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# GENERATING PAIRS OF 2-BRIDGE KNOT GROUPS

MICHAEL HEUSENER AND RICHARD WEIDMANN

ABSTRACT. We study Nielsen equivalence classes of generating pairs of Kleinian groups and HNN-extensions. We establish the following facts:

- (1) Hyperbolic 2-bridge knot groups have infinitely many Nielsen classes of generating pairs.
- (2) For any  $n \in \mathbb{N}$  there is a closed hyperbolic 3-manifold whose fundamental group has  $n$  distinct Nielsen classes of generating pairs.
- (3) Two pairs of elements of a fundamental group of an HNN-extension are Nielsen equivalent iff they are so for the obvious reasons.

## INTRODUCTION

The main purpose of this note is to study Nielsen equivalence classes of generating pairs of fundamental groups of hyperbolic 2-bridge knot spaces and of closed hyperbolic 3-manifolds obtained from those spaces by Dehn fillings.

It is a result of T. Delzant (following Gromov) [D] that any torsion-free hyperbolic group has only finitely many Nielsen classes of generating pairs. In the case of closed hyperbolic 3-manifolds Delzant's proof actually provides an explicit upper bound for this number in terms of the injectivity radius as observed by I. Agol [A], see also [Sou]. The finiteness for generating tuples of arbitrary size was established in [KW].

The examples constructed in this article show that this finiteness fails for cusped hyperbolic 3-manifolds. We establish the following:

**Theorem 0.1.** *Let  $\mathfrak{k}$  be a hyperbolic 2-bridge knot with knot exterior  $M$ . Then  $\pi_1(M)$  has infinitely many Nielsen classes of generating pairs.*

We further show that there is no uniform bound on the number of Nielsen classes of generating pairs of fundamental groups of closed hyperbolic 3-manifolds if the assumption on the injectivity radius is dropped. The constructed manifolds are obtained from hyperbolic 2-bridge knot complements by increasingly complicated Dehn fillings.

**Theorem 0.2.** *For any  $n$  there exists a closed hyperbolic 3-manifold  $M$  such that  $\pi_1(M)$  has at least  $n$  distinct Nielsen classes of generating pairs.*

The proofs rely on some simple facts about Nielsen equivalence of generating pairs due to Nielsen and in the case of Theorem 0.1 some basic hyperbolic geometry. For the proof of Theorem 0.2 we further exploit the geometric convergence of manifolds obtained by increasingly complicated Dehn surgery on a 2-bridge knot to the hyperbolic knot complement.

After discussing some essential material on Nielsen equivalences of generating pairs we first prove a simple theorem about generating pairs of HNN-extensions. The argument in this case is easier but similar to the argument needed in the proofs of the two theorems discussed above.

### 1. NIELSEN EQUIVALENCE OF PAIRS OF ELEMENTS

Let  $G$  be a group and  $\mathcal{T} = (g_1, \dots, g_k)$  and  $\mathcal{T}' = (h_1, \dots, h_k)$  be two  $k$ -tuples of elements of  $G$ . We say that  $\mathcal{T}$  and  $\mathcal{T}'$  are *elementarily equivalent* if one of the following holds:

- (1)  $h_i = g_{\sigma(i)}$  for  $1 \leq i \leq k$  and some  $\sigma \in S_k$ .
- (2)  $h_i = g_i^{-1}$  for some  $i \in \{1, \dots, k\}$  and  $h_j = g_j$  for  $j \neq i$ .
- (3)  $h_i = g_i g_j^\varepsilon$  for some  $i \neq j$  with  $\varepsilon \in \{-1, 1\}$  and  $h_l = g_l$  for  $l \neq i$ .

Two tuples are further called *Nielsen equivalent* if one can be transformed into the other by a finite sequence of elementary equivalences. Note that the elementary equivalences are also called *Nielsen transformations* or *Nielsen moves*.

The fact that  $\text{Aut } F_n$  is generated by so-called elementary Nielsen automorphisms implies that the above definition of Nielsen equivalence can be rephrased in the following way.

Let  $F_k = F(x_1, \dots, x_k)$  be the free group of rank  $k$ . Then two  $k$ -tuples  $(g_1, \dots, g_k)$  and  $(h_1, \dots, h_k)$  are Nielsen equivalent iff there exists a homomorphism  $\phi : F_k \rightarrow G$  and an automorphism  $\alpha$  of  $F_k$  such that the following hold:

- (1)  $g_i = \phi(x_i)$  for  $1 \leq i \leq k$ .
- (2)  $h_i = \phi \circ \alpha(x_i)$  for  $1 \leq i \leq k$ .

Deciding Nielsen equivalence of two tuples or classifying all Nielsen equivalence classes is usually a very difficult problem and undecidable in general. However in the case of pairs of elements the situation tends to be much easier. The reason is that the automorphism group of  $F_2$  and the structure of primitive elements in  $F_2$  are particularly easy to understand.

Nielsen [N] observed that any automorphism of  $F(a, b)$  preserves the commutator  $[a, b] = aba^{-1}b^{-1}$  up to conjugation and inversion. This is easily verified by checking that it holds for the elementary Nielsen automorphisms. As a consequence we get the following simple and much used test for Nielsen equivalence of pairs of elements.

**Proposition 1.1.** *Let  $G$  be a group and  $(x, y)$  and  $(x', y')$  be two pairs of elements. If  $(x, y) \sim (x', y')$  then  $[x, y]$  is conjugate to  $[x', y']$  or  $[x', y']^{-1}$ .*

While convenient the above criterion is in general not sufficient to distinguish all Nielsen classes. Another useful fact in distinguishing Nielsen classes of pairs is that primitive elements of  $F(a, b)$  are well understood, in fact in [OZ] Osborne and Zieschang gave a complete description of primitive elements of the free groups of rank 2; recall that an element of a free group is called primitive if it is part of some basis. The proof in [OZ] relies on the fact already observed by Nielsen [N] that to any primitive element  $p$  in the abelianization of  $F(a, b)$  there is a unique conjugacy class of primitive elements in  $F(a, b)$  that is mapped to  $p$ .

An immediate consequence of their description is the proposition below, see also [CMZ]. We give a proof of the weaker statement that we need as we can without breaking a sweat, note that  $\varepsilon$  and  $\eta$  below are simply the signs of the exponents of  $a$  and  $b$  in the abelianization of  $g$ .

**Proposition 1.2.** *Let  $g$  be a primitive element of  $F(a, b)$ . Then there exist  $\varepsilon, \eta \in \{-1, 1\}$  such that  $g$  is conjugate to an element represented by a positive word in  $a^\varepsilon$  and  $b^\eta$ .*

*Proof.* As the proof in [OZ] we rely on the fact that for any primitive element  $z_1a + z_2b$  in the abelianization we have a unique conjugacy class of primitive elements in  $F(a, b)$  that maps to  $z_1a + z_2b$ .

We assume that  $g$  maps to  $na + mb$  in the homology where  $n, m \geq 0$ , the other cases are analogous. We need to show that there exists a primitive element that can be written as a positive word in  $a^\varepsilon$  and  $b^\eta$  that maps to  $na + mb$ .

Choose  $r, s \geq 0$  such that  $na + mb$  and  $ra + sb$  form a basis of the homology. It is easily checked that we can transform this basis into the basis  $a, b$  by only applying elementary Nielsen transformations of type (1) and of type (3) with  $\varepsilon = -1$  such that all intermediate elements only have positive coefficients. We recover the original basis by running the inverse transformation in inverse order, here all transformation are of type (1) or of type (3) with  $\varepsilon = 1$ .

Now we can run the same sequence of Nielsen transformations in  $F(a, b)$  starting with  $a, b$ . We obtain a basis of  $F(a, b)$  whose first element maps to  $na + mb$  in the homology. As no inverses are introduced in this sequence of Nielsen transformations it follows that this first basis element is a positive word and must be conjugate to  $g$ . This proves the claim.  $\square$

## 2. GENERATING PAIRS OF HNN-EXTENSIONS

In the following we assume that  $\mathbb{A}$  is a graph of groups with non-trivial edge groups. It is well known that any tuple that generates a non-free subgroup is Nielsen equivalent to a tuple containing an elliptic element, i.e. an element conjugate to an element of one of the vertex groups, see [St], [Z], [PR] and [W1] for various levels of generality. The tuple containing an elliptic element can be obtained from the original one by a sequence of Nielsen reductions or, equivalently, by a folding sequence. If the underlying graph is not a tree, i.e. if  $\mathbb{A}$  has an HNN-component then it is not possible that both generators are elliptic as they would both lie in the kernel of the projection to the fundamental group of the underlying graph.

This justifies in the theorem below to only consider pairs of elements  $(x, y)$  such that  $x$  is elliptic and  $y$  is hyperbolic. The following is a simple consequence of Proposition 1.2.

**Theorem 2.1.** *Let  $\mathbb{A}$  be a graph of groups whose underlying graph  $A$  is not a tree. Let  $G = \pi_1(\mathbb{A})$ . Let  $(x, y)$  and  $(x', y')$  be generating tuples of  $G$  such that  $x$  and  $x'$  are elliptic.*

*Then  $(x, y) \sim (x', y')$  iff there exist  $g \in G$ ,  $k \in \mathbb{Z}$  and  $\varepsilon, \eta \in \{-1, 1\}$  such that*

$$x' = gx^\varepsilon g^{-1} \text{ and } y' = gy^\eta x^k g^{-1}.$$

*Proof.* Choose a basepoint  $v_0$  of  $A$  such that (after conjugation)  $x$  is represented by the  $\mathbb{A}$ -path  $[a]$  as an element of  $\pi_1(\mathbb{A}, v_0)$ . Choose a reduced  $\mathbb{A}$ -path  $\gamma = a_0, e_1, \dots, e_k, a_k$  that represents  $y$ , i.e. that  $y = [\gamma] \in \pi_1(\mathbb{A}, v_0)$ .

To show that  $x' = gx^\varepsilon g^{-1}$  holds note first that no element that can be represented by a (positive) word  $w$  in  $x^\varepsilon$  and  $y^\eta$  for fixed  $\varepsilon, \eta \in \{-1, 1\}$  and with at least one occurrence of  $y^\eta$  can be elliptic. To see this note first that any such word is conjugate to a word of type

$$w = y^\eta x^{n_1 \varepsilon} \cdot \dots \cdot y^\eta x^{n_k \varepsilon} = [\gamma^\eta][a^{n_1 \varepsilon}] \cdot \dots \cdot [\gamma^\eta][a^{n_k \varepsilon}]$$

with  $n_i \geq 0$  for  $1 \leq i \leq k$ . If the element represented by  $w$  is elliptic then the (reduced form) length of the reduced form of any positive power of  $w$  can be at most as large as the length of a reduced form of  $w$ . This implies that  $w$  has a subword  $y^\eta x^{n_i \varepsilon} y^\eta = [\gamma^\eta][a^{n_i \varepsilon}][\gamma^\eta]$  such that at least half of both occurrence of  $\gamma^\eta$  reduce when reducing a power of  $w$ .

This however implies that half of  $y^\eta x^{n_i \varepsilon}$  cancels in  $(y^\eta x^{n_i \varepsilon})(y^\eta x^{n_i \varepsilon})$  which implies that  $y^\eta x^{n_i \varepsilon}$  is elliptic. This however is impossible as  $(x, y)$  is Nielsen equivalent to  $(x, y^\eta x^{n_i \varepsilon})$  but  $\pi_1(\mathbb{A})$  is not generate by two elliptic elements as explained above.

Thus Proposition 1.2 implies that if  $(x, y)$  is Nielsen equivalent to  $(x', y')$  with  $x'$  elliptic then  $x'$  is conjugate to a power of  $x$ , thus  $x'$  is conjugate to  $x$  or  $x^{-1}$  as proper powers are not primitive. The second

part is now immediate as any element  $g$  of  $F(a, b)$  such that  $\{a, g\}$  forms a basis must be of type  $g = a^n b^\eta a^m$  with  $n, m \in \mathbb{Z}$ ,  $\eta = \pm 1$  and  $a^n b^\eta a^m$  is conjugate to  $b^\eta a^{m-n}$ .  $\square$

### 3. PIECEWISE GEODESICS IN HYPERBOLIC SPACE

Throughout this section all paths are paths in  $\mathbb{H}^n$ .

A  $(N, \alpha)$ -piecewise geodesic is a path that is composed of geodesic segments  $[x_i, x_{i+1}]$  of length at least  $N$  such that the angle  $\theta_{i+1} \in [0, \pi]$  between  $[x_i, x_{i+1}]$  and  $[x_{i+1}, x_{i+2}]$  at  $x_{i+1}$  is at least  $\alpha$ .

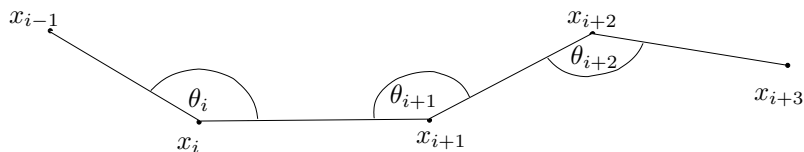


FIGURE 1. A section of a piecewise geodesic

It is easily verified that for large  $N$  and  $\alpha$  close to  $\pi$  any  $(N, \alpha)$ -piecewise geodesic  $\gamma$  is a quasi-geodesic. This implies in particular that its ends determine two point on the ideal boundary if  $\gamma$  is bi-infinite. Standard arguments easily show the following.

**Lemma 3.1.** *For any  $\varepsilon > 0$  there exist  $B > 0$  and  $\alpha \in (0, \pi)$  such that any bi-infinite  $(B, \alpha)$ -piecewise geodesic a  $(1 + \varepsilon, 0)$ -quasigeodesic that is  $\varepsilon$ -Hausdorff-close to the geodesic with the same ends.*

### 4. HYPERBOLIC KNOT COMPLEMENTS AS LIMITS OF CLOSED HYPERBOLIC 3-MANIFOLDS

It is a deep insight of Thurston that a cusped finite volume hyperbolic 3-manifold  $M$  occurs as the geometric limit of closed hyperbolic manifolds which are topologically obtained from  $M$  by Dehn fillings along increasingly complicated slopes.

Let now  $M$  be the complement of a hyperbolic knot  $\mathfrak{k}$  and let  $m$  and  $l$  denote the meridian and the longitude respectively of  $\mathfrak{k}$ . Let  $\rho: \pi_1(M) \rightarrow PSL(2, \mathbb{C})$  denote the holonomy of the complete hyperbolic structure. The image of  $m$  and  $l$  are commuting parabolic elements and we can assume that their common fixed point is  $\infty$  i.e.  $Stab_\infty = \langle m, l \rangle$  and that

$$\rho(m)(z) = z + 1 \text{ and } \rho(l)(z) = z + \tau_0$$

where  $\tau_0 \in \mathbb{C}$  is the cusp parameter which has non zero imaginary part.

The deformation space of hyperbolic structures on  $M$  can be holomorphically parametrised by a complex parameter  $u$  in a neighborhood  $U$  of  $0 \in \mathbb{C}$ . More precisely, there is an analytic family  $\rho_u$ ,  $u \in U$ ,

of representations  $\rho_u: \pi_1(M) \rightarrow PSL(2, \mathbb{C})$  and an analytic function  $v = v(u)$  such that  $u$  and  $v$  are the complex translation length of  $\rho_u(m)$  and  $\rho_u(l)$  respectively. The function  $\tau(u) = v(u)/u$  is analytic and  $\tau(u) = \tau(0) + O(u^2)$  where  $\tau(0) = \tau_0$  is the cusp parameter. The representation  $\rho_u$  is the holonomy of a non complete hyperbolic structure  $M_u$  on  $M$ . If  $u \neq 0$  then the equation

$$up + vq = 2\pi i$$

has a unique solution for  $(p, q) \in \mathbb{R}^2$ . If  $(p, q)$  are coprime integers then  $\rho_u$  factors through  $\pi_1(M(p/q))$  and the metric completion of  $M_u$  is homeomorphic to  $M(p/q)$ . Here and in the sequel  $M(p/q)$  denotes the manifold obtained from  $M$  by Dehn filling along the slope  $pm + ql$ . By Mostow-Prasad rigidity the faithful discrete representations of  $\pi_1(M)$  and  $\pi_1(M(p/q))$  in  $PSL(2, \mathbb{C})$  are unique up to conjugation.

We are mainly interested in the manifolds  $M_n = M(1/n)$ . By Thurston's Dehn filling theorem almost all of these manifolds carry a hyperbolic structure and the hyperbolic manifold  $M_n$  contains a new geodesic, the core of the filling solid torus, which is represented by  $l$ .

Now we let  $u_n$  denote the parameter satisfying  $u_n + nv_n = 2\pi i$  where  $v_n = v(u_n)$ . Then we have

$$u_n = \frac{2\pi i}{1 + n\tau_n} \text{ and } v_n = \frac{2\pi i\tau_n}{1 + n\tau_n}$$

where  $\tau_n = v_n/u_n$ . Note that  $\lim_{n \rightarrow \infty} \tau_n = \tau(0) = \tau_0$ . Then  $\rho_n := \rho_{u_n}$  determines the holonomy of the complete hyperbolic structure on  $M_n$ . We follow now the discussion in [M, 4.9]. Hence we can assume that  $\rho_n(m) = U_n$  and  $\rho_n(l) = V_n$  where

$$U_n(z) = a_n z + \tau_n \frac{a_n - 1}{b_n - 1} \text{ and } V_n(z) = b_n z + \tau_n$$

where  $a_n = \exp(u_n)$  and  $b_n = \exp(v_n)$ . Note that  $a_n = b_n^{-n}$  implies  $V_n^{-n} = U_n$  and that the common fixed points of  $V_n$  and  $U_n$  are  $\infty$  and  $\tau_n/(1 - b_n)$ .

In this setting we obtain the following:

- (1) The sequence of groups  $\{\rho_n(\pi_1(M))\}$  converges geometrically to  $\rho(\pi_1(M))$ .
- (2) Furthermore the sequence of subgroups  $\langle \rho_n(l) \rangle$  generated by the core of the filling solid torus converges geometrically to the peripheral subgroup  $P = \rho(\pi_1(\partial M))$ .
- (3) The elements  $\rho_n(l)$  are loxodromic isometries. The fixed points of  $\rho_n(l)$  converge to the fixed point  $\infty$  of  $P$ .
- (4)  $\rho_n(l)$  converges to  $\rho(l)$  and  $\rho_n(m) = \rho_n(l)^{-n}$  converges to  $m$ .

In the sequel we shall use the following convention: we shall identify  $\pi_1(M)$  with the image  $\rho(\pi_1(M)) \subset PSL(2, \mathbb{C})$  and for each  $g \in \pi_1(M)$  we write  $g_n = \rho_n(g)$  for short. We shall denote by  $\gamma_n \subset M_n$  the new geodesic i.e. the core of the filling solid torus.

The following proposition gives some more information about the geometry of the limiting process. Note that the translation lengths of an element  $g$  on some  $g$ -invariant subset  $Y$  of  $\mathbb{H}^3$  is defined to be

$$|g|_Y := \inf_{y \in Y} d_{\mathbb{H}^3}(y, gy).$$

In particular the translation length of a parabolic element  $g$  on a  $g$ -invariant horosphere  $S$  is measured with respect to the metric of  $\mathbb{H}^3$  rather than the Euclidean path metric of  $H$ .

**Proposition 4.1.** *For any horoball  $H$  at  $\infty$  there exists a sequence  $(r_n)_{n \in \mathbb{N}}$  of real numbers such that the following hold where  $N_n := N_{r_n}(A_{l_n})$  is the  $r_n$ -neighborhood of the axis  $A_{l_n}$  of  $l_n$  in  $\mathbb{H}^3$ .*

- (1) *For any fixed  $k, N$  we have  $\lim_{n \rightarrow \infty} (|l_n^{N-k \cdot n}|_{\partial N_n}) = |l^N m^k|_{\partial H}$ .*
- (2) *If  $N \in \mathbb{N}$  such that  $|l^N m^k|_{\partial H} > C > 0$  for all  $k \in \mathbb{Z}$ , then for any  $\eta > 0$  there exists some  $n'$  such that  $|l_n^{N-k \cdot n}|_{\partial N_n} \geq C - \eta$  for all  $n \geq n'$  and all  $k \in \mathbb{Z}$ .*

Moreover if  $(g_n)$  is a sequence of elements with  $g_n \in \rho_n(M_n)$  that converges to a hyperbolic element  $g \in \rho(M)$  and  $C > 0$  then  $H$  can be chosen such that the following hold:

- (a) *For sufficiently large  $n$  the geodesic segment  $[x_n, y_n]$  between  $N_n$  and  $g_n N_n$  is of length at least  $C$*
- (b) *For sufficiently large  $n$  the  $C$ -neighborhood of  $[x_n, y_n]$  does not intersect any translates of  $N_n$  except  $N_n$  and  $g_n N_n$ .*

*Proof.* Let  $H = \{(x, t) \in \mathbb{H} \mid t \geq t_0\}$  be given. Then the translation length  $|l|_{\partial H}$  is given by

$$\cosh(|l|_{\partial H}) = 1 + \frac{|\tau_0|^2}{2t_0^2}.$$

We now define  $r_n := 0$  if  $|\Re(v_n)| > |l|_{\partial H}$ . If  $|\Re(v_n)| < |l|_{\partial H}$  then  $r_n$  is defined to be the unique positive real number satisfying

$$(1) \quad \sinh^2(r_n) = \frac{\cosh(|l|_{\partial H}) - \cosh(\Re(v_n))}{|\cosh(v_n) - 1|}.$$

This definition implies that  $|l_n|_{\partial N_n} = |l|_{\partial H}$  if  $|\Re(v_n)| < |l|_{\partial H}$  is satisfied. In fact, the translation length  $|l_n|_{\partial N_n}$  on the cylinder  $\partial N_n$  is determined by the complex length  $v_n$  and the radius  $r_n$  of  $N_n$ . More precisely, we have

$$\cosh(|l_n|_{\partial N_n}) = \cosh(\Re(v_n)) - \sinh^2(r_n) |\cosh(v_n) - 1|$$

(see [Mey87, 9. Lemma 2]).

Fix now  $N$  and  $k$ . The complex translation length of  $l_n^{N-kn}$  is  $(N - kn)v_n$  and the equation  $u_n + nv_n = 2\pi i$  implies

$$(N - kn)v_n \equiv Nv_n + ku_n \pmod{2\pi i}.$$



Hence the translation length  $|l_n^{N-kn}|_{\partial N_n}$  is given by

$$(2) \quad \cosh(|l_n^{N-kn}|_{\partial N_n}) = \cosh(\Re(Nv_n + ku_n)) - \sinh^2(r_n) |\cosh(Nv_n + ku_n) - 1|.$$

Now

$$\frac{|\cosh(Nv_n + ku_n) - 1|}{|\cosh(v_n) - 1|} = \frac{|N\tau_n + k|^2}{|\tau_n|^2} (1 + O(|v_n|^2))$$

and hence

$$\lim_{n \rightarrow \infty} \cosh(|l_n^{N-kn}|_{\partial N_n}) = 1 + \frac{|N\tau_0 + k|^2}{2t_0^2} = \cosh(|l^N m^k|_{\partial H}).$$

This proves the first point.

Suppose that  $N \in \mathbb{N}$  satisfies  $|l^N m^k|_{\partial H} > C > 0$  for all  $k \in \mathbb{Z}$  and let  $\eta > 0$  be given. We choose  $n'' = n''(N, C)$  such that for all  $n \geq n''$  the following holds:

$$\left| N \frac{\Re(v_n)}{|v_n|} \right| < 1, \quad |v_n| \cdot \cosh(r_n) \geq \frac{|\tau_0|}{2t_0}$$

and

$$\frac{|\Re(u_n)|}{|v_n|} > \frac{1}{2} \frac{|\Im(\tau_0)|}{|\tau_0|}.$$

This is always possible since  $\lim_n |v_n| \cosh(r_n) = |\tau_0|/t_0$  and

$$\frac{u_n}{|u_n|} = i \frac{1 + n\tau_n}{1 + n\tau_n}$$

implies that

$$\lim_{n \rightarrow \infty} \frac{u_n}{|u_n|} = i \frac{\bar{\tau}_0}{|\tau_0|}, \quad \lim_{n \rightarrow \infty} \frac{v_n}{|v_n|} = i \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{u_n}{|v_n|} = i \frac{\bar{\tau}_0}{|\tau_0|^2}.$$

Note that  $\Im(\tau_0) \neq 0$ .

In order to prove the second point we start again with formula (2):

$$\begin{aligned} \cosh(|l_n^{N-kn}|_{\partial N_n}) &= \cosh(\Re(Nv_n + ku_n)) \\ &\quad - \sinh^2(r_n) |\cosh(Nv_n + ku_n) - 1| \\ &= \cosh(\Re(Nv_n + ku_n)) \cosh^2(r_n) \\ &\quad - \cos(\Im(Nv_n + ku_n)) \sinh^2(r_n) \\ &> (\cosh(\Re(Nv_n + ku_n)) - 1) \cosh^2(r_n) \\ &\geq \frac{1}{2} (\Re(Nv_n + ku_n))^2 \cosh^2(r_n) \\ &= \frac{1}{2} \left( \Re\left(N \frac{v_n}{|v_n|} + k \frac{u_n}{|v_n|}\right) \right)^2 (|v_n| \cosh(r_n))^2 \end{aligned}$$

and for  $n \geq n''$  we obtain

$$\cosh(|l_n^{N-kn}|_{\partial N_n}) > \frac{|\tau_0|^2}{8t_0^2} \cdot \left| \left| k \frac{\Re(u_n)}{|v_n|} \right| - \left| N \frac{\Re(v_n)}{|v_n|} \right| \right|^2.$$

Now for

$$|k| \geq C' := 2 \frac{|\tau_0|}{|\Im \tau_0|} \left( 2\sqrt{2} \frac{t_0}{|\tau_0|} \cosh(C)^{1/2} + 1 \right)$$

we obtain  $|l_n^{N-kn}|_{\partial N_n} \geq C$ . Since there are only finitely many  $k \in \mathbb{Z}$  such that  $|k| < C'$  it follows from the first part of the proposition that we can find  $n'''$  such that for all  $n \geq n'''$  and all  $|k| < C'$  the equation

$$|l_n^{N-kn}|_{\partial N_n} \geq C - \eta$$

holds. The second point follows for  $n' = \max(n'', n''')$ .

In order to prove (a) and (b) we shall use some results of Meyerhoff. By [Mey87, Sec. 3], we know that  $|\cosh(v_n) - 1| < \sqrt{2} - 1$  implies the existence of a solid tube  $V_n$  around  $A_{l_n}$  with radius  $r_n$  satisfying

- for all  $g_n \in \rho_n(\pi_1(M)) \setminus \langle l_n \rangle$  we have that  $g(V_n) \cap V_n = \emptyset$  i.e. the geodesic  $\gamma_n$  has a solid tube neighborhood of radius  $r_n$  in  $M_n$ .
- $\lim_{n \rightarrow \infty} r_n = \infty$  i.e. if  $v_n$  approaches 0 then the radius of the solid tube approaches infinity.

Moreover, there exists  $\epsilon > 0$  such that  $d_{\mathbb{H}}(p, l_n(p)) \leq \epsilon$  implies  $p \in V_n$  for all  $p \in \mathbb{H}$ . Note also that  $\{p \in \mathbb{H} \mid d_{\mathbb{H}}(p, l(p)) \leq \epsilon\}$  gives an embedded cusp neighborhood in  $M$  for  $\epsilon$  sufficiently small (see [Mey87]). Now we define

$$N_n := \{p \in \mathbb{H} \mid d_{\mathbb{H}}(p, l_n(p)) \leq \epsilon\} \text{ and } H := \{p \in \mathbb{H} \mid d_{\mathbb{H}}(p, l(p)) \leq \epsilon\}.$$

To prove (a) observe that the geodesic segment  $[x_n, y_n]$  is part of the common normal of the lines  $A_{l_n}$  and  $A_{g_n l_n g_n^{-1}} = g_n A_{l_n}$ . Hence it is easy to see that the segment  $[x_n, y_n]$  converges to  $[x, y]$ . By choosing the above  $\epsilon$  smaller we can increase the length of the segment  $[x, y]$  i.e. we can assume that the length of  $[x, y]$  is  $C + 1$ . Hence for  $n$  sufficiently large we get that the length of  $[x_n, y_n]$  is at least  $C$ .

To prove (b) we shall use the following fact: if  $(z_n)$  is a sequence of points in  $\mathbb{H}^3$  such that each  $z_n$  is contained in a translate of  $N_n$  and such that  $(z_n)$  converges to a point  $z \in \mathbb{H}^3$ , then there exists a  $h \in \pi_1(M)$  such that  $z_n \in h_n N_n$  for  $n$  sufficiently large.

Next note that there can be only finitely many translates  $hH$ ,  $h \in \pi_1(M)$ ,  $h \notin P \cup gP$ , such that the intersection of  $hH$  and the  $(C + 1)$ -neighborhood of  $[x, y]$  is non empty. This follows from the fact that the  $(C + 1)$ -neighborhood of  $[x, y]$  is compact and that the translates of  $H$  are disjoint. Thus by choosing  $\epsilon$  smaller we obtain that the  $(C + 1)$ -neighborhood of  $[x, y]$  does not intersect  $hH$  for  $h \notin P \cup gP$ . Moreover, note that the  $C$ -neighborhood of  $[x_n, y_n]$  converges to the  $C$ -neighborhood of  $[x, y]$ . Hence for  $n$  sufficiently large we have that the

$C$ -neighborhood of  $[x_n, y_n]$  is contained in the  $(C + 1)$ -neighborhood of  $[x, y]$ .

Now suppose that there is a sequence  $(j_i)$  with  $\lim j_i = \infty$  such that a translate of  $N_{j_i}$  intersects the  $C$ -neighborhood of  $[x_{j_i}, y_{j_i}]$  and let  $z_i$  be a point of this intersection. For  $i$  sufficiently large the  $C$ -neighborhood of  $[x_{j_i}, y_{j_i}]$  is contained in the compact  $(C + 1)$ -neighborhood of  $[x, y]$ . Thus after passing to a subsequence we can assume that  $(z_i)$  converges and such that each  $z_i$  is contained in a translate of  $N_{j_i}$ . This implies the existence of an element  $h \in \pi_1(M)$  such that  $z_i \in h_{j_i} N_{j_i}$  for all sufficiently large  $i$ . Hence  $\lim z_i = z \in hH$  and  $z$  is contained in the  $(C + 1)$ -neighborhood of  $[x, y]$ . This gives that  $h \in P \cup gP$  and  $z_i \in N_{j_i}$  or  $z_i \in g_{j_i} N_{j_i}$ .  $\square$

**Lemma 4.2.** *For any  $\beta \in (0, \pi/2)$  there exist  $\kappa(\beta)$  and  $r(\beta)$  such that if  $\gamma$  is a geodesic in  $\mathbb{H}^3$ ,  $r \geq r(\beta)$  and  $x, y \in \partial(N_r(\gamma))$  such that  $d(x, y) \geq \kappa(\beta)$  then the angles enclosed by  $[x, y]$  and  $\partial(N_r(\gamma))$  are at least  $\beta$ .*

*Proof.* The proof of the lemma is by calculation. We follow the setup of [GMM, Section 2]. We choose an orientation of  $\gamma$  and define coordinates on  $\partial(N_r(\gamma))$  as follows: we choose an origin  $(0, 0) \in \partial(N_r(\gamma))$  and for  $\delta \in \mathbb{R}$  and  $\phi \in [0, 2\pi)$  let  $(\delta, \phi)$  be the image of  $(0, 0)$  under the loxodromic motion with axis  $\gamma$  and complex length  $\delta + i\phi$ .

We shall do the calculation in the Klein hyperboloid model of  $\mathbb{H}^3$ . In this model,  $\mathbb{H}^3$  is the hypersurface

$$\{(t, x, y, z) \in \mathbb{R}^4 \mid -t^2 + x^2 + y^2 + z^2 = -1\}$$

We assume that  $\gamma$  is the intersection of  $\mathbb{H}^3$  with the plane  $\{x = z = 0\}$ . The loxodromic motion along  $\gamma$  with complex length  $\delta + i\phi$  is given by

$$\begin{pmatrix} \cosh(\delta) & 0 & \sinh(\delta) & 0 \\ 0 & \cos(\phi) & 0 & -\sin(\phi) \\ \sinh(\delta) & 0 & \cosh(\delta) & 0 \\ 0 & \sin(\phi) & 0 & \cos(\phi) \end{pmatrix}.$$

We fix the point  $x = (\cosh(r), \sinh(r), 0, 0)$  as origin on  $\partial(N_r(\gamma))$  i.e.  $x = (0, 0)$ .

The inward unit normal vector  $\vec{n} \in T_x \mathbb{H}^3$  to  $\partial(N_r(\gamma))$  is given by  $\vec{n} = (-\sinh(r), -\cosh(r), 0, 0)$ . For every  $y \in \mathbb{H}^3$  the unit vector  $\vec{m} \in T_x \mathbb{H}^3$  pointing into the direction of  $y$  is given by

$$\vec{m} = \frac{y + \langle y, x \rangle x}{(\langle y, x \rangle^2 - 1)^{1/2}}.$$

Here  $\langle x, y \rangle$  denotes the Lorentz scalar product.

Let now  $y \in \partial N_r(\gamma)$  be the point  $y = (\delta, \phi)$  and denote by  $\beta(y)$ ,  $0 \leq \beta(y) \leq \pi/2$ , the angle enclosed by  $[x, y]$  and  $\partial(N_r(\gamma))$ . By elementary

calculations which are similar to the calculations in [GMM, Section 2] we obtain

$$\begin{aligned} \sin(\beta(y)) &= \cos\left(\frac{\pi}{2} - \beta(y)\right) = \langle \vec{m}, \vec{n} \rangle \\ &= \frac{\cosh(r) \sinh(r)}{\sinh(d(x, y))} (\cosh(\delta) - \cos(\phi)). \end{aligned}$$

Moreover we have

$$\cosh(d(x, y)) = \cosh(\delta) \cosh^2(r) - \cos(\phi) \sinh^2(r)$$

and hence

$$\sin(\beta(y)) = \tanh(r) \frac{\cosh(d(x, y)) - \cos(\phi)}{\sinh(d(x, y))}.$$

For a fixed distance  $d = d(x, y)$  the angle  $\beta(y)$  becomes minimal if  $y = (\delta, 0)$ . Therefore

$$\sin(\beta(y)) \geq \tanh(r) \frac{\cosh(d(x, y)) - 1}{\sinh(d(x, y))} = \tanh(r) \tanh\left(\frac{d(x, y)}{2}\right).$$

Now let  $\beta$ ,  $0 < \beta < \pi/2$ , be given. We choose  $r(\beta) > 0$  such that  $\sin(\beta) < \sin(\beta) \coth(r(\beta)) = q < 1$  and  $\kappa(\beta)$  such that  $q = \tanh(\kappa(\beta)/2)$ . Hence for  $r \geq r(\beta)$  and for  $y \in \partial N_r$  such that  $d(x, y) \geq \kappa(\beta)$  we obtain

$$\begin{aligned} \sin(\beta(y)) &\geq \tanh(r) \tanh(d(x, y)/2) \\ &\geq \tanh(r(\beta)) \tanh(\kappa(\beta)/2) = \sin(\beta). \end{aligned}$$

Therefore we have for all  $r \geq r(\beta)$  and all  $y \in \partial N_r$  such that  $d(x, y) \geq \kappa(\beta)$  that  $\beta(y) \geq \beta$ . This proves the Lemma.  $\square$

## 5. GENERATING PAIRS OF 2-BRIDGE KNOT GROUPS

In this section we prove that hyperbolic 2-bridge knot groups have infinitely many Nielsen classes of generating pairs. Moreover we prove that there exist closed hyperbolic 3-manifolds that have arbitrarily many Nielsen classes of generating pairs. Those manifolds are obtained by Dehn surgery of  $S^3$  at 2-bridge knots.

The infinity of Nielsen classes of generating pairs of fundamental group of Seifert fibered 2-bridge knot spaces has been known for a long time. For the trefoil knot this is due to Dunwoody and Pietrowsky [DP] and the general case is due to Zieschang [Z2] who in fact gives a complete classification of Nielsen classes of generating pairs.

Let  $M$  the exterior of a hyperbolic 2-bridge knot  $\mathfrak{k}$ . Choose  $m, l \in \pi_1(M)$  such that  $m$  represents the meridian, that  $l$  represents the longitude and that  $\langle m, l \rangle \cong \mathbb{Z}^2$  is a maximal peripheral subgroup. Inspecting the Wirtinger presentation shows that 2-bridge knot groups are generated by two meridional element, i.e. there exists some  $g$  such that  $\pi_1(M) = \langle m, gm g^{-1} \rangle$ .

As  $(gl^N) \cdot m \cdot (gl^N)^{-1} = g \cdot l^N m l^{-N} \cdot g^{-1} = gmg^{-1}$  it follows that  $P_N := (m, gl^N)$  is a generating pair for  $\pi_1(M)$  for all  $N \in \mathbb{Z}$ . Clearly the  $P_N$  can also be considered as generating pairs of the fundamental groups of manifolds obtained from  $M$  by a Dehn filling.

The following theorem is the main theorem of this article.

**Theorem 5.1.** *Let  $M$  the exterior of a hyperbolic 2-bridge knot  $\mathfrak{k}$  and  $M_n = M(\frac{1}{n})$ . There exists  $N_0 \in \mathbb{N}$  such that for  $N, N' \geq N_0$  there exists  $n_0$  such that for  $n \geq n_0$  the generating pairs  $P_N$  and  $P_{N'}$  of  $\pi_1(M_n)$  are not Nielsen equivalent.*

Note that all generating pairs  $P_N$  and  $P_{N'}$  have the same commutator invariant

$$[m, gl^N] = m \cdot gl^N \cdot m^{-1} \cdot l^{-N} g^{-1} = [m, g].$$

Hence the commutator invariant can not distinguish the Nielsen equivalence classes of  $P_N$  and  $P_{N'}$ .

**Corollary 5.2.** *For any  $n$  there exists a closed hyperbolic 3-manifold  $M$  such that  $\pi_1(M)$  has at least  $n$  distinct Nielsen classes of generating pairs.*

As further Nielsen-equivalent tuples cannot become non-equivalent when passing to a factor group we also get the following.

**Corollary 5.3.** *Let  $\mathfrak{k}$  be a hyperbolic 2-bridge knot with knot exterior  $M$ . Then  $\pi_1(M)$  has infinitely many Nielsen classes of generating pairs.*

We prove two lemmas before giving the proof of Theorem 5.1. In the following we will, on the level of notation, not distinguish between an element of  $\pi_1(M)$  and an element of  $\pi_1(M_n)$ .

**Lemma 5.4.** *There exists  $N_1$  such that for all  $N \geq N_1$  there exists  $n_1 \in \mathbb{N}$  such that for  $n \geq n_1$  and  $\varepsilon, \eta \in \{-1, 1\}$  a positive word  $w$  in  $m^\eta$  and  $(gl^N)^\varepsilon$  represents (in  $\pi_1(M_n)$ ) an element conjugate to  $m^{\pm 1}$  iff  $w = m^\eta$ .*

*Proof.* Choose  $B \geq 100$  and  $\alpha$  such the conclusion of Lemma 3.1 is fulfilled for  $\varepsilon = \frac{1}{2}$ . Furthermore choose  $\kappa := \kappa(\alpha - \pi/2)$  and  $r := r(\alpha - \pi/2)$  as in Lemma 4.2.

Choose a horoball  $H$  at the fixed-point of  $P$  such that the conclusions (a) and (b) of Proposition 4.1 hold for  $C = B$ . Thus there exist  $\hat{n}$  such that  $d(N_n, g_n N_n) \geq B$  for  $n \geq \hat{n}$ . Denote the geodesic segment between  $H$  and  $gH$  by  $[x, y]$  and the segment between  $N_n$  and  $g_n N_n$  by  $[x_n, y_n]$ .

Choose  $t$  such that  $d(g_n x_n, y_n) \leq t$  for all  $n$ . Such  $t$  clearly exists as the segments  $[x_n, y_n]$  converge to  $[x, y]$  and  $g_n$  to  $g$ . Choose further  $N_1$  such that  $|l^N m^k|_{\partial H} \geq B + t + \kappa + 1$  for all  $N \geq N_1$  and  $k \in \mathbb{Z}$ .

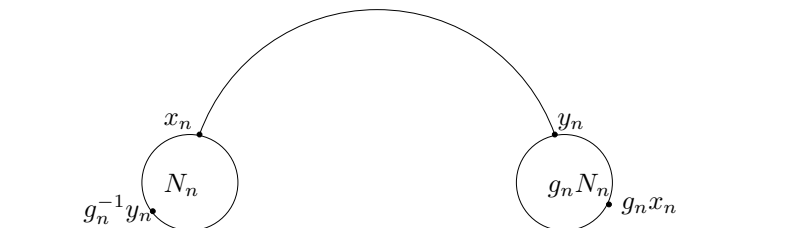


FIGURE 2. The horoball approximation  $N_n$  and its translate by  $g_n$

Now fix  $N \geq N_1$ . It now follows from Proposition 4.1 (3) (by choosing  $\eta$  sufficiently small) that there exists some  $\tilde{n} \geq \hat{n}$  such that  $|l_n^N m_n^k|_{\partial N_n} \geq B + t + \kappa$  for all  $k \in \mathbb{Z}$  and  $n \geq \tilde{n}$ .

Now choose  $n_1 \geq \tilde{n}$  such that  $|m_n|_{\mathbb{H}^3} < 1$  and  $r_n > r(\alpha - \pi/2)$  for all  $n \geq n_1$ ; this is clearly possible as the element  $m_n$  converges to the parabolic element  $m$  and  $r_n$  tends to infinity as  $n$  tends to infinity. We need to show for  $n \geq n_1$  and  $\varepsilon, \eta \in \{-1, 1\}$  a positive word  $w$  in  $m_n^\eta$  and  $(g_n l_n^N)^\varepsilon$  represents (in  $\pi_1(M_n)$ ) an element conjugate to  $m_n^{\pm 1}$  iff  $w = m_n^\eta$ .

We check the case that  $\eta = \varepsilon = 1$ , the other cases are analogous. Thus either  $w$  is a power of  $m_n$  in which case there is nothing to show or  $w$  is (after a cyclic permutation) of type

$$(g_n l_n^N) m_n^{b_1} \cdot \dots \cdot (g_n l_n^N) m_n^{b_r}$$

with  $b_i \geq 0$  for  $1 \leq i \leq r$ . Thus  $w$  can be rewritten as a product

$$(g_n p_1) \cdot \dots \cdot (g_n p_r)$$

where  $p_i = l_n^N m_n^{b_i}$  for  $1 \leq i \leq r$ . By the above choices the translation length of all  $p_i$  on  $\partial N_n$  is at least  $B + t + \kappa$ .

We now construct a  $w$ -invariant bi-infinite  $(B, \alpha)$ -piecewise geodesic  $\gamma_w$  containing  $x_n$ . We first construct a  $(B, \alpha)$ -piecewise geodesic  $\gamma_0$  from  $x_n$  to  $w x_n$  and put  $\gamma_i = w^i \gamma_0$ . We then put

$$\gamma_w := \dots \gamma_{-2} \cdot \gamma_{-1} \cdot \gamma_0 \cdot \gamma_1 \cdot \gamma_2 \dots$$

which clearly implies the  $w$ -invariance. The fact that  $\gamma_w$  is also a  $(B, \alpha)$ -piecewise geodesic follows immediately from the construction.

For  $1 \leq i \leq r$  put  $w_i = (g_n p_1) \cdot \dots \cdot (g_n p_i)$ ,  $x_n^i = w_i x_n$  and  $y_n^i = w_i y_n$ . We then put

$$\gamma_0 := [x_n, y_n] \cdot [y_n, x_n^1] \cdot [x_n^1, y_n^1] \cdot [y_n^1, x_n^2] \cdot \dots \cdot [x_n^{r-1}, y_n^{r-1}] \cdot [y_n^{r-1}, x_n^r = w x_n].$$

Now  $x_n^i \in \partial(w_i N_n) = w_i \partial N_n$  and  $y_n^i \in \partial(w_{i+1} N_n)$ . The segments  $[x_n^i, y_n^i]$  are of length at least  $B$  and are perpendicular to the respective translates of  $\partial N_n$ . As the segments  $[y_n^i, x_n^{i+1}]$  are of length at least  $B + \kappa$ , it follows from the choice of  $\kappa$  and Lemma 4.2 that they enclose angles greater or equal than  $\alpha - \pi/2$  with the respective translates

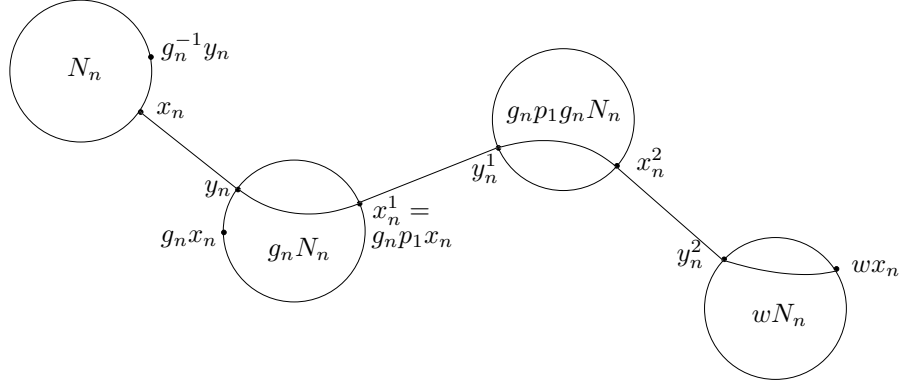


FIGURE 3. The piecewise geodesic  $\gamma_0$  for  $w = (g_n p_1)(g_n p_2)(g_n p_3)$

of  $\partial N_n$ . This proves that  $\gamma_i$  and therefore  $\gamma_w$  is a  $(B, \alpha)$ -piecewise geodesic.

Note that the piecewise geodesic  $\gamma_0$  has path length greater than  $2B \geq 200$ . This makes  $\gamma_0$  a  $(1 + \frac{1}{2}, \frac{1}{2})$ -quasigeodesic by Lemma 3.1, it follows that  $d(x_n, w x_n) \geq 50$ .

The axis  $A_w$  of  $w$  must be the geodesic with the same ends as  $\gamma_w$ . Thus  $\gamma_w$  and  $A_w$  are  $\frac{1}{2}$ -Hausdorff close by the choice of  $B$  and  $\alpha$ , see Lemma 3.1. It follows that the projection of  $x_n$  to  $A_w$  is moved under the action of  $w$  by at least 48, in particular the translation length of  $w$  on  $\mathbb{H}^3$  is at least 48. Thus  $w$  cannot be conjugate to  $m_n^{\pm 1}$  as  $m_n$  is assumed to have translation length at most 1, the lemma is proven.  $\square$

**Lemma 5.5.** *There exists  $N_2$  such that for all  $N \neq N' \geq N_2$  there exists  $n_2 \in \mathbb{N}$  such that for  $n \geq n_2$  and  $k \in \mathbb{Z}$  the element  $g_n l_n^N$  and  $(g_n l_n^{N'+nk})^\epsilon$  are not conjugate in  $\pi_1(M_n)$  for  $\epsilon \in \{\pm 1\}$ .*

*Proof.* The proof is similar to the proof of Lemma 5.4. Choose the constants  $B, \alpha, \kappa, t, N_1$  and the horoball  $H$  as in the proof of Lemma 5.4 and put  $N_2 := N_1$ . Choose  $N \neq N' \geq N_2$ . Now by choosing  $n_2 > N + N'$  sufficiently large it follows as before that

$$|l_n^{N+kn}|_{\partial N_n}, |l_n^{N'+kn}|_{\partial N_n} \geq B + t + \kappa$$

for all  $n \geq n_2$  and  $k \in \mathbb{Z}$ . Put  $w_1 = g_n l_n^N$  and  $w_2 = g_n l_n^{N'+nk}$ , note that  $w_1 \neq w_2$  as  $l_n$  is of infinite order and  $n > N + N'$ . Thus we can construct the  $w_1$ -, respectively  $w_2$ -, invariant  $(B, \alpha)$ -piecewise geodesics  $\gamma_{w_1}$  and  $\gamma_{w_2}$  with the same properties as before. Note that by Theorem 4.1 (b) we can further assume that the 1-neighborhood of  $[x_n, y_n]$  does not intersect any translate of  $N_n$  except  $N_n$  and  $g_n N_n$ .

Now the axes  $A_{w_1}$  and  $A_{w_2}$  are  $\frac{1}{2}$ -Hausdorff-close to  $\gamma_{w_1}$  and  $\gamma_{w_2}$ . By construction the translates of  $N_n$  intersected by  $A_{w_i}$  are precisely the translates  $w_i^k N_n$  for  $i = 1, 2$ . Note that  $N_n$  and  $g_n N_n$  are intersected by both  $A_{w_1}$  and  $A_{w_2}$ . Now if  $w_1$  and  $w_2^\epsilon$  are conjugate then there



must exist some  $h \in \pi_1(M_n)$  such that  $w_2^\epsilon = hw_1h^{-1}$  which implies that  $hA_{w_1} = A_{w_2^\epsilon} = A_{w_2}$ ;  $h$  must in particular map the translates intersected by  $A_{w_1}$  to those intersected by  $A_{w_2}$ .

After possible right multiplication with a power of  $w_1$  we can assume that  $h$  fixes both  $N_n$  and  $gN_n$  (note that it cannot exchange them as it would otherwise fix the midpoint of  $[x_n, y_n]$  and therefore be elliptic). As the intersection of the stabilizers of  $N_n$  and  $gN_n$  is trivial this implies that  $h = 1$  i.e. that  $w_1 = w_2$ . This is clearly a contradiction, thus  $w_1$  and  $w_2$  are not conjugate.  $\square$

*Proof of Theorem 5.1.* Let  $N_1, n_1, N_2, n_2$  be as before and put  $N_0 = \max(N_1, N_2)$  and  $n_0 = \max(n_1, n_2)$ . Let  $N \neq N' \geq N_0$ . We show that  $P_N$  and  $P_{N'}$  are not Nielsen equivalent in  $\pi_1(M_n)$  for  $n \geq n_0$ . This clearly proves the theorem.

Suppose that  $P_N$  and  $P_{N'}$  are Nielsen equivalent. Thus there exists a basis  $\{b_1(a, b), b_2(a, b)\}$  of  $F(a, b)$  such that  $m_n = b_1(m_n, g_n l_n^{N'})$  and  $g_n l_n^N = b_2(m_n, g_n l_n^{N'})$ . It now follows from Proposition 1.2 and Lemma 5.4 that  $b_1$  is in  $F(a, b)$  conjugate to  $a$ , i.e. that  $b_1(a, b) = u(a, b)au(a, b)^{-1}$ .

It follows in particular that  $b_2(a, b) = u(a, b)a^{k_1}b^\epsilon a^{k_2}u(a, b)^{-1}$  for some  $k_1, k_2 \in \mathbb{Z}$  and  $\epsilon \in \{\pm 1\}$ . Thus  $g_n l_n^N = hm_n^{k_1}(g_n l_n^{N'})^\epsilon m_n^{k_2}h^{-1}$  which implies that  $g_n l_n^N$  is conjugate to  $(g_n l_n^{N'})^\epsilon m_n^{k_1+k_2}$  which contradicts Lemma 5.5. Thus  $P_N$  and  $P_{N'}$  are not Nielsen equivalent in  $\pi_1(M_n)$ .  $\square$

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