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Michael Heusener, Richard Weidmann

GENERATING PAIRS OF 2-BRIDGE KNOT GROUPS

MICHAEL HEUSENER AND RICHARD WEIDMANN

Abstract. We study Nielsen equivalence classes of generating pairs of Kleinian groups and HNN-extensions. We establish the following facts:

(1) Hyperbolic 2-bridge knot groups have infinitely many Nielsen classes of generating pairs.
(2) For any $n \in \mathbb{N}$ there is a closed hyperbolic 3-manifold whose fundamental group has n distinct Nielsen classes of generating pairs.
(3) Two pairs of elements of a fundamental group of an HNN-extension are Nielsen equivalent iff they are so for the obvious reasons.

Introduction

The main purpose of this note is to study Nielsen equivalence classes of generating pairs of fundamental groups of hyperbolic 2-bridge knot spaces and of closed hyperbolic 3-manifolds obtained from those spaces by Dehn fillings.

It is a result of Delzant (following Gromov) [D] that any torsion-free hyperbolic group has only finitely many Nielsen classes of generating pairs. In the case of closed hyperbolic 3-manifolds Delzant's proof actually provides an explicit upper bound for this number in terms of the injectivity radius as observed by Agol, see [Sou1] for an account of Agol's ideas. The finiteness of Nielsen classes of generating tuples of fundamental groups of closed hyperbolic 3-manifolds of arbitrary size was established in [KW].

The examples constructed in this article show that this finiteness fails for cusped hyperbolic 3-manifolds. We establish the following:

Theorem 0.1. *Let $\mathfrak{k}$ be a hyperbolic 2-bridge knot with knot exterior M. Then $\pi_1(M)$ has infinitely many Nielsen classes of generating pairs.*

We further show that there is no uniform bound on the number of Nielsen classes of generating pairs of fundamental groups of closed hyperbolic 3-manifolds if the assumption on the injectivity radius is dropped. The constructed manifolds are obtained from hyperbolic 2-bridge knot complements by increasingly complicated Dehn fillings.

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Theorem 0.2. *For any n there exists a closed hyperbolic 3-manifold M such that $\pi_1(M)$ has at least n distinct Nielsen classes of generating pairs.*

The non-uniqueness of Nielsen classes of generating tuples of fundamental groups of hyperbolic 3-manifolds is not new. It is an immediate consequence of the work of Lustig and Moriah [LM] that there exist closed hyperbolic 3-manifolds whose fundamental groups are of rank r and have at least $2^r - 2$ Nielsen classes of generating r-tuples. Note that while the distinct Nielsen classes exhibited by Lustig and Moriah are geometric and therefore correspond to non-isotopic Heegaard splittings this is not true in the current setting. Indeed by Kobayshi's work [Ko] it is known that 2-bridge knot exteriors admit at most 6 isotopy classes of Heegaard splittings of genus 2, thus almost all of the generating pairs exhibited in this note are non-geometric.

The proofs rely on some simple facts about Nielsen equivalence of generating pairs due to Nielsen and in the case of Theorem 0.1 some basic hyperbolic geometry. For the proof of Theorem 0.2 we further exploit the geometric convergence of manifolds obtained by increasingly complicated Dehn surgery on a 2-bridge knot to the hyperbolic knot complement.

After discussing some basic material on Nielsen equivalences of generating pairs we first prove a simple theorem about generating pairs of HNN-extensions. The argument in this case is easier but similar to the argument needed in the proofs of the two theorems discussed above. We will then establish a simple fact about about piecewise geodesics in hyperbolic space before we proceed with the proof of the main theorem.

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1. Nielsen equivalence of pairs of elements

Let G be a group and $\mathcal{T} = (g_1, \dots, g_k)$ and $\mathcal{T}' = (h_1, \dots, h_k)$ be two k-tuples of elements of G. We say that $\mathcal{T}$ and $\mathcal{T}'$ are *elementarily equivalent* if one of the following holds:

(1) $h_i = g_{\sigma(i)}$ for $1 \leq i \leq k$ and some $\sigma \in S_k$.
(2) $h_i = g_i^{-1}$ for some $i \in \{1, \dots, k\}$ and $h_j = g_j$ for $j \neq i$.
(3) $h_i = g_i g_j^{\varepsilon}$ for some $i \neq j$ with $\varepsilon \in \{-1, 1\}$ and $h_l = g_l$ for $l \neq i$.

Two tuples are further called *Nielsen equivalent* if one can be transformed into the other by a finite sequence of elementary equivalences. Note that the elementary equivalences are also called *Nielsen transformations* or *Nielsen moves*.

The fact that Aut F_n is generated by so-called elementary Nielsen automorphisms implies that the above definition of Nielsen equivalence can be rephrased in the following way.

Let $F_k = F(x_1, \dots, x_k)$ be the free group of rank k. Then two k-tuples $\mathcal{T} = (g_1, \dots, g_k)$ and $\mathcal{T}' = (h_1, \dots, h_k)$ are Nielsen equivalent iff there exists a homomorphism $\phi : F_k \to G$ and an automorphism α of F_k such that the following hold:

(1) $g_i = \phi(x_i)$ for $1 \leq i \leq k$.
(2) $h_i = \phi \circ \alpha(x_i)$ for $1 \leq i \leq k$.

Deciding Nielsen equivalence of two tuples or classifying all Nielsen equivalence classes is usually a very difficult problem and undecidable in general. However in the case of pairs of elements the situation tends to be much easier. The reason is that the automorphism group of F_2 and the structure of primitive elements in F_2 are particularly easy to understand.

Nielsen [N] observed that any automorphism of $F(a, b)$ preserves the commutator $[a, b] = aba^{-1}b^{-1}$ up to conjugation and inversion. This is easily verified by checking that it holds for the elementary Nielsen automorphisms. As a consequence we get the following simple and much used test for Nielsen equivalence of pairs of elements.

Proposition 1.1. *Let G be a group and (x, y) and (x', y') be two pairs of elements. If $(x, y) \sim (x', y')$ then $[x, y]$ is conjugate to $[x', y']$ or $[x', y']^{-1}$.*

While convenient the above criterion is in general not sufficient to distinguish all Nielsen classes. Another useful fact in distinguishing Nielsen classes of pairs is that primitive elements of $F(a, b)$ are well understood, in fact in [OZ] Osborne and Zieschang gave a complete description of primitive elements of the free groups of rank 2; recall that an element of a free group is called primitive if it is part of some basis. The proof in [OZ] relies on the fact already observed by Nielsen [N] that for any primitive element p in the abelianization of $F(a, b)$ there is a unique conjugacy class of primitive elements in $F(a, b)$ that is mapped to p.

An immediate consequence of their description is the proposition below, see also [CMZ]. We give a proof of the weaker statement that we need as we can without breaking a sweat, note that ε and η below are simply the signs of the exponents of a and b in the abelianization of g.

Proposition 1.2. *Let g be a primitive element of $F(a, b)$. Then there exist $\varepsilon, \eta \in \{-1, 1\}$ such that g is conjugate to an element represented by a positive word in a^ε and b^η.*

Proof. As the proof in [OZ] we rely on the fact that for any primitive element $z_1a + z_2b$ in the abelianization we have a unique conjugacy class of primitive elements in $F(a,b)$ that maps to $z_1a + z_2b$.

We assume that g maps to $na+mb$ in the homology where $n, m \geq 0$, the other cases are analogous. We need to show that there exists a primitive element that can be written as a positive word in a and b that maps to $na + mb$.

Choose $r, s \geq 0$ such that $na + mb$ and $ra + sb$ form a basis of the homology. It is easily checked that we can transform this basis into the basis a, b by only applying elementary Nielsen transformations of type (1) and of type (3) with $\varepsilon = -1$ such that all intermediate elements only have positive coefficients. We recover the original basis by running the inverse transformation in inverse order, here all transformation are of type (1) or of type (3) with $\varepsilon = 1$.

Now we can run the same sequence of inverse Nielsen transformations in $F(a,b)$ starting with a, b. We obtain a basis of $F(a,b)$ whose first element maps to $na+nb$ in the homology. As no inverses are introduced in this sequence of Nielsen transformations it follows that this first basis element is a positive word in a and b and must be conjugate to g. This proves the claim. □

As automorphisms of free groups map bases to bases and therefore primitive elements to primitive elements this immediately implies the following.

Corollary 1.3. *Let G be a group and (x,y) and (x',y') be two pairs of elements. If $(x,y) \sim (x',y')$ then x' is conjugate to an element represented by a positive word in x^ε and y^η where $\varepsilon, \eta \in \{-1, 1\}$.*

2. Generating pairs of HNN-extensions

In the following we assume that $\mathbb{A}$ is a graph of groups with underlying graph A. The vertex group of a vertex v is denoted by A_v. It is well know that any tuple that generates a non-free subgroup is Nielsen equivalent to a tuple containing an elliptic element, i.e. an element conjugate to an element of one of the vertex groups, see [St], [Z], [PR] and [W1] for various levels of generality. The tuple containing an elliptic element can be obtained from the original one by a sequence of length reducing Nielsen moves. If the underlying graph is not a tree, i.e. if $\mathbb{A}$ has an HNN-component then it is not possible that all generators are elliptic as they would all lie in the kernel of the projection to the fundamental group of the underlying graph.

This justifies in the theorem below to only consider pairs of elements (x,y) such that x is elliptic and y is hyperbolic. Recall that an element of the fundamental group of a graph of groups $\mathbb{A}$ with respect to a base point v_0 is a homotopy class $[p]$ of $\mathbb{A}$-paths, were an $\mathbb{A}$-path p is a

sequence $a_0, e_1, a_1, \dots, e_k, a_k$ where $e_1, \dots, e_k$ is a closed edge path in A based at v_0, $a_0 \in A_{v_0}$ and $a_i \in A_{\omega(e_i)}$ for $1 \leq i \leq k$, see [S] or [KMW].

Theorem 2.1. *Let $\mathbb{A}$ be a graph of groups whose underlying graph A is not a tree. Let $G = \pi_1(\mathbb{A})$. Suppose that (x, y) and (x', y') are generating pairs of G such that x and x' are elliptic.*

Then $(x, y) \sim (x', y')$ iff there exist $g \in G$, $k \in \mathbb{Z}$ and $\varepsilon, \eta \in \{-1, 1\}$ such that

$$x' = gx^{\varepsilon}g^{-1} \text{ and } y' = gy^{\eta}x^{k}g^{-1}.$$

Proof. Choose a basepoint v_0 of A such that (after conjugation) $x = [a]$, i.e. that x is represented by the degenerate $\mathbb{A}$-path a as an element of $\pi_1(\mathbb{A}, v_0)$. Choose a reduced $\mathbb{A}$-path $\gamma = a_0, e_1, \dots, e_k, a_k$ that represents y, i.e. that $y = [\gamma] \in \pi_1(\mathbb{A}, v_0)$.

To show that $x' = gx^{\varepsilon}g^{-1}$ holds note first that an elliptic element cannot be represented by a (positive) word w in x^{ε} and y^{η} for fixed $\varepsilon, \eta \in \{-1, 1\}$ and with at least one occurence of y^{η}. To see this note first that any such word is conjugate to a word of type

$$w = y^{\eta}x^{n_1\varepsilon} \cdot \ldots \cdot y^{\eta}x^{n_k\varepsilon} = [\gamma^{\eta}][a^{n_1\varepsilon}] \cdot \ldots \cdot [\gamma^{\eta}][a^{n_k\varepsilon}]$$

with $n_i \geq 0$ for $1 \leq i \leq k$. If the element represented by w is elliptic then the length of the reduced form of any positive power of w can be at most as large as the length of a reduced form of w. This implies that a power w^m of w has a subword $y^{\eta}x^{n_i\varepsilon}y^{\eta} = [\gamma^{\eta}][a^{n_i\varepsilon}][\gamma^{\eta}]$ such that at least half of both occurences of γ^{η} is cancelled when reducing w^m.

This however implies that half of $y^{\eta}x^{n_i\varepsilon}$ cancels in $(y^{\eta}x^{n_i\varepsilon})(y^{\eta}x^{n_i\varepsilon})$ which implies that $y^{\eta}x^{n_i\varepsilon}$ is elliptic. This however is impossible as (x, y) is Nielsen equivalent to $(x, y^{\eta}x^{n_i\varepsilon})$ but $\pi_1(\mathbb{A})$ is not generated by two elliptic elements as explained above.

Thus Corollary 1.3 implies that if (x, y) is Nielsen equivalent to (x', y') with x' elliptic then x' is conjugate to a power of x, thus x' is conjugate to x or x^{-1} as proper powers are not primitive. The second part is now immediate as any element g of $F(a, b)$ such that $\{a, g\}$ forms a basis must be of type $g = a^n b^{\eta} a^m$ with $n, m \in \mathbb{Z}$, $\eta = \pm 1$ and $a^n b^{\eta} a^m$ is conjugate to $b^{\eta} a^{m+n}$. □

3. Piecewise geodesics in hyperbolic space

In this section we introduce piecewise geodesics and establish some basic properties needed later on. Throughout this section all paths are paths in $\mathbb{H}^3$.

A (N, α)-piecewise geodesic is a path that is composed of geodesic segments $[x_i, x_{i+1}]$ of length at least N such that the angle $\theta_{i+1} \in [0, \pi]$ between $[x_i, x_{i+1}]$ and $[x_{i+1}, x_{i+2}]$ at x_{i+1} is at least α.

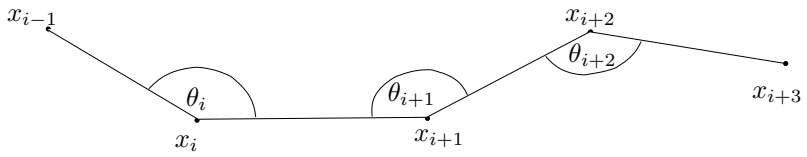

FIGURE 1. A section of a piecewise geodesic

We will need the following basic fact about piecewise geodesics; as its proof is entirely standard we merely sketch it. For definitions of quasigeodesics and local quasigeodesics and their basic properties used in the proof below the reader is referred to [CDP].

Lemma 3.1. *For any $\varepsilon > 0$ there exist $B > 0$ and $\alpha \in [0, \pi)$ such that any bi-infinite (B, α)-piecewise geodesic γ is ε-Hausdorff-close to some geodesic β. Moreover γ is a quasigeodesic with the same ends as β.*

Proof. For any geodesic β and $x \in \mathbb{H}^3$ we denote the nearest point projection of x to β by $p_\beta(x)$.

Note first that for any $\eta > 0$ there exists some angle $\theta \in [0, \pi)$ such that for any geodesic triangle ABC in $\mathbb{H}^3$ whose angle at A is greater or equal to θ the sides AB and AC lie in the η-neighborhood of BC. This is most easily seen in the Poincare disk model by choosing A to be the center. As the angle at this vertex tends to π the opposite side of the triangle comes arbitrarily close the center, independently of the length of the sides of the triangle.

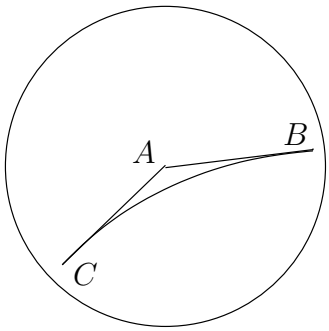

FIGURE 2. As the angle at A increases $d(A, BC)$ decreases

If θ_0 is chosen such that the above holds for $\eta = \min(\frac{\varepsilon}{2}, \frac{1}{2})$ then it is immediate that any (B, α)-piecewise geodesic with $\alpha \geq \theta_0$ is a $(B, 1, 1)$-local quasigeodesic. If moreover B_0 is chosen sufficiently large then the local-to-global phenomenon for quasigeodesics implies that any (B, α)-piecewise geodesic with $B \geq B_0$ and $\alpha \geq \theta_0$ is a (λ, c)-quasigeodesic for some fixed $\lambda \geq 1$ and $c \geq 0$.

As quasigeodics stay within bounded distance of geodesics this implies that any (B,α)-piecewise geodesic γ with $B \geq B_0$ and $\alpha \geq \theta_0$ stays within bounded distance of the geodesic β that has the same ends in $\partial\mathbb{H}^3$ as γ. This bound C is uniform, i.e. it only depends on B_0 and θ_0.

Now observe that there exists $B_1 \geq B_0$ such that for any (B,α)-piecewise geodesic γ with $B \geq B_1$ and $\alpha \geq \theta_0$ the midpoints m_i of the segments $[x_i, x_{i+1}]$ lie in the $\frac{\varepsilon}{2}$-neighborhood of β. This is true as the quadrilaterals spanned by x_i, x_{i+1}, $p_\beta(x_{i+1})$ and $p_\beta(x_i)$ can be assured to be arbitrarily thin provided that the distance between x_i and x_{i+1} is sufficently large. This is true as $d(x_i, p_\beta(x_i))$ and $d(x_{i+1}, p_\beta(x_{i+1}))$ are bounded from above by C, see Figure 3.

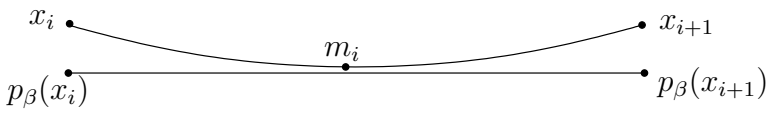

FIGURE 3. m_i gets arbitrarily close to β as $d(x_i, x_{i+1})$ increases

Note further that the convexity of the distance function to the geodesic β then immediately implies that the geodesics $[m_i, m_{i+1}]$ also lie in the $\frac{\varepsilon}{2}$-neigborhood of β for all i.

To conclude it clearly suffices to show that the piecewise geodesic $[m_i, x_{i+1}] \cup [x_{i+1}, m_{i+1}]$ lies in the $\frac{\varepsilon}{2}$-neighborhood of $[m_i, m_{i+1}]$. This however is true by our choice of θ_0.

Thus we have shown that there exist $B_1 > 0$ and $\theta_0 \in [0, \pi)$ such that any (B,α)-piecewise geodesic γ with $B \geq B_1$ and $\alpha \geq \theta_0$ is a quasigeodesic that remains within distance ε of the geodesic with the same ends. □

4. Hyperbolic knot complements as limits of closed hyperbolic 3-manifolds

It is a deep insight of Thurston that a cusped finite volume hyperbolic 3-manifold M occurs as the geometric limit of closed hyperbolic manifolds which are topologically obtained from M by Dehn fillings along increasingly complicated slopes. For definitions and details concerning algebraic and geometric convergence see [M, Capter 4] and [MT98, Chapter 7].

Let now M be the complement of a hyperbolic knot $\mathfrak{k}$ and let m and l denote the meridian/longitude pair for $\mathfrak{k}$. Let $\rho\colon \pi_1(M) \to PSL(2,\mathbb{C})$ denote the holonomy of the complete hyperbolic structure. The image of m and l are commuting parabolic elements and we can assume that their common fixed point is ∞ i.e. $Stab_\infty = \langle \rho(m), \rho(l) \rangle$ and that

$$\rho(m)(z) = z + 1 \text{ and } \rho(l)(z) = z + \tau_0\,.$$

The complex number $\tau_0 \in \mathbb{C} \setminus \mathbb{R}$ is called the *cusp parameter* of $\mathfrak{k}$.

The deformation space of hyperbolic structures on M can be holomorphically parametrised by a complex parameter u in a neighborhood U of $0 \in \mathbb{C}$. Details about the deformation space and Thurston's hyperbolic Dehn filling theorem can be found in the Appendix B of [BP01]. See also [Th, Chapter 5].

The following facts can be found in [BP01, B.1.2]: there is an analytic family ρ_u, $u \in U$, of representations $\rho_u \colon \pi_1(M) \to PSL(2,\mathbb{C})$ and an analytic function $v = v(u)$ such that u and v are the complex translation length of $\rho_u(m)$ and $\rho_u(l)$ respectively and $v(0) = 0$. The function $\tau(u) = v(u)/u$ is analytic and $\tau(u) = \tau(0) + O(|u|^2)$ where $\tau(0) = \tau_0$ is the cusp parameter. For $u \in U$ the generalized *Dehn filling coefficient* of the cusp is the defined to be the element of $\mathbf{R}^2 \cup \infty \cong S^2$ defined by

$$\begin{cases} \infty & \text{if } u = 0 \\ (p,q) \text{ s.t. } up + vq = 2\pi i & \text{if } u \neq 0. \end{cases}$$

The representation ρ_0 is the holonomy of the complete hyperbolic structure of M. If $u \neq 0$ then the representation ρ_u is the holonomy of a non complete hyperbolic structure M_u on M and the metric completiton of M_u is described by the Dehn filling parameters (see [BP01, B.1]). We are only interested in the case that p and q are coprime integers. Then ρ_u factors through $\pi_1(M(p/q))$ and the metric completion of M_u is homeomorphic to $M(p/q)$. Here and in the sequel $M(p/q)$ denotes the manifold obtained from M by Dehn filling along the slope $pm+ql$. By Mostow-Prasad rigidity the faithful discrete representations of $\pi_1(M)$ and $\pi_1(M(p/q))$ in $PSL(2,\mathbb{C})$ are unique up to conjugation.

We are mainly interested in the manifolds $M_n = M(1/n)$. By Thurston's Dehn filling theorem almost all of these manifolds carry a hyperbolic structure and the hyperbolic manifold M_n contains a new geodesic, the core of the filling solid torus, which is represented by l.

Now we let u_n denote the parameter satisfying $u_n + nv_n = 2\pi i$ where $v_n = v(u_n)$. Then we have

$$u_n = \frac{2\pi i}{1 + n\tau_n} \text{ and } v_n = \frac{2\pi i \tau_n}{1 + n\tau_n} \tag{1}$$

where $\tau_n = v_n/u_n$. Note that $\lim_{n\to\infty} \tau_n = \tau(0) = \tau_0$. Then $\rho_n := \rho_{u_n}$ determines the holonomy of the complete hyperbolic strucure on M_n. On the peripheral subgroup $\langle m, l\rangle$ of the knot $\mathfrak{k}$ the representation is given by:

$$\rho_n(m)(z) = e^{u_n} z + 1 \text{ and } \rho_n(l)(z) = e^{v_n} z + \frac{e^{v_n} - 1}{e^{u_n} - 1}$$

(see [BP01, B.1]). By conjugation of ρ_n by the parabolic transformation A_n given by

$$A_n(z) = z + \frac{\tau_n}{1-e^{v_n}} - \frac{1}{1-e^{u_n}}$$

we can assume that $\rho_n(m) = W_n$ and $\rho_n(l) = V_n$ where

$$W_n(z) = e^{u_n}z + \tau_n\frac{e^{u_n}-1}{e^{v_n}-1} \text{ and } V_n(z) = e^{v_n}z + \tau_n\,.$$

Note that A_n converges to A given by $A(z) = z+(\tau_0-1)/2$. Note also that the Dehn Surgery Theorem (see [Th, Chapter 5] and [PePo]) implies that the sequence of groups $\{\rho_n(\pi_1(M))\}$ converges geometrically to $\rho(\pi_1(M))$.

A simple calculation shows that

$$V_n^k(z) = e^{kv_n}z + \frac{e^{kv_n}-1}{e^{v_n}-1}\tau_n$$

and hence $V_n^{-n} = W_n$ since $u_n + nv_n = 2\pi i$ and therefore $e^{-nv_n} = e^{u_n}$. Moreover the fixed points of the loxodromic transformation V_n are ∞ and $\tau_n/(1-e^{v_n})$. The above facts allow us to conclude as in the discussion in [M, 4.9] that the following hold:

(1) The elements $\rho_n(l)$ are loxodromic isometries. The fixed points of $\rho_n(l)$ converge to the fixed point ∞ of the parabolic subgroup $P = \rho(\pi_1(\partial M))$.
(2) $\rho_n(l)$ converges uniformly on compact sets to $\rho(l)$ and $\rho_n(m) = \rho_n(l)^{-n}$ converges to $\rho(m)$.
(3) Furthermore the sequence of subgroups $\langle\rho_n(l)\rangle$ generated by the core of the filling solid torus converges geometrically to the peripheral subgroup P.

In the sequel we shall use the following convention: we shall identify $\pi_1(M)$ with the image $\rho(\pi_1(M)) \subset PSL(2,\mathbb{C})$ and for each $g \in \pi_1(M)$ we write $g_n = \rho_n(g)$ for short. We shall denote by $\gamma_n \subset M_n$ the new geodesic i.e. the core of the filling solid torus.

The following proposition gives some more information about the geometry of the limiting process. Note that the translation lengths of an element g on some g-invariant subset Y of $\mathbb{H}^3$ is defined to be

$$|g|_Y := \inf_{y\in Y} d_{\mathbb{H}^3}(y, gy).$$

In particular the translation length of a parabolic element g on a g-invariant horosphere S is measured with respect to the metric of $\mathbb{H}^3$ rather than the Euclidean path metric of H. We will need the following lemma, see [Mey87, Sec. 9, Lemma 2]).

Lemma 4.1. *Let $g \in Isom(\mathbb{H}^3)$ be a hyperbolic isometry with complex translation length $a+ib$ and let $N_r = N_r(A_g)$ be the r neighborhood*

of the axis of g. Then

$$\cosh(|g|_{\partial N_r}) = \cosh(a) + \sinh^2(r)(\cosh(a) - \cos(b)) =$$
$$\cosh(a) + \sinh^2(r)|\cosh(a+ib) - 1|.$$

Note that the two formulae in Lemma 4.1 are equivalent via the identity

$$|\cosh(a+ib) - 1| = \cosh(a) - \cos(b). \tag{2}$$

Proposition 4.2. *For any horoball H at ∞ there exists a sequence $(r_n)_{n\in\mathbb{N}}$ of real numbers such that the following hold where $N_n := N_{r_n}(A_{l_n})$ is the r_n-neighborhood of the axis A_{l_n} of l_n in $\mathbb{H}^3$.*

1. *For any fixed k, N we have $\lim_{n\to\infty}\left(|l_n^{N-k\cdot n}|_{\partial N_n}\right) = |l^N m^k|_{\partial H}$.*
2. *If $N \in \mathbb{N}$ such that $|l^N m^k|_{\partial H} > C > 0$ for all $k \in \mathbb{Z}$, then for any $\eta > 0$ there exists some n' such that $|l_n^{N-k\cdot n}|_{\partial N_n} \geq C - \eta$ for all $n \geq n'$ and all $k \in \mathbb{Z}$.*

Moreover if (g_n) is a sequence of elements with $g_n \in \rho_n(\pi_1(M_n))$ that converges to a hyperbolic element $g \in \rho(\pi_1(M))$ and $C > 0$ then H can be chosen such that the following hold:

(a) *For sufficiently large n we have $d(N_n, h_n N_n) \geq C$ for all $h_n \notin \langle l_n\rangle$.*
(b) *For sufficiently large n the C-neighborhood of the geodesic segment $[x_n, y_n]$ between N_n and $g_n N_n$ does not intersect any translates of N_n except N_n and $g_n N_n$.*

Proof. Let $H = \{(x,t) \in \mathbb{C}\times\mathbb{R} = \mathbb{H}^3 \mid t \geq t_0\}$ be the upper half space model of the hyperbolic space. For a complex number z we will denote by $\Re(z)$ and $\Im(z)$ the real and the imaginary part of z.

The translation length $|l|_{\partial H}$ is given by

$$\cosh(|l|_{\partial H}) = 1 + \frac{|\tau_0|^2}{2t_0^2}\,. \tag{3}$$

We now define $r_n := 0$ if $|\Re(v_n)| \geq |l|_{\partial H}$. If $|\Re(v_n)| < |l|_{\partial H}$ then we define r_n to be the unique positive real number satisfying

$$\sinh^2(r_n) = \frac{\cosh(|l|_{\partial H}) - \cosh(\Re(v_n))}{|\cosh(v_n) - 1|}\,. \tag{4}$$

This definition implies that $|l_n|_{\partial N_n} = |l|_{\partial H}$ if $|\Re(v_n)| < |l|_{\partial H}$. Indeed this follows from Lemma 4.1 and the fact that v_n is the complex translation length of l_n.

Fix now N and k. The complex tanslation length of l_n^{N-kn} is $(N - kn)v_n$ and the equation $u_n + nv_n = 2\pi i$ implies

$$(N-kn)v_n \equiv Nv_n + ku_n \mod 2\pi i\,.$$

Hence, by Lemma 4.1, the translation length $|l_n^{N-kn}|_{\partial N_n}$ is given by

$$\cosh(|l_n^{N-kn}|_{\partial N_n}) = \cosh(\Re(Nv_n + ku_n)) + \sinh^2(r_n) \cdot |\cosh(Nv_n + ku_n) - 1| \,. \tag{5}$$

Now it follows from the low order asymptotics of the hyperbolic cosine and its Taylor expansion that

$$\frac{|\cosh(Nv_n + ku_n) - 1|}{|\cosh(v_n) - 1|} = \frac{|N\tau_n + k|^2}{|\tau_n|^2}\big(1 + O(|v_n|^2)\big) \,.$$

Moreover equation (4) implies

$$\lim_{n\to\infty} |\cosh(v_n) - 1| \cdot \sinh^2(r_n) = \cosh(|l|_{\partial H}) - 1 \,.$$

The last two equations together with equation (5) imply:

$$\begin{aligned}
\lim_{n\to\infty} \cosh(|l_n^{N-kn}|_{\partial N_n}) &= \cosh(\Re(Nv_n + ku_n)) \\
&\quad + \sinh^2(r_n) \cdot |\cosh(v_n) - 1| \frac{|\cosh(Nv_n + ku_n) - 1|}{|\cosh(v_n) - 1|} \\
&\overset{n\to\infty}{\longrightarrow} 1 + (\cosh(|l|_{\partial H}) - 1)\frac{|N\tau_0 + k|^2}{|\tau_0|^2} \\
&= 1 + \frac{|N\tau_0 + k|^2}{2t_0^2} \\
&= \cosh(|l^N m^k|_{\partial H}) \,.
\end{aligned}$$

This proves the first point.

In order to prove the second point we will make use of the following limits

$$\lim_{n\to\infty} \frac{|\Re(v_n)|}{|v_n|} = 0 \text{ and } \lim_{n\to\infty} \frac{|\Re(u_n)|}{|v_n|} = \frac{|\Im(\tau_0)|}{|\tau_0|^2} > 0 \,. \tag{6}$$

and

$$\lim_{n\to\infty} |v_n| \cosh(r_n) = \frac{|\tau_0|}{t_0} \tag{7}$$

The first two follow easily from (1) to verify the second one observe first that (4), the identity $\sinh^2(z) = \cosh^2(z) - 1$ and multiplication by $|v_n|^2$ imply that

$$|v_n|^2 \cosh^2(r_n) = |v_n|^2 \frac{\cosh(|l|_{\partial H}) - \cosh(\Re(v_n)) + |\cosh(v_n) - 1|}{|\cosh(v_n) - 1|}$$

applying (2) and (3) then gives

$$\begin{aligned}
|v_n|^2 \cosh^2(r_n) &= |v_n|^2 \frac{\cosh(|l|_{\partial H}) - \cos(\Im(v_n))}{|\cosh(v_n) - 1|} \\
&= |v_n|^2 \frac{1 + \frac{|\tau_0|^2}{2t_0^2} - \cos(\Im(v_n))}{|\cosh(v_n) - 1|} .
\end{aligned}$$

Now as $\lim_{z\to 0}\frac{|z|^2}{|\cosh^2(z)-1|}=2$ this implies that $\lim_{n\to\infty}|v_n|^2\cosh^2(r_n)=\frac{|\tau_0|^2}{t_0^2}$ which clearly proves the claim.

Suppose now that $N\in\mathbb{N}$ satisfies $|l^N m^k|_{\partial H}>C>0$ for all $k\in\mathbb{Z}$ and let $\eta>0$ be given. We choose $n''=n''(N,C)$ such that for all $n\geq n''$ the following holds:

$$\left|N\frac{\Re(v_n)}{|v_n|}\right|<1,\quad |v_n|\cdot\cosh(r_n)\geq\frac{|\tau_0|}{2t_0}$$

and

$$\frac{|\Re(u_n)|}{|v_n|}>\frac{1}{2}\frac{|\Im(\tau_0)|}{|\tau_0|^2}\,.$$

In order to prove the second point we start again with formula (5):

$$\begin{aligned}\cosh(|l_n^{N-kn}|_{\partial N_n}) &= \cosh\big(\Re(Nv_n+ku_n)\big)\cosh^2(r_n)\\ &\quad-\cos\big(\Im(Nv_n+ku_n)\big)\sinh^2(r_n)\\ &> \big(\cosh(\Re(Nv_n+ku_n))-1\big)\cosh^2(r_n)\\ &\geq \frac{1}{2}\big(\Re(Nv_n+ku_n)\big)^2\cosh^2(r_n)\\ &= \frac{1}{2}\big(\Re(N\frac{v_n}{|v_n|}+k\frac{u_n}{|v_n|})\big)^2(|v_n|\cosh(r_n))^2\end{aligned}$$

and for $n\geq n''$ we obtain

$$\cosh(|l_n^{N-kn}|_{\partial N_n})>\frac{|\tau_0|^2}{8t_0^2}\cdot\big||k\frac{\Re(u_n)}{|v_n|}|-|N\frac{\Re(v_n)}{|v_n|}|\big|^2.$$

Hence there exists a constant C' such that $|k|\geq C'$ implies

$$|l_n^{N-kn}|_{\partial N_n}\geq C\,.$$

Since there are only finitely many $k\in\mathbb{Z}$ such that $|k|<C'$ it follows from the first part of the proposition that we can find n''' such that for all $n\geq n'''$ and all $|k|<C'$ the equation

$$\big|l_n^{N-kn}\big|_{\partial N_n}\geq C-\eta$$

holds. The second point follows for $n'=\max(n'',n''')$.

In order to prove (a) and (b) we shall use some results of Meyerhoff. By [Mey87, Sec. 3&9], we know that if we choose H such that $\varepsilon=|l|_{\partial H}$ is sufficiently small then the sets

$$N_n=N_{r_n}(A_{l_n})=\{p\in\mathbb{H}^3\mid d_{\mathbb{H}^3}(p,l_n(p))\leq\varepsilon\}$$

have the property that $h_n(N_n)\cap N_n=\emptyset$ for all $h_n\notin\langle l_n\rangle$.

By further reducing ε we can further assume that $d(N_n,h_nN_n)\geq C$ for all $h_n\notin\langle l_n\rangle$ as the radii r_n decrease uniformly as ε decreases as follows from (4). This proves (a).

We now put $\tilde{H}=GH$ and $\tilde{N}_n=G_nN_n$ where $G=\rho(\pi_1(M))$ and $G_n=\rho_n(\pi_1(M))$. It now follows from the geometric convergence of

(G_n) to G that for all $\varepsilon > 0$ and compact $K \subset \mathbb{H}^3$ there exists n' such that for all $n \geq n'$

$$d_H(K \cap \tilde{H}, K \cap \tilde{N}_n) \leq \varepsilon.$$

Here d_H denotes the Hausdorff distance. Moreover for any $h \in G$ we have $d_H(K \cap hH, K \cap h_n N_n) \leq \varepsilon$.

Let $[x, y]$ denote the geodesic segment between H and hH and let $[x_n, y_n]$ denote the geodesic segment between N_n and $h_n N_n$.

Note that there can be only finitely many translates hH, $h \in \pi_1(M)$, $h \notin P \cup gP$, such that the intersection of hH and the $(C+1)$-neighborhood of $[x, y]$ is non-empty. This follows form the fact that the $(C+1)$-neighborhood of $[x, y]$ is compact and that the translates of H are disjoint. Thus after decreasing ε we can assume that the $(C+1)$-neighborhood of $[x, y]$ does not intersect hH for $h \notin P \cup gP$.

Let now K be the $(C+1)$-neighborhood of $[x, y]$. The above remark applies to K. Hence we know that for sufficiently large n the Hausdorff distances between $H \cap K$ and $N_n \cap K$ and between $K \cap gH$ and $K \cap g_n N_n$ are arbitrarily small. This implies that the segments $[x_n, y_n]$ converge to $[x, y]$ thus $d_H([x, y], [x_n, y_n]) \leq \frac{1}{2}$ for large n. We also see that the $(C + \frac{1}{2})$-neighborhood of $[x, y]$ does not meet any translates of N_n besides N_n and $g_n N_n$ for large n. Thus we have shown that the C-neighborhood of $[x_n, y_n]$ does not meet any translates of N_n besides N_n and $g_n N_n$. This proves (b). □

Lemma 4.3. *For any $\beta \in (0, \pi/2)$ there exist $\kappa(\beta)$ and $r(\beta)$ such that if γ is a geodesic in $\mathbb{H}^3$, $r \geq r(\beta)$ and $x, y \in \partial(N_r(\gamma))$ such that $d(x, y) \geq \kappa(\beta)$ then the angles enclosed by $[x, y]$ and $\partial(N_r(\gamma))$ are at least β.*

Proof. The proof of the lemma is by calculation, we follow the setup of [GMM, Section 2]. We perform all calculations in the Klein hyperboloid model of $\mathbb{H}^3$. In this model, $\mathbb{H}^3$ is the hypersurface

$$\{(x_0, x_1, x_2, x_3) \in \mathbb{R}^4 \mid -x_0^2 + x_1^2 + x_2^2 + x_3^2 = -1\}$$

We assume that γ is the intersection of $\mathbb{H}^3$ with the plane $\{x_1 = x_3 = 0\}$. Let g be the loxodromic motion along γ with complex length $\delta + i\phi$. The isometry g is represented by the matrix

$$M_g := \begin{pmatrix} \cosh(\delta) & 0 & \sinh(\delta) & 0 \\ 0 & \cos(\phi) & 0 & -\sin(\phi) \\ \sinh(\delta) & 0 & \cosh(\delta) & 0 \\ 0 & \sin(\phi) & 0 & \cos(\phi) \end{pmatrix}.$$

We fix the point $x = (\cosh(r), \sinh(r), 0, 0)$. Let $\overrightarrow{n} \in T_x\mathbb{H}^3$ be the inward normal vector to $\partial(N_r(\gamma))$. Thus

$$\overrightarrow{n} = (-\sinh(r), -\cosh(r), 0, 0).$$

For every $y \in \mathbb{H}^3$ the unit vector $\overrightarrow{m}_y \in T_x\mathbb{H}^3$ pointing into the direction of y is given by

$$\overrightarrow{m}_y = \frac{y + \langle y, x\rangle x}{(\langle y, x\rangle^2 - 1)^{1/2}}\,.$$

Here $\langle x, y\rangle$ denotes the Minkowski inner product. Denote by $\beta(y)$, $0 \leq \beta(y) \leq \pi/2$, the angle enclosed by $[x, y]$ and $\partial(N_r(\gamma))$. We need to show that $\beta(y)$ is arbitrarily close to $\pi/2$ provided that r and $d(x, y)$ is sufficiently large.

Let now $y = M_g x \in \partial N_r(\gamma)$ be the image of x under g. Elementary calculations similar to those in [GMM, Section 2] show that

$$\begin{aligned}\sin(\beta(y)) = \cos(\frac{\pi}{2} - \beta(y)) &= \langle \overrightarrow{m}_y, \overrightarrow{n}\rangle \\ &= \frac{\cosh(r)\sinh(r)}{\sinh(d(x, y))}\big(\cosh(\delta) - \cos(\phi)\big)\,.\end{aligned}$$

Moreover we have

$$\cosh(d(x, y)) = \cosh(\delta)\cosh^2(r) - \cos(\phi)\sinh^2(r)$$

and hence

$$\sin(\beta(y)) = \tanh(r)\frac{\cosh(d(x, y)) - \cos(\phi)}{\sinh(d(x, y))}\,.$$

For a fixed distance $d = d(x, y)$ the angle $\beta(y)$ becomes minimal if $\phi = 0$. Therefore

$$\sin(\beta(y)) \geq \tanh(r)\frac{\cosh(d(x, y)) - 1}{\sinh(d(x, y))} = \tanh(r)\tanh\left(\frac{d(x, y)}{2}\right).$$

Now let β, $0 < \beta < \pi/2$, be given. We choose $r(\beta) > 0$ such that $\sin(\beta) < \sin(\beta)\coth(r(\beta)) = q < 1$ and $\kappa(\beta)$ such that $q = \tanh(\kappa(\beta)/2)$. Hence for $r \geq r(\beta)$ and for $y \in \partial N_r$ such that $d(x, y) \geq \kappa(\beta)$ we obtain

$$\begin{aligned}\sin(\beta(y)) &\geq \tanh(r)\tanh\big(d(x, y)/2\big) \\ &\geq \tanh(r(\beta))\tanh\big(\kappa(\beta)/2\big) = \sin(\beta)\,.\end{aligned}$$

Therefore we have for all $r \geq r(\beta)$ and all $y \in \partial N_r$ such that $d(x, y) \geq \kappa(\beta)$ that $\beta(y) \geq \beta$. This proves the Lemma. □

5. Generating pairs of 2-bridge knot groups

In this section we prove that hyperbolic 2-bridge knot groups have infinitely many Nielsen classes of generating pairs. Moreover we prove that there exist closed hyperbolic 3-manifolds that have arbitrarily many Nielsen classes of generating pairs. Those manifolds are obtained by Dehn surgery on S^3 at 2-bridge knots.

The infinity of Nielsen classes of generating pairs of fundamental groups of Seifert fibered 2-bridge knot spaces has been known for a long time. For the trefoil knot this is due to Dunwoody and Pietrowsky

[DP] and the general case is due to Zieschang [Z2] who in fact gives a complete classification of Nielsen classes of generating pairs.

Let M be the exterior of a hyperbolic 2-bridge knot $\mathfrak{k}$. Choose $m, l \in \pi_1(M)$ such that m represents the meridian, that l represents the longitude and that $\langle m, l\rangle \cong \mathbb{Z}^2$ is a maximal peripheral subgroup. Inspecting the Wirtinger presentation shows that 2-bridge knot groups are generated by two meridional elements, i.e. that there exists some g such that $\pi_1(M) = \langle m, gmg^{-1}\rangle$.

As $(gl^N)\cdot m\cdot (gl^N)^{-1} = g\cdot l^N m l^{-N}\cdot g^{-1} = gmg^{-1}$ it follows that

$$P^N := (m, gl^N)$$

is a generating pair for $\pi_1(M)$ for all $N \in \mathbb{Z}$. As in Section 4 we put $M_n := M(1/n)$ and denote the image of an element $h \in \pi_1(M)$ in $\pi_1(M_n)$ by h_n. In particular $P_n^N = (m_n, g_n l_n^N)$ is a generating pair of $\pi_1(M_n)$ for all $n \in \mathbb{N}$,

The following theorem is the main theorem of this article.

Theorem 5.1. *There exists $N_0 \in \mathbb{N}$ such that for any $N, N' \geq N_0$ there exists some n_0 such that for $n \geq n_0$ the generating pairs P_n^N and $P_n^{N'}$ of $\pi_1(M_n)$ are not Nielsen equivalent.*

Note that for all n, N and N' the generating pairs P_n^N and $P_n^{N'}$ have the same commutator as

$$[m_n, g_n l_n^N] = m_n \cdot g_n l_n^N \cdot m_n^{-1} \cdot l_n^{-N} g_n^{-1} = m_n g_n m_n^{-1} g_n^{-1} = [m_n, g_n].$$

Thus we cannot apply Proposition 1.1 to distinguish the Nielsen equivalence classes of P_n^N and $P_n^{N'}$. We get the two results stated in the introduction as immediate corollaries.

Corollary 5.2. *For any n there exists a closed hyperbolic 3-manifold M such that $\pi_1(M)$ has at least n distinct Nielsen classes of generating pairs.*

As further Nielsen-equivalent tuples cannot become non-equivalent when passing to a factor group we also get the following.

Corollary 5.3. *Let $\mathfrak{k}$ be a hyperbolic 2-bridge knot with knot exterior M. Then $\pi_1(M)$ has infinitely many Nielsen classes of generating pairs.*

We will prove two lemmas before giving the proof of Theorem 5.1. We use the same notations as in Section 4.

Lemma 5.4. *There exists N_1 such that for any $N \geq N_1$ there exists some $n_1 \in \mathbb{N}$ such that for $n \geq n_1$ and $\varepsilon, \eta \in \{-1, 1\}$ a positive word w in m_n^η and $(g_n l_n^N)^\varepsilon$ represents an element that is in $\pi_1(M_n)$ conjugate to $m_n^{\pm 1}$ iff $w = m_n^\eta$.*

Proof. Choose $B \geq 100$ and α such that the conclusion of Lemma 3.1 is fulfilled for $\varepsilon = \frac{1}{2}$. Furthermore choose $\kappa := \kappa(\alpha - \pi/2)$ and $r := r(\alpha - \pi/2)$ as in Lemma 4.3.

Choose a horoball H at the fixed-point of $\langle m, l\rangle$ such that the conclusions (a) and (b) of Proposition 4.2 hold for $C = B$ and the sequence (g_n). Thus there exist $\hat{n}$ such that $d(N_n, g_n N_n) \geq B$ for $n \geq \hat{n}$. Denote the geodesic segment between H and gH by $[x, y]$ and the segment between N_n and $g_n N_n$ by $[x_n, y_n]$.

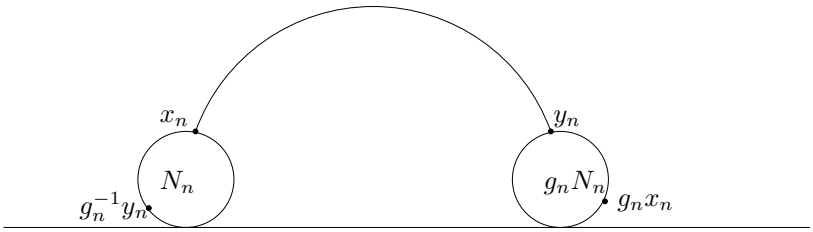

FIGURE 4. The horoball approximation N_n and its translate by g_n

Choose t such that $d(g_n x_n, y_n) \leq t$ for all n. Such t clearly exists as the segments $[x_n, y_n]$ converge to the segment $[x, y]$ and g_n to g. Choose further N_1 such that $|l^N m^k|_{\partial H} \geq B + t + \kappa + 1$ for all $N \geq N_1$ and $k \in \mathbb{Z}$. Such N_1 clearly exists as for any constant K there only finitely many elements $h \in \langle m, l\rangle$ such that $|h|_{\partial H} \leq K$.

Now fix $N \geq N_1$. It now follows from Proposition 4.2 (2) (by chosing η sufficiently small) that there exists some $\tilde{n} \geq \hat{n}$ such that $|l_n^N m_n^k|_{\partial N_n} \geq B + t + \kappa$ for all $k \in \mathbb{Z}$ and $n \geq \tilde{n}$.

Now choose $n_1 \geq \tilde{n}$ such that $|m_n|_{\mathbb{H}^3} < 1$ and $r_n > r(\alpha - \pi/2)$ for all $n \geq n_1$; this is clearly possible as the element m_n converges to the parabolic element m and r_n tends to infinity as n tends to infinity. We need to show for $n \geq n_1$ and $\varepsilon, \eta \in \{-1, 1\}$ a positive word w in m_n^η and $(g_n l_n^N)^\varepsilon$ represents an element that is in $\pi_1(M_n)$ conjugate to $m_n^{\pm 1}$ if and only if $w = m_n^\eta$.

We check the case that $\eta = \varepsilon = 1$, the other cases are analogous. Thus either w is a power of m_n in which case there is nothing to show as a hyperbolic element is never conjugate to a proper power of itself or w is (after a cyclic permutation) of type

$$(g_n l_n^N) m_n^{b_1} \cdot \ldots \cdot (g_n l_n^N) m_n^{b_r}$$

with $b_i \geq 0$ for $1 \leq i \leq r$. Thus w can be rewritten as a product

$$(g_n p_1) \cdot \ldots \cdot (g_n p_r)$$

where $p_i = l_n^N m_n^{b_i}$ for $1 \leq i \leq r$. By the above choices the translation length of all p_i on ∂N_n is at least $B + t + \kappa$.

We now construct a w-invariant bi-infinite (B,α)-piecewise geodesic γ_w containing x_n. We first construct a (B,α)-piecewise geodesic γ_0 from x_n to wx_n and put $\gamma_i = w^i\gamma_0$. We then put

$$\gamma_w := \ldots\gamma_{-2}\cdot\gamma_{-1}\cdot\gamma_0\cdot\gamma_1\cdot\gamma_2\ldots$$

which clearly implies the w-invariance of γ_w. The fact that γ_w is also a (B,α)-piecewise geodesic follows immediately from the construction.

For $1 \le i \le r$ put $w_i = (g_np_1)\cdot\ldots\cdot(g_np_i)$, $x_n^i = w_ix_n$ and $y_n^i = w_iy_n$. We then put

$$\gamma_0 := [x_n, y_n]\cdot[y_n, x_n^1]\cdot[x_n^1, y_n^1]\cdot[y_n^1, x_n^2]\cdot\ldots\cdot[x_n^{r-1}, y_n^{r-1}]\cdot[y_n^{r-1}, x_n^r = wx_n].$$

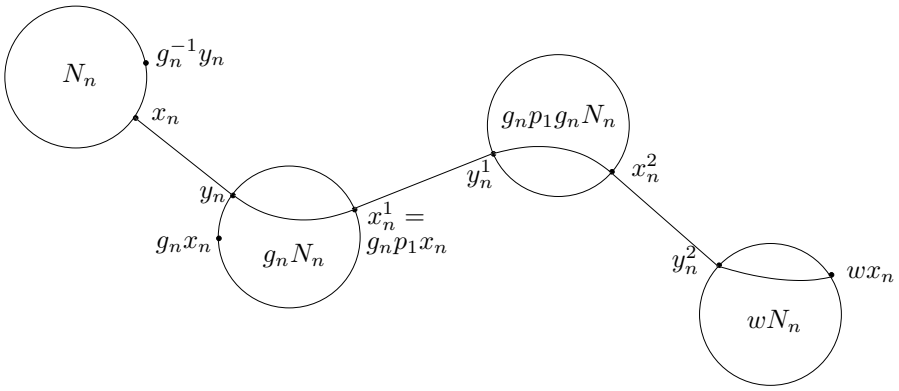

FIGURE 5. The piecewise geodesic γ_0 for $w = (g_np_1)(g_np_2)(g_np_3)$

Now $x_n^i \in \partial(w_iN_n) = w_i\partial N_n$ and $y_n^i \in \partial(w_{i+1}N_n)$. The segments $[x_n^i, y_n^i]$ are of length at least B by assumption and are perpendicular to the respective translates of ∂N_n. Note further that the segments $[y_n^i, x_n^{i+1}]$ are of length at least $B+\kappa$. Indeed this follows from the triangle inequality and the fact that

$$d(w_ig_nx_n, y_n^i) = d(w_ig_nx_n, w_iy_n) = d(g_nx_n, y_n) \le t$$

and

$$d(x_n^{i+1}, w_ig_nx_n) = d(w_{i+1}x_n, w_ig_nx_n) = d(w_ig_np_{i+1}x_n, w_ig_nx_n) =$$
$$= d(p_{i+1}x_n, x_n) \ge B + t + \kappa.$$

It follows from the choice of κ and Lemma 4.3 that the segments $[y_n^i, x_n^{i+1}]$ enclose angles greater or equal than $\alpha - \pi/2$ with the respective translates of ∂N_n. This proves that γ_i and therefore γ_w is a (B,α)-piecewise geodesic.

By Lemma 3.1 γ_w is a quasigeodesic that lies in the $\frac{1}{2}$-neighborhood of the geodesic β that has the same ends. Clearly β is invariant under the action of w, thus we must have $\beta = A_w$ where A_w is the axis of w.

We argue that the translation length of w must be at least 198. Recall that γ_0 is a (B,α)-piecewise geodesic consisting of at least 2 segments of length at least $B \ge 100$. As γ_0 lies in the $\frac{1}{2}$-neighborhood

of β it follows that each of the geodesic segment projects under p_β to a geodesic segment of length at least 99. Thus

$$d(p_\beta(x_n), p_\beta(wx_n)) = d(p_\beta(x_n), wp_\beta(x_n)) \geq 2 \cdot 99 = 198.$$

Now $d(p_\beta(x_n), wp_\beta(x_n))$ is the translation length of w thus the claim follows.

It follows that w cannot be conjugate to $m_n^{\pm 1}$ as m_n is assumed to have translation length at most 1, the lemma is proven. □

Lemma 5.5. *There exists N_2 such that for all $N \neq N' \geq N_2$ there exists $n_2 \in \mathbb{N}$ such that for $n \geq n_2$ and $k \in \mathbb{Z}$ the elements $g_n l_n^N$ and $(g_n l_n^{N'+nk})^\varepsilon$ are not conjugate in $\pi_1(M_n)$ for $\varepsilon \in \{\pm 1\}$.*

Proof. The proof is similar to the proof of Lemma 5.4. Choose the constants B, α, κ, t, N_1, the segments $[x_n, y_n]$ and the horoball H as in the proof of Lemma 5.4 and put $N_2 := N_1$. Choose $N \neq N' \geq N_2$. Now by choosing $n_2 > N + N'$ sufficiently large it follows as before that

$$|l_n^{N+kn}|_{\partial N_n}, |l_n^{N'+kn}|_{\partial N_n} \geq B + t + \kappa$$

for all $n \geq n_2$ and $k \in \mathbb{Z}$. Put $w_1 = g_n l_n^N$ and $w_2 = g_n l_n^{N'+nk}$, note that $w_1 \neq w_2$ as l_n is of infinite order, $N \neq N'$ and $n > N + N'$. Thus we can construct the w_1-, w_2-, and $w_1 w_2$-invariant (B, α)-piecewise geodesics γ_{w_1}, γ_{w_2} and $\gamma_{w_1 w_2}$ with the same properties as before. Note that by Theorem 4.2 (a) and (b) and as $C = B \geq 100$ we can further assume that the 1-neighborhoods of $[x_n, y_n]$, N_n and $g_n N_n$ do not intersect any translate of N_n except N_n and $g_n N_n$. Note that the existence of the piecewise geodesic $\gamma_{w_1 w_2}$ on which $w_1 w_2$ acts non-trivially implies that $w_1 w_2 \neq 1$, i.e. that $w_1 \neq w_2^{-1}$.

Now the axes A_{w_1} and A_{w_2} are $\frac{1}{2}$-Hausdorff-close to γ_{w_1} and γ_{w_2}. As the 1-neighborhoods of γ_{w_i} does not meet any translates of N_n except the $w_i^k N_n$ this implies that the translates of N_n intersected by the $\frac{1}{2}$-neighborhood $\bar{A}_{w_i}$ of A_{w_i} are precisely the translates $w_i^k N_n$ for $i = 1, 2$. Note that N_n and $g_n N_n$ are intersected by both $\bar{A}_{w_1}$ and $\bar{A}_{w_2}$. Now if w_1 and w_2^ε are conjugate then there must exist some $h \in \pi_1(M_n)$ such that $w_2^\varepsilon = h w_1 h^{-1}$ which implies that $h A_{w_1} = A_{w_2^\varepsilon} = A_{w_2}$; h must in particular map the translates intersected by $\bar{A}_{w_1}$ to those intersected by $\bar{A}_{w_2}$.

After replacing h by $h w_1^l$ for some $l \in \mathbb{Z}$ we can assume that h fixes both N_n and gN_n, indeed h cannot exchange N_n and gN_n as it would otherwise fix the midpoint of $[x_n, y_n]$ and therefore be elliptic. Note that this replacement does not alter the fact that $w_2^\varepsilon = h w_1 h^{-1}$. As the intersection of the stabilizers of N_n and $g_n N_n$ is trivial this implies that $h = 1$ i.e. that $w_1 = w_2^\varepsilon$. This is clearly a contradiction, thus w_1 and w_2^ε are not conjugate. □

Proof of Theorem 5.1. Let N_1, n_1, N_2, n_2 be as Lemma 5.4 and Lemma 5.5 and put $N_0 = \max(N_1, N_2)$ and $n_0 = \max(n_1, n_2)$. Let $N \neq$

$N' \geq N_0$. We show that P_n^N and $P_n^{N'}$ are not Nielsen equivalent in $\pi_1(M_n)$ for $n \geq n_0$. This clearly proves the theorem.

Suppose that P_n^N and $P_n^{N'}$ are Nielsen equivalent. Thus there exists a basis $\{b_1(a,b), b_2(a,b)\}$ of $F(a,b)$ such that $m_n = b_1(m_n, g_n l_n^{N'})$ and $g_n l_n^N = b_2(m_n, g_n l_n^{N'})$. It now follows from Proposition 1.2 and Lemma 5.4 that b_1 is in $F(a,b)$ conjugate to a, i.e. that $b_1(a,b) = u(a,b)au(a,b)^{-1}$.

It follows in particular that $b_2(a,b) = u(a,b)a^{k_1}b^{\varepsilon}a^{k_2}u(a,b)^{-1}$ for some $k_1, k_2 \in \mathbb{Z}$ and $\varepsilon \in \{\pm 1\}$. Thus $g_n l_n^N = h m_n^{k_1} (g_n l_n^{N'})^{\varepsilon} m_n^{k_2} h^{-1}$ which implies that $g_n l_n^N$ is conjugate to $(g_n l_n^{N'})^{\varepsilon} m_n^{k_1+k_2}$ which contradicts Lemma 5.5. Thus P_n^N and $P_n^{N'}$ are not Nielsen equivalent in $\pi_1(M_n)$. □

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