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► **To cite this version:**

Alexandre Masquelein, Anne Quéguiner-Mathieu, Jean-Pierre Tignol. Quadratic forms of dimension 8 with trivial discriminant and Clifford algebra of index 4.. 2009. hal-00359156

HAL Id: hal-00359156

<https://hal.science/hal-00359156>

Preprint submitted on 6 Feb 2009

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QUADRATIC FORMS OF DIMENSION 8 WITH TRIVIAL DISCRIMINANT AND CLIFFORD ALGEBRA OF INDEX 4.

ALEXANDRE MASQUELEIN, ANNE QUÉGUINER-MATHIEU,
AND JEAN-PIERRE TIGNOL

ABSTRACT. Izhboldin and Karpenko proved in [IK00, Thm 16.10] that any quadratic form of dimension 8 with trivial discriminant and Clifford algebra of index 4 is isometric to the transfer, with respect to some quadratic étale extension, of a quadratic form similar to a two-fold Pfister form. We give a new proof of this result, based on a theorem of decomposability for degree 8 and index 4 algebras with orthogonal involution.

Let WF denote the Witt ring of a field F of characteristic different from 2. As explained in [Lam05, X.5 and XII.2], one would like to describe those quadratic forms whose Witt class belongs to the n th power $I^n F$ of the fundamental ideal IF of WF . By the Arason-Pfister Hauptsatz, such a form is hyperbolic if it has dimension $< 2^n$ and similar to a Pfister form if it has dimension 2^n . More generally, Vishik's Gap Theorem gives the possible dimensions of anisotropic forms in $I^n F$.

In addition, one may describe explicitly, for some small values of n , low dimensional anisotropic quadratic forms in $I^n F$. This is the case, in particular, for $n = 2$, that is for even-dimensional quadratic forms with trivial discriminant. In dimension 6, it is well known that such a form is similar to an Albert form, and uniquely determined up to similarity by its Clifford invariant. In dimension 8, if the index of the Clifford algebra is ≤ 4 , Izhboldin and Karpenko proved in [IK00, Thm 16.10] that it is isometric to the transfer, with respect to some quadratic étale extension, of a quadratic form similar to a two-fold Pfister form.

The purpose of this paper is to give a new proof of Izhboldin and Karpenko's result. Our proof is in the framework of algebras with involution, and does not use Rost's description of 14-dimensional forms in $I^3 F$ (see [IK00, Rmk 16.11.2]). More precisely, we use triality [KMRT98, (42.3)] to translate the question into a question on algebras of degree 8 and index 4 with orthogonal involution. Our main tool then is a decomposability theorem (Thm. 1.1), proven in § 3. We also use a refinement of a statement of Arason [Ara75, 4.18] describing the even part of the Clifford algebra of a transfer (see Prop. 2.1 below).

1. NOTATIONS AND STATEMENT OF THE THEOREM

Throughout the paper, we work over a base field F of characteristic different from 2. We refer the reader to [KMRT98] and [Lam05] for background information on algebras with involution and on quadratic forms. However, we depart from the notation in [Lam05] by using $\langle\langle a_1, \dots, a_n \rangle\rangle$ to denote the n -fold Pfister form $\otimes_{i=1}^n \langle 1, -a_i \rangle$. For any quadratic space (V, ϕ) over F , we let Ad_ϕ be the algebra

Date: February 6, 2009.

The third author is supported in part by the F.R.S.-FNRS (Belgium).

with involution $(\text{End}_F(V), \text{ad}_\phi)$, where ad_ϕ is the adjoint involution with respect to ϕ , denoted by σ_ϕ in [KMRT98].

For any field extension L/F , we denote by $GP_n(L)$ the set of quadratic forms that are similar to n -fold Pfister forms. This notation extends to the quadratic étale extension $F \times F$ by $GP_n(F \times F) = GP_n(F) \times GP_n(F)$. For any quadratic form ψ over L , let $\mathcal{C}(\psi)$ be its full Clifford algebra, with even part $\mathcal{C}_0(\psi)$. Both $\mathcal{C}(\psi)$ and $\mathcal{C}_0(\psi)$ are endowed with a canonical involution, which is the identity on the underlying vector space, denoted by γ (see [KMRT98, p.89]). If ψ has even dimension and trivial discriminant, then its even Clifford algebra splits as a direct product $\mathcal{C}_+(\psi) \times \mathcal{C}_-(\psi)$, for some isomorphic central simple algebras $\mathcal{C}_+(\psi)$ and $\mathcal{C}_-(\psi)$ over F (see [Lam05, V, Thm 2.5]). Those algebras are Brauer-equivalent to the full Clifford algebra of ψ and their Brauer class is the Clifford invariant of ψ . Assume moreover that $\dim(\psi) \equiv 0 \pmod{4}$. As explained in [KMRT98, (8.4)], the involution γ then induces an involution on each factor of $\mathcal{C}_0(\psi)$, and one may easily check that the isomorphism between the two factors described in the proof of [Lam05, V, Thm 2.5] preserves the involution, so that we actually get a decomposition $(\mathcal{C}_0(\psi), \gamma) \simeq (\mathcal{C}_+(\psi), \gamma_+) \times (\mathcal{C}_-(\psi), \gamma_-)$, with $(\mathcal{C}_+(\psi), \gamma_+) \simeq (\mathcal{C}_-(\psi), \gamma_-)$.

Let L/F be a quadratic field extension. For any quadratic form ψ over L , we let $\text{tr}_*(\psi)$ be the transfer of ψ , associated to the trace map $\text{tr} : L \rightarrow F$, as defined in [Lam05, VII.1.2]. This definition extends to the split étale case $L = F \times F$ and leads to $\text{tr}_*(\psi, \psi') = \psi + \psi'$. On the other hand, for any algebra A over L , we let $N_{L/F}(A)$ be its norm, as defined in [KMRT98, §3.B]. Recall that the Brauer class of $N_{L/F}(A)$ is the corestriction of the Brauer class of A . Moreover, if A is endowed with an involution of the first kind σ , then the tensor product $\sigma \otimes \sigma$ restricts to an involution $N_{L/F}(\sigma)$ on $N_{L/F}(A)$. We use the following notation: $N_{L/F}(A, \sigma) = (N_{L/F}(A), N_{L/F}(\sigma))$. In the split étale case, we get $N_{F \times F/F}((A, \sigma), (A', \sigma')) = (A, \sigma) \otimes (A', \sigma')$ (see [KMRT98, §15.B]).

Let (A, σ) be a degree 8 algebra with orthogonal involution. We assume that (A, σ) is *totally decomposable*, that is, isomorphic to a tensor product of three quaternion algebras with involution,

$$(A, \sigma) = \otimes_{i=1}^3 (Q_i, \sigma_i).$$

If A is split (resp. has index 2), then (A, σ) admits a decomposition as above in which each quaternion algebra (resp. each but one) is split (see [Bec08]). Our main result is the following theorem:

Theorem 1.1. *Let (A, σ) be a degree 8 totally decomposable algebra with orthogonal involution. If the index of A is ≤ 4 , then there exists $\lambda \in F^\times$ and a biquaternion algebra with orthogonal involution (D, θ) such that*

$$(A, \sigma) \simeq (D, \theta) \otimes \text{Ad}_{\langle\langle \lambda \rangle\rangle}.$$

The theorem readily follows from Becher's results mentioned above if A has index 1 or 2; it is proven in § 3 for algebras of index 4. For algebras of index ≤ 2 , we may even assume that (D, θ) decomposes as a tensor product of two quaternion algebras with involution; this is not the case anymore if A has index 4, as was shown by Sivatski [Siv05, Prop. 5].

Using triality, we easily deduce the following from Theorem 1.1:

Theorem 1.2 (Izhboldin-Karpenko). *Let ϕ be an 8-dimensional quadratic form over F . The following are equivalent:*

- (i) ϕ has trivial discriminant and Clifford invariant of index ≤ 4 ;
 (ii) there exists a quadratic étale extension L/F and a form $\psi \in GP_2(L)$ such that $\phi = \text{tr}_*(\psi)$.

If $\phi = \text{tr}_*(\psi)$ for some $\psi \in GP_2(L)$, it follows from some direct computation made in [IK00, §16] that ϕ has trivial discriminant and Clifford invariant of index ≤ 4 .

Assume conversely that ϕ has trivial discriminant. By the Arason-Pfister Hauptsatz, ϕ is in $GP_3(F)$ if and only if it has trivial Clifford invariant. More generally, it is well-known that ϕ decomposes as $\phi = \langle\langle a \rangle\rangle q$ for some $a \in F^\times$ and some 4-dimensional quadratic form q over F if and only if its Clifford invariant has index ≤ 2 (see for instance [Kne77, Ex 9.12]). Hence, in both cases, ϕ decomposes as a sum $\phi = \pi_1 + \pi_2$ of two forms $\pi_1, \pi_2 \in GP_2(F)$. This proves that condition (ii) holds with $L = F \times F$.

In section 4 below, we finish this proof by treating the index 4 case. This part of the proof differs from the argument given in [IK00]. In particular, we do not use Rost's description of 14-dimensional forms in $I^3 F$.

2. CLIFFORD ALGEBRA OF THE TRANSFER OF A QUADRATIC FORM

Let L/F be a quadratic field extension. By Arason [Ara75, 4.18], for any quadratic form $\psi \in GP_2(L)$, the Clifford invariant of the transfer $\text{tr}_*(\psi)$ coincides with the corestriction of the Clifford invariant of ψ . In this section, we extend this result, taking into account the algebras with involution rather than just the Brauer classes. More precisely, we prove:

Proposition 2.1. *Let $L = F[X]/(X^2 - d)$ be a quadratic étale extension of F . Consider a quadratic form ψ over L with $\dim(\psi) \equiv 0 \pmod{4}$ and $d_\pm(\psi) = 1$, so that its even Clifford algebra decomposes as*

$$(\mathcal{C}_0(\psi), \gamma) \simeq (\mathcal{C}_+(\psi), \gamma_+) \times (\mathcal{C}_-(\psi), \gamma_-), \text{ with } (\mathcal{C}_+(\psi), \gamma_+) \simeq (\mathcal{C}_-(\psi), \gamma_-).$$

For any $\lambda \in L^\times$ represented by ψ , the two components of the even Clifford algebra of the transfer of ψ are both isomorphic to

$$(\mathcal{C}_+(\text{tr}_*(\psi)), \gamma_+) \simeq \text{Ad}_{\langle\langle -dN_{L/F}(\lambda) \rangle\rangle} \otimes_{N_{L/F}} (\mathcal{C}_+(\psi), \gamma_+).$$

Proof. In the split étale case $L = F \times F$, the quadratic form ψ is a couple (ϕ, ϕ') of two quadratic forms over F with

$$\dim(\phi) = \dim(\phi') \equiv 0 \pmod{4} \quad \text{and} \quad d_\pm(\phi) = d_\pm(\phi') = 1 \in F^*/F^{*2}.$$

Pick λ and λ' in F respectively represented by ϕ and ϕ' ; the norm $N_{F \times F/F}(\lambda, \lambda')$ is $\lambda\lambda'$. So the following lemma proves the proposition in that case:

Lemma 2.2. *Let ϕ and ϕ' be two quadratic forms over F of the same dimension $n \equiv 0 \pmod{4}$ and trivial discriminant. For any λ and $\lambda' \in F^\times$, respectively represented by ϕ and ϕ' , the components of the even Clifford algebra of the orthogonal sum $\phi + \phi'$ are isomorphic to*

$$(\mathcal{C}_+(\phi + \phi'), \gamma_+) \simeq \text{Ad}_{\langle\langle -\lambda\lambda' \rangle\rangle} \otimes (\mathcal{C}_+(\phi), \gamma_+) \otimes (\mathcal{C}_+(\phi'), \gamma_+).$$

Proof of Lemma 2.2. Denote by V and V' the underlying quadratic spaces. The natural embeddings $V \hookrightarrow V \oplus V'$ and $V' \hookrightarrow V \oplus V'$ induce F -algebra homomorphisms

$$\mathcal{C}(\phi) \rightarrow \mathcal{C}(\phi + \phi') \quad \text{and} \quad \mathcal{C}(\phi') \rightarrow \mathcal{C}(\phi + \phi').$$

One may easily check that the images of the even parts centralize each other, so that we get an F -algebra homomorphism

$$(\mathcal{C}_0(\phi), \gamma) \otimes (\mathcal{C}_0(\phi'), \gamma) \rightarrow (\mathcal{C}_0(\phi + \phi'), \gamma).$$

Pick orthogonal bases (e_1, \dots, e_n) of (V, ϕ) and (e'_1, \dots, e'_n) of (V', ϕ') . The basis of $\mathcal{C}_0(\phi + \phi')$ consisting of products of an even number of vectors of the set $\{e_1, \dots, e_n, e'_1, \dots, e'_n\}$ as described in [Lam05, V, cor 1.9] clearly contains the image of a basis of $\mathcal{C}_0(\phi) \otimes \mathcal{C}_0(\phi')$, so that the homomorphism above is injective. In the sequel, we will identify $\mathcal{C}_0(\phi)$ and $\mathcal{C}_0(\phi')$ with their images in $\mathcal{C}_0(\phi + \phi')$.

Consider the element $z = e_1 \dots e_n \in \mathcal{C}_0(\phi)$. As explained in [Lam05, V, Thm2.2], for any $v \in V$, one has $vz = -zv \in \mathcal{C}(\phi)$ and z generates the center of $\mathcal{C}_0(\phi)$. Since ϕ has dimension 0 mod 4 and trivial discriminant, this element z is γ -symmetric, and multiplying e_1 by a scalar if necessary, we may assume $z^2 = 1$. The two components of $\mathcal{C}_0(\phi)$ are $\mathcal{C}_+(\phi) = \mathcal{C}_0(\phi) \frac{1+z}{2}$ and $\mathcal{C}_-(\phi) = \mathcal{C}_0(\phi) \frac{1-z}{2}$. Consider similarly $z' = e'_1 \dots e'_n$, with $\gamma(z') = z'$ and assume $z'^2 = 1$. The product zz' also has square 1 and generates the center of $\mathcal{C}_0(\phi + \phi')$. We denote by ε the idempotent $\varepsilon = \frac{1+zz'}{2}$, so that $\mathcal{C}_+(\phi + \phi') = \mathcal{C}_0(\phi + \phi')\varepsilon$ and $\mathcal{C}_-(\phi + \phi') = \mathcal{C}_0(\phi + \phi')(1 - \varepsilon)$.

Let us now fix two vectors $v \in V$ and $v' \in V'$ such that $\phi(v) = \lambda$ and $\phi'(v') = \lambda'$. Since $\frac{1+z}{2}v^{-1} = v^{-1}\frac{1-z}{2}$, we have $v xv^{-1} \in \mathcal{C}_-(\phi)$ for any $x \in \mathcal{C}_+(\phi)$. Using this identification between the two components, we may diagonally embed $\mathcal{C}_+(\phi)$ in $\mathcal{C}_0(\phi)$ by considering $x \in \mathcal{C}_+(\phi) \mapsto x + v xv^{-1} \in \mathcal{C}_0(\phi)$. Similarly, we may embed $\mathcal{C}_+(\phi')$ in $\mathcal{C}_0(\phi')$ by $x' \in \mathcal{C}_+(\phi') \mapsto x' + v' x' v'^{-1} \in \mathcal{C}_0(\phi')$. Combining those two maps with the morphism

$$\mathcal{C}_0(\phi) \otimes \mathcal{C}_0(\phi') \rightarrow \mathcal{C}_0(\phi + \phi'),$$

and the projection

$$y \in \mathcal{C}_0(\phi + \phi') \mapsto y\varepsilon \in \mathcal{C}_+(\phi + \phi'),$$

we get an algebra homomorphism

$$\begin{aligned} \mathcal{C}_+(\phi) \otimes \mathcal{C}_+(\phi') &\rightarrow \mathcal{C}_+(\phi + \phi'), \\ x \otimes x' &\mapsto (x + v xv^{-1})(x' + v' x' v'^{-1})\varepsilon. \end{aligned}$$

One may easily check on generators that this map is not trivial; hence it is injective. To conclude the proof, it only remains to identify the centralizer of the image, which by dimension count has degree 2. It clearly contains $\frac{z+z'}{2}\varepsilon$ and $vv'\varepsilon$. Moreover, these two elements anticommute, have square ε and $-\lambda\lambda'\varepsilon$, and are respectively symmetric and skew-symmetric under γ . Hence they generate a split quaternion algebra, with orthogonal involution of discriminant $-\lambda\lambda'$, which is isomorphic to $\text{Ad}_{\langle\langle -\lambda\lambda' \rangle\rangle}$. \square

This concludes the proof in the split étale case. Until the end of this section, we assume L is a quadratic field extension of F , with non-trivial F -automorphism denoted by ι . To prove the proposition in this case, we will use the following description of the transfer of a quadratic form and its Clifford algebra.

Let ψ be any quadratic form over L , defined on the vector space V . We consider its conjugate ${}^tV = \{{}^t v, v \in V\}$ with the following operations ${}^t v_1 + {}^t v_2 = {}^t(v_1 + v_2)$ and $\lambda \cdot {}^t v = {}^t(\iota(\lambda) \cdot v)$, for any v_1, v_2 and v in V and $\lambda \in L$. Clearly, ${}^t\psi({}^t v) = \iota(\psi(v))$ is a quadratic form on tV . One may easily check from the definition given in [Lam05, VII §1] that the quadratic form $\text{tr}_\star(\psi)$ is nothing but the restriction of $\psi + {}^t\psi$ to

the F -vector space of fixed points $(V \oplus {}^tV)^s$, where s is the switch semi-linear automorphism defined on the direct sum $V \oplus {}^tV$ by $s(v_1 + {}^tv_2) = v_2 + {}^tv_1$.

Moreover, s induces a semi-linear automorphism of order 2 of the tensor algebra $T(V \oplus {}^tV)$ which preserves the ideal generated by the elements

$$(v_1 + {}^tv_2) \otimes (v_1 + {}^tv_2) - (\psi(v_1) + {}^t\psi({}^tv_2)).$$

Hence, we get a semi-linear automorphism s of order 2 on the Clifford algebra $\mathcal{C}(\psi + {}^t\psi)$, which commutes with the canonical involution. The set of fixed points $(\mathcal{C}(\psi + {}^t\psi))^s$ is an F -algebra; the involution γ restricts to an F -linear involution which we denote by γ_s . We then have:

Lemma 2.3. *The natural embedding $(V \oplus {}^tV) \hookrightarrow \mathcal{C}(\psi + {}^t\psi)$, restricted to $(V + {}^tV)^s$, induces an isomorphism of graded algebras*

$$(\mathcal{C}(\mathrm{tr}_*(\psi)), \gamma) \xrightarrow{\sim} ((\mathcal{C}(\psi + {}^t\psi))^s, \gamma_s).$$

Proof of Lemma 2.3. The natural embedding $(V \oplus {}^tV) \hookrightarrow \mathcal{C}(\psi + {}^t\psi)$ restricts to an injective map $i : (V + {}^tV)^s \hookrightarrow \mathcal{C}(\psi + {}^t\psi)^s$, which clearly satisfies

$$i(w)^2 = (\psi + {}^t\psi)(w) \quad \text{for any } w \in (V \oplus {}^tV)^s.$$

By the universal property of Clifford algebras, it extends to a non-trivial algebra homomorphism $\mathcal{C}(\mathrm{tr}_*(\psi)) \mapsto \mathcal{C}(\psi + {}^t\psi)^s$, which clearly preserves the grading. Since $\mathcal{C}(\mathrm{tr}_*(\psi))$ is simple, and both algebras have the same dimension, it is an isomorphism. Clearly, γ coincides with γ_s under this isomorphism. \square

Hence, we want to describe one component of $\mathcal{C}_0(\mathrm{tr}_*(\psi)) \simeq (\mathcal{C}_0(\psi + {}^t\psi))^s$. We proceed as in the split étale case. Fix an orthogonal basis e_1, \dots, e_n of V over L such that $\psi(e_n) = \lambda$. The elements ${}^te_1, \dots, {}^te_n$ are an orthogonal basis of tV and ${}^t\psi({}^te_n) = \iota(\lambda)$. We may moreover assume that $z = e_1 \dots e_n$ and ${}^tz = {}^te_1 \dots {}^te_n$ have square 1. Since the idempotent $\varepsilon = \frac{1+z'z}{2} \in \mathcal{C}_0(\psi + {}^t\psi)$ satisfies $s(\varepsilon) = \varepsilon$, the semilinear automorphism s preserves each factor $\mathcal{C}_+(\psi + {}^t\psi)$ and $\mathcal{C}_-(\psi + {}^t\psi)$. Hence, the components of $\mathcal{C}_0(\mathrm{tr}_*(\psi))$ are

$$\mathcal{C}_0(\mathrm{tr}_*(\psi)) = (\mathcal{C}_+(\psi + {}^t\psi))^s \times (\mathcal{C}_-(\psi + {}^t\psi))^s.$$

Moreover, by Lemma 2.2, we have

$$\mathcal{C}_+(\psi + {}^t\psi) \simeq \mathrm{Ad}_{\langle -\lambda, \iota(\lambda) \rangle} \otimes (\mathcal{C}_+(\psi), \gamma) \otimes (\mathcal{C}_+({}^t\psi), \gamma),$$

and it remains to understand the action of the switch automorphism on this tensor product. First, one may identify $\mathcal{C}_+({}^t\psi)$ with the algebra ${}^t\mathcal{C}_+(\psi)$ defined by

$${}^t\mathcal{C}_+(\psi) = \{{}^tx, x \in \mathcal{C}_+(\psi)\},$$

with the operations

$${}^tx + {}^ty = {}^t(x + y), \quad {}^tx{}^ty = {}^t(xy) \quad \text{and} \quad {}^t(\lambda x) = \iota(\lambda){}^tx,$$

for all $x, y \in \mathcal{C}_+(\psi)$ and $\lambda \in L$. Clearly, the switch automorphism acts on the tensor product

$$\mathcal{C}_+(\psi) \otimes \mathcal{C}_+({}^t\psi) \simeq \mathcal{C}_+(\psi) \otimes {}^t\mathcal{C}_+(\psi),$$

by

$$s(x \otimes {}^ty) = y \otimes {}^tx,$$

and by definition of the corestriction (see [KMRT98, 3.B]), the F -subalgebra of fixed points is

$$((\mathcal{C}_+(\psi), \gamma) \otimes ({}^t\mathcal{C}_+(\psi), \gamma))^s = N_{L/F}(\mathcal{C}_+(\psi), \gamma).$$

It remains to understand the action of the switch on the centralizer, which is the split quaternion algebra over L generated by $x = \frac{z+\iota}{2}\varepsilon$ and $y = e_n{}^\iota e_n\varepsilon$. The element x clearly is s -symmetric, while y satisfies $s(y) = -y$. Let δ be a generator of the quadratic extension L/F , so that $\iota(\delta) = -\delta$ and $\delta^2 = d$. Since the switch map s is L/F semi-linear, we may replace y by δy which now satisfies $s(\delta y) = \delta y$. Hence, the set of fixed points under s is the split quaternion algebra over F generated by x and δy . Since $(\delta y)^2 = -dN_{L/F}(\lambda)$, it is isomorphic to $\text{Ad}_{\langle\langle -dN_{L/F}(\lambda) \rangle\rangle}$. \square

3. PROOF OF THE DECOMPOSABILITY THEOREM

In this section, we finish the proof of Theorem 1.1. Let $(A, \sigma) = \otimes_{i=1}^3 (Q_i, \sigma_i)$ be a product of three quaternion algebras with orthogonal involution. We assume that A has index 4, so that it is Brauer-equivalent to a biquaternion division algebra D . We have to prove that (A, σ) is isomorphic to $(D, \theta) \otimes \text{Ad}_{\langle\langle \lambda \rangle\rangle}$ for a well chosen involution θ on D and some $\lambda \in F^\times$.

The algebra D is endowed with an orthogonal involution τ , and we may represent

$$(A, \sigma) = (\text{End}_D(M), \text{ad}_h),$$

for some 2-dimensional hermitian module (M, h) over (D, τ) . Let us consider a diagonalisation $\langle a_1, a_2 \rangle$ of h , and define

$$\theta = \text{Int}(a_1^{-1}) \circ \tau.$$

With respect to this new involution, we get another representation

$$(A, \sigma) = (\text{End}_D(M), \text{ad}_{h'}),$$

where h' is a hermitian form over (D, θ) which diagonalises as $h' = \langle 1, -a \rangle$ for some θ -symmetric element $a \in D^\times$. The theorem now follows from the following lemma:

Lemma 3.1. *The involutions θ and $\theta' = \text{Int}(a^{-1}) \circ \theta$ of the biquaternion algebra D are conjugate.*

Indeed, assume there exists $u \in A^\times$ such that $\theta = \text{Int}(u) \circ \theta' \circ \text{Int}(u^{-1})$. We then have $\theta = \text{Int}(ua^{-1}) \circ \theta \circ \text{Int}(u^{-1}) = \theta \circ \text{Int}(\theta(u)^{-1}au^{-1})$. Hence, there exists $\lambda \in F^\times$ such that $\theta(u)^{-1}au^{-1} = \lambda$, that is $a = \lambda\theta(u)u$. This implies that the hermitian form $h' = \langle 1, -a \rangle$ over (D, θ) is isometric to $\langle 1, -\lambda \rangle$. Since $\lambda \in F^\times$, we get $(A, \sigma) = (\text{End}_D(M), \text{ad}_{\langle 1, -\lambda \rangle}) = (D, \theta) \otimes \text{Ad}_{\langle\langle \lambda \rangle\rangle}$, and it only remains to prove the lemma.

Proof of Lemma 3.1. We want to compare the orthogonal involutions θ and θ' of the biquaternion algebra D . By [LT99, Prop. 2], they are conjugate if and only if their Clifford algebras \mathcal{C} and \mathcal{C}' are isomorphic as F -algebras. This can be proven as follows.

Since (A, σ) is a product of three quaternion algebras with involution, we know from [KMRT98, (42.11)] that the discriminant of σ is 1 and its Clifford algebra has one split component.

On the other hand, the representation $(A, \sigma) = (\text{End}_D(M), \text{ad}_{\langle 1, -a \rangle})$ tells us that (A, σ) is an orthogonal sum, as in [Dej95], of (D, θ) and (D, θ') . Hence its invariants can be computed in terms of those of (D, θ) and (D, θ') . By [Dej95, Prop. 2.3.3], the discriminant of σ is the product of the discriminants of θ and θ' . So θ and θ' have the same discriminant, and we may identify the centers Z and Z' of their Clifford algebras in two different ways. We are in the situation described in [LT99, p. 265], where the Clifford algebra of such an orthogonal sum is computed. In

particular, since one component of the Clifford algebra of (A, σ) is split, it follows from [LT99, Lem 1] that

$$\mathcal{C} \simeq \mathcal{C}' \quad \text{or} \quad \mathcal{C} \simeq {}'\mathcal{C}',$$

depending on the chosen identification between Z and Z' . In both cases, \mathcal{C} and \mathcal{C}' are isomorphic as F -algebras, and this concludes the proof. \square

4. A NEW PROOF OF IZHBOLDIN AND KARPENKO'S THEOREM

Let ϕ be an 8-dimensional quadratic form over F with trivial discriminant and Clifford invariant of index 4. We denote by (A, σ) one component of its even Clifford algebra, so that

$$(\mathcal{C}_0(\phi), \gamma) \simeq (A, \sigma) \times (A, \sigma),$$

where A is an index 4 central simple algebra over F , with orthogonal involution σ .

By triality [KMRT98, (42.3)], the involution σ has trivial discriminant and its Clifford algebra is

$$\mathcal{C}(A, \sigma) = \text{Ad}_\phi \times (A, \sigma).$$

In particular, it has a split component, so that the algebra with involution (A, σ) is isomorphic to a tensor product of three quaternion algebras with involution (see [KMRT98, (42.11)]). Hence we can apply our decomposability theorem 1.1, and write $(A, \sigma) = (D, \theta) \otimes \text{Ad}_{\langle\langle\lambda\rangle\rangle}$ for some biquaternion division algebra with orthogonal involution (D, θ) and some $\lambda \in F^\times$.

Let us denote by d the discriminant of θ , and let $L = F[X]/(X^2 - d)$ be the corresponding quadratic étale extension. Consider the image δ of X in L . By Tao's computation of the Clifford algebra of a tensor product [Tao95, Thm. 4.12], the components of $\mathcal{C}(A, \sigma)$ are Brauer-equivalent to the quaternion algebra (d, λ) over F and the tensor product $(d, \lambda) \otimes A$. Since A has index 4, the split component has to be (d, λ) , so that λ is a norm of L/F , say $\lambda = N_{L/F}(\mu)$.

Consider now the Clifford algebra of (D, θ) . It is a quaternion algebra Q over L , endowed with its canonical (symplectic) involution γ . Denote by n_Q the norm form of Q , that is $n_Q = \langle\langle\alpha, \beta\rangle\rangle$ if $Q = (\alpha, \beta)_L$. It is a 2-fold Pfister form and for any $\ell \in L^\times$, $(\mathcal{C}_+(\langle\ell\rangle n_Q), \gamma_+) \simeq (Q, \gamma)$. Moreover, by the equivalence of categories $A_1^2 \equiv D_2$ described in [KMRT98, (15.7)], the algebra with involution (D, θ) is canonically isomorphic to $N_{L/F}(Q, \gamma)$.

Hence we get that $(A, \sigma) = N_{L/F}(Q, \gamma) \otimes \text{Ad}_{\langle\langle -dN_{L/F}(\delta\mu) \rangle\rangle}$. By Proposition 2.1, this implies that

$$(A, \sigma) \times (A, \sigma) \simeq (\mathcal{C}_0(\text{tr}_*(\psi)), \gamma),$$

where $\psi = \langle\delta\mu\rangle n_Q$. Applying again triality [KMRT98, (42.3)], we get that the split component Ad_ϕ of the Clifford algebra of (A, σ) also is isomorphic to $\text{Ad}_{\text{tr}_*(\psi)}$, so that the quadratic forms ϕ and $\text{tr}_*(\psi)$ are similar. This concludes the proof since ψ belongs to $GP_2(L)$.

Remark. Let ϕ and (A, σ) be as above, and let $L = F[X]/(X^2 - d)$ be a fixed quadratic étale extension of F . It follows from the proof that the quadratic form ϕ is isometric to the transfer of a form $\psi \in GP_2(L)$ if and only if (A, σ) admits a decomposition $(A, \sigma) = \text{Ad}_{\langle\langle\lambda\rangle\rangle} \otimes (D, \theta)$, with $d_\pm(\theta) = d$. In particular, the quadratic form ϕ is a sum of two forms similar to 2-fold Pfister forms exactly when the algebra with involution (A, σ) admits a decomposition as $(D, \theta) \otimes \text{Ad}_{\langle\langle\lambda\rangle\rangle}$ with θ of

discriminant 1, that is when it decomposes as a tensor product of three quaternion algebras with involution, with one split factor.

Such a decomposition does not always exist, as was shown by Sivatski [Siv05, Prop 5]. This reflects the fact that 8-dimensional quadratic forms ϕ with trivial discriminant and Clifford algebra of index ≤ 4 do not always decompose as a sum of two forms similar to two-fold Pfister forms (see [IK00, §16] and [HT98] for explicit examples).

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