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ALMOST SURE CONVERGENCE OF A STOCHASTIC APPROXIMATION PROCESS IN A CONVEX SET

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Abstract : We consider a stochastic approximation process in a convex set K of \mathbb{R}^k : $X_{n+1} = \Pi(X_n - A_n Y_n)$, with $E[A_n Y_n | T_n] = a_n M_n(X_n)$, where Π is the projection operator on K , A_n a random matrix, a_n a positive number, M_n a function from K into \mathbb{R}^k and T_n the sub- σ -algebra generated by the events before time n . We prove two theorems of almost sure convergence in the case where the equation $M_n(x) = 0$ has a set of solutions and give two applications.

AMS Subj. Classification : 62L20

Key Words : stochastic approximation, linear regression

1. Introduction

We define a stochastic approximation process (X_n) in a non-empty closed convex subset K of \mathbb{R}^k , named parameter space ; we consider :

- . for $n \geq 1$, an observable random variable Y_n in \mathbb{R}^p , named observation space ; remark that the observation space may be different from the parameter space ;
- . for $n \geq 1$, a (k, p) random matrix A_n ;
- . the projection operator Π on K ;
- . the process (X_n) in K defined recursively by

$$X_{n+1} = \Pi(X_n - A_n Y_n)$$

All random variables are defined on a probability space (Ω, \mathcal{A}, P) . Denote T_n the sub- σ -algebra of \mathcal{A} generated by the events before time n ; $X_1, \dots, X_n, A_1, \dots, A_n, Y_1, \dots, Y_{n-1}$ are measurable with respect to T_n .

Suppose that, for $n \geq 1$, there exists a measurable function M_n from K into \mathbb{R}^k and a positive number a_n such that

$$E[A_n Y_n | T_n] = A_n E[Y_n | T_n] = a_n M_n(X_n) \text{ a.s.}$$

Let B_n be a set of solutions of the equation $M_n(x) = 0$. Define a distance $d(x, B)$ from x in \mathbb{R}^k to a subset B .

We give in Section 2 two almost sure convergence theorems of $d(X_n, B_n)$ to 0. An application of each theorem is given in Section 3, concerning the estimation of a quantile interval of an unknown probability distribution and the estimation of a linear regression parameter under convex constraints.

In the following, $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ are respectively the usual inner product and norm in \mathbb{R}^k ; A' denotes the transposed matrix of A , $\lambda_{\min}(B)$ the smallest eigenvalue of B ; the abbreviation *a.s.* means almost surely.

2. Lemmas

Let (X_n) be a stochastic process in a subset K of \mathbb{R}^k . Let (F_n) and (φ_n) be two sequences of measurable functions from K into \mathbb{R}^+ and (a_n) a sequence in \mathbb{R}^+ . Suppose :

(H1a) There exists a random variable T in \mathbb{R}^+ such that $F_n(X_n) \longrightarrow T$ *a.s.*

(H1b) $\sum_1^\infty a_n \varphi_n(X_n) < \infty$ *a.s.*

(H2a) Whatever $0 < \epsilon < 1$, $\sum_1^\infty a_n \inf_{\{x \in K, \epsilon < F_n(x) < \frac{1}{\epsilon}\}} \varphi_n(x) = +\infty$.

Lemma 1 *Assume H1a, b and H2a hold ; then $F_n(X_n) \longrightarrow 0$ a.s.*

Proof. $\omega \in \Omega$ is fixed throughout the proof, belonging to the intersection of the defined *a.s.* convergence sets. Suppose $T(\omega) \neq 0$ and suppress ω writing.

By H1a, there exist $0 < \epsilon_1 < 1$ and an integer $N(\epsilon_1)$ such that for $n > N(\epsilon_1)$, $\epsilon_1 < F_n(X_n) < \frac{1}{\epsilon_1}$.

This implies $\varphi_n(X_n) \geq \inf_{\{x \in K, \epsilon_1 < F_n(x) < \frac{1}{\epsilon_1}\}} \varphi_n(x)$; then by H2a,

$$\sum_1^\infty a_n \varphi_n(X_n) = \infty,$$

a contradiction with H1b. Thus $T(\omega) = 0$. ■

Suppose now :

(H1c) $\|X_{n+1} - X_n\| \longrightarrow 0$ *a.s.*

(H3a) For all $0 < \epsilon_1 < 1$, for all $\epsilon > 0$, there exists $\eta > 0$ such that

$$(\|x_1 - x_2\| < \eta) \Rightarrow \left(\sup_n \sup_{\{\epsilon_1 < F_n(x_1) < \frac{1}{\epsilon_1}\}} |\varphi_n(x_1) - \varphi_n(x_2)| < \epsilon \right)$$

(H3b) There exist a positive integer r , a sequence of integers (n_l) , for all $0 < \epsilon < 1$ an integer $L(\epsilon)$ such that $n_{l+1} \leq n_l + r$ and

$$b(\epsilon) = \inf_{l > L(\epsilon)} \inf_{\{x \in K, \epsilon < F_{n_l}(x) < \frac{1}{\epsilon}\}} \sum_{j \in I_l} \varphi_j(x) > 0$$

with $I_l = \{n_l, n_l + 1, \dots, n_{l+1} - 1\}$

(H2b) $\sum_l \min_{j \in I_l} a_j = \infty$.

Lemma 2 Assume H1a, b, c, H2b and H3a, b hold ; then $F_n(X_n) \longrightarrow 0$ *a.s.*

Proof. $\omega \in \Omega$ is fixed throughout the proof, belonging to the intersection of the defined *a.s.* convergence sets. Suppose $T(\omega) \neq 0$. Below ω is omitted.

By H1a, there exist $0 < \epsilon_1 < 1$ and an integer $N(\epsilon_1)$ such that for $n > N(\epsilon_1)$, $\epsilon_1 < F_n(X_n) < \frac{1}{\epsilon_1}$.

By H3b, there exists an integer $L(\epsilon_1)$ such that for $l > L(\epsilon_1)$,

$$\sum_{j \in I_l} \varphi_j(X_{n_l}) > b(\epsilon_1).$$

It follows that there exists $m_l \in I_l$ such that

$$\varphi_{m_l}(X_{n_l}) > \frac{b(\epsilon_1)}{r}.$$

Consider the decomposition

$$\begin{aligned} \varphi_{m_l}(X_{m_l}) &= \varphi_{m_l}(X_{n_l}) + \varphi_{m_l}(X_{m_l}) - \varphi_{m_l}(X_{n_l}). \\ \varphi_{m_l}(X_{m_l}) &> \frac{b(\epsilon_1)}{r} - |\varphi_{m_l}(X_{m_l}) - \varphi_{m_l}(X_{n_l})|. \end{aligned}$$

Let $\epsilon > 0$; by H3a, there exists $\eta > 0$ corresponding to ϵ_1 and ϵ ; by H1c, we have for l sufficiently large :

$$\|X_{m_l} - X_{n_l}\| < \eta \quad ; \quad \epsilon_1 < F_{m_l}(X_{m_l}) < \frac{1}{\epsilon_1}.$$

By H3a, this implies :

$$|\varphi_{m_l}(X_{m_l}) - \varphi_{m_l}(X_{n_l})| < \epsilon ;$$

$$\varphi_{m_l}(X_{m_l}) > \frac{b(\epsilon_1)}{r} - \epsilon.$$

Choose $\epsilon < \frac{b(\epsilon_1)}{r}$. By H2b, $\sum_l a_{m_l} \varphi_{m_l}(X_{m_l}) = +\infty$. Then

$$\sum_n a_n \varphi_n(X_n) = +\infty,$$

a contradiction with H1b. Thus $T(\omega) = 0$. ■

3. Theorems of almost sure convergence

Consider the process (X_n) as defined in section 1 :

$$\begin{aligned} X_{n+1} &= \Pi(X_n - A_n Y_n) \\ E[A_n Y_n | T_n] &= a_n M_n(X_n) \text{ a.s.} \end{aligned}$$

Denote $d(x, B)$ a distance from $x \in \mathbb{R}^k$ to a subset B .

For all n , let F_n be a function from \mathbb{R}^k into \mathbb{R}^+ twice continuously differentiable, with gradient G_n and hessian matrix H_n ; by the Taylor formula, there exists $0 < \mu_n < 1$ such that

$$F_n(X_n - A_n Y_n) = F_n(X_n) - \langle G_n(X_n), A_n Y_n \rangle + \frac{1}{2} \langle A_n Y_n, H_n(X_n - \mu_n A_n Y_n) A_n Y_n \rangle$$

$$\text{Denote } V_n = \frac{1}{2} E[\langle A_n Y_n, H_n(X_n - \mu_n A_n Y_n) A_n Y_n \rangle | T_n].$$

Suppose:

(H4a) For all n , F_n is twice continuously differentiable

(H4b) For all $\epsilon > 0$, there exists $\nu(\epsilon) > 0$ and for all n , there exists a subset B_n of K such that

$$\inf_n \inf_{\{d(x, B_n) > \epsilon\}} F_n(x) > \nu(\epsilon)$$

(H4c) There exist two sequences of positive numbers (γ_n) and (δ_n) such that $\sum_1^\infty \gamma_n < \infty$, $\sum_1^\infty \delta_n < \infty$ and for all n and x ,

$$F_{n+1}(\Pi x) \leq (1 + \delta_n) F_n(x) + \gamma_n$$

(H5) For all n , there exist two random variables D_n and E_n in \mathbb{R}^+ , measurable with respect to T_n , such that

$$\begin{aligned} \sum_1^\infty D_n &< \infty, \quad \sum_1^\infty E_n < \infty, \\ V_n &\leq D_n F_n(X_n) + E_n \text{ a.s.} \end{aligned}$$

$$(H6) \sum_1^\infty \langle G_n(X_n), a_n M_n(X_n) \rangle^- < \infty \quad a.s.$$

$$(H7) \text{ For all } 0 < \epsilon < 1, \sum_1^\infty a_n \inf_{\{x \in K, \epsilon < F_n(x) < \frac{1}{\epsilon}\}} \langle G_n(x), M_n(x) \rangle^+ < \infty.$$

Remark that in the case where B_n is reduced to a single element θ of \mathbb{R}^k not depending on n , if we take $F_n(x) = d^2(x, \theta) = \|x - \theta\|^2$, then assumptions H4a, b, c hold and $G_n(x) = 2(x - \theta)$, $H_n(x) = 2I$ (I : identity matrix), $V_n = E[\|A_n Y_n\|^2 | T_n]$.

Theorem 3 Assume H4a, b, c, H5, H6 and H7 hold ; then $F_n(X_n) \longrightarrow 0$ and $d(X_n, B_n) \longrightarrow 0$ a.s.

We use in the proof the Robbins-Siegmund lemma [4] :

Lemma 4 Let (Ω, \mathcal{A}, P) be a probability space and (T_n) an increasing sequence of sub- σ -algebras of \mathcal{A} . For $n \geq 1$, let z_n, β_n, ξ_n and ζ_n be non-negative T_n -measurable random variables such that $E[z_{n+1} | T_n] \leq z_n(1 + \beta_n) + \xi_n - \zeta_n$. Suppose $\sum_1^\infty \beta_n < \infty$, $\sum_1^\infty \xi_n < \infty$ a.s. Then $\lim_{n \rightarrow \infty} z_n$ exists and is finite and $\sum_1^\infty \zeta_n < \infty$ a.s.

Proof. By H4a, c and H5, we have :

$$\begin{aligned} F_{n+1}(X_{n+1}) &\leq (1 + \delta_n)F_n(X_n - A_n Y_n) + \gamma_n. \\ E[F_{n+1}(X_{n+1}) | T_n] &\leq (1 + \delta_n)(F_n(X_n) - \langle G_n(X_n), a_n M_n(X_n) \rangle + V_n) + \gamma_n \\ &\leq (1 + \delta_n)(1 + D_n)F_n(X_n) + (1 + \delta_n)E_n \\ &\quad + (1 + \delta_n) \langle G_n(X_n), a_n M_n(X_n) \rangle^- + \gamma_n \\ &\quad - (1 + \delta_n) \langle G_n(X_n), a_n M_n(X_n) \rangle^+ \quad a.s. \end{aligned}$$

By H4c, H5 and H6, the assumptions of the preceding lemma hold ; then there exists a random variable T in \mathbb{R}^+ such that $F_n(X_n) \longrightarrow T$ a.s. and $\sum_1^\infty \langle G_n(X_n), a_n M_n(X_n) \rangle^+ < \infty$ a.s.

Let $\varphi_n(x) = \langle G_n(x), M_n(x) \rangle^+$. The assumptions H1a, b and H2a of lemma 1 hold. Then $F_n(X_n) \longrightarrow 0$ a.s.

By H4b, it follows that $d(X_n, B_n) \longrightarrow 0$ a.s. ■

Prove now a second theorem.

Suppose :

$$(H4d) \text{ For all } 0 < \epsilon < 1, \sup_n \sup_{\{\epsilon < F_n(x) < \frac{1}{\epsilon}\}} \|G_n(x)\| < \infty$$

$$(H4e) \text{ For all } \epsilon > 0, \text{ there exists } \eta > 0 \text{ such that} \\ (\|x_1 - x_2\| < \eta) \Rightarrow (\sup_n \|G_n(x_1) - G_n(x_2)\| < \epsilon)$$

$$(H8a) \text{ For all } 0 < \epsilon < 1, \sup_n \sup_{\{\epsilon < F_n(x) < \frac{1}{\epsilon}\}} \|M_n(x)\| < \infty$$

(H8b) For all $\epsilon > 0$, there exists $\eta > 0$ such that
 $(\|x_1 - x_2\| < \eta) \Rightarrow (\sup_n \|M_n(x_1) - M_n(x_2)\| < \epsilon)$

(H8c) There exist a positive integer r , a sequence of integers (n_l) , for all $0 < \epsilon < 1$ an integer $L(\epsilon)$ such that $n_{l+1} \leq n_l + r$ and

$$b(\epsilon) = \inf_{l > L(\epsilon)} \inf_{\{x \in K, \epsilon < F_{n_l}(x) < \frac{1}{\epsilon}\}} \sum_{j \in I_l} \langle G_j(x), M_j(x) \rangle^+ > 0$$

with $I_l = \{n_l, n_l + 1, \dots, n_{l+1} - 1\}$

$$(H2b) \sum_l \min_{j \in I_l} a_j = \infty.$$

Remark that in the case where $B_n = \{\theta\}$ and $F_n(x) = \|x - \theta\|^2$, assumptions H4d, e hold.

Theorem 5 *Assume H2b, H4a, b, c, d, e, H5, H6, H8a, b, c hold ; then in the set $\{A_n Y_n \longrightarrow 0\}$, $F_n(X_n) \longrightarrow 0$ and $d(X_n, B_n) \longrightarrow 0$ a.s.*

Proof. Following the proof of theorem 3, we have by H4a, c, H5, H6 :
 $F_n(X_n) \longrightarrow T$ and $\sum_1^\infty \langle G_n(X_n), a_n M_n(X_n) \rangle^+ < \infty$ a.s.

Apply lemma 2 with $\varphi_n(x) = \langle G_n(x), M_n(x) \rangle^+$.

H1a, b and H3b hold. H1c holds in the set $\{A_n Y_n \longrightarrow 0\}$ as

$$\|X_{n+1} - X_n\| = \|\Pi(X_n - A_n Y_n) - \Pi X_n\| \leq \|X_n - A_n Y_n - X_n\| = \|A_n Y_n\|$$

As $|a^+ - b^+| \leq |a - b|$, we have :

$$\begin{aligned} |\varphi_n(x_1) - \varphi_n(x_2)| &\leq |\langle G_n(x_1), M_n(x_1) \rangle - \langle G_n(x_2), M_n(x_2) \rangle| \\ &\leq |\langle G_n(x_1), M_n(x_1) - M_n(x_2) \rangle| \\ &\quad + |\langle G_n(x_1) - G_n(x_2), M_n(x_2) - M_n(x_1) \rangle| \\ &\quad + |\langle G_n(x_1) - G_n(x_2), M_n(x_1) \rangle|. \end{aligned}$$

By H4d, e and H8a, b, assumption H3a holds.

Then $F_n(X_n) \longrightarrow 0$ a.s. By H4b, $d(X_n, B_n) \longrightarrow 0$ a.s. ■

4. Application to the estimation of a quantile interval

Let Z be a real random variable whose distribution function $F(t) = P(Z < t)$ is unknown. Suppose that there exists an interval (a, b) , which is eventually reduced to a single point, such that : $F(t) = \alpha \Leftrightarrow t \in (a, b)$.

Let $m \geq 1$ be an integer and $(Z_{nj}, n \geq 1, j = 1, \dots, m)$ a set of mutually independent random variables which have the same law as Z . For all x , define the random variables $I_{nj}(x)$ and $F_{nm}(x)$ such that :

$$\begin{aligned} I_{nj}(x) &= 1 \text{ if } Z_{nj} < x, I_{nj}(x) = 0 \text{ otherwise} \\ F_{nm}(x) &= \frac{1}{m} \sum_{j=1}^m I_{nj}(x). \end{aligned}$$

Then $E[F_{nm}(x)] = E[I_{nj}(x)] = F(x)$.

Define the stochastic approximation process (X_n) such that

$$X_{n+1} = X_n - a_n(F_{nm}(X_n) - \alpha).$$

If z_{nj} is the observed value of Z_{nj} and x_n the value of X_n , $F_{nm}(x_n)$ is the proportion of elements of $\{z_{n1}, \dots, z_{nm}\}$ which are smaller than x_n .

Suppose :

$$\begin{aligned} \text{(H2b')} \quad &\sum_1^\infty a_n = \infty \\ \text{(H2c)} \quad &\sum_1^\infty a_n^2 < \infty. \end{aligned}$$

Theorem 6 *Let $d(x, (a, b)) = \inf_{\{y \in (a, b)\}} |x - y|$. Assume H2b', c hold ; then $d(X_n, (a, b)) \longrightarrow 0$ a.s.*

Proof. Define the function f such that

$$\begin{aligned} f(x) &= (x - a)^2 \text{ if } x < a \\ f(x) &= 0 \text{ if } a \leq x \leq b \\ f(x) &= (x - b)^2 \text{ if } x > b. \end{aligned}$$

H4a, b, c hold for $F_n = f$ and $B_n = (a, b)$.

$|f''(x)| \leq 2$, $|F_{nm}(x) - \alpha| \leq 1$; then $V_n \leq a_n^2$; H5 holds.

$M_n(X_n) = E[F_{nm}(X_n) - \alpha | T_n] = F(X_n) - \alpha$;

$f'(x)(F(x) - \alpha) \geq 0$, $\inf_{\{\epsilon < f(x) < \frac{1}{\epsilon}\}} f'(x)(F(x) - \alpha) > 0$; H6 and H7 hold.

Applying theorem 3 gives $d(X_n, (a, b)) \longrightarrow 0$ a.s. ■

5. Application to linear regression under convex constraints

Consider a sequence (Z_n) of observable mutually independent real random variables.

Suppose that there exist an unknown vector θ in \mathbb{R}^k , for all n a known vector b_n in \mathbb{R}^k and a real random variable R_n with $E[R_n] = 0$ such that

$$Z_n = b'_n \theta + R_n.$$

Suppose moreover that θ belongs to a non-empty closed convex set K of \mathbb{R}^k . For instance :

- 1) $\|\theta\|$ is bounded ;
- 2) the components of θ are non-negative.

Consider the stochastic approximation process (X_n) such that :

$$X_{n+1} = \Pi \left(X_n - a_n \frac{b_n}{\|b_n\|^2} (b'_n X_n - Z_n) \right).$$

Suppose :

$$(H2b) \sum_1^\infty \min_{j \in I_l} a_j = \infty$$

$$(H2c) \sum_1^\infty a_n^2 < \infty$$

$$(H2d) \sum_1^\infty a_n^2 \frac{E[R_n^2]}{\|b_n\|^2} < \infty$$

$$(H9) \lambda = \inf_l \lambda_{\min} \left(\sum_{j \in I_l} \frac{b_j b'_j}{\|b_j\|^2} \right) > 0.$$

Theorem 7 Assume H2b, c, d and H9 hold ; then $X_n \longrightarrow \theta$ a.s.

This theorem completes in the case of linear regression results of Albert and Gardner [1] (p. 103, conjectured theorem).

Proof. Let $Y_n = b'_n X_n - Z_n = b'_n (X_n - \theta) - R_n$ and $A_n = a_n \frac{b_n}{\|b_n\|^2}$.

As $E[R_n | T_n] = E[R_n] = 0$, $M_n(X_n) = \frac{b_n b'_n}{\|b_n\|^2} (X_n - \theta)$ a.s.

Remark that, for fixed n , equation $M_n(x) = 0$ has an infinity of solutions. Denote I an identity matrix. Define $F_n(x) = \|x - \theta\|^2$; then :

$$G_n(x) = 2(x - \theta), H_n(x) = 2I, V_n = E [a_n^2 \|Y_n\|^2 | T_n].$$

Assumptions H4a, b, c, d, e, H6, H8a, b hold.

$$V_n = E [a_n^2 \|Y_n\|^2 | T_n] = a_n^2 \|X_n - \theta\|^2 + a_n^2 \frac{E[R_n^2]}{\|b_n\|^2}.$$

By H2d, assumption H5 holds.

By H9, assumption H8c holds as

$$\begin{aligned} \sum_{j \in I_l} \langle G_j(x), M_j(x) \rangle^+ &= 2 \sum_{j \in I_l} \left\langle x - \theta, \frac{b_j b'_j}{\|b_j\|^2} (x - \theta) \right\rangle \\ &\geq 2\lambda \|x - \theta\|^2. \end{aligned}$$

Furthermore, as $E [R_n | T_n] = 0$:

$$\begin{aligned} E [\|X_{n+1} - \theta\|^2 | T_n] &= \|X_n - \theta\|^2 + a_n^2 E [\|Y_n\|^2 | T_n] \\ &\quad - 2a_n \left\langle X_n - \theta, \frac{b_n b_n'}{\|b_n\|^2} (X_n - \theta) \right\rangle \\ &\leq (1 + a_n^2) \|X_n - \theta\|^2 + a_n^2 \frac{E [R_n^2]}{\|b_n\|^2}. \\ E [\|X_{n+1} - \theta\|^2] &\leq (1 + a_n^2) E [\|X_n - \theta\|^2] + a_n^2 \frac{E [R_n^2]}{\|b_n\|^2}. \end{aligned}$$

By H2c, d, there exists $t \geq 0$ such that $E [\|X_n - \theta\|^2] \longrightarrow t$. Then :

$$\begin{aligned} \sum_1^\infty E [a_n^2 \|Y_n\|^2] &= \sum_1^\infty \left(a_n^2 E [\|X_n - \theta\|^2] + a_n^2 \frac{E [R_n^2]}{\|b_n\|^2} \right) < \infty ; \\ \sum_1^\infty a_n^2 \|Y_n\|^2 &< \infty \text{ a.s. ; } a_n Y_n \longrightarrow 0 \text{ a.s.} \end{aligned}$$

Applying theorem 5 gives $X_n \longrightarrow \theta$ a.s. ■

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