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## Witness sets

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#### Abstract

Given a set $C$ of binary $n$-tuples and $c \in C$, how many bits of $c$ suffice to distinguish it from the other elements in $C$ ? We shed new light on this old combinatorial problem and improve on previously known bounds.


## 1 Introduction

Let $C \subset\{0,1\}^{n}$ be a set of distinct binary vectors that we will call a code, and denote by $[n]=\{1,2, \ldots n\}$ the set of coordinate positions. It is standard in coding theory to ask for codes (or sets) $C$ such that every codeword $c \in C$ is as different as possible from all the other codewords. The most usual interpretation of this is that every codeword $c$ has a large Hamming distance to all other codewords, and the associated combinatorial question is to determine the maximum size of a code that has a given minimal Hamming distance $d$. The point of view of the present paper is to consider that "a codeword $c$ is as different as possible from all the other codewords" means that there exists a small subset $W \subset[n]$ of coordinates such that $c$ differs from every other codeword in $W$. Put differently, it is possible to single out $c$ from all the other codewords by focusing attention on a small subset of coordinates. More precisely, for $x \in\{0,1\}^{n}$, and $W \subset[n]$ let us define the projection $\pi_{W}$

$$
\begin{aligned}
\pi_{W}:\{0,1\}^{[n]} & \rightarrow\{0,1\}^{W} \\
x & \mapsto\left(x_{i}\right)_{i \in W}
\end{aligned}
$$

and let us say that $W$ is a witness set (or a witness for short) for $c \in C$ if $\pi_{W}(c) \neq \pi_{W}\left(c^{\prime}\right)$ for every $c^{\prime} \in C, c \neq c^{\prime}$. Codes for which every codeword has a small witness set arise in a variety of contexts, in particular in machine learning theory [1. 3. (4] where a witness set is also called a specifying set or a discriminant: see [5, Ch. 12] for a short survey of known results and also [2] and references therein for a more recent discussion of this topic and some variations.

Let us now say that a code has the $w$-witness property, or is a $w$-witness code, if every one of its codewords has a witness set of size $w$. Our concern is to study the maximum possible cardinality $f(n, w)$ of a $w$-witness code of length $n$. We shall give improved upper and lower bounds on $f(n, w)$ that almost meet.

The paper is organised as follows. Section 2 gives some easy facts for reference. Section 3 is devoted to upper bounds on $f(n, w)$ and introduces our main result, namely Theorem 2. Section 1 is devoted to constant weight $w$-witness codes, and we derive precise values of the cardinality of optimal codes. Section 5 studies mean values for the number of witness sets of a codeword and the number of codewords that have a given witness set. Section 6 is devoted to constructions of large $w$-witness codes, sometimes giving improved lower values of $f(n, w)$. Finally, Section concludes with some open problems.

## 2 Easy and known facts

Let us start by mentioning two self-evident facts

- If $C$ is a $w$-witness code, so is any translate $C+x$,
- $f(n, w)$ is an increasing function of $n$ and $w$.

Continue with the following example. Let $C$ be the set of all $n$ vectors of length $n$ and weight 1 . Then every codeword of $C$ has a witness of size 1 , namely its support. Note the dramatic change for the slightly different code $C \cup\{\mathbf{0}\}$. Now the all-zero vector $\mathbf{0}$ has no witness set of size less than $n$. Bondy [3] shows however that if $|C| \leq n$, then $C$ is a $w$-witness code with $w \leq|C|-1$ and furthermore $C$ is a uniform $w$-witness code, meaning that there exists a single subset of $[n]$ of size $w$ that is a witness set for all codewords.

We clearly have the upper bound $|C| \leq 2^{w}$ for uniform $w$-witness codes. For ordinary $w$-witness codes however, the best known upper bound is, 肘, Proposition 12.2],

$$
\begin{equation*}
f(n, w) \leq 2^{w}\binom{n}{w} \tag{1}
\end{equation*}
$$

The proof is simple and consists in applying the pigeon-hole principle. A subset of $[n]$ can be a witness set for at most $2^{w}$ codewords and there are at most $\binom{n}{w}$ witness sets.

We also have the following lower bound on $f(n, w)$, based on a trivial construction of a $w$-witness code.

Proposition 1. We have: $f(n, w) \geq\binom{ n}{w}$.
Proof. Let $C=\binom{[n]}{w}$ be the set of all vectors of weight $w$. Notice that for all $c \in C, W(c)=\operatorname{support}(c)$ is a witness set of $c$.

Note that the problem is essentially solved for $w \geq n / 2$; since $f(n, w)$ is increasing with $w$, we then have:

$$
2^{n} \geq f(n, w) \geq f(n, n / 2) \geq\binom{ n}{n / 2} \geq 2^{n} /(2 n)^{1 / 2} .
$$

We shall therefore focus in the sequel on the case $w \leq n / 2$.
In the next section we improve the upper bound (1) to a quantity that comes close to the lower bound of Proposition [1.

## 3 An improved upper bound

The key result is the following.
Theorem 1. Let $g(n, w)=f(n, w) /\binom{n}{w}$. Then, for fixed $w, g(n, w)$ is a decreasing function of $n$. That is:

$$
n \geq v \geq w \quad \Rightarrow \quad g(n, w) \leq g(v, w)
$$

Proof. Let $C$ be a binary code of length $n$ having the $w$-witness property, with maximal cardinality $|C|=f(n, w)$. Fix a choice function $\phi: C \rightarrow\binom{[n]}{w}$ such that for any $c \in C, \phi(c)$ is a witness for $c$. For any $V \in\binom{[n]}{v}$, denote by $C_{V}$ the subset of $C$ formed by the $c$ satisfying $\phi(c) \subset V$. Remark that the projection $\pi_{V}$ is injective on $C_{V}$, since each element of $C_{V}$ has a witness in $V$. Then $\pi_{V}\left(C_{V}\right)$ also has the $w$-witness property.

Remark now that if $V$ is uniformly distributed in $\binom{[n]}{v}$ and $W$ is uniformly distributed in $\binom{[n]}{w}$ and independent from $V$, then for any function $\psi:\binom{[n]}{w} \rightarrow \mathbb{R}$ one has

$$
\begin{equation*}
E_{W}(\psi(W))=E_{V}\left(E_{W}(\psi(W) \mid W \subset V)\right) \tag{2}
\end{equation*}
$$

where we denote by $E_{W}(\psi(W))$ the mean value (or expectation) of $\psi(W)$ as $W$ varies in $\binom{[n]}{w}$, and so on.

We apply this with $\psi(W)=\left|\phi^{-1}(W)\right|$ to find

$$
\begin{aligned}
g(n, w) & =\binom{n}{w}^{-1}|C|=\binom{n}{w}^{-1} \sum_{W \in\binom{(n n)}{w}}\left|\phi^{-1}(W)\right| \\
& =E_{W}\left(\left|\phi^{-1}(W)\right|\right) \\
& =E_{V}\left(E_{W}\left(\left|\phi^{-1}(W)\right| \mid W \subset V\right)\right) \\
& =E_{V}\left(\binom{v}{w}^{-1} \sum_{W \in\binom{V}{w}}\left|\phi^{-1}(W)\right|\right) \\
& =E_{V}\left(\binom{v}{w}^{-1}\left|C_{V}\right|\right) \\
& =E_{V}\left(\binom{v}{w}^{-1}\left|\pi_{V}\left(C_{V}\right)\right|\right) \\
& \leq g(v, w)
\end{aligned}
$$

the last inequality because $\pi_{V}\left(C_{V}\right)$ is a binary code of length $v$ having the $w$ witness property.

Remark: It would be interesting to try to improve Theorem 1 using some unexploited aspects of the above proof, such as the fact that the choice function $\phi$ may be non-unique, or the fact that the last inequality not only holds in mean value, but for all $V$. For instance, suppose there is a codeword $c \in C$ (with $C$ optimal as in the proof) that admits two distinct witnesses $W$ and $W^{\prime}$, with $W \not \subset W^{\prime}$. Let $\phi$ be a choice function with $\phi(c)=W$, and let $\phi^{\prime}$ be the choice function that coincides everywhere with $\phi$, except for $\phi^{\prime}(c)=W^{\prime}$. Let $V$ contain $W^{\prime}$ but not $W$. If we denote by $C_{V}^{\prime}$ the subcode obtained as $C_{V}$ but using $\phi^{\prime}$ as choice function, then $C_{V}^{\prime}=C_{V} \cup\{c\}$ (disjoint union), so $\left|\pi_{V}\left(C_{V}\right)\right|=\left|\pi_{V}\left(C_{V}^{\prime}\right)\right|-1<f(v, w)$, and $g(n, w)<g(v, w)$.

Theorem has a number of consequences: the following is straightforward.
Corollary 1. For fixed $w$, the limit

$$
\lim _{n \rightarrow \infty} g(n, w)=\frac{f(n, w)}{\binom{n}{w}}
$$

exists.
The following theorem gives an improved upper bound on $f(n, w)$.
Theorem 2. For $w \leq n / 2$, we have the upper bound:

$$
f(n, w) \leq 2 w^{1 / 2}\binom{n}{w}
$$

Proof. Choose $v=2 w$ and use $f(v, w) \leq 2^{v}$; then $f(n, w) \leq\binom{ n}{w} f(2 w, w) /\binom{2 w}{w}$ and the result follows by Stirling's approximation.

Set $w=\omega n$ and denote by $h(x)$ the binary entropy function

$$
h(x)=-x \log _{2} x-(1-x) \log _{2}(1-x) .
$$

Theorem 2 together with Proposition 11 yield:

## Corollary 2. We have

$$
\begin{aligned}
\lim _{n \rightarrow \infty} \frac{1}{n} \log _{2} f(n, \omega n) & =h(\omega) & & \text { for } 0 \leq \omega \leq 1 / 2 \\
& =1 & & \text { for } 1 / 2 \leq \omega \leq 1
\end{aligned}
$$

## 4 Constant-weight codes

Denote now by $f(n, w, k)$ the maximal size of a $w$-witness code with codewords of weight $k$. The following result is proved using a folklore method usually attributed to Bassalygo and Elias, valid when the required property is invariant under some group operation.

Proposition 2. We have:

$$
\max _{k} f(n, w, k) \leq f(n, w) \leq \min _{k} \frac{f(n, w, k) 2^{n}}{\binom{n}{k}}
$$

Proof. The lower bound is trivial.
For the upper bound, fix $k$, pick an optimal $w$-witness code $C$ and consider its $2^{n}$ translates by all possible vectors. Every $n$-tuple, in particular those of weight $k$, occurs exactly $|C|$ times in the union of the translates; hence there exists a translate (also an optimal $w$-witness code of size $f(n, w)$ - see the remark at the beginning of Section (2) containing at least the average number $\left.|C| \begin{array}{l}n \\ k\end{array}\right) 2^{-n}$ of vectors of weight $k$. Since $k$ was arbitrary, the result follows.

We now deduce from the previous proposition the exact value of the function $f(n, w, k)$ in some cases.

Corollary 3. For constant-weight codes we have:

- If $k \leq w \leq n / 2$ then $f(n, w, k)=\binom{n}{k}$ and an optimal code is given by $S_{k}(\mathbf{0})$, the Hamming sphere of radius $k$ centered on $\mathbf{0}$.
- If $n-k \leq w \leq n / 2$, then $f(n, w, n-k)=\binom{n}{k}$ and an optimal code is given by the sphere $S_{k}(\mathbf{1})$.

Proof. If $k \leq w \leq n / 2$, we have the following series of inequalities:

$$
\binom{n}{k} \leq f(n, k, k) \leq f(n, w, k) \leq\binom{ n}{k}
$$

If $n-k \leq w \leq n / 2$, perform wordwise complementation.

## 5 Some mean values

Let $C$ be a binary code of length $n$ (not necessarily having the $w$-witness property). Let

$$
\mathcal{W}_{C, w}: C \rightarrow 2^{\binom{[n]}{w}}, \quad \mathcal{W}_{C, w}(c)=\left\{W \in\binom{[n]}{w}: W \text { is a witness for } c\right\}
$$

and symmetrically,

$$
\mathcal{C}_{C, w}:\binom{[n]}{w} \rightarrow 2^{C}, \quad \mathcal{C}_{C, w}(W)=\{c \in C: W \text { is a witness for } c\} .
$$

Remark that if $C^{\prime} \subset C$ is a subcode, then $\mathcal{W}_{C^{\prime}, w}(c) \supset \mathcal{W}_{C, w}(c)$ for any $c \in C^{\prime}$, while $\mathcal{C}_{C^{\prime}, w}(W) \supset\left(C^{\prime} \cap \mathcal{C}_{C, w}(W)\right)$ for any $W \in\binom{[n]}{w}$.

Lemma 1. With these notations, the mean values of $\left|\mathcal{W}_{C, w}\right|$ and $\left|\mathcal{C}_{C, w}\right|$ are related by

$$
|C| E_{c}\left(\left|\mathcal{W}_{C, w}(c)\right|\right)=\binom{n}{w} E_{W}\left(\left|\mathcal{C}_{C, w}(W)\right|\right)
$$

or equivalently

$$
\frac{|C|}{\binom{n}{w}}=\frac{E_{W}\left(\left|\mathcal{C}_{C, w}(W)\right|\right)}{E_{c}\left(\left|\mathcal{W}_{C, w}(c)\right|\right)} .
$$

Proof. Double count the set $\left\{(W, c) \in\binom{[n]}{w} \times C: W\right.$ is a witness for $\left.c\right\}$.
Now let $\gamma(C, w)=E_{W}\left(\left|\mathcal{C}_{C, w}(W)\right|\right)$ and let $\gamma^{+}(n, w)$ be the maximum possible value of $\gamma(C, w)$ for $C$ a binary code of length $n$, and $\gamma^{++}(n, w)$ be the maximum possible value of $\gamma(C, w)$ for $C$ a binary code of length $n$ having the $w$-witness property.

Lemma 2. With these notations, one has $\gamma^{+}(n, w)=\gamma^{++}(n, w)$.
Proof. By construction $\gamma^{+}(n, w) \geq \gamma^{++}(n, w)$. On the other hand, let $C$ be a binary code of length $n$ with $\gamma(C, w)=\gamma^{+}(n, w)$, and let then $C^{\prime}$ be the subcode of $C$ formed by the $c$ having at least one witness of size $w$, i.e. $C^{\prime}=$ $\bigcup_{W \in\binom{[n]}{w}} \mathcal{C}_{C, w}(W)$. Then $C^{\prime}$ has the $w$-witness property, and

$$
\gamma^{++}(n, w) \geq \gamma\left(C^{\prime}, w\right) \geq \gamma(C, w)=\gamma^{+}(n, w)
$$

The technique of the proof of Proposition immediately adapts to give:
Proposition 3. With these notations, $w$ being fixed, $\gamma^{+}(n, w)$ is a decreasing function of $n$. That is:

$$
n \geq v \geq w \quad \Rightarrow \quad \gamma^{+}(n, w) \leq \gamma^{+}(v, w)
$$

Proof. Let $C$ be a binary code of length $n$ with $\gamma(C, w)=\gamma^{+}(n, w)$. For $V \in$ $\binom{[n]}{v}$, denote by $C_{V}$ the subset of $C$ formed by the $c$ having at least one witness of size $w$ included in $V$, i.e. $C_{V}^{\prime}=\bigcup_{W \in\binom{V}{w}} \mathcal{C}_{C, w}(W)$. Then $C_{V}^{\prime}$ has the $w$-witness property, $\mathcal{C}_{C, w}(W) \subset \mathcal{C}_{C_{V}^{\prime}, w}(W)$ for any $W \subset V$, and $\pi_{V}$ is injective on $C_{V}^{\prime}$. Using this and (2), one gets:

$$
\begin{aligned}
\gamma^{+}(n, w) & =E_{W}\left(\left|\mathcal{C}_{C, w}(W)\right|\right) \\
& =E_{V}\left(E_{W}\left(\left|\mathcal{C}_{C, w}(W)\right| \mid W \subset V\right)\right) \\
& \leq E_{V}\left(E_{W}\left(\left|\mathcal{C}_{C_{V}^{\prime}, w}^{\prime}(W)\right| \mid W \subset V\right)\right) \\
& =E_{V}\left(E_{W}\left(\left|\mathcal{C}_{\pi_{V}\left(C_{V}^{\prime}\right), w}(W)\right| \mid W \subset V\right)\right) \\
& =E_{V}\left(\gamma\left(\pi_{V}\left(C_{V}^{\prime}\right), w\right)\right) \\
& \leq \gamma^{+}(v, w)
\end{aligned}
$$

## 6 Constructions

### 6.1 A generic construction

Let $\mathcal{F} \subset\binom{[n]}{\leq w}$ be a set of subsets of $\{1, \ldots, n\}$ all having cardinality at most $w$.
Let $C_{\mathcal{F}} \subset\{0,1\}^{n}$ be the set of words having support included in one and only one $W \in \mathcal{F}$. Then:

Proposition 4. With these notations, $C_{\mathcal{F}}$ has the $w$-witness property.
Proof. For each $c \in C_{\mathcal{F}}$, let $W_{c}$ be the unique $W \in \mathcal{F}$ containing the support of $c$. Then $W_{c}$ is a witness for $c$.

Example 1. For $\mathcal{F}=\binom{[n]}{w}$ we find $C_{\mathcal{F}}=S_{w}(\mathbf{0})$, and

$$
f(n, w) \geq\left|C_{\mathcal{F}}\right|=\binom{n}{w}
$$

Example 1'. Suppose $w \geq n / 2$. Then for $\mathcal{F}=\binom{[n]}{n / 2}$ we find $C_{\mathcal{F}}=S_{n / 2}(\mathbf{0})$, and

$$
f(n, w) \geq\left|C_{\mathcal{F}}\right|=\binom{n}{n / 2}
$$

(where for ease of notation we write $n / 2$ instead of $\lfloor n / 2\rfloor$ ).
Example 2. For $\mathcal{F}=\{W\}$ with $|W| \leq w$ we find $C_{\mathcal{F}}=\{0,1\}^{W}$ (where we see $\{0,1\}^{W}$ as a subset of $\{0,1\}^{n}$ by extension by 0 on the other coordinates), and

$$
f(n, w) \geq\left|C_{\mathcal{F}}\right|=2^{w}
$$

Exemple 3. Let $\mathcal{F}$ be the set of (supports of) words of a code with constant weight $w$ and minimal distance $d$ (one can suppose $d$ even). Then for all distinct $W, W^{\prime} \in \mathcal{F}$ one has $\left|W \cap W^{\prime}\right| \leq w-d / 2$, so for all $W \in \mathcal{F}$, the code $C_{\mathcal{F}}$ contains all words of weight larger than $w-d / 2$ supported in $W$. This implies :

Corollary 4. For all $d$ one has

$$
f(n, w) \geq A(n, d, w) B(w, d / 2-1)
$$

where:

- $A(n, d, w)$ is the maximal cardinality of a code of length $n$ with minimal distance at least $d$ and constant weight $w$
$-B(w, r)=\Sigma_{1 \leq i \leq r}\binom{w}{i}$ is the cardinality of the ball of radius $r$ in $\{0,1\}^{w}$.
For $d=2$, this construction gives the sphere again. For $d=4$, this gives $f(n, w) \geq(1+w) A(n, d, w)$. We consider the following special values:
$-n=4, d=4, w=2: A(4,4,2)=2$
$-n=8, d=4, w=4: A(8,4,4)=14$
$-n=12, d=4, w=6: A(12,4,6)=132$
the last two being obtained with $\mathcal{F}$ the Steiner system $S(3,4,8)$ and $S(5,6,12)$ respectively.

The corresponding codes $C_{\mathcal{F}}$ have same cardinality as the sphere $(2 \times 3=6$, $14 \times 5=70$ and $132 \times 7=924$ respectively), but they are not translates of a sphere. Indeed, when $C$ is a (translate of a) sphere with $w=n / 2$, one has $\mathcal{C}_{C, w}(W)=2$ for any window $W \in\binom{[n]}{w}$. On the other hand, for $C=C_{\mathcal{F}}$ as above, one has by construction $\mathcal{C}_{C, w}(W)=w+1$ for $W \in \mathcal{F}$.

### 6.2 Another construction

Let $D \subset\{0,1\}^{w}$ be a binary (non-linear) code of length $w>n / 2$ and minimal weight at least $2 w-n$.

Let $C_{1}$ be the code of length $n$ obtained by taking all words of length $w$ that do not belong to $D$, and completing them with 0 on the last $n-w$ coordinates. Thus $\left|C_{1}\right|=2^{w}-|D|$.

Let $C_{2}$ be the code of length $n$ formed by the words $c$ of weight exactly $w$, and such that the projection of $c$ on the first $w$ coordinates belongs to $D$. Thus if $n_{k}$ is the number of codewords of weight $k$ in $D$, one finds $\left|C_{2}\right|=\sum_{k} n_{k}\binom{n-w}{w-k}$.

Now let $C$ be the (disjoint!) union of $C_{1}$ and $C_{2}$. Then $C$ has the $w$-witness property. Indeed, let $c \in C$. Then if $c \in C_{1}, c$ admits $[w]$ as witness, while if $c \in C_{2}, c$ admits its support as witness.

As an illustration, let $D$ be the sphere of radius $w-t$ in $\{0,1\}^{w}$, for $t \in$ $\left\{1, \ldots, \frac{n-w}{2}\right\}$. Then

$$
f(n, w) \geq|C|=2^{w}+\binom{w}{w-t}\left(\binom{n-w}{t}-1\right)
$$

If $w$ satisfies $2^{w}>\binom{n}{n / 2}$ but $w<n-1$, this improves on examples $1,1^{\prime}$, and 2 of the last subsection, in that one finds then

$$
f(n, w) \geq|C|>\max \left(\binom{n}{w},\binom{n}{n / 2}, 2^{w}\right) .
$$

On the other hand, remark that $C_{1} \subset\{0,1\}^{[w]}$ and $C_{2} \subset S_{w}(\mathbf{0})$, so that $|C| \leq 2^{w}+\binom{n}{w}$.

## 7 Conclusion and open problems

We have determined the asymptotic size of optimal $w$-witness codes. A few issues remain open in the non-asymptotic case, among which:

- When is the sphere $S_{w}(\mathbf{0})$ the/an optimal $w$-witness code? Do we have $f(n, w)=\binom{n}{w}$ for $w \leq n / 2$ ? In particular do we have $f(2 w, w)=\binom{2 w}{w}$ ?
- For $w>n / 2$, do we have $f(n, w) \leq \max \left(\binom{n}{n / 2}, 2^{w}+\binom{n}{w}\right)$ ?
- Denoting by $f(n, w, \geq d)$ the maximal size of a $w$-witness code with minimum distance $d$, can the asymptotics of Proposition 2 be improved to

$$
\frac{1}{n} \log _{2} f(n, \omega n, \geq \delta n)<h(\omega) ?
$$

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