{i \in I_G}, (x_{ij})_{(i,j) \in I_G^2} \right) = \left( (V'_i)_{i \in \tilde{I}}, (x'_{ij})_{(i,j) \in \tilde{I}^2} \right)$ . By functoriality of  $\varphi_{(i,\rho),(j,\sigma)}$ , a morphism  $f : \left( (V_i)_{i \in I_G}, (x_{ij})_{(i,j) \in I_G^2} \right) \rightarrow \left( (W_i)_{i \in I_G}, (y_{ij})_{(i,j) \in I_G^2} \right)$  induces a morphism  $\Psi(f) : \Psi \left( (V_i)_{i \in I_G}, (x_{ij})_{(i,j) \in I_G^2} \right) \rightarrow \Psi \left( (W_i)_{i \in I_G}, (y_{ij})_{(i,j) \in I_G^2} \right)$  by setting for each  $i \in \tilde{I}$

$$\Psi(f)_i = \bigoplus_{\rho \in \text{irr}(G_i)} \text{Id}_{\rho} \otimes f_{(i,\rho)}.$$

Hence one gets a functor  $\Phi$  from the category of representations of  $Q_G$  to the category of representations of  $\tilde{Q}_G$ . It is easily seen to be faithful and essentially surjective (because  $\varphi_{(i,\rho),(j,\sigma)}$  is an isomorphism). As  $k$  is algebraically closed and its characteristic does not divide the cardinality of  $G$ , for each  $(i, \rho) \in I_G$ ,  $\text{Hom}_{\mathbb{C}} \left( V_{(i,\rho)}, W_{(i,\rho)} \right) \rightarrow \text{Hom}_{G_i} \left( \rho \otimes_k V_{(i,\rho)}, \rho \otimes_k W_{(i,\rho)} \right)$ ,  $f \mapsto \text{Id}_{\rho} \otimes f$  is an isomorphism and therefore,  $\Phi$  is full.  $\square$

Let  $(i, \rho), (j, \sigma) \in I_G^2$ . For  $f \in A_{(i,\rho),(j,\sigma)}^* = \text{Hom}_{G_j}(\tilde{A}_{ij} \otimes_{G_i} \rho, \sigma)$  and  $g \in A_{(j,\sigma),(i,\rho)}^* = \text{Hom}_{G_i}(\tilde{A}_{ji} \otimes_{G_j} \sigma, \rho)$ , define

$$\langle f, g \rangle'' = \sum_{a \in \tilde{\mathcal{B}}_{ij}} \text{tr} \left( (s \mapsto g(a^* \otimes s)) \circ (r \mapsto f(a \otimes r)) \right)$$

which is clearly bilinear symmetric. Remark also that for every  $r \in \rho$ ,

$$\sum_{a \in \tilde{\mathcal{B}}_{ij}} g(a^* \otimes f(a \otimes r)) = \frac{\langle f, g \rangle''}{\dim \rho} r$$

by comparing the traces and by observing that the left hand side is a morphism from  $\rho$  to  $\rho$  that must be a multiple of the identity because  $\rho$  is simple. Moreover, the map that sends  $f$  to  $r \mapsto \sum_{a \in \tilde{\mathcal{B}}_{ij}} a^* \otimes f(a \otimes r)$  is an isomorphism from  $\text{Hom}_{G_j}(\tilde{A}_{ij} \otimes_{G_i} \rho, \sigma)$  to  $\text{Hom}_{G_i}(\rho, \tilde{A}_{ji} \otimes_{G_j} \sigma)$

as observed in the definition of the preprojective relations. In particular if  $f \neq 0$  then  $r \mapsto \sum_{a \in \tilde{\mathcal{B}}_{ij}} a^* \otimes f(a \otimes r)$  is not null and  $\langle f, - \rangle'' \neq 0$ . So that  $\langle, \rangle''$  is not degenerate. By biduality, one has also a non degenerate pairing between  $A'_{(i,\rho),(j,\sigma)}$  and  $A'_{(j,\sigma),(i,\rho)}$  which will be also denoted by  $\langle, \rangle''$ . Define also

$$\varepsilon''_{(j,\sigma),(i,\rho)}(f) = f \circ \left( {}^t \varepsilon'_{ji} \otimes \text{Id}_\rho \right) = -f \circ (\varepsilon'_{ij} \otimes \text{Id}_\rho).$$

One then has for every  $r \in \rho$

$$\begin{aligned} \left\langle \varepsilon''_{(j,\sigma),(i,\rho)}(f), g \right\rangle'' r &= -\dim(\rho) \sum_{a \in \tilde{\mathcal{B}}_{ij}} g(a^* \otimes f(\varepsilon'_{ij}(a) \otimes r)) \\ &= -\dim(\rho) \sum_{a \in \tilde{\mathcal{B}}_{ij}} g\left(\left(\varepsilon'^{-1}_{ij}(a)\right)^* \otimes f(a \otimes r)\right) \\ &= -\dim(\rho) \sum_{a \in \tilde{\mathcal{B}}_{ij}} g\left({}^t \varepsilon'_{ij}(a^*) \otimes f(a \otimes r)\right) \\ &= -\left\langle f, \varepsilon''_{(i,\rho),(j,\sigma)}(g) \right\rangle'' r \end{aligned}$$

so that  $\varepsilon''_{(j,\sigma),(i,\rho)} = -{}^t \varepsilon''_{(i,\rho),(j,\sigma)}$ . The automorphism  $\varepsilon''_{(i,\rho),(j,\sigma)}$  of  $A'^*_{(j,\sigma),(i,\rho)}$  can be seen as an automorphism  $A'_{(i,\rho),(j,\sigma)}$  through the pairing  $\langle, \rangle''$  and therefore,  $Q_G$  has a structure of double quiver.

**Proposition 20.** *The category of representations of  $Q_G$  satisfying the preprojective relations is equivalent to the category of representations of  $\tilde{Q}_G$  satisfying the preprojective relations.*

*Proof.* For every  $(i, \rho), (j, \sigma) \in I_G^2$ , let  $\mathcal{B}_{(i,\rho),(j,\sigma)}$  be a basis of  $\text{Hom}_{G_j}(\tilde{A}_{ij} \otimes_{G_i} \rho, \sigma)$  such that  $\mathcal{B}_{(i,\rho),(j,\sigma)}$  and  $\mathcal{B}_{(j,\sigma),(i,\rho)}$  are dual basis for  $\langle, \rangle''$ .

Let  $V = \left( (V_i)_{i \in I_G}, (x_{ij})_{(i,j) \in I_G^2} \right) \in \text{mod } kQ_G$ . Then, using the same notation as in proposition 19,  $\Psi(V) = \left( (W_i)_{i \in \tilde{I}}, (y_{ij})_{(i,j) \in \tilde{I}^2} \right)$  satisfies the preprojective relations if and only if, for all  $i \in \tilde{I}$ ,  $\rho \in \text{irr}(G_i)$  and  $r \otimes v \in \rho \otimes V_{(i,\rho)} \subset W_i$ ,

$$\begin{aligned} 0 &= \sum_{j \in \tilde{I}} y_{ji} \left( (\varepsilon'_{ji} \otimes \text{Id}_{W_j})(\tilde{y}_{ij}(r \otimes v)) \right) = \sum_{j \in \tilde{I}} y_{ji} \left( (\varepsilon'_{ji} \otimes \text{Id}_{W_j}) \left( \sum_{a \in \tilde{\mathcal{B}}_{ij}} a^* \otimes y_{ij}(a \otimes r \otimes v) \right) \right) \\ &= \sum_{j \in \tilde{I}} y_{ji} \left( (\varepsilon'_{ji} \otimes \text{Id}_{W_j}) \left( \sum_{a \in \tilde{\mathcal{B}}_{ij}} a^* \otimes \sum_{\sigma \in \text{irr}(G_j)} \sum_{\alpha \in \mathcal{B}_{(i,\rho),(j,\sigma)}} \alpha(a \otimes r) \otimes x_{(i,\rho),(j,\sigma)}(\alpha^* \otimes v) \right) \right) \\ &= \sum_{j \in \tilde{I}} \sum_{a \in \tilde{\mathcal{B}}_{ij}} \sum_{\sigma \in \text{irr}(G_j)} \sum_{\alpha \in \mathcal{B}_{(i,\rho),(j,\sigma)}} y_{ji} \left( \varepsilon'_{ji}(a^*) \otimes \alpha(a \otimes r) \otimes x_{(i,\rho),(j,\sigma)}(\alpha^* \otimes v) \right) \\ &= \sum_{(j,\sigma) \in I_G} \sum_{a \in \tilde{\mathcal{B}}_{ij}} \sum_{\tau \in \text{irr}(G_i)} \sum_{\substack{\alpha \in \mathcal{B}_{(i,\rho),(j,\sigma)} \\ \beta \in \mathcal{B}_{(j,\sigma),(i,\tau)}}} \beta(\varepsilon'_{ji}(a^*) \otimes \alpha(a \otimes r)) \otimes x_{(j,\sigma),(i,\rho)}(\beta^* \otimes x_{(i,\rho),(j,\sigma)}(\alpha^* \otimes v)) \end{aligned}$$

For  $(j, \sigma) \in I_G$ ,  $\tau \in \text{irr}(G_i)$ ,  $\alpha \in \mathcal{B}_{(i,\rho),(j,\sigma)}$ ,  $\beta \in \mathcal{B}_{(j,\sigma),(i,\tau)}$  and  $r \in \rho$ , define

$$\varphi_{\alpha,\beta}(r) = \sum_{a \in \tilde{\mathcal{B}}_{ij}} \beta(\varepsilon'_{ji}(a^*) \otimes \alpha(a \otimes r))$$

which does not depend on the basis  $\tilde{\mathcal{B}}_{ij}$  of  $\tilde{A}_{ij}$ . One then has, for every  $g \in G_i$ ,

$$\begin{aligned}\varphi_{\alpha,\beta}(gr) &= \sum_{a \in \tilde{\mathcal{B}}_{ij}} \beta(\varepsilon'_{ji}(a^*) \otimes \alpha(a \otimes gr)) = \sum_{a \in \tilde{\mathcal{B}}_{ij}} \beta(\varepsilon'_{ji}(a^*) \otimes \alpha(ag \otimes r)) \\ &= \sum_{a \in \tilde{\mathcal{B}}_{ij}g} \beta(\varepsilon'_{ji}((ag^{-1})^*) \otimes \alpha(a \otimes r)) = \sum_{a \in \tilde{\mathcal{B}}_{ij}g} \beta(\varepsilon'_{ji}(ga^*) \otimes \alpha(a \otimes r)) \\ &= g \sum_{a \in \tilde{\mathcal{B}}_{ij}g} \beta(\varepsilon'_{ji}(a^*) \otimes \alpha(a \otimes r)) = g\varphi_{\alpha,\beta}(r)\end{aligned}$$

so that  $\varphi_{\alpha,\beta}$  is a morphism from  $\rho$  to  $\tau$ . Therefore, if  $\tau \neq \rho$ ,  $\varphi_{\alpha,\beta} = 0$ . One also has

$$\begin{aligned}\varphi_{\alpha,\varepsilon''_{ij}^{-1}(\beta)}(r) &= \sum_{a \in \tilde{\mathcal{B}}_{ij}} \varepsilon''_{ij}^{-1}(\beta)(\varepsilon'_{ji}(a^*) \otimes \alpha(a \otimes r)) \\ &= - \sum_{a \in \tilde{\mathcal{B}}_{ij}} (\beta \circ (\varepsilon'_{ji}{}^{-1} \otimes \text{Id}_\sigma))(\varepsilon'_{ji}(a^*) \otimes \alpha(a \otimes r)) \\ &= - \sum_{a \in \tilde{\mathcal{B}}_{ij}} \beta(a^* \otimes \alpha(a \otimes r)) = -\frac{\langle \alpha, \beta \rangle''}{\dim \rho} r\end{aligned}$$

Then  $\Psi(V)$  satisfies the preprojective relations if and only if, for all  $(i, \rho) \in I_G$  and  $r \otimes v \in \rho \otimes V_{(i,\rho)} \subset W_i$ ,

$$\begin{aligned}0 &= \sum_{(j,\sigma) \in I_G} \sum_{\substack{\alpha \in \mathcal{B}_{(i,\rho),(j,\sigma)} \\ \beta \in \mathcal{B}_{(j,\sigma),(i,\rho)}}} \varphi_{\alpha,\beta}(r) \otimes x_{(j,\sigma),(i,\rho)}(\beta^* \otimes x_{(i,\rho),(j,\sigma)}(\alpha^* \otimes v)) \\ &= \sum_{(j,\sigma) \in I_G} \sum_{\substack{\alpha \in \mathcal{B}_{(i,\rho),(j,\sigma)} \\ \beta \in \mathcal{B}_{(j,\sigma),(i,\rho)}}} \varphi_{\alpha,\varepsilon''_{ij}^{-1}(\beta)}(r) \otimes x_{(j,\sigma),(i,\rho)}\left(\left(\varepsilon''_{ij}^{-1}(\beta)\right)^* \otimes x_{(i,\rho),(j,\sigma)}(\alpha^* \otimes v)\right) \\ &= - \sum_{(j,\sigma) \in I_G} \sum_{\substack{\alpha \in \mathcal{B}_{(i,\rho),(j,\sigma)} \\ \beta \in \mathcal{B}_{(j,\sigma),(i,\rho)}}} \frac{\langle \alpha, \beta \rangle''}{\dim \rho} r \otimes x_{(j,\sigma),(i,\rho)}\left({}^t\varepsilon''_{ij}(\beta^*) \otimes x_{(i,\rho),(j,\sigma)}(\alpha^* \otimes v)\right) \\ &= \frac{r}{\dim \rho} \otimes \sum_{(j,\sigma) \in I_G} \sum_{\alpha \in \mathcal{B}_{(i,\rho),(j,\sigma)}} x_{(j,\sigma),(i,\rho)}(\varepsilon''_{ji}(\alpha) \otimes x_{(i,\rho),(j,\sigma)}(\alpha^* \otimes v)) \\ &= \frac{r}{\dim \rho} \otimes \sum_{(j,\sigma) \in I_G} x_{(j,\sigma),(i,\rho)}\left((\varepsilon''_{ji} \otimes \text{Id})(\tilde{x}_{(i,\rho),(j,\sigma)}(v))\right)\end{aligned}$$

which concludes the proof.  $\square$

To conclude the proof of theorem 2, we need to relate its hypothesis to the condition (\*) given before lemma 16.

**Lemma 21.** *Let  $Q = (I, (A_{ij})_{(i,j) \in I^2})$  be a quiver endowed with a structure of double quiver given by  $\langle, \rangle$  and  $\varepsilon$ . Suppose that a finite group  $G$  acts on  $kQ$  by stabilizing  $I$ . Then the following are equivalent*

- (1) *The preprojective relation is stable under the action of  $G$ , in the sense that for every  $g \in G$ ,*

$$\sum_{(i,j) \in I^2} \sum_{a \in \mathcal{B}_{ij}} \varepsilon_{ji}(a^*) a = \left( \sum_{(i,j) \in I^2} \sum_{a \in \mathcal{B}_{ij}} \varepsilon_{ji}(a^*) a \right)^g$$

(note that each side does not depend on the choice of the basis  $\mathcal{B}_{ij}$  of  $A_{ij}$ ).

- (2) *There exists some double quiver structure  $(\langle, \rangle', \varepsilon')$  on  $Q$  such that the preprojective relation for  $\langle, \rangle'$  and  $\varepsilon'$  is the same than the one for  $\langle, \rangle$  and  $\varepsilon$  satisfying, for every  $g \in G$ , every  $(i, j) \in I^2$  and every  $(a_1, a_2) \in A_{ij} \times A_{ji}$ ,*

$$\langle a_1^g, a_2^g \rangle'_{gi, gj} = \langle a_1, a_2 \rangle'_{ij} \quad \text{and} \quad \varepsilon'_{gi, gj}(a_1^g) = \varepsilon'_{ij}(a_1)^g.$$

*Proof.* First of all, if the second condition is satisfied, then one can suppose that  $\langle, \rangle = \langle, \rangle'$  and  $\varepsilon = \varepsilon'$ . One then has

$$\left( \sum_{(i,j) \in I^2} \sum_{a \in \mathcal{B}_{ij}} \varepsilon_{ji}(a^*) a \right)^g = \sum_{(i,j) \in I^2} \sum_{a \in \mathcal{B}_{ij}} \varepsilon_{gj, gi}((a^*)^g) a^g = \sum_{(i,j) \in I^2} \sum_{a \in \mathcal{B}_{ij}} \varepsilon_{gj, gi}((a^g)^*) a^g$$

so that the first condition is satisfied because it does not depend on the choice of the basis  $\mathcal{B}_{ij}$ .

Now, suppose that the first condition is satisfied. It is easy to construct  $\langle, \rangle'$  that is  $G$ -invariant in the sens given before. Now, by the proof of the lemma 13,  $\varepsilon'$  can be chosen such that  $\langle, \rangle'$  and  $\varepsilon'$  leads to the same preprojective relation as  $\langle, \rangle$  and  $\varepsilon$ . We suppose now that  $\langle, \rangle' = \langle, \rangle$  and  $\varepsilon' = \varepsilon$ . One has

$$\sum_{(i,j) \in I^2} \sum_{a \in \mathcal{B}_{ij}} \varepsilon_{ji}(a^*) a = \sum_{(i,j) \in I^2} \sum_{a \in \mathcal{B}_{ij}} \varepsilon_{ji}(a^*)^g a^g = \sum_{(i,j) \in I^2} \sum_{a \in \mathcal{B}_{ij}} \varepsilon_{ji} \left( (a^{g^{-1}})^* \right)^g a$$

which implies, because of the algebra structure of  $kQ$  that for every  $(i, j) \in I^2$  and  $a \in \mathcal{B}_{ij}$ ,

$$\varepsilon_{g^{-1}j, g^{-1}i} \left( (a^{g^{-1}})^* \right)^g = \varepsilon_{ij}(a^*)$$

which is equivalent to  $\varepsilon_{gj, gi}((a^g)^*) = \varepsilon_{ij}(a^*)^g$ . As  $\langle, \rangle$  is  $G$ -invariant,  $(a^g)^* = (a^*)^g$  and therefore  $\varepsilon$  satisfies the searched relation.  $\square$

The following lemma leads to the computation of  $(\overline{Q})_G$  when  $\overline{Q}$  is the double quiver of a quiver  $Q$  and the action of  $G$  on  $k\overline{Q}$  comes from an action of  $G$  on  $kQ$ . This lemma relies corollary 3 with theorem 2.

**Lemma 22.** *Suppose that  $Q = (I, (A_{ij})_{(i,j) \in I^2})$  is a quiver and that  $G$  acts on  $kQ$  by stabilizing  $I$ . Then  $G$  acts in a natural way on the double quiver  $\overline{Q}$  and stabilizes the preprojective relation. Moreover,  $\widetilde{\overline{Q}}_G = \overline{\widetilde{Q}}_G$  and  $(\overline{Q})_G = \overline{Q}_G$  where  $\widetilde{\overline{Q}}_G$  and  $\overline{Q}_G$  are as in section 4.*

*Proof.* If  $a \oplus f \in \overline{A}_{ij} = A_{ij} \oplus A_{ji}^*$  and  $g \in G$ , let  $(a \oplus f)^g = a^g \oplus \left( a' \mapsto f(a'^{g^{-1}}) \right) \in \overline{A}_{gi, gj}$ . With this action, all the rest of the lemma is clear.  $\square$

## ACKNOWLEDGMENTS

The author would like to thank his PhD advisor Bernard Leclerc for his advices and corrections. He would also like to thank Christof Geiß, Bernhard Keller and Idun Reiten for interesting discussions and comments on the topic.

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