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Alexandre Masquelein, Anne Quéguiner-Mathieu, Jean-Pierre Tignol. Quadratic forms of dimension 8 with trivial discriminant and Clifford algebra of index 4.. 2009. hal-00359156

**HAL Id: hal-00359156**

**<https://hal.science/hal-00359156>**

Preprint submitted on 6 Feb 2009

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# QUADRATIC FORMS OF DIMENSION 8 WITH TRIVIAL DISCRIMINANT AND CLIFFORD ALGEBRA OF INDEX 4.

ALEXANDRE MASQUELEIN, ANNE QUÉGUINER-MATHIEU,  
AND JEAN-PIERRE TIGNOL

ABSTRACT. Izhboldin and Karpenko proved in [IK00, Thm 16.10] that any quadratic form of dimension 8 with trivial discriminant and Clifford algebra of index 4 is isometric to the transfer, with respect to some quadratic étale extension, of a quadratic form similar to a two-fold Pfister form. We give a new proof of this result, based on a theorem of decomposability for degree 8 and index 4 algebras with orthogonal involution.

Let  $WF$  denote the Witt ring of a field  $F$  of characteristic different from 2. As explained in [Lam05, X.5 and XII.2], one would like to describe those quadratic forms whose Witt class belongs to the  $n$ th power  $I^n F$  of the fundamental ideal  $IF$  of  $WF$ . By the Arason-Pfister Hauptsatz, such a form is hyperbolic if it has dimension  $< 2^n$  and similar to a Pfister form if it has dimension  $2^n$ . More generally, Vishik's Gap Theorem gives the possible dimensions of anisotropic forms in  $I^n F$ .

In addition, one may describe explicitly, for some small values of  $n$ , low dimensional anisotropic quadratic forms in  $I^n F$ . This is the case, in particular, for  $n = 2$ , that is for even-dimensional quadratic forms with trivial discriminant. In dimension 6, it is well known that such a form is similar to an Albert form, and uniquely determined up to similarity by its Clifford invariant. In dimension 8, if the index of the Clifford algebra is  $\leq 4$ , Izhboldin and Karpenko proved in [IK00, Thm 16.10] that it is isometric to the transfer, with respect to some quadratic étale extension, of a quadratic form similar to a two-fold Pfister form.

The purpose of this paper is to give a new proof of Izhboldin and Karpenko's result. Our proof is in the framework of algebras with involution, and does not use Rost's description of 14-dimensional forms in  $I^3 F$  (see [IK00, Rmk 16.11.2]). More precisely, we use triality [KMRT98, (42.3)] to translate the question into a question on algebras of degree 8 and index 4 with orthogonal involution. Our main tool then is a decomposability theorem (Thm. 1.1), proven in § 3. We also use a refinement of a statement of Arason [Ara75, 4.18] describing the even part of the Clifford algebra of a transfer (see Prop. 2.1 below).

## 1. NOTATIONS AND STATEMENT OF THE THEOREM

Throughout the paper, we work over a base field  $F$  of characteristic different from 2. We refer the reader to [KMRT98] and [Lam05] for background information on algebras with involution and on quadratic forms. However, we depart from the notation in [Lam05] by using  $\langle\langle a_1, \dots, a_n \rangle\rangle$  to denote the  $n$ -fold Pfister form  $\otimes_{i=1}^n \langle 1, -a_i \rangle$ . For any quadratic space  $(V, \phi)$  over  $F$ , we let  $\text{Ad}_\phi$  be the algebra

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*Date:* February 6, 2009.

The third author is supported in part by the F.R.S.-FNRS (Belgium).

with involution  $(\text{End}_F(V), \text{ad}_\phi)$ , where  $\text{ad}_\phi$  is the adjoint involution with respect to  $\phi$ , denoted by  $\sigma_\phi$  in [KMRT98].

For any field extension  $L/F$ , we denote by  $GP_n(L)$  the set of quadratic forms that are similar to  $n$ -fold Pfister forms. This notation extends to the quadratic étale extension  $F \times F$  by  $GP_n(F \times F) = GP_n(F) \times GP_n(F)$ . For any quadratic form  $\psi$  over  $L$ , let  $\mathcal{C}(\psi)$  be its full Clifford algebra, with even part  $\mathcal{C}_0(\psi)$ . Both  $\mathcal{C}(\psi)$  and  $\mathcal{C}_0(\psi)$  are endowed with a canonical involution, which is the identity on the underlying vector space, denoted by  $\gamma$  (see [KMRT98, p.89]). If  $\psi$  has even dimension and trivial discriminant, then its even Clifford algebra splits as a direct product  $\mathcal{C}_+(\psi) \times \mathcal{C}_-(\psi)$ , for some isomorphic central simple algebras  $\mathcal{C}_+(\psi)$  and  $\mathcal{C}_-(\psi)$  over  $F$  (see [Lam05, V, Thm 2.5]). Those algebras are Brauer-equivalent to the full Clifford algebra of  $\psi$  and their Brauer class is the Clifford invariant of  $\psi$ . Assume moreover that  $\dim(\psi) \equiv 0 \pmod{4}$ . As explained in [KMRT98, (8.4)], the involution  $\gamma$  then induces an involution on each factor of  $\mathcal{C}_0(\psi)$ , and one may easily check that the isomorphism between the two factors described in the proof of [Lam05, V, Thm 2.5] preserves the involution, so that we actually get a decomposition  $(\mathcal{C}_0(\psi), \gamma) \simeq (\mathcal{C}_+(\psi), \gamma_+) \times (\mathcal{C}_-(\psi), \gamma_-)$ , with  $(\mathcal{C}_+(\psi), \gamma_+) \simeq (\mathcal{C}_-(\psi), \gamma_-)$ .

Let  $L/F$  be a quadratic field extension. For any quadratic form  $\psi$  over  $L$ , we let  $\text{tr}_*(\psi)$  be the transfer of  $\psi$ , associated to the trace map  $\text{tr} : L \rightarrow F$ , as defined in [Lam05, VII.1.2]. This definition extends to the split étale case  $L = F \times F$  and leads to  $\text{tr}_*(\psi, \psi') = \psi + \psi'$ . On the other hand, for any algebra  $A$  over  $L$ , we let  $N_{L/F}(A)$  be its norm, as defined in [KMRT98, §3.B]. Recall that the Brauer class of  $N_{L/F}(A)$  is the corestriction of the Brauer class of  $A$ . Moreover, if  $A$  is endowed with an involution of the first kind  $\sigma$ , then the tensor product  $\sigma \otimes \sigma$  restricts to an involution  $N_{L/F}(\sigma)$  on  $N_{L/F}(A)$ . We use the following notation:  $N_{L/F}(A, \sigma) = (N_{L/F}(A), N_{L/F}(\sigma))$ . In the split étale case, we get  $N_{F \times F/F}((A, \sigma), (A', \sigma')) = (A, \sigma) \otimes (A', \sigma')$  (see [KMRT98, §15.B]).

Let  $(A, \sigma)$  be a degree 8 algebra with orthogonal involution. We assume that  $(A, \sigma)$  is *totally decomposable*, that is, isomorphic to a tensor product of three quaternion algebras with involution,

$$(A, \sigma) = \otimes_{i=1}^3 (Q_i, \sigma_i).$$

If  $A$  is split (resp. has index 2), then  $(A, \sigma)$  admits a decomposition as above in which each quaternion algebra (resp. each but one) is split (see [Bec08]). Our main result is the following theorem:

**Theorem 1.1.** *Let  $(A, \sigma)$  be a degree 8 totally decomposable algebra with orthogonal involution. If the index of  $A$  is  $\leq 4$ , then there exists  $\lambda \in F^\times$  and a biquaternion algebra with orthogonal involution  $(D, \theta)$  such that*

$$(A, \sigma) \simeq (D, \theta) \otimes \text{Ad}_{\langle\langle \lambda \rangle\rangle}.$$

The theorem readily follows from Becher's results mentioned above if  $A$  has index 1 or 2; it is proven in § 3 for algebras of index 4. For algebras of index  $\leq 2$ , we may even assume that  $(D, \theta)$  decomposes as a tensor product of two quaternion algebras with involution; this is not the case anymore if  $A$  has index 4, as was shown by Sivatski [Siv05, Prop. 5].

Using triality, we easily deduce the following from Theorem 1.1:

**Theorem 1.2** (Izhboldin-Karpenko). *Let  $\phi$  be an 8-dimensional quadratic form over  $F$ . The following are equivalent:*

- (i)  $\phi$  has trivial discriminant and Clifford invariant of index  $\leq 4$ ;
- (ii) there exists a quadratic étale extension  $L/F$  and a form  $\psi \in GP_2(L)$  such that  $\phi = \text{tr}_*(\psi)$ .

If  $\phi = \text{tr}_*(\psi)$  for some  $\psi \in GP_2(L)$ , it follows from some direct computation made in [IK00, §16] that  $\phi$  has trivial discriminant and Clifford invariant of index  $\leq 4$ .

Assume conversely that  $\phi$  has trivial discriminant. By the Arason-Pfister Hauptsatz,  $\phi$  is in  $GP_3(F)$  if and only if it has trivial Clifford invariant. More generally, it is well-known that  $\phi$  decomposes as  $\phi = \langle\langle a \rangle\rangle q$  for some  $a \in F^\times$  and some 4-dimensional quadratic form  $q$  over  $F$  if and only if its Clifford invariant has index  $\leq 2$  (see for instance [Kne77, Ex 9.12]). Hence, in both cases,  $\phi$  decomposes as a sum  $\phi = \pi_1 + \pi_2$  of two forms  $\pi_1, \pi_2 \in GP_2(F)$ . This proves that condition (ii) holds with  $L = F \times F$ .

In section 4 below, we finish this proof by treating the index 4 case. This part of the proof differs from the argument given in [IK00]. In particular, we do not use Rost's description of 14-dimensional forms in  $I^3 F$ .

## 2. CLIFFORD ALGEBRA OF THE TRANSFER OF A QUADRATIC FORM

Let  $L/F$  be a quadratic field extension. By Arason [Ara75, 4.18], for any quadratic form  $\psi \in GP_2(L)$ , the Clifford invariant of the transfer  $\text{tr}_*(\psi)$  coincides with the corestriction of the Clifford invariant of  $\psi$ . In this section, we extend this result, taking into account the algebras with involution rather than just the Brauer classes. More precisely, we prove:

**Proposition 2.1.** *Let  $L = F[X]/(X^2 - d)$  be a quadratic étale extension of  $F$ . Consider a quadratic form  $\psi$  over  $L$  with  $\dim(\psi) \equiv 0 \pmod{4}$  and  $d_\pm(\psi) = 1$ , so that its even Clifford algebra decomposes as*

$$(\mathcal{C}_0(\psi), \gamma) \simeq (\mathcal{C}_+(\psi), \gamma_+) \times (\mathcal{C}_-(\psi), \gamma_-), \text{ with } (\mathcal{C}_+(\psi), \gamma_+) \simeq (\mathcal{C}_-(\psi), \gamma_-).$$

*For any  $\lambda \in L^\times$  represented by  $\psi$ , the two components of the even Clifford algebra of the transfer of  $\psi$  are both isomorphic to*

$$(\mathcal{C}_+(\text{tr}_*(\psi)), \gamma_+) \simeq \text{Ad}_{\langle\langle -dN_{L/F}(\lambda) \rangle\rangle} \otimes_{N_{L/F}} (\mathcal{C}_+(\psi), \gamma_+).$$

*Proof.* In the split étale case  $L = F \times F$ , the quadratic form  $\psi$  is a couple  $(\phi, \phi')$  of two quadratic forms over  $F$  with

$$\dim(\phi) = \dim(\phi') \equiv 0 \pmod{4} \quad \text{and} \quad d_\pm(\phi) = d_\pm(\phi') = 1 \in F^\times/F^{\star 2}.$$

Pick  $\lambda$  and  $\lambda'$  in  $F$  respectively represented by  $\phi$  and  $\phi'$ ; the norm  $N_{F \times F/F}(\lambda, \lambda')$  is  $\lambda\lambda'$ . So the following lemma proves the proposition in that case:

**Lemma 2.2.** *Let  $\phi$  and  $\phi'$  be two quadratic forms over  $F$  of the same dimension  $n \equiv 0 \pmod{4}$  and trivial discriminant. For any  $\lambda$  and  $\lambda' \in F^\times$ , respectively represented by  $\phi$  and  $\phi'$ , the components of the even Clifford algebra of the orthogonal sum  $\phi + \phi'$  are isomorphic to*

$$(\mathcal{C}_+(\phi + \phi'), \gamma_+) \simeq \text{Ad}_{\langle\langle -\lambda\lambda' \rangle\rangle} \otimes (\mathcal{C}_+(\phi), \gamma_+) \otimes (\mathcal{C}_+(\phi'), \gamma_+).$$

*Proof of Lemma 2.2.* Denote by  $V$  and  $V'$  the underlying quadratic spaces. The natural embeddings  $V \hookrightarrow V \oplus V'$  and  $V' \hookrightarrow V \oplus V'$  induce  $F$ -algebra homomorphisms

$$\mathcal{C}(\phi) \rightarrow \mathcal{C}(\phi + \phi') \text{ and } \mathcal{C}(\phi') \rightarrow \mathcal{C}(\phi + \phi').$$

One may easily check that the images of the even parts centralize each other, so that we get an  $F$ -algebra homomorphism

$$(\mathcal{C}_0(\phi), \gamma) \otimes (\mathcal{C}_0(\phi'), \gamma) \rightarrow (\mathcal{C}_0(\phi + \phi'), \gamma).$$

Pick orthogonal bases  $(e_1, \dots, e_n)$  of  $(V, \phi)$  and  $(e'_1, \dots, e'_n)$  of  $(V', \phi')$ . The basis of  $\mathcal{C}_0(\phi + \phi')$  consisting of products of an even number of vectors of the set  $\{e_1, \dots, e_n, e'_1, \dots, e'_n\}$  as described in [Lam05, V, cor 1.9] clearly contains the image of a basis of  $\mathcal{C}_0(\phi) \otimes \mathcal{C}_0(\phi')$ , so that the homomorphism above is injective. In the sequel, we will identify  $\mathcal{C}_0(\phi)$  and  $\mathcal{C}_0(\phi')$  with their images in  $\mathcal{C}_0(\phi + \phi')$ .

Consider the element  $z = e_1 \dots e_n \in \mathcal{C}_0(\phi)$ . As explained in [Lam05, V, Thm2.2], for any  $v \in V$ , one has  $vz = -zv \in \mathcal{C}(\phi)$  and  $z$  generates the center of  $\mathcal{C}_0(\phi)$ . Since  $\phi$  has dimension 0 mod 4 and trivial discriminant, this element  $z$  is  $\gamma$ -symmetric, and multiplying  $e_1$  by a scalar if necessary, we may assume  $z^2 = 1$ . The two components of  $\mathcal{C}_0(\phi)$  are  $\mathcal{C}_+(\phi) = \mathcal{C}_0(\phi)^{\frac{1+z}{2}}$  and  $\mathcal{C}_-(\phi) = \mathcal{C}_0(\phi)^{\frac{1-z}{2}}$ . Consider similarly  $z' = e'_1 \dots e'_n$ , with  $\gamma(z') = z'$  and assume  $z'^2 = 1$ . The product  $zz'$  also has square 1 and generates the center of  $\mathcal{C}_0(\phi + \phi')$ . We denote by  $\varepsilon$  the idempotent  $\varepsilon = \frac{1+zz'}{2}$ , so that  $\mathcal{C}_+(\phi + \phi') = \mathcal{C}_0(\phi + \phi')\varepsilon$  and  $\mathcal{C}_-(\phi + \phi') = \mathcal{C}_0(\phi + \phi')(1 - \varepsilon)$ .

Let us now fix two vectors  $v \in V$  and  $v' \in V'$  such that  $\phi(v) = \lambda$  and  $\phi'(v') = \lambda'$ . Since  $\frac{1+z}{2}v^{-1} = v^{-1}\frac{1-z}{2}$ , we have  $v xv^{-1} \in \mathcal{C}_-(\phi)$  for any  $x \in \mathcal{C}_+(\phi)$ . Using this identification between the two components, we may diagonally embed  $\mathcal{C}_+(\phi)$  in  $\mathcal{C}_0(\phi)$  by considering  $x \in \mathcal{C}_+(\phi) \mapsto x + v xv^{-1} \in \mathcal{C}_0(\phi)$ . Similarly, we may embed  $\mathcal{C}_+(\phi')$  in  $\mathcal{C}_0(\phi')$  by  $x' \in \mathcal{C}_+(\phi') \mapsto x' + v' x' v'^{-1} \in \mathcal{C}_0(\phi')$ . Combining those two maps with the morphism

$$\mathcal{C}_0(\phi) \otimes \mathcal{C}_0(\phi') \rightarrow \mathcal{C}_0(\phi + \phi'),$$

and the projection

$$y \in \mathcal{C}_0(\phi + \phi') \mapsto y\varepsilon \in \mathcal{C}_+(\phi + \phi'),$$

we get an algebra homomorphism

$$\begin{aligned} \mathcal{C}_+(\phi) \otimes \mathcal{C}_+(\phi') &\rightarrow \mathcal{C}_+(\phi + \phi'), \\ x \otimes x' &\mapsto (x + v xv^{-1})(x' + v' x' v'^{-1})\varepsilon. \end{aligned}$$

One may easily check on generators that this map is not trivial; hence it is injective. To conclude the proof, it only remains to identify the centralizer of the image, which by dimension count has degree 2. It clearly contains  $\frac{z+z'}{2}\varepsilon$  and  $vv'\varepsilon$ . Moreover, these two elements anticommute, have square  $\varepsilon$  and  $-\lambda\lambda'\varepsilon$ , and are respectively symmetric and skew-symmetric under  $\gamma$ . Hence they generate a split quaternion algebra, with orthogonal involution of discriminant  $-\lambda\lambda'$ , which is isomorphic to  $\text{Ad}_{\langle\langle -\lambda\lambda' \rangle\rangle}$ .  $\square$

This concludes the proof in the split étale case. Until the end of this section, we assume  $L$  is a quadratic field extension of  $F$ , with non-trivial  $F$ -automorphism denoted by  $\iota$ . To prove the proposition in this case, we will use the following description of the transfer of a quadratic form and its Clifford algebra.

Let  $\psi$  be any quadratic form over  $L$ , defined on the vector space  $V$ . We consider its conjugate  ${}^tV = \{{}^t v, v \in V\}$  with the following operations  ${}^t v_1 + {}^t v_2 = {}^t(v_1 + v_2)$  and  $\lambda \cdot {}^t v = {}^t(\iota(\lambda) \cdot v)$ , for any  $v_1, v_2$  and  $v$  in  $V$  and  $\lambda \in L$ . Clearly,  ${}^t\psi({}^t v) = \iota(\psi(v))$  is a quadratic form on  ${}^tV$ . One may easily check from the definition given in [Lam05, VII §1] that the quadratic form  $\text{tr}_*(\psi)$  is nothing but the restriction of  $\psi + {}^t\psi$  to

the  $F$ -vector space of fixed points  $(V \oplus {}^tV)^s$ , where  $s$  is the switch semi-linear automorphism defined on the direct sum  $V \oplus {}^tV$  by  $s(v_1 + {}^t v_2) = v_2 + {}^t v_1$ .

Moreover,  $s$  induces a semi-linear automorphism of order 2 of the tensor algebra  $T(V \oplus {}^tV)$  which preserves the ideal generated by the elements

$$(v_1 + {}^t v_2) \otimes (v_1 + {}^t v_2) - (\psi(v_1) + {}^t \psi({}^t v_2)).$$

Hence, we get a semi-linear automorphism  $s$  of order 2 on the Clifford algebra  $\mathcal{C}(\psi + {}^t \psi)$ , which commutes with the canonical involution. The set of fixed points  $(\mathcal{C}(\psi + {}^t \psi))^s$  is an  $F$ -algebra; the involution  $\gamma$  restricts to an  $F$ -linear involution which we denote by  $\gamma_s$ . We then have:

**Lemma 2.3.** *The natural embedding  $(V \oplus {}^tV) \hookrightarrow \mathcal{C}(\psi + {}^t \psi)$ , restricted to  $(V + {}^tV)^s$ , induces an isomorphism of graded algebras*

$$(\mathcal{C}(\text{tr}_*(\psi)), \gamma) \xrightarrow{\sim} ((\mathcal{C}(\psi + {}^t \psi))^s, \gamma_s).$$

*Proof of Lemma 2.3.* The natural embedding  $(V \oplus {}^tV) \hookrightarrow \mathcal{C}(\psi + {}^t \psi)$  restricts to an injective map  $i : (V + {}^tV)^s \hookrightarrow \mathcal{C}(\psi + {}^t \psi)^s$ , which clearly satisfies

$$i(w)^2 = (\psi + {}^t \psi)(w) \quad \text{for any } w \in (V \oplus {}^tV)^s.$$

By the universal property of Clifford algebras, it extends to a non-trivial algebra homomorphism  $\mathcal{C}(\text{tr}_*(\psi)) \mapsto \mathcal{C}(\psi + {}^t \psi)^s$ , which clearly preserves the grading. Since  $\mathcal{C}(\text{tr}_*(\psi))$  is simple, and both algebras have the same dimension, it is an isomorphism. Clearly,  $\gamma$  coincides with  $\gamma_s$  under this isomorphism.  $\square$

Hence, we want to describe one component of  $\mathcal{C}_0(\text{tr}_*(\psi)) \simeq (\mathcal{C}_0(\psi + {}^t \psi))^s$ . We proceed as in the split étale case. Fix an orthogonal basis  $e_1, \dots, e_n$  of  $V$  over  $L$  such that  $\psi(e_n) = \lambda$ . The elements  ${}^t e_1, \dots, {}^t e_n$  are an orthogonal basis of  ${}^t V$  and  ${}^t \psi({}^t e_n) = {}^t(\lambda)$ . We may moreover assume that  $z = e_1 \dots e_n$  and  ${}^t z = {}^t e_1 \dots {}^t e_n$  have square 1. Since the idempotent  $\varepsilon = \frac{1+z{}^t z}{2} \in \mathcal{C}_0(\psi + {}^t \psi)$  satisfies  $s(\varepsilon) = \varepsilon$ , the semilinear automorphism  $s$  preserves each factor  $\mathcal{C}_+(\psi + {}^t \psi)$  and  $\mathcal{C}_-(\psi + {}^t \psi)$ . Hence, the components of  $\mathcal{C}_0(\text{tr}_*(\psi))$  are

$$\mathcal{C}_0(\text{tr}_*(\psi)) = (\mathcal{C}_+(\psi + {}^t \psi))^s \times (\mathcal{C}_-(\psi + {}^t \psi))^s.$$

Moreover, by Lemma 2.2, we have

$$\mathcal{C}_+(\psi + {}^t \psi) \simeq \text{Ad}_{\langle -\lambda {}^t(\lambda) \rangle} \otimes (\mathcal{C}_+(\psi), \gamma) \otimes (\mathcal{C}_+({}^t \psi), \gamma),$$

and it remains to understand the action of the switch automorphism on this tensor product. First, one may identify  $\mathcal{C}_+({}^t \psi)$  with the algebra  ${}^t \mathcal{C}_+(\psi)$  defined by

$${}^t \mathcal{C}_+(\psi) = \{{}^t x, x \in \mathcal{C}_+(\psi)\},$$

with the operations

$${}^t x + {}^t y = {}^t(x + y), \quad {}^t x {}^t y = {}^t(xy) \quad \text{and} \quad {}^t(\lambda x) = {}^t(\lambda) {}^t x,$$

for all  $x, y \in \mathcal{C}_+(\psi)$  and  $\lambda \in L$ . Clearly, the switch automorphism acts on the tensor product

$$\mathcal{C}_+(\psi) \otimes \mathcal{C}_+({}^t \psi) \simeq \mathcal{C}_+(\psi) \otimes {}^t \mathcal{C}_+(\psi),$$

by

$$s(x \otimes {}^t y) = y \otimes {}^t x,$$

and by definition of the corestriction (see [KMRT98, 3.B]), the  $F$ -subalgebra of fixed points is

$$((\mathcal{C}_+(\psi), \gamma) \otimes ({}^t \mathcal{C}_+(\psi), \gamma))^s = N_{L/F}(\mathcal{C}_+(\psi), \gamma).$$

It remains to understand the action of the switch on the centralizer, which is the split quaternion algebra over  $L$  generated by  $x = \frac{z+\iota}{2}\varepsilon$  and  $y = e_n \iota e_n \varepsilon$ . The element  $x$  clearly is  $s$ -symmetric, while  $y$  satisfies  $s(y) = -y$ . Let  $\delta$  be a generator of the quadratic extension  $L/F$ , so that  $\iota(\delta) = -\delta$  and  $\delta^2 = d$ . Since the switch map  $s$  is  $L/F$  semi-linear, we may replace  $y$  by  $\delta y$  which now satisfies  $s(\delta y) = \delta y$ . Hence, the set of fixed points under  $s$  is the split quaternion algebra over  $F$  generated by  $x$  and  $\delta y$ . Since  $(\delta y)^2 = -dN_{L/F}(\lambda)$ , it is isomorphic to  $\text{Ad}_{\langle\langle -dN_{L/F}(\lambda) \rangle\rangle}$ .  $\square$

### 3. PROOF OF THE DECOMPOSABILITY THEOREM

In this section, we finish the proof of Theorem 1.1. Let  $(A, \sigma) = \otimes_{i=1}^3 (Q_i, \sigma_i)$  be a product of three quaternion algebras with orthogonal involution. We assume that  $A$  has index 4, so that it is Brauer-equivalent to a biquaternion division algebra  $D$ . We have to prove that  $(A, \sigma)$  is isomorphic to  $(D, \theta) \otimes \text{Ad}_{\langle\langle \lambda \rangle\rangle}$  for a well chosen involution  $\theta$  on  $D$  and some  $\lambda \in F^\times$ .

The algebra  $D$  is endowed with an orthogonal involution  $\tau$ , and we may represent

$$(A, \sigma) = (\text{End}_D(M), \text{ad}_h),$$

for some 2-dimensional hermitian module  $(M, h)$  over  $(D, \tau)$ . Let us consider a diagonalisation  $\langle a_1, a_2 \rangle$  of  $h$ , and define

$$\theta = \text{Int}(a_1^{-1}) \circ \tau.$$

With respect to this new involution, we get another representation

$$(A, \sigma) = (\text{End}_D(M), \text{ad}_{h'}),$$

where  $h'$  is a hermitian form over  $(D, \theta)$  which diagonalises as  $h' = \langle 1, -a \rangle$  for some  $\theta$ -symmetric element  $a \in D^\times$ . The theorem now follows from the following lemma:

**Lemma 3.1.** *The involutions  $\theta$  and  $\theta' = \text{Int}(a^{-1}) \circ \theta$  of the biquaternion algebra  $D$  are conjugate.*

Indeed, assume there exists  $u \in A^\times$  such that  $\theta = \text{Int}(u) \circ \theta' \circ \text{Int}(u^{-1})$ . We then have  $\theta = \text{Int}(ua^{-1}) \circ \theta \circ \text{Int}(u^{-1}) = \theta \circ \text{Int}(\theta(u)^{-1}au^{-1})$ . Hence, there exists  $\lambda \in F^\times$  such that  $\theta(u)^{-1}au^{-1} = \lambda$ , that is  $a = \lambda\theta(u)u$ . This implies that the hermitian form  $h' = \langle 1, -a \rangle$  over  $(D, \theta)$  is isometric to  $\langle 1, -\lambda \rangle$ . Since  $\lambda \in F^\times$ , we get  $(A, \sigma) = (\text{End}_D(M), \text{ad}_{\langle 1, -\lambda \rangle}) = (D, \theta) \otimes \text{Ad}_{\langle\langle \lambda \rangle\rangle}$ , and it only remains to prove the lemma.

*Proof of Lemma 3.1.* We want to compare the orthogonal involutions  $\theta$  and  $\theta'$  of the biquaternion algebra  $D$ . By [LT99, Prop. 2], they are conjugate if and only if their Clifford algebras  $\mathcal{C}$  and  $\mathcal{C}'$  are isomorphic as  $F$ -algebras. This can be proven as follows.

Since  $(A, \sigma)$  is a product of three quaternion algebras with involution, we know from [KMRT98, (42.11)] that the discriminant of  $\sigma$  is 1 and its Clifford algebra has one split component.

On the other hand, the representation  $(A, \sigma) = (\text{End}_D(M), \text{ad}_{\langle 1, -a \rangle})$  tells us that  $(A, \sigma)$  is an orthogonal sum, as in [Dej95], of  $(D, \theta)$  and  $(D, \theta')$ . Hence its invariants can be computed in terms of those of  $(D, \theta)$  and  $(D, \theta')$ . By [Dej95, Prop. 2.3.3], the discriminant of  $\sigma$  is the product of the discriminants of  $\theta$  and  $\theta'$ . So  $\theta$  and  $\theta'$  have the same discriminant, and we may identify the centers  $Z$  and  $Z'$  of their Clifford algebras in two different ways. We are in the situation described in [LT99, p. 265], where the Clifford algebra of such an orthogonal sum is computed. In

particular, since one component of the Clifford algebra of  $(A, \sigma)$  is split, it follows from [LT99, Lem 1] that

$$\mathcal{C} \simeq \mathcal{C}' \quad \text{or} \quad \mathcal{C} \simeq {}^t\mathcal{C}',$$

depending on the chosen identification between  $Z$  and  $Z'$ . In both cases,  $\mathcal{C}$  and  $\mathcal{C}'$  are isomorphic as  $F$ -algebras, and this concludes the proof.  $\square$

#### 4. A NEW PROOF OF IZHBOLDIN AND KARPENKO'S THEOREM

Let  $\phi$  be an 8-dimensional quadratic form over  $F$  with trivial discriminant and Clifford invariant of index 4. We denote by  $(A, \sigma)$  one component of its even Clifford algebra, so that

$$(\mathcal{C}_0(\phi), \gamma) \simeq (A, \sigma) \times (A, \sigma),$$

where  $A$  is an index 4 central simple algebra over  $F$ , with orthogonal involution  $\sigma$ .

By triality [KMRT98, (42.3)], the involution  $\sigma$  has trivial discriminant and its Clifford algebra is

$$\mathcal{C}(A, \sigma) = \text{Ad}_\phi \times (A, \sigma).$$

In particular, it has a split component, so that the algebra with involution  $(A, \sigma)$  is isomorphic to a tensor product of three quaternion algebras with involution (see [KMRT98, (42.11)]). Hence we can apply our decomposability theorem 1.1, and write  $(A, \sigma) = (D, \theta) \otimes \text{Ad}_{\langle\langle\lambda\rangle\rangle}$  for some biquaternion division algebra with orthogonal involution  $(D, \theta)$  and some  $\lambda \in F^\times$ .

Let us denote by  $d$  the discriminant of  $\theta$ , and let  $L = F[X]/(X^2 - d)$  be the corresponding quadratic étale extension. Consider the image  $\delta$  of  $X$  in  $L$ . By Tao's computation of the Clifford algebra of a tensor product [Tao95, Thm. 4.12], the components of  $\mathcal{C}(A, \sigma)$  are Brauer-equivalent to the quaternion algebra  $(d, \lambda)$  over  $F$  and the tensor product  $(d, \lambda) \otimes A$ . Since  $A$  has index 4, the split component has to be  $(d, \lambda)$ , so that  $\lambda$  is a norm of  $L/F$ , say  $\lambda = N_{L/F}(\mu)$ .

Consider now the Clifford algebra of  $(D, \theta)$ . It is a quaternion algebra  $Q$  over  $L$ , endowed with its canonical (symplectic) involution  $\gamma$ . Denote by  $n_Q$  the norm form of  $Q$ , that is  $n_Q = \langle\langle\alpha, \beta\rangle\rangle$  if  $Q = (\alpha, \beta)_L$ . It is a 2-fold Pfister form and for any  $\ell \in L^\star$ ,  $(\mathcal{C}_+(\langle\ell\rangle n_Q), \gamma_+) \simeq (Q, \gamma)$ . Moreover, by the equivalence of categories  $A_1^2 \equiv D_2$  described in [KMRT98, (15.7)], the algebra with involution  $(D, \theta)$  is canonically isomorphic to  $N_{L/F}(Q, \gamma)$ .

Hence we get that  $(A, \sigma) = N_{L/F}(Q, \gamma) \otimes \text{Ad}_{\langle\langle -dN_{L/F}(\delta\mu) \rangle\rangle}$ . By Proposition 2.1, this implies that

$$(A, \sigma) \times (A, \sigma) \simeq (\mathcal{C}_0(\text{tr}_\star(\psi)), \gamma),$$

where  $\psi = \langle\delta\mu\rangle n_Q$ . Applying again triality [KMRT98, (42.3)], we get that the split component  $\text{Ad}_\phi$  of the Clifford algebra of  $(A, \sigma)$  also is isomorphic to  $\text{Ad}_{\text{tr}_\star(\psi)}$ , so that the quadratic forms  $\phi$  and  $\text{tr}_\star(\psi)$  are similar. This concludes the proof since  $\psi$  belongs to  $GP_2(L)$ .

*Remark.* Let  $\phi$  and  $(A, \sigma)$  be as above, and let  $L = F[X]/(X^2 - d)$  be a fixed quadratic étale extension of  $F$ . It follows from the proof that the quadratic form  $\phi$  is isometric to the transfer of a form  $\psi \in GP_2(L)$  if and only if  $(A, \sigma)$  admits a decomposition  $(A, \sigma) = \text{Ad}_{\langle\langle\lambda\rangle\rangle} \otimes (D, \theta)$ , with  $d_\pm(\theta) = d$ . In particular, the quadratic form  $\phi$  is a sum of two forms similar to 2-fold Pfister forms exactly when the algebra with involution  $(A, \sigma)$  admits a decomposition as  $(D, \theta) \otimes \text{Ad}_{\langle\langle\lambda\rangle\rangle}$  with  $\theta$  of



discriminant 1, that is when it decomposes as a tensor product of three quaternion algebras with involution, with one split factor.

Such a decomposition does not always exist, as was shown by Sivatski [Siv05, Prop 5]. This reflects the fact that 8-dimensional quadratic forms  $\phi$  with trivial discriminant and Clifford algebra of index  $\leq 4$  do not always decompose as a sum of two forms similar to two-fold Pfister forms (see [IK00, §16] and [HT98] for explicit examples).

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