ISOLATED BOUNDARY SINGULARITIES OF SEMILINEAR ELLIPTIC EQUATIONS

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Abstract. Given a smooth domain $\Omega \subset \mathbb{R}^N$ such that $0 \in \partial\Omega$ and given a nonnegative smooth function ζ on $\partial\Omega$, we study the behavior near 0 of positive solutions of $-\Delta u = u^q$ in Ω such that $u = \zeta$ on $\partial\Omega \setminus \{0\}$. We prove that if $\frac{N+1}{N-1} < q < \frac{N+2}{N-2}$, then $u(x) \le C\,|x|^{-\frac{2}{q-1}}$ and we compute the limit of $|x|^{\frac{2}{q-1}}\,u(x)$ as $x \to 0$. We also investigate the case $q = \frac{N+1}{N-1}$. The proofs rely on the existence and uniqueness of solutions of related equations on spherical domains.

Contents

1. Introduction and main results

Let Ω be a smooth open subset of $\mathbb{R}^N$, with $N \ge 2$, such that $0 \in \partial\Omega$. Given $q > 1$ and $\zeta \in C^\infty(\partial\Omega)$ with $\zeta \ge 0$ on $\partial\Omega$, consider the problem

$$\begin{cases} -\Delta u = u^q & \text{in } \Omega, \\ u \ge 0 & \text{in } \Omega, \\ u = \zeta & \text{on } \partial\Omega \setminus \{0\}. \end{cases} \tag{1.1}$$

By a solution of (1.1) we mean a function $u \in C^2(\Omega) \cap C(\overline{\Omega} \setminus \{0\})$ which satisfies (1.1) in the classical sense. A solution may develop an isolated singularity at 0. Our main goal in this paper is to describe the behavior of u in a neighborhood of this point.

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In the study of boundary singularities of (1.1), one finds three critical exponents; namely,

$$q_1 = \tfrac{N+1}{N-1}, \quad q_2 = \tfrac{N+2}{N-2} \quad \text{and} \quad q_3 = \tfrac{N+1}{N-3}.$$

When $1 < q < q_1$, it is proved by Bidaut-Véron–Vivier [6] that for every solution u of (1.1) there exists $\alpha \geq 0$ (depending on u) such that

$$u(x) = \alpha\, |x|^{-N} \operatorname{dist}(x, \partial\Omega)\, \big(1 + o(1)\big) \quad \text{as } x \to 0.$$

In this paper we mainly investigate the case $q_1 \leq q < q_3$.

The counterpart of (1.1) for an interior singularity,

$$-\Delta u = u^q \quad \text{in } \Omega \setminus \{x_0\},$$

where $x_0 \in \Omega$, was studied by P.-L. Lions [16] in the subcritical case $1 < q < \frac{N}{N-2}$, by Aviles [2] when $q = \frac{N}{N-2}$ and by Gidas-Spruck [11] in the range $\frac{N}{N-2} < q < \frac{N+2}{N-2}$. We prove some counterparts of the works of Gidas-Spruck and Aviles in the framework of boundary singularities.

When (1.1) is replaced by an equation with an absortion term,

$$-\Delta u + u^q = 0 \quad \text{in } \Omega, \tag{1.2}$$

the problem has been first adressed by Gmira-Véron [12] (and later to nonsmooth domains in [10]). These results are important in the theory of boundary trace of positive solutions of (1.2) which was developed by Marcus-Véron [17, 18, 19] using analytic tools and by Le Gall [15] and Dynkin-Kuznetsov [8, 9] with a probabilistic approach. We refer the reader to Véron [23] for the case of interior singularities of (1.2).

Let us first consider the case where Ω is the upper-half space $\mathbb{R}^N_+$, and we look for solutions of (1.1) of the form

$$u(x) = |x|^{-\frac{2}{q-1}} \omega\big(\tfrac{x}{|x|}\big).$$

By an easy computation, ω must satisfy

$$\begin{cases} -\Delta'\omega = \ell_{N,q}\omega + \omega^q & \text{in } S^{N-1}_+, \\ \omega \geq 0 & \text{in } S^{N-1}_+, \\ \omega = 0 & \text{on } \partial S^{N-1}_+, \end{cases} \tag{1.3}$$

where Δ' denotes the Laplace-Beltrami operator in the unit sphere S^{N-1},

$$\ell_{N,q} = \tfrac{2(N - q(N-2))}{(q-1)^2} \quad \text{and} \quad S^{N-1}_+ = S^{N-1} \cap \mathbb{R}^N_+.$$

Concerning equation (1.3), we prove

Theorem 1.1.

(*i*) *If* $1 < q \leq q_1$, *then* (1.3) *admits no positive solution.*
(*ii*) *If* $q_1 < q < q_3$, *then* (1.3) *admits a unique positive solution.*
(*iii*) *If* $q \geq q_3$, *then* (1.3) *admits no positive solution.*

In Section 3 we study uniqueness of solutions of (1.3) with $\ell_{N,q}$ replaced by any $\ell \in \mathbb{R}$. The proofs are inspired from some interesting ideas taken from Kwong [13] and Kwong-Li [14]. The nonexistence of solutions of (1.3) when $q \geq q_3$ is based on a Pohožaev identity for spherical domains; see Theorem 2.1 below.

We now consider the case where $\Omega \subset \mathbb{R}^N$ is a smooth domain such that $0 \in \partial\Omega$. Without loss of generality, we may assume that $-\mathbf{e}_N$ is the outward unit normal vector of $\partial\Omega$ at 0. We prove the following classification of isolated singularities of solutions of (1.1):

Theorem 1.2. *Assume that $q_1 < q < q_2$. If u satisfies (1.1), then either u can be continuously extended at 0 or for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $x \in \Omega \setminus \{0\}$, $\frac{x}{|x|} \in S^{N-1}_+$ and $|x| < \delta$,*

$$\left| |x|^{\frac{2}{q-1}} u(x) - \omega\left(\tfrac{x}{|x|}\right) \right| < \varepsilon, \tag{1.4}$$

where ω is the unique positive solution of (1.3).

When $q_2 < q < q_3$, we have a similar conclusion provided u satisfies the estimate

$$u(x) \le C|x|^{-\frac{2}{q-1}} \quad \forall x \in \Omega,$$

for some constant $C > 0$; see Proposition 8.1 below. In the critical case $q = q_1$ there is a superposition of the linear and nonlinear effects since their characteristic exponents $\frac{2}{q-1}$ and $N-1$ coincide. The counterpart of Theorem 1.2 in this case is the following:

Theorem 1.3. *Assume that $q = q_1$. If u satisfies (1.1), then either u can be continuously extended at 0 or for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $x \in \Omega \setminus \{0\}$ and $|x| < \delta$,*

$$\left| |x|^{N-1} \big(\log \tfrac{1}{|x|}\big)^{\frac{N-1}{2}} u(x) - \kappa \tfrac{x_N}{|x|} \right| < \varepsilon, \tag{1.5}$$

where κ is a positive constant depending only on the dimension N.

Our characterization of boundary isolated singularities is complemented by the existence of singular solutions which has been recently obtained by del Pino-Musso-Pacard [21]. We recall their result:

Theorem 1.4. *Assume that $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain. There exists $p > q_1$ such that for every $q_1 \le q < p$ and for every $\xi_1, \xi_2, \dots, \xi_k \in \partial\Omega$, there exists a solution of (1.1) with $\zeta \equiv 0$ such that*

$$u(x) \to +\infty \quad \text{as } x \to \xi_j \text{ nontangentially for every } i = 1, 2, \dots, k.$$

In view of Theorems 1.2 and 1.3 any such solution must have the singular behavior we have obtained therein. In [21], the authors conjecture that such solutions exist for every $q_1 \le q < q_2$.

Some of the main ingredients in the proofs of Theorems 1.2 and 1.3 are Theorem 1.1 above concerning existence and uniqueness of positive solutions of (1.3), a removable singularity result (see Theorems 7.1 and 7.2 below) and the following a priori bound of solutions of (1.1):

Theorem 1.5. *Assume that $1 < q < q_2$. Then, every solution of (1.1) satisfies*

$$u(x) \le C|x|^{-\frac{2}{q-1}} \quad \forall x \in \Omega, \tag{1.6}$$

for some constant $C > 0$ independent of the solution.

We establish this estimate using a topological argument, called the *Doubling lemma* (see Lemma 5.1 below), introduced by Poláčik-Quittner-Souplet [22].

Theorems 1.2, 1.3 and 1.5 have been announced in [4].

2. Pohožaev identity in spherical domains

We first prove the following Pohožaev identity in spherical domains.

Theorem 2.1. *Let $q > 1$, $\ell \in \mathbb{R}$ and S be a smooth domain in S_+^{N-1}. If $v \in C^2(S) \cap C(\overline{S})$ satisfies*

$$\begin{cases} -\Delta' v = \ell v + |v|^{q-1} v & \text{in } S, \\ v = 0 & \text{on } \partial S, \end{cases} \tag{2.1}$$

then

$$\left(\tfrac{N-3}{2} - \tfrac{N-1}{q+1}\right) \int_S |\nabla' v|^2 \, \phi \, d\sigma - \tfrac{N-1}{2} \left(\tfrac{\ell q + N - 1}{q+1}\right) \int_S v^2 \phi \, d\sigma = \tfrac{1}{2} \int_{\partial S} |\nabla' v|^2 \, \langle \nabla' \phi, \nu \rangle \, d\tau,$$

where ν is the outward unit normal vector on ∂S, ∇' the tangential gradient to S^{N-1}, and ϕ is a first eigenfunction of the Laplace-Beltrami operator $-\Delta'$ in $W_0^{1,2}(S_+^{N-1})$.

We recall that the first eigenvalue of $-\Delta'$ in $W_0^{1,2}(S_+^{N-1})$ is $N-1$ and the eigenspace associated to this eigenvalue is spanned by the function $\phi(x) = \frac{x_N}{|x|}$.

Proof. Let

$$P = \langle \nabla' \phi, \nabla' v \rangle \nabla' v.$$

By the Divergence theorem,

$$\int_S \operatorname{div} P \, d\sigma = \int_{\partial S} \langle P, \nu \rangle \, d\tau. \tag{2.2}$$

Note that

$$\operatorname{div} P = \langle \nabla' v, \nabla' \phi \rangle \Delta' v + D^2 v(\nabla' v, \nabla' \phi) + D^2 \phi(\nabla' v, \nabla' v).$$

where $D^2 v$ is the Hessian operator. Now,

$$D^2 v(\nabla' v, \nabla' \phi) = \frac{1}{2} \langle \nabla' \, |\nabla' v|^2 \, , \nabla' \phi \rangle.$$

Using the classical identity

$$D^2 \phi + g \phi = 0$$

where $g = (g_{i,j})$ is the metric tensor on S^{N-1}, we get

$$D^2 \phi(\nabla' v, \nabla' v) = -g(\nabla' v, \nabla' v) \phi = -|\nabla' v|^2 \phi.$$

We replace these identities in the expression of $\operatorname{div} P$,

$$\operatorname{div} P = -\langle \nabla' v, \nabla' \phi \rangle \left(\ell v + |v|^{q-1} v\right) + \frac{1}{2} \langle \nabla' \, |\nabla' v|^2 \, , \nabla' \phi \rangle - |\nabla' v|^2 \, \phi.$$

Integrating over S, we obtain

$$\int_S \operatorname{div} P \, d\sigma = -\int_S \langle \nabla' v, \nabla' \phi \rangle \left(\ell v + |v|^{q-1} v\right) d\sigma + \frac{1}{2} \int_S \langle \nabla' \, |\nabla' v|^2 \, , \nabla' \phi \rangle \, d\sigma - \int_S |\nabla' v|^2 \, \phi \, d\sigma.$$

Note that

$$\begin{aligned} \int_S \langle \nabla' v, \nabla' \phi \rangle \left(\ell v + |v|^{q-1} v\right) d\sigma &= \int_S \left\langle \nabla' \left(\tfrac{\ell}{2} v^2 + \tfrac{1}{q+1} |v|^{q+1} \right), \nabla' \phi \right\rangle d\sigma \\ &= -\int_S \left(\tfrac{\ell}{2} v^2 + \tfrac{1}{q+1} |v|^{q+1} \right) \Delta' \phi \, d\sigma \\ &= (N-1) \int_S \left(\tfrac{\ell}{2} v^2 + \tfrac{1}{q+1} |v|^{q+1} \right) \phi \, d\sigma, \end{aligned}$$

and

$$\int_S \langle \nabla' |\nabla' v|^2, \nabla' \phi\rangle \, d\sigma = -\int_S |\nabla' v|^2 \, \Delta' \phi \, d\sigma + \int_{\partial S} |\nabla' v|^2 \, \langle \nabla' \phi, \nu\rangle \, d\tau$$
$$= (N-1) \int_S |\nabla' v|^2 \, \phi \, d\sigma + \int_{\partial S} |\nabla' v|^2 \, \langle \nabla' \phi, \nu\rangle \, d\tau.$$

These identities imply

(2.3)
$$\int_S \operatorname{div} P \, d\sigma = -\tfrac{\ell(N-1)}{2} \int_S v^2 \phi \, d\sigma - \tfrac{N-1}{q+1} \int_S |v|^{q+1} \, \phi \, d\sigma + \tfrac{N-3}{2} \int_S |\nabla' v|^2 \, \phi \, d\sigma +$$
$$+ \tfrac{1}{2} \int_{\partial S} |\nabla' v|^2 \, \langle \nabla' \phi, \nu\rangle \, d\tau.$$

On the other hand, since v satisfies (2.1),

$$\int_S \big(\ell v^2 + |v|^{q+1}\big) \phi \, d\sigma = -\int_S (\Delta' v) v \phi \, d\sigma$$
$$= \int_S \langle \nabla' v, \nabla'(v\phi)\rangle \, d\sigma = \int_S |\nabla' v|^2 \, \phi \, d\sigma + \int_S \langle \nabla' v, \nabla' \phi\rangle v \, d\sigma.$$

Since $v \nabla' v = \frac{1}{2} \nabla'(v^2)$ and $\Delta' \phi = -(N-1)\phi$,

$$\int_S \langle \nabla' v, \nabla' \phi\rangle v \, d\sigma = \frac{1}{2} \int_S \langle \nabla'(v^2), \nabla' \phi\rangle \, d\sigma = \tfrac{N-1}{2} \int_S v^2 \phi \, d\sigma.$$

Thus,

$$\int_S \big(\ell v^2 + |v|^{q+1}\big) \phi \, d\sigma = \int_S |\nabla' v|^2 \, \phi \, d\sigma + \tfrac{N-1}{2} \int_S v^2 \phi \, d\sigma.$$

This implies

$$\int_S |v|^{q+1} \, \phi \, d\sigma = \int_S |\nabla' v|^2 \, \phi \, d\sigma + \big(\tfrac{N-1}{2} - \ell\big) \int_S v^2 \phi \, d\sigma.$$

Inserting this identity in (2.3), we obtain

(2.4)
$$\int_S \operatorname{div} P \, d\sigma = \big(\tfrac{N-3}{2} - \tfrac{N-1}{q+1}\big) \int_S |\nabla' v|^2 \, \phi \, d\sigma - \big(\tfrac{\ell(N-1)}{2} + \tfrac{N-1}{q+1} \big(\tfrac{N-1}{2} - \ell\big)\big) \int_S v^2 \phi \, d\sigma +$$
$$+ \tfrac{1}{2} \int_{\partial S} |\nabla' v|^2 \, \langle \nabla' \phi, \nu\rangle \, d\tau.$$

Since v vanishes on ∂S, $\nabla' v = \langle \nabla' v, \nu\rangle \nu$ and, in particular, $|\nabla' v| = |\langle \nabla' v, \nu\rangle|$. Thus,

(2.5)
$$\int_{\partial S} \langle P, \nu\rangle \, d\tau = \int_{\partial S} \langle \nabla' \phi, \nabla' v\rangle \langle \nabla' v, \nu\rangle \, d\tau = \int_{\partial S} \big(\langle \nabla' v, \nu\rangle\big)^2 \langle \nabla' \phi, \nu\rangle \, d\tau$$
$$= \int_{\partial S} |\nabla' v|^2 \, \langle \nabla' \phi, \nu\rangle \, d\tau.$$

Combining (2.2), (2.4) and (2.5), we get the Pohožaev identity. □

Using the Pohožaev identity on S_+^{N-1} we can prove that the Dirichlet problem (2.1) can only have trivial solutions for suitable values of q and ℓ.

Corollary 2.1. *Let $N \geq 4$. If $q \geq q_3$ and $\ell \leq -\frac{N-1}{q-1}$, then the function identically zero is the only solution in $C^2(S^{N-1}) \cap C^2(\overline{S^{N-1}})$ of the Dirichlet problem*

$$\begin{cases} -\Delta' v = \ell v + |v|^{q-1} v & \text{in } S^{N-1}_+, \\ v = 0 & \text{on } \partial S^{N-1}_+. \end{cases}$$

Proof. Let v be a solution of the Dirichlet problem. Applying the Pohožaev identity with $\phi(x) = \frac{x_N}{|x|}$, then the left-hand side of the Pohožaev identity is nonnegative, while its right-hand side is nonpositive. Thus, both sides are zero. If at least one of the inequalitites $q \geq q_3$ or $\ell \leq -\frac{N-1}{q-1}$ is strict, then we immediately deduce that $v = 0$ in S^{N-1}_+.
If $q = q_3$ and $\ell = -\frac{N-1}{q-1}$, then

$$\int\limits_{\partial S^{N-1}_+} |\nabla' v|^2 \langle \nabla' \phi, \nu \rangle \, d\tau = 0.$$

Since $\langle \nabla' \phi, \nu \rangle < 0$ on ∂S^{N-1}_+, we conclude that $\nabla' v = 0$ on ∂S^{N-1}_+. Define the function $\tilde{v} : S^{N-1} \to \mathbb{R}$ by

$$\tilde{v}(x) = \begin{cases} v(x) & \text{if } x \in S^{N-1}_+, \\ 0 & \text{otherwise.} \end{cases}$$

Then, $\tilde{v}$ satisfies (in the sense of distributions)

$$-\Delta' \tilde{v} = \ell \tilde{v} + |\tilde{v}|^{q-1} \tilde{v} \quad \text{in } S^{N-1}.$$

Since $\tilde{v}$ vanishes in an open subset of S^{N-1}, by the unique continuation principle we have $\tilde{v} = 0$ in S^{N-1} and the conclusion follows. □

3. Uniqueness of solutions of a pde in S^{N-1}_+

In this section we address the question of uniqueness of positive solutions of the Dirichlet problem

$$\begin{cases} -\Delta' v = \ell v + v^q & \text{in } S^{N-1}_+, \\ v \geq 0 & \text{in } S^{N-1}_+, \\ v = 0 & \text{on } \partial S^{N-1}_+, \end{cases} \tag{3.1}$$

where $\ell \in \mathbb{R}$. A solution of (3.1) is understood in the classical sense.

We shall prove the following results:

Theorem 3.1. *Assume that $N = 2$. If $q > 1$, then for every $\ell \in \mathbb{R}$ the Dirichlet problem* (3.1) *has at most one positive solution.*

Theorem 3.2. *Assume that $N \geq 4$. If $1 < q < q_3$, then for every $\ell \in \mathbb{R}$ the Dirichlet problem* (3.1) *has at most one positive solution.*

Theorem 3.3. *Assume that $N = 3$. Then, the Dirichlet problem* (3.1) *has at most one positive solution under one of the following assumptions:*

- *for every $1 < q \leq 5$ and $\ell \in \mathbb{R}$,*
- *for every $q > 5$ and $\ell \leq \frac{2(3-q)}{(q+3)(q-1)}$.*

Remark 3.1. In dimension $N = 3$ we do not know whether the Dirichlet problem (3.1) has a unique positive solution if $q > 5$ and $\ell > \frac{2(3-q)}{(q+3)(q-1)}$.

We first show that the graphs of two positive solutions of (3.1) must cross.

Lemma 3.1. *Assume that v_1 and v_2 are positive solutions of* (3.1). *If $v_1 \le v_2$ in S_+^{N-1}, then $v_1 = v_2$.*

Proof. Multiplying by v_2 the equation satisfied by v_1 and integrating by parts, we get

$$\int\limits_{S_+^{N-1}} \langle \nabla v_1, \nabla v_2 \rangle \, d\sigma = \int\limits_{S_+^{N-1}} \big(\ell v_1 + (v_1)^q\big) v_2 \, d\sigma.$$

Reversing the roles of v_1 and v_2, we also have

$$\int\limits_{S_+^{N-1}} \langle \nabla v_2, \nabla v_1 \rangle \, d\sigma = \int\limits_{S_+^{N-1}} \big(\ell v_2 + (v_2)^q\big) v_1 \, d\sigma.$$

Subtracting these identities, we have

$$\int\limits_{S_+^{N-1}} \big({v_1}^{q-1} - {v_2}^{q-1}\big) v_1 v_2 \, d\sigma = 0.$$

Since the integrand is nonnegative we must have ${v_1}^{q-1} - {v_2}^{q-1} = 0$ and the conclusion follows. □

We now establish the uniqueness result in the case $N = 2$.

Proof of Theorem 3.1. Denoting by

$$\theta = \arccos \tfrac{x_2}{|x|},$$

then a solution of (3.1) satisfies

$$\begin{cases} v_{\theta\theta} + \ell v + v^q = 0 \quad \text{in } \big(0, \frac{\pi}{2}\big), \\ v_\theta(0) = 0, \quad v(\frac{\pi}{2}) = 0. \end{cases}$$

Moreover, for every $\theta \in (0, \frac{\pi}{2}]$, $v_\theta(\theta) < 0$ (this can be established using for example the moving plane method). Thus, v is decreasing. Let $V : [0, v(0)] \to \mathbb{R}$ be the function defined by

$$V(\xi) = v_\theta(v^{-1}(\xi)). \tag{3.2}$$

Then, V is of class C^1 in $[0, v(0))$. Since for every $\xi \in [0, v(0))$,

$$(v^{-1})_\xi(\xi) = \frac{1}{v_\theta(v^{-1}(\xi))} = \frac{1}{V(\xi)},$$

we deduce that

$$(V^2)_\xi = 2VV_\xi = 2V(v_{\theta\theta} \circ v^{-1})(v^{-1})_\xi = 2(v_{\theta\theta} \circ v^{-1}) = -2(\ell\xi + \xi^q). \tag{3.3}$$

Assume by contradiction that (3.1) has two distinct positive solutions, say v_1 and v_2. We may assume they are both defined in terms of the variable θ. Then, there exists $c_1 \in (0, \frac{\pi}{2})$ such that $v_1(c_1) = v_2(c_1)$. Let $c_2 \in (c_1, \frac{\pi}{2}]$ be the smallest number

such that $v_1(c_2) = v_2(c_2)$ (this point c_2 exists since $v_{1\theta}(c_1) \neq v_{2\theta}(c_1)$). Without loss of generality, we may assume that, for every $\theta \in (c_1, c_2)$,

$$v_1(\xi) < v_2(\xi).$$

Let V_1 and V_2 be the functions given by (3.2) corresponding to v_1 and v_2, respectively. For $i \in \{1, 2\}$, let

$$\alpha_i = v_1(c_i) = v_2(c_i).$$

By (3.3), for every $\xi \in (\alpha_2, \alpha_1)$,

$$({V_1}^2)_\xi(\xi) = -2(\ell\xi + \xi^q) = ({V_2}^2)_\xi(\xi).$$

Hence, the function ${V_1}^2 - {V_2}^2$ is constant. On the other hand, since $v_1 < v_2$ and v_1, v_2 are both decreasing, by uniqueness of the Cauchy problem,

$$v_{1\theta}(c_1) < v_{2\theta}(c_1) < 0 \quad \text{and} \quad v_{2\theta}(c_2) < v_{1\theta}(c_2) < 0.$$

Thus,

$${V_1}^2(\alpha_1) - {V_2}^2(\alpha_1) > 0 \quad \text{and} \quad {V_1}^2(\alpha_2) - {V_2}^2(\alpha_2) < 0.$$

This is a contradiction. We conclude that problem (3.1) cannot have more than one positive solution. □

In order to study (3.1) in the case of higher dimensions, the first step is to rewrite the Dirichlet problem in terms of an ODE. By an adaptation of the moving planes method to S^{N-1} (see [20]), any positive solution v of (3.1) depends only on the geodesic distance to the North pole:

$$\theta = \arccos \tfrac{x_N}{|x|}$$

and v decreasing with respect to θ. Since in this case

$$\Delta' v = \frac{1}{(\sin\theta)^{N-2}} \frac{d}{d\theta} \left((\sin\theta)^{N-2} v_\theta \right),$$

every solution of (3.1) satisfies the following ODE in terms of the variable θ:

$$\begin{cases} v_{\theta\theta} + (N-2)\cot\theta\, v_\theta + \ell v + v^q = 0 \quad \text{in } \left(0, \tfrac{\pi}{2}\right), \\ v_\theta(0) = 0, \quad v(\tfrac{\pi}{2}) = 0. \end{cases} \tag{3.4}$$

The heart of the matter is then to apply some ideas from Kwong [13] and Kwong-Li [14], originally dealing with positive solutions of

$$\begin{cases} u_{rr} + (N-2)\dfrac{1}{r}\, u_r + \ell u + u^q = 0 \quad \text{in } (0, a), \\ u_r(0) = 0, \quad u(a) = 0. \end{cases} \tag{3.5}$$

By Lemma 3.1 and the discussion above, the graphs of two positive solutions of (3.4) must intersect in $(0, \frac{\pi}{2})$. Of course, the number of intersection points could be arbitrarily large (but always finite in view of the uniqueness of the Cauchy problem). The next lemma allows us to reduce the problem to the case where there could be only one intersection point. The argument is standard and relies on the shooting method; we only give a sketch of the proof.

Lemma 3.2. *Assume that* (3.4) *has two distinct positive solutions. Then, there exist two positive solutions of* (3.4) *which intersect only once in the interval* $(0, \frac{\pi}{2})$.

Sketch of the proof. Let v_1 and v_2 be two distinct solutions of (3.4). We may assume that $v_1(0) > v_2(0)$. For each $\alpha \in [0, v_2(0)]$ let v^α be the (unique) maximal positive solution of

$$\begin{cases} v_{\theta\theta} + (N-2)\cot\theta\, v_\theta + \ell v + v^q = 0 \quad \text{in } \big(0, \frac{\pi}{2}\big), \\ v_\theta(0) = 0, \quad v(0) = \alpha. \end{cases}$$

Note that $v^\alpha = v_2$ and $v^0 = 0$. As we decrease α from $v_2(0)$ to 0,

- the first intersection point σ_1^α between v_1 and v^α cannot tend to 0 since the derivatives of v^α in the interval $[0, \sigma_1^\alpha]$ are uniformly bounded;
- two consecutive intersection points between v^α and v_1 cannot get arbitrarily close for otherwise there would exist $\gamma \in [0, v_2(0)]$ and $\sigma \in [0, \frac{\pi}{2}]$ such that $v^\gamma(\sigma) = v_1(\sigma)$ and $v^\gamma{}_\theta(\sigma) = v_{1\theta}(\sigma)$, but this is impossible.

Combining these facts, we can make the number of intersection points between v_1 and v^α decrease one by one, until there is left only one intersection point. For the first α where this happens, we must have $v^\alpha(\frac{\pi}{2}) = 0$. □

The next result is standard but we present a proof for the convenience of the reader.

Lemma 3.3. *Assume that v_1 and v_2 are positive solutions of (3.4) whose graphs coincide at a single point of $(0, \frac{\pi}{2})$. If $v_1(0) > v_2(0)$, then the function*

$$\theta \in (0, \tfrac{\pi}{2}) \longmapsto \frac{v_2(\theta)}{v_1(\theta)}$$

is increasing.

Proof. Let $J : [0, \frac{\pi}{2}] \to \mathbb{R}$ be the function defined as $J = v_1 v_{2\theta} - v_2 v_{1\theta}$. To prove the lemma, it suffices to show that $J > 0$ in $(0, \frac{\pi}{2})$. Using the equations satisfied by v_1 and v_2, one finds

$$J_\theta = -(N-2)\cot\theta J + \big({v_1}^{q-1} - {v_2}^{q-1}\big) v_1 v_2.$$

Thus,

$$\frac{1}{(\sin\theta)^{N-2}}\big((\sin\theta)^{N-2} J\big)_\theta = \big({v_1}^{q-1} - {v_2}^{q-1}\big) v_1 v_2.$$

Let $\sigma \in (0, \frac{\pi}{2})$ be such that $v_1(\sigma) = v_2(\sigma)$. Since $v_{1\theta}(\sigma) \neq v_{2\theta}(\sigma)$, we have $v_1 > v_2$ in $(0, \sigma)$ and $v_1 < v_2$ in $(\sigma, \frac{\pi}{2})$, we conclude that the function

$$\theta \in [0, \tfrac{\pi}{2}] \longmapsto (\sin\theta)^{N-2} J(\theta)$$

is increasing in $(0, \sigma)$ and decreasing in $(\sigma, \frac{\pi}{2})$. Since it vanishes at 0 and $\frac{\pi}{2}$, we have

$$(\sin\theta)^{N-2} J > 0 \quad \text{in } (0, \tfrac{\pi}{2}).$$

Thus $J > 0$ in $(0, \frac{\pi}{2})$ and the conclusion follows. □

The following identity will be needed in the proofs of Theorems 3.2 and 3.3.

Lemma 3.4. *Let v be a solution of (3.4), $\alpha = \frac{2(N-2)}{q+3}$ and $\beta = \frac{2(N-2)(q-1)}{q+3}$. Set*

$$w(\theta) = (\sin\theta)^\alpha\, v(\theta) \tag{3.6}$$

Then,

$$\frac{d}{d\theta}\left((\sin\theta)^\beta \frac{(w_\theta)^2}{2} + G(\theta)\frac{w^2}{2} + \frac{w^{q+1}}{q+1}\right) = G_\theta(\theta)\frac{w^2}{2}, \tag{3.7}$$

where $G : (0, \frac{\pi}{2}) \to \mathbb{R}$ *is defined by*

$$G(\theta) = \Big(\big(\alpha(N-2-\alpha)+\ell\big)(\sin\theta)^2 + \alpha(\alpha+3-N)\Big)(\sin\theta)^{\beta-2}. \tag{3.8}$$

Proof. Let $w : (0, \frac{\pi}{2}) \to \mathbb{R}$ be the function defined by (3.6). Then,

$$w_{\theta\theta} + (N-2-2\alpha)\cot\theta\, w_\theta + \left(\alpha(N-2-\alpha)+\ell + \frac{\alpha(\alpha+3-N)}{(\sin\theta)^2}\right) w + \frac{w^q}{(\sin\theta)^{\alpha(q-1)}} = 0.$$

Multiplying this identity by $(\sin\theta)^\beta$, we get

$$(\sin\theta)^\beta\, w_{\theta\theta} + (N-2-2\alpha)(\sin\theta)^{\beta-1}\cos\theta\, w_\theta + G(\theta)\, w + (\sin\theta)^{\beta-\alpha(q-1)} w^q = 0$$

where G is defined by (3.8). We now observe that α and β satisfy

$$N-2-2\alpha = \frac{\beta}{2} \quad \text{and} \quad \beta - \alpha(q-1) = 0.$$

The identity satisfied by w becomes

$$(\sin\theta)^\beta\, w_{\theta\theta} + \frac{\beta}{2}(\sin\theta)^{\beta-1}\cos\theta\, w_\theta + G(\theta)\, w + w^q = 0.$$

Since

$$\frac{d}{d\theta}\left((\sin\theta)^\beta \frac{(w_\theta)^2}{2}\right) = \left((\sin\theta)^\beta\, w_{\theta\theta} + \frac{\beta}{2}(\sin\theta)^{\beta-1}\cos\theta\, w_\theta\right) w_\theta$$

and

$$\frac{d}{d\theta}\left(G(\theta)\frac{w^2}{2}\right) = G(\theta) w w_\theta + G_\theta(\theta)\frac{w^2}{2}$$

identity (3.7) follows. □

The following proof is inspired from Kwong-Li [14].

Proof of Theorem 3.2. We use the notation of Lemma 3.4. Let $E : (0, \frac{\pi}{2}) \to \mathbb{R}$ be the function defined as

$$E(\theta) = (\sin\theta)^\beta \frac{(w_\theta)^2}{2} + G(\theta)\frac{w^2}{2} + \frac{w^{q+1}}{q+1}.$$

Then, by Lemma 3.4,

$$E_\theta = G_\theta w^2.$$

We observe that E can be continuously extended at 0 and $\frac{\pi}{2}$. This is clear at $\frac{\pi}{2}$, where we take

$$E(\tfrac{\pi}{2}) = \frac{(w_\theta(\frac{\pi}{2}))^2}{2} = \frac{(v_\theta(\frac{\pi}{2}))^2}{2}.$$

To reach the conclusion at 0, it suffices to observe that for every $\theta \in (0, \frac{\pi}{2})$,

$$\begin{aligned}(\sin\theta)^\beta (w_\theta(\theta))^2 &= (\sin\theta)^\beta \Big(\alpha(\sin\theta)^{\alpha-1}\cos\theta\, v(\theta) + (\sin\theta)^\alpha\, v_\theta(\theta)\Big)^2 \\ &= (\sin\theta)^{2\alpha+\beta-2}\Big(\alpha\cos\theta\, v(\theta) + \sin\theta\, v_\theta(\theta)\Big)^2.\end{aligned}$$

Since $N \geq 4$,

$$2\alpha+\beta-2 = \tfrac{2(N-3)}{q+3}\big(q + \tfrac{N-5}{N-3}\big) > 0,$$

the right-hand side of the previous expression converges to 0 as $\theta \to 0$. We can then define

$$E(0) = 0.$$

Notice that

$$G_\theta(\theta) = \Big[\big(\alpha(N-2-\alpha)+\ell\big)\beta(\sin\theta)^2 + \alpha(\alpha+3-N)(\beta-2)\Big](\sin\theta)^{\beta-3}\cos\theta.$$

By the choices of α and β,

$$\alpha(\alpha+3-N)(\beta-2) = \tfrac{4(N-2)(N-3)^2}{(q+3)^3}\big(q+\tfrac{N-5}{N-3}\big)\big(\tfrac{N+1}{N-3}-q\big).$$

Since $N \geq 4$ and $1 < q < \frac{N+1}{N-3}$, this quantity is positive. Hence, there exists $\varepsilon > 0$ such that

$$G_\theta(\theta) > 0 \quad \forall \theta \in (0, \varepsilon).$$

In view of the expression of G_θ, we have the following possibilities: either

(i) $G_\theta > 0$ in $(0, \frac{\pi}{2})$,

or

(ii) there exists $c \in (0, \frac{\pi}{2})$ such that $G_\theta > 0$ in $(0, c)$ and $G_\theta < 0$ in $(c, \frac{\pi}{2})$.

Assume by contradiction that (3.1) has more than one solution, hence by Lemma 3.2 problem (3.4) has two positive solutions v_1 and v_2 whose graphs intersect exactly once in the interval $(0, \frac{\pi}{2})$. Without loss of generality, we may assume that $v_1(0) > v_2(0)$. For $i \in \{1, 2\}$, define w_i and E_i accordingly.

Assume that G satisfies property (i) above. Let

$$\gamma = \frac{v_{2\theta}(\frac{\pi}{2})}{v_{1\theta}(\frac{\pi}{2})}.$$

We have

$$(E_2 - \gamma^2 E_1)(0) = 0$$

and, since $v_2 > v_1$ in a neighborhood of $\frac{\pi}{2}$,

$$(E_2 - \gamma^2 E_1)(0) = \frac{(v_{2\theta}(\frac{\pi}{2}))^2 - \gamma^2 (v_{1\theta}(\frac{\pi}{2}))^2}{2} > 0.$$

Thus, by the Mean value theorem, there exists $\sigma \in (0, \frac{\pi}{2})$ such that

$$(E_2 - \gamma^2 E_1)_\theta(\sigma) > 0. \tag{3.9}$$

On the other hand, by Lemma 3.3 the function

$$\theta \in (0, \tfrac{\pi}{2}) \longmapsto \frac{v_2(\theta)}{v_1(\theta)}$$

is increasing. In particular, for every $\theta \in [0, \frac{\pi}{2})$,

$$\frac{v_2(\theta)}{v_1(\theta)} < \lim_{\theta \to \frac{\pi}{2}-} \frac{v_2(\theta)}{v_1(\theta)} = \frac{v_{2\theta}(\frac{\pi}{2})}{v_{1\theta}(\frac{\pi}{2})} = \gamma.$$

Hence,

$$(w_2)^2 - \gamma^2 (w_1)^2 = (\sin\theta)^{2\alpha}\big((v_2)^2 - \gamma^2 (v_1)^2\big) < 0 \quad \text{in } \big(0, \tfrac{\pi}{2}\big).$$

Thus, by Lemma 3.4 and by assumption (i), we have for every $\theta \in (0, \frac{\pi}{2})$,

$$(E_2 - \gamma^2 E_1)_\theta(\theta) = G_\theta(\theta)\big((w_2)^2 - \gamma^2 (w_1)^2\big) < 0.$$

This contradicts (3.9). Therefore, problem (3.1) cannot have two distinct positive solutions if G satisfies (i).

We now assume that G satisfies property (ii) for some point c. Let

$$\tilde{\gamma} = \frac{v_2(c)}{v_1(c)}.$$

As in the previous case,

$$(E_2 - \tilde{\gamma}^2 E_1)(0) = 0 \quad \text{and} \quad (E_2 - \tilde{\gamma}^2 E_1)(\tfrac{\pi}{2}) > 0.$$

Thus, by the Mean value theorem, there exists $\tilde{\sigma} \in (0, \frac{\pi}{2})$ such that

$$(E_2 - \tilde{\gamma}^2 E_1)_\theta(\tilde{\sigma}) > 0. \tag{3.10}$$

By Lemma 3.3, we have

$$\frac{v_2}{v_1} < \tilde{\gamma} \quad \text{in } (0, c) \quad \text{and} \quad \frac{v_2}{v_1} > \tilde{\gamma} \quad \text{in } (c, \tfrac{\pi}{2}).$$

Thus, by Lemma 3.4 and by assumption (ii), we have for every $\theta \in (0, \frac{\pi}{2})$,

$$(E_2 - \tilde{\gamma}^2 E_1)_\theta(\theta) = G_\theta(\theta)\big((w_2)^2 - \tilde{\gamma}^2 (w_1)^2\big) < 0.$$

This contradicts (3.10). Therefore, if G satisfies (ii), then problem (3.1) has a unique positive solution. The proof of Theorem 3.2 is complete. □

When $N = 2$, the proof of uniqueness of positive solutions of (3.1) is inspired from Kwong-Li [14] (Case 1 below) and Kwong [13] (Case 2 below).

Proof of Theorem 3.3. We split the proof in two cases:
Case 1. $q > 1$ and $\ell \leq \frac{2(3-q)}{(q+3)(q-1)}$.

Let $G : (0, \frac{\pi}{2}) \to \mathbb{R}$ be the function defined by (3.8). Since $N = 3$, we have $\alpha = \frac{2}{q+3}$ and $\beta = \frac{2(q-1)}{q+3}$. Thus,

$$\alpha(\alpha + 3 - N)(\beta - 2) = \alpha^2(\beta - 2) = -\tfrac{32}{(q+3)^3} < 0.$$

Moreover, since by assumption $\ell \leq \frac{2(3-q)}{(q+3)(q-1)}$, we have

$$\big(\alpha(N - 2 - \alpha) + \ell\big)\beta + \alpha(\alpha + 3 - N)(\beta - 2) = \tfrac{2(q-1)}{q+3}\Big[\tfrac{2(q-3)}{(q+3)(q-1)} + \ell\Big] \leq 0.$$

Therefore, G satisfies

(iii) $G_\theta < 0$ in $(0, \frac{\pi}{2})$.

Let $E : (0, \frac{\pi}{2}) \to \mathbb{R}$ be the function defined as

$$E(\theta) = (\sin\theta)^\beta \frac{(w_\theta)^2}{2} + G(\theta)\frac{w^2}{2} + \frac{w^{q+1}}{q+1}.$$

Then, by Lemma 3.4,

$$E_\theta = G_\theta w^2.$$

We observe that E can be continuously extended at $\frac{\pi}{2}$, so that

$$E(\tfrac{\pi}{2}) = \frac{(v_\theta(\frac{\pi}{2}))^2}{2},$$

but not at 0 since $E(\theta)$ diverges to $+\infty$ as $\theta \to 0$.

Assume by contradiction that (3.1) has more than one solution, hence by Lemma 3.2 problem (3.4) has two positive solutions v_1 and v_2 whose graphs intersect exactly once in the interval $(0, \frac{\pi}{2})$. Without loss of generality, we may assume that $v_1(0) > v_2(0)$. For $i \in \{1, 2\}$, define w_i and E_i accordingly.

Let

$$\hat{\gamma} = \frac{v_2(0)}{v_1(0)}.$$

By Lemma 3.3 the function

$$\theta \in (0, \tfrac{\pi}{2}) \longmapsto \frac{v_2(\theta)}{v_1(\theta)}$$

is increasing. Thus,

$$(w_2)^2 - \hat{\gamma}^2 (w_1)^2 = (\sin\theta)^{2\alpha}\big((v_2)^2 - \hat{\gamma}^2 (v_1)^2\big) > 0 \quad \text{in } (0, \tfrac{\pi}{2}).$$

By Lemma 3.4 and by assumption (iii), we have for every $\theta \in (0, \frac{\pi}{2})$,

$$(E_2 - \hat{\gamma}^2 E_1)_\theta(\theta) = G_\theta(\theta)\big((w_2)^2 - \hat{\gamma}^2 (w_1)^2\big) < 0. \tag{3.11}$$

Since $v_2 > v_1$ in a neighborhood of $\frac{\pi}{2}$,

$$(E_2 - \hat{\gamma}^2 E_1)(\tfrac{\pi}{2}) = \frac{(v_{2\theta}(\frac{\pi}{2}))^2 - \hat{\gamma}^2 (v_{1\theta}(\frac{\pi}{2}))^2}{2} > 0.$$

Although E_1 and E_2 cannot be continuously extended at 0, one checks that

$$\lim_{\theta \to 0} \big(E_2(\theta) - \hat{\gamma}^2 E_1(\theta)\big) = 0.$$

Thus, the function $E_2 - \hat{\gamma}^2 E_1$ can be continuously extended at 0. By the Mean value theorem, there exists $\hat{\sigma} \in (0, \frac{\pi}{2})$ such that

$$(E_2 - \hat{\gamma}^2 E_1)_\theta(\hat{\sigma}) > 0.$$

This contradicts (3.11). Therefore, equation (3.1) has at most one positive solution if $q > 1$ and $\ell \leq \frac{2(3-q)}{(q+3)(q-1)}$.

Case 2. $1 < q \leq 5$ and $\ell > \frac{2(3-q)}{(q+3)(q-1)}$.

Since $1 < q \leq 5$, we have

$$\tfrac{2(3-q)}{(q+3)(q-1)} + \tfrac{1}{8} = \tfrac{(5-q)(9-q)}{(q+3)(q-1)} \geq 0.$$

In particular, under the assumptions on q and ℓ,

$$\ell \geq -\frac{1}{4}.$$

(The remaining of the argument only requires $1 < q \leq 5$ and $\ell \geq -\frac{1}{4}$.)
Let $z : (0, \frac{\pi}{2}) \to \mathbb{R}$ be the function defined as

$$z(\theta) = (\sin\theta)^{\frac{1}{2}}\, v(\theta).$$

Then, z satisfies

$$z_{\theta\theta} + \left(\ell + \frac{1}{4} + \frac{1}{4(\sin\theta)^2}\right) z + \frac{z^q}{(\sin\theta)^{\frac{q-1}{2}}} = 0. \tag{3.12}$$

Assume by contradiction that equation (3.1) has two positive distinct solutions. Denote by v_1 and v_2 two solutions of (3.4) and define z_1 and z_2 accordingly. By Lemma 3.1, the graphs of v_1 and v_2 intersect in $(0, \frac{\pi}{2})$. Let $\sigma_1 < \sigma_2$ be two consecutive points where the graphs of z_1 and z_2 intersect. Without loss of generality, we may assume that

$$z_1 < z_2 \quad \text{in } (\sigma_1, \sigma_2).$$

Let

$$\xi_1 = z_1(\sigma_1) = z_2(\sigma_1) \quad \text{and} \quad \xi_2 = z_1(\sigma_2) = z_2(\sigma_2).$$

We first show that z_1 and z_2 cannot be both decreasing in $[\sigma_1, \sigma_2]$. Assume by contradiction that z_1 and z_2 are decreasing in $[\sigma_1, \sigma_2]$. In particular, $\sigma_1 > 0$. We may consider their inverses $z_i^{-1} : [\xi_2, \xi_1] \to [\sigma_1, \sigma_2]$. For $i \in \{1, 2\}$, let $Z_i : [\xi_2, \xi_1] \to \mathbb{R}$ be the function given by

$$Z_i(\xi) = z_{i\theta}(z_i^{-1}(\xi))$$

(Z_i is well-defined since $\sigma_1 > 0$). Since

$$z_{1\theta}(\sigma_1) < z_{2\theta}(\sigma_1) < 0 \quad \text{and} \quad z_{2\theta}(\sigma_2) < z_{1\theta}(\sigma_2) < 0,$$

we have

$$(Z_1(\xi_1))^2 > (Z_2(\xi_1))^2 \quad \text{and} \quad (Z_1(\xi_2))^2 < (Z_2(\xi_2))^2.$$

From the Mean value theorem, there exists $\eta \in (\xi_2, \xi_1)$ such that

$$({Z_1}^2)_\xi(\eta) > ({Z_2}^2)_\xi(\eta). \tag{3.13}$$

On the other hand, for $i \in \{1, 2\}$ and for every $\xi \in (\xi_2, \xi_1)$,

$$Z_i Z_{i\xi} = z_{i\theta\theta}(z_i^{-1}(\xi)) = -\left(\ell + \frac{1}{4} + \frac{1}{4(\sin z_i^{-1}(\xi))^2}\right)\xi - \frac{\xi^q}{(\sin z_i^{-1}(\xi))^{\frac{q-1}{2}}}.$$

Since $z_1^{-1} < z_2^{-1}$ in (ξ_2, ξ_1), we deduce that

$$({Z_1}^2)_\xi = 2Z_1 Z_{1\xi} < 2Z_2 Z_{2\xi} = ({Z_2}^2)_\xi.$$

This contradicts (3.13). Therefore, z_1 and z_2 cannot be both decreasing in (σ_1, σ_2).

We now show that z_1 and z_2 cannot be both increasing in (σ_1, σ_2). Assume by contradiction that z_1 and z_2 are increasing in (σ_1, σ_2). In particular, $\sigma_2 < \frac{\pi}{2}$. We may consider their inverses $z_i^{-1} : [\xi_1, \xi_2] \to [\sigma_1, \sigma_2]$. For $i \in \{1, 2\}$, let $Y_i : [\xi_2, \xi_1] \to \mathbb{R}$ be the function defined as

$$Y_i(\xi) = z_{i\theta}(z_i^{-1}(\xi))(\sin z_i^{-1}(\xi)).$$

If $\xi_1 = \sigma_1 = 0$, this formula is meaningless (since z_i is not differentiable at 0) but Y_i can be continuously extended to 0 by taking $Y_i(0) = 0$. Since $z_1 < z_2$ in (σ_1, σ_2), we have

$$(Y_1(\xi_1))^2 \le (Y_2(\xi_1))^2 \quad \text{and} \quad (Y_1(\xi_2))^2 > (Y_2(\xi_2))^2 \tag{3.14}$$

(equality may happen at ξ_1 if $\xi_1 = 0$). On the other hand, for $i \in \{1, 2\}$,

$$\begin{aligned} Y_{i\xi} &= \Big(z_{i\theta\theta}(z_i^{-1}(\xi))(\sin z_i^{-1}(\xi)) + z_{i\theta}(z_i^{-1}(\xi))(\cos z_i^{-1}(\xi))\Big)\frac{1}{z_{i\theta}(z_i^{-1}(\xi))} \\ &= \Big(z_{i\theta\theta}(z_i^{-1}(\xi))(\sin z_i^{-1}(\xi))^2\Big)\frac{1}{Y_i} + \cos z_i^{-1}(\xi). \end{aligned}$$

Thus,

$$Y_i Y_{i\xi} - Y_i \cos z_i^{-1}(\xi) = -\Big((\ell + \tfrac{1}{4})(\sin z_i^{-1}(\xi))^2 + \tfrac{1}{4}\Big)\xi - (\sin z_i^{-1}(\xi))^{\frac{5-q}{2}}\xi^q.$$

Since $z_1^{-1} > z_2^{-1}$ in (ξ_1, ξ_2) and $\ell \ge -\frac{1}{4}$,

$$(\ell + \tfrac{1}{4})(\sin z_1^{-1}(\xi))^2 \ge (\ell + \tfrac{1}{4})(\sin z_2^{-1}(\xi))^2.$$

Since $q \le 5$,

$$(\sin z_1^{-1}(\xi))^{\frac{5-q}{2}} \ge (\sin z_2^{-1}(\xi))^{\frac{5-q}{2}}.$$

We deduce that

$$Y_1 Y_{1\xi} - Y_1 \cos z_1^{-1}(\xi) \le Y_2 Y_{2\xi} - Y_2 \cos z_2^{-1}(\xi).$$

Hence,

$$\begin{aligned}\big((Y_1)^2-(Y_2)^2\big)_\xi &\le 2(Y_1\cos z_1^{-1}(\xi)-Y_2\cos z_2^{-1}(\xi))\\ &\le \frac{2\cos z_2^{-1}(\xi)}{2}(Y_1-Y_2)\\ &\le \frac{2\cos z_2^{-1}(\xi)}{Y_1+Y_2}\big((Y_1)^2-(Y_2)^2\big).\end{aligned}$$

Let $f:(\xi_1,\xi_2)\to\mathbb{R}$ be the function defined by

$$f(\xi)=\frac{2\cos z_2^{-1}(\xi)}{Y_1(\xi)+Y_2(\xi)}.$$

Using this notation,

$$\big((Y_1)^2-(Y_2)^2\big)_\xi \le f(\xi)\,\big((Y_1)^2-(Y_2)^2\big).$$

Thus, for every $\xi\in[\xi_1,\xi_2]$,

$$\big((Y_1)^2-(Y_2)^2\big)(\xi)\ge\big((Y_1)^2-(Y_2)^2\big)(\xi_2)\,\mathrm{e}^{\int_\xi^{\xi_2} f(\tau)\,d\tau}.$$

Since, by (3.14),

$$\big((Y_1)^2-(Y_2)^2\big)(\xi_1)\le 0\quad\text{and}\quad\big((Y_1)^2-(Y_2)^2\big)(\xi_2)>0,$$

we have reached a contradiction. Therefore, z_1 and z_2 cannot be both increasing in (σ_1,σ_2).

It follows from equation (3.12) that both z_1 and z_2 are concave. By Lemma 3.1 their graphs must intersect in $(0,\frac{\pi}{2})$. Since z_1 and z_2 cannot be simultaneously increasing or decreasing between two intersection points, their graphs intersect exactly once in $(0,\frac{\pi}{2})$. Let $\sigma\in(0,\frac{\pi}{2})$ be such that $z_1(\sigma)=z_2(\sigma)$. Without loss of generality, we may assume that

$$z_1>z_2\quad\text{in }(0,\sigma)\quad\text{and}\quad z_1<z_2\quad\text{in }(\sigma,\tfrac{\pi}{2}).$$

In particular,

$$z_{1\theta}(\sigma)<z_{2\theta}(\sigma).$$

Since z_1 and z_2 cannot be both increasing in $(0,\sigma)$ and both are concave, we must have $z_{1\theta}(\sigma)<0$. A similar argument using the interval $(\sigma,\frac{\pi}{2})$ implies that $z_{2\theta}(\sigma)>0$. Therefore, the maximum of z_1 is achieved in $(0,\sigma)$ while the maximum of z_2 is achieved in $(\sigma,\frac{\pi}{2})$.

Denote the maximum of z_i by m_i. We first show that $m_2>m_1$. Indeed, assume by contradiction that $m_2\le m_1$. Let $\tilde\sigma_2\in(\sigma,\frac{\pi}{2})$ be such that

$$z_2(\tilde\sigma_2)=m_2.$$

Let $\tilde\sigma_1$ be the largest number in $(0,\frac{\pi}{2})$ such that

$$z_1(\tilde\sigma_1)=m_2.$$

The restrictions $z_i:[\tilde\sigma_i,\frac{\pi}{2}]\to[0,m_2]$ are both decreasing. Let $\tilde Z_i:[0,m_2]\to\mathbb{R}$ be the function defined as

$$\tilde Z_i(\xi)=z_{i\theta}(z_i^{-1}(\xi)).$$

In the interval $[\tilde\sigma_i,\frac{\pi}{2}]$ we have $z_1^{-1}<z_2^{-1}$, thus

$$(\tilde Z_1^2)_\xi<(\tilde Z_2^2)_\xi.$$

Since

$$(\tilde{Z}_1(0))^2 < (\tilde{Z}_2(0))^2 \quad \text{and} \quad (\tilde{Z}_1(m_2))^2 > 0 = (\tilde{Z}_2(m_2))^2,$$

we have a contradiction.

We now show that $m_1 > m_2$. Assume by contradiction that $m_1 \leq m_2$. Let $\hat{\sigma}_1 \in (0, \sigma)$ be such that

$$z_1(\hat{\sigma}_1) = m_1.$$

Let $\hat{\sigma}_2$ be the smallest number in $(0, \frac{\pi}{2})$ such that

$$z_2(\hat{\sigma}_1) = m_1.$$

The restrictions $z_i : [0, \hat{\sigma}_i] \to [0, m_1]$ are both increasing. Let $\hat{Y}_i : [0, m_1] \to \mathbb{R}$ be the function defined as

$$\hat{Y}_i(\xi) = z_{i\theta}(z_i^{-1}(\xi))(\sin z_i^{-1}(\xi))$$

if $\xi \neq 0$ and $\hat{Y}_i(0) = 0$. Then, $\hat{Y}_i$ is continuous. In the interval $[0, \hat{\sigma}_i]$ we have $z_1^{-1} < z_2^{-1}$, thus

$$\big((\hat{Y}_2)^2 - (\hat{Y}_1)^2\big)_\xi \leq \frac{2\cos z_1^{-1}(\xi)}{\hat{Y}_2 + \hat{Y}_1}\big((\hat{Y}_2)^2 - (\hat{Y}_1)^2\big).$$

As before, this contradicts

$$\big((\hat{Y}_2)^2 - (\hat{Y}_1)^2\big)(0) \leq 0 \quad \text{and} \quad \big((\hat{Y}_2)^2 - (\hat{Y}_1)^2\big)(m_1) > 0.$$

Since $m_1 > m_2$ and $m_1 < m_2$ we have reached a contradiction. Therefore, problem (3.1) can have at most one positive solution. □

4. Proof of Theorem 1.1

Proof of (i). Assume that $1 < q \leq q_1$. Let ϕ be a positive eigenfunction of $-\Delta'$ in $W_0^{1,2}(S_+^{N-1})$ associated to the first eigenvalue $N-1$, and let ω be a solution of (1.3). Using ϕ as test function, we get

$$\int_{S_+^{N-1}} \langle \nabla'\omega, \nabla'\phi\rangle \, d\sigma = \int_{S_+^{N-1}} (\ell_{N,q}\omega + \omega^q)\phi \, d\sigma.$$

On the other hand, since ϕ is an eigenfunction of $-\Delta'$,

$$\int_{S_+^{N-1}} \langle \nabla'\omega, \nabla'\phi\rangle \, d\sigma = (N-1)\int_{S_+^{N-1}} \omega\phi \, d\sigma.$$

Thus,

$$(N - 1 - \ell_{N,q})\int_{S_+^{N-1}} \omega\phi \, d\sigma = \int_{S_+^{N-1}} \omega^q\phi \, d\sigma. \tag{4.1}$$

Since $q \leq q_1$, we have

$$N - 1 - \ell_{N,q} = \tfrac{(N-1)(q+1)}{(q-1)^2}\big(q - \tfrac{N+1}{N-1}\big) \leq 0.$$

Hence, the left-hand side of (4.1) is nonpositive while the right-hand side is nonnegative. Thus,

$$\int_{S_+^{N-1}} \omega^q\phi \, d\sigma = 0$$

We conclude that $\omega = 0$ in S_+^{N-1}. Hence, problem (1.3) has no positive solution. □

Proof of (ii). Since $q > q_1$,

$$N - 1 - \ell_{N,q} = \tfrac{(N-1)(q+1)}{(q-1)^2}\big(q - \tfrac{N+1}{N-1}\big) > 0.$$

Thus, the functional $\mathcal{J} : W_0^{1,2}(S_+^{N-1}) \to \mathbb{R}$ defined by

$$\mathcal{J}(w) = \int\limits_{S_+^{N-1}} \big(|\nabla' w|^2 - \ell_{N,q} w^2\big)\, d\sigma$$

is bounded from below by 0. On the other hand, since $q < q_3$ we can minimize $\mathcal{J}$ over the set

$$\Big\{ w \in W_0^{1,2}(S_+^{N-1}) \ ; \int\limits_{S_+^{N-1}} (w^+)^{q+1}\, d\sigma = 1 \Big\}.$$

Let w be a minimizer. Then, w^+ is also a minimizer, whence $w = w^+$ and this function satisfies

$$-\Delta' w - \ell_{N,q} w = \lambda w^q \quad \text{in } S_+^{N-1}$$

for some $\lambda > 0$. By standard elliptic regularity theory, w is smooth and vanishes on ∂S_+^{N-1} in the classical sense. The function $\lambda^{\frac{1}{q-1}} w$ is therefore a solution of (1.3). □

Proof of (iii). We may assume that $N \geq 4$, for otherwise there is nothing to prove. Note that if $q \geq q_3$,

$$\tfrac{N-1}{q-1} - \ell_{N,q} = -\tfrac{N-3}{(q-1)^2}\big(q - \tfrac{N+1}{N-3}\big) \leq 0.$$

Applying Corollary 2.1, we deduce that (1.3) has no positive solution. □

5. The a priori estimate

In this section we establish Theorem 1.5 whose proof is based on the following result.

Proposition 5.1. *Assume that* $1 < q < q_2$. *Let* $0 < r < \frac{1}{2}\operatorname{diam}\Omega$ *and* $\zeta \in C^\infty(\partial\Omega)$ *with* $\zeta \geq 0$ *on* $\partial\Omega$. *Then, every solution of*

$$\begin{cases} -\Delta u = u^q & \text{in } \Omega \cap (B_{2r} \setminus \overline{B}_r), \\ u \geq 0 & \text{in } \Omega \cap (B_{2r} \setminus \overline{B}_r), \\ u = \zeta & \text{on } \partial\Omega \cap (B_{2r} \setminus \overline{B}_r), \end{cases} \tag{5.1}$$

satisfies

$$u(x) \leq C\big[\operatorname{dist}(x, \Gamma_r)\big]^{-\frac{2}{q-1}} \quad \forall x \in \Omega \cap (B_{2r} \setminus \overline{B}_r), \tag{5.2}$$

where $\Gamma_r = \overline{\Omega} \cap (\partial B_{2r} \cup \partial B_r)$ *and* $C > 0$ *is a constant independent of* u.

We denote by B_r the ball of radius r centered at 0. The proof of this estimate is based on two results: a Liouville theorem for the equation $-\Delta u = u^q$ in $\mathbb{R}^N$ or in $\mathbb{R}^N_+$ (see [7]) and the Doubling lemma of Poláčik-Quittner-Souplet [22] which we recall:

Lemma 5.1. *Let (X,d) be a complete metric space, $\Gamma \subsetneq X$ and $\gamma : X \setminus \Gamma \to (0,+\infty)$. Assume that γ is bounded on all compact subsets of $X \setminus \Gamma$. Given $k>0$, let $y \in X \setminus \Gamma$ be such that*

$$\gamma(y)\ \mathrm{dist}(y,\Gamma) > 2k.$$

Then, there exists $x \in X \setminus \Gamma$ such that

- $\gamma(x)\ \mathrm{dist}(x,\Gamma) > 2k$;
- $\gamma(x) \geq \gamma(y)$;
- $2\gamma(x) \geq \gamma(z)$, $\forall z \in B_{k/\gamma(x)}(x)$.

Proof of Proposition 5.1. To simplify the notation we may assume that $\zeta \equiv 0$. Assume by contradiction that (5.2) is false. Then, for every integer $k \geq 1$ there exist $0 < r_k < \frac{1}{2}\,\mathrm{diam}\,\Omega$, a solution u_k of (5.1) with $r = r_k$, and $y_k \in \Omega \cap (B_{2r_k} \setminus \overline{B}_{r_k})$ such that

$$u_k(y_k) > (2k)^{\frac{2}{q-1}} \big[\,\mathrm{dist}(y_k, \Gamma_{r_k})\big]^{-\frac{2}{q-1}}.$$

Applying the previous lemma with

$$X = \overline{\Omega} \cap (\overline{B}_{2r_k} \setminus B_{r_k}) \quad \text{and} \quad \gamma = u_k^{\frac{q-1}{2}},$$

one finds $x_k \in X \setminus \Gamma_{r_k}$ such that

(*i*) $u_k(x_k) > (2k)^{\frac{2}{q-1}} \big[\,\mathrm{dist}(x_k, \Gamma_{r_k})\big]^{-\frac{2}{q-1}}$;
(*ii*) $u_k(x_k) \geq u_k(y_k)$;
(*iii*) $2^{\frac{2}{q-1}} u_k(x_k) \geq u_k(z)$, $\forall z \in B_{R_k}(x_k) \cap \Omega$, with $R_k = k[u_k(x_k)]^{-\frac{q-1}{2}}$.

By (*i*) we have $R_k < \frac{1}{2}\,\mathrm{dist}(x_k, \Gamma_{r_k})$ and thus

$$B_{R_k}(x_k) \cap \Gamma_{r_k} = \emptyset.$$

Since $\mathrm{dist}(x_k, \Gamma_{r_k}) \leq \frac{1}{2} r_k < \frac{1}{4}\,\mathrm{diam}\,\Omega$, we also deduce from (*i*) that

$$u_k(x_k) \geq \left(\frac{8k}{\mathrm{diam}\,\Omega}\right)^{\frac{2}{q-1}}.$$

In particular,

$$u_k(x_k) \to +\infty \quad \text{as } k \to +\infty.$$

For every $k \geq 1$, let

$$t_k = [u_k(x_k)]^{-\frac{q-1}{2}},$$
$$D_k = \Big\{\xi \in \mathbb{R}^N;\ |\xi| \leq k \ \text{ and } \ x_k + t_k\xi \in \Omega\Big\}$$

and

$$v_k(\xi) = \frac{1}{u_k(x_k)}\, u_k\big(x_k + t_k\xi\big) \quad \forall \xi \in D_k.$$

Then, v_k satisfies

$$-\Delta v_k = v_k^q, \quad 0 \leq v_k \leq 2^{\frac{2}{q-1}} \quad \text{and} \quad v_k(0) = 1.$$

Passing to a subsequence if necessary, we may assume that either

(*A*) for every $a > 0$ there exists $k_0 \geq 1$ such that if $k \geq k_0$, then $B_{at_k}(x_k) \cap \partial\Omega = \emptyset$,

or

(*B*) there exists $a_0 > 0$ such that for every $k \geq 1$, $B_{a_0 t_k}(x_k) \cap \partial\Omega \neq \emptyset$.

Since the sequence (v_k) is uniformly bounded, it follows that (Δv_k) is also uniformly bounded. In both cases, by elliptic (interior and boundary) estimates, we have for every $1 < p < +\infty$ and every $s > 0$,

$$\|v_k\|_{W^{2,p}(D_k \cap B_s)} \leq C_{s,p}.$$

If (A) holds, then up to a subsequence (v_k) converges locally uniformly in $\mathbb{R}^N$ to some smooth function v such that

$$-\Delta v = v^q, \quad 0 \leq v \leq 2^{\frac{2}{q-1}} \quad \text{and} \quad v(0) = 1.$$

On the other hand, if (B) holds, then up to a subsequence and a rotation of the domain there exists some smooth function v defined in $\mathbb{R}^N_+$ such that (v_k) converges locally uniformly to v. Since the sequence (v_k) is equicontinuous and for every $k \geq 1$, $v_k(0) = 0$, we have $v(0) = 1$.

In both cases, we deduce that v is a nontrivial bounded solution of

$$-\Delta v = v^q$$

in $\mathbb{R}^N$ or in $\mathbb{R}^N_+$, which is impossible (see [7]). Therefore, estimate (5.2) must hold. □

Proof of Theorem 1.5. It suffices to establish (1.6) if $x \in \Omega$ and $|x| < \frac{3}{4} \operatorname{diam} \Omega$. For this purpose, we apply Proposition 5.1 with $r = \frac{2}{3}|x|$. Since $\operatorname{dist}(x, \Gamma_r) = \frac{1}{3}r$, we deduce that

$$u(x) \leq C\big[\operatorname{dist}(x, \Gamma_r)\big]^{-\frac{2}{q-1}} = C\left(\frac{r}{3}\right)^{-\frac{2}{q-1}} = \widetilde{C}\,|x|^{-\frac{2}{q-1}}.$$

This establishes the result. □

6. The geometric and analytic framework

We recall some of the preliminaries and the geometric framework in [12] which will be used in the remaining of the paper.

We denote by $(x_1, \dots, x_N)$ the coordinates of $x \in \mathbb{R}^N$ and by $\mathcal{B} = \{\mathbf{e}_1, \dots, \mathbf{e}_N\}$ the canonical orthonormal basis in $\mathbb{R}^N$. Since we are assuming that the outward unit normal vector is $-\mathbf{e}_N$, $\partial\Omega$ is the graph of a smooth function in a neighborhood of 0. In other words, there exist a neighborhood G of 0 and a smooth function $\phi : G \cap T_0\Omega \to \mathbb{R}$ such that

$$G \cap \partial\Omega = \Big\{(x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} \ ; \ x' \in G \cap T_0\Omega \text{ and } x_N = \phi(x')\Big\}.$$

Furthermore,

$$\phi(0) = 0 \quad \text{and} \quad \nabla\phi(0) = 0.$$

Setting $\Phi(x) = y$, with $y_i = x_i$ if $i = 1, \dots, N-1$ and $y_N = x_N - \phi(x')$, we can assume that Φ is a C^∞ diffeomorphism from G to $\tilde{G} = \Phi(G)$, and $\Phi(\Omega \cap G) = \tilde{G} \cap \mathbb{R}^N_+$. To avoid introducing some additional notation, we will assume that

$$\tilde{G} = B_1.$$

Given $\zeta \in C^\infty(\partial\Omega)$, let z be the harmonic extension of ζ in Ω. For every solution u of (1.1), we denote

$$u(x) - z(x) = \tilde{u}(y), \quad z(x) = \tilde{z}(y) \quad \text{and} \quad \zeta(x) = \tilde{\zeta}(y),$$

for every $x = \Phi^{-1}(y)$ with $y \in \tilde{G} \cap \mathbb{R}^N_+$. Since u is superharmonic and $u = z$ on $\partial\Omega$, we have $\tilde{u} \geq 0$. On the other hand, a straightforward computation yields

$$\Delta u = \Delta\tilde{u} + |\nabla\phi|^2 \tilde{u}_{y_N,y_N} - 2\langle\nabla\phi, \nabla\tilde{u}_{y_N}\rangle - \tilde{u}_{y_N}\Delta\phi$$

Thus, $\tilde{u}$ satisfies the equation

$$-\Delta\tilde{u} - |\nabla\phi|^2 \tilde{u}_{y_N,y_N} + 2\langle\nabla\phi, \nabla\tilde{u}_{y_N}\rangle + \tilde{u}_{y_N}\Delta\phi = (\tilde{u} + \tilde{z})^q.$$

Rewriting this equation in terms of spherical coordinates, one obtains

$$(1+\eta_1)\,\tilde{u}_{rr} + \frac{1}{r^2}\Delta'\tilde{u} + \frac{N-1+\eta_2}{r}\,\tilde{u}_r + (\tilde{u}+\tilde{z})^q = \\ = \frac{1}{r^2}\langle\nabla'\tilde{u}, \overrightarrow{\eta_3}\rangle + \frac{1}{r}\langle\nabla'\tilde{u}_r, \overrightarrow{\eta_4}\rangle + \frac{1}{r^2}\langle\nabla'\langle\nabla'\tilde{u}, \mathbf{e}_N\rangle, \overrightarrow{\eta_5}\rangle.$$

where

$$\eta_1 = -2\phi_r\langle\mathbf{n}, \mathbf{e}_N\rangle + |\nabla\phi|^2 \langle\mathbf{n}, \mathbf{e}_N\rangle^2,$$
$$\eta_2 = -r\langle\mathbf{n}, \mathbf{e}_N\rangle\Delta\phi - 2\langle\nabla'\langle\mathbf{n}, \mathbf{e}_N\rangle, \nabla'\phi\rangle + r\,|\nabla\phi|^2 \langle\nabla'\langle\mathbf{n}, \mathbf{e}_N\rangle, \mathbf{e}_N\rangle,$$
$$\overrightarrow{\eta_3} = -\left(2\phi_r - |\nabla\phi|^2 \langle\mathbf{n}, \mathbf{e}_N\rangle - r\Delta\phi\right)\mathbf{e}_N,$$
$$\overrightarrow{\eta_4} = -\left(|\nabla\phi|^2 \langle\mathbf{n}, \mathbf{e}_N\rangle - 2\phi_r\right)\mathbf{e}_N + \frac{2}{r}\langle\mathbf{n}, \mathbf{e}_N\rangle\nabla'\phi,$$
$$\overrightarrow{\eta_5} = -|\nabla\phi|^2 \mathbf{e}_N + \frac{2}{r}\nabla'\phi.$$

Taking into account the fact that $\phi(0) = 0$ and $\nabla\phi(0) = 0$,

$$|\phi(x)| \leq Cr^2, \quad |D\phi(x)| \leq Cr \quad \text{and} \quad \left|D^2\phi\right| \leq C.$$

Thus, for every $j = 1, \ldots, 5$,

$$\|\eta_j(r,\cdot)\|_{L^\infty} \leq Cr \quad \forall r \in (0,1).$$

Lemma 6.1. *Let*

$$t = \log\tfrac{1}{r}, \quad v(t,\sigma) = r^{\frac{2}{q-1}}\,\tilde{u}(r,\sigma) \quad \textit{and} \quad \alpha(t,\sigma) = r^{\frac{2}{q-1}}\,\tilde{z}(r,\sigma). \tag{6.1}$$

Then, v satisfies

$$(1+\epsilon_1)\,v_{tt} + \Delta' v - \left(N - \tfrac{2(q+1)}{q-1} + \epsilon_2\right)v_t + (\ell_{N,q} + \epsilon_3)\,v + (v+\alpha)^q = \\ = \langle\nabla' v, \overrightarrow{\epsilon_4}\rangle + \langle\nabla' v_t, \overrightarrow{\epsilon_5}\rangle + \langle\nabla'\langle\nabla' v, \mathbf{e}_N\rangle, \overrightarrow{\epsilon_6}\rangle, \tag{6.2}$$

where ϵ_j are functions defined in $(0,+\infty)\times S^{N-1}_+$ satisfying the estimates

$$\|\epsilon_j(t,\cdot)\|_{L^\infty} \leq C\mathrm{e}^{-t} \quad \forall t \geq 0, \tag{6.3}$$

for every $j = 1, \ldots, 6$.

We refer the reader to [12] for the proof of Lemma 6.1 and for the explicit expressions of the functions ϵ_j.

For every $T \geq 0$ and $\delta > 0$, let

$$Q_T = (T, +\infty)\times S^{N-1}_+ \quad \text{and} \quad Q_{T,\delta} = (T-\delta, T+\delta)\times S^{N-1}_+.$$

We have the following $W^{2,p}$-estimates satisfied by v:

Proposition 6.1. *Let v be defined as in Lemma 6.1. If v is uniformly bounded in Q_0, then for every $1 < p < +\infty$,*

$$\|v\|_{W^{2,p}(Q_{T,1})} \le C\Big(\|v\|_{L^2(Q_{T,2})} + \mathrm{e}^{-\frac{2T}{q-1}}\Big) \quad \forall T \ge 2, \tag{6.4}$$

for some positive constant depending on $\|v\|_{L^\infty}$ and on p.

Proof. Since Δ' is uniformly elliptic and Φ is a diffeomorphism, the operator L given by

$$L(v) = (1+\epsilon_1)\, v_{tt} + \Delta' v - \Big(N - \tfrac{2(q+1)}{q-1} + \epsilon_2\Big) v_t + \\ - \langle \nabla' v, \overrightarrow{\epsilon_4}\rangle - \langle \nabla' v_t, \overrightarrow{\epsilon_5}\rangle - \langle \nabla' \langle \nabla' v, \mathbf{e}_N\rangle, \overrightarrow{\epsilon_6}\rangle$$

is uniformly elliptic. Let $\delta > 0$. By the Agmon-Douglis-Nirenberg estimates (see [1]) applied to the restriction of v on the set $Q_{T,1+\delta}$,

$$\|v\|_{W^{2,p}(Q_{T,1+\delta})} \le C\Big(\|v\|_{L^p(Q_{T,1+2\delta})} + \|(\alpha+v)^q\|_{L^p(Q_{T,1+2\delta})}\Big).$$

Since α and v are uniformly bounded in Q_0, for every $s \in (1,2)$ we have

$$\begin{aligned}\|(\alpha+v)^q\|_{L^p(Q_{T,s})} &\le \|\alpha+v\|^{q-1}_{L^\infty(Q_{T,s})}\|\alpha+v\|_{L^p(Q_{T,s})} \\ &\le C\Big(\|\alpha\|_{L^p(Q_{T,s})} + \|v\|_{L^p(Q_{T,s})}\Big)\end{aligned}$$

Since $\tilde z$ is uniformly bounded in Ω,

$$\|\alpha\|_{L^p(Q_{T,s})} \le C\mathrm{e}^{-\frac{2T}{q-1}}\|\tilde z\|_{L^\infty(\Omega)} \le C\mathrm{e}^{-\frac{2T}{q-1}}.$$

Thus,

$$\|v\|_{W^{2,p}(Q_{T,1+\delta})} \le C\Big(\|v\|_{L^p(Q_{T,1+2\delta})} + \mathrm{e}^{-\frac{2T}{q-1}}\Big). \tag{6.5}$$

In particular,

$$\|v\|_{W^{2,p}(Q_{T,1})} \le C\Big(\|v\|_{L^p(Q_{T,\frac{3}{2}})} + \mathrm{e}^{-\frac{2T}{q-1}}\Big).$$

By a bootstrap argument based on the estimate (6.5) above and the Sobolev imbedding, we also have

$$\|v\|_{L^p(Q_{T,\frac{3}{2}})} \le C\Big(\|v\|_{L^2(Q_{T,2})} + \mathrm{e}^{-\frac{2T}{q-1}}\Big).$$

Combining these inequalities, the estimate follows. □

7. Removable singularities at 0

The goal of this section is to show that solutions of (1.1) which are not too large in a neighborhood of 0 must be continuous at 0.

Theorem 7.1. *Let $q > q_1$ and let u be a solution of* (1.1)*. If*

$$\lim_{x\to 0} |x|^{\frac{2}{q-1}}\, u(x) = 0, \tag{7.1}$$

then u can be continuously extended at 0*.*

Proof. Let v be the function given by (6.1). By assumption (7.1), we have

$$\lim_{t\to+\infty} v(t,\cdot) = 0 \quad \text{uniformly in } S^{N-1}_+. \tag{7.2}$$

We now rewrite (6.2) under the form

$$v_{tt} - \left(N - \tfrac{2(q+1)}{q-1}\right) v_t + \ell_{N,q} v + \Delta' v + (v+\alpha)^q = H, \tag{7.3}$$

where H is given by

$$H = -\epsilon_1 v_{tt} + \epsilon_2 v_t - \epsilon_3 v + \langle \nabla' v, \overrightarrow{\epsilon_4}\rangle + \langle \nabla' v_t, \overrightarrow{\epsilon_5}\rangle + \langle \nabla' \langle \nabla' v, \mathbf{e}_N\rangle, \overrightarrow{\epsilon_6}\rangle. \tag{7.4}$$

Thus,

$$\int\limits_{S^{N-1}_+} v v_{tt}\, d\sigma - \left(N - \tfrac{2(q+1)}{q-1}\right) \int\limits_{S^{N-1}_+} v v_t\, d\sigma + \ell_{N,q} \int\limits_{S^{N-1}_+} v^2\, d\sigma + \int\limits_{S^{N-1}_+} v\Delta' v\, d\sigma + \\ + \int\limits_{S^{N-1}_+} v(v+\alpha)^q\, d\sigma = \int\limits_{S^{N-1}_+} vH d\sigma. \tag{7.5}$$

Let

$$X(t) = \|v(t,\cdot)\|_{L^2(S^{N-1}_+)} \quad \forall t \geq 0.$$

Note that for every $t > 0$,

$$XX_t = \int\limits_{S^{N-1}_+} v v_t\, d\sigma. \tag{7.6}$$

Using Hölder's inequality we have

$$|XX_t| \leq \|v(t,\cdot)\|_{L^2(S^{N-1}_+)} \|v_t(t,\cdot)\|_{L^2(S^{N-1}_+)}.$$

Thus,

$$|X_t| \leq \|v_t(t,\cdot)\|_{L^2(S^{N-1}_+)}. \tag{7.7}$$

Computing the derivative with respect to t on both sides of identity (7.6), we get

$$(X_t)^2 + XX_{tt} = \int\limits_{S^{N-1}_+} (v_t)^2\, d\sigma + \int\limits_{S^{N-1}_+} v v_{tt}\, d\sigma = \|v_t(t,\cdot)\|^2_{L^2(S^{N-1}_+)} + \int\limits_{S^{N-1}_+} v v_{tt}\, d\sigma.$$

From this identity and estimate (7.7), we deduce that

$$XX_{tt} \geq \int\limits_{S^{N-1}_+} v v_{tt}\, d\sigma. \tag{7.8}$$

On the other hand, since the first eigenvalue of the Laplace-Beltrami operator $-\Delta'$ in $W^{1,2}_0(S^{N-1}_+)$ is $N-1$,

$$(N-1)X^2 \leq \int\limits_{S^{N-1}_+} |\nabla' v|^2\, d\sigma = -\int\limits_{S^{N-1}_+} v\, \Delta' v\, d\sigma.$$

By Hölder's inequality,

$$\int\limits_{S^{N-1}_+} vH\, d\sigma \leq X \|H(t,\cdot)\|_{L^2(S^{N-1}_+)}.$$

From the elementary inequality

$$(v+\alpha)^q \le 2^q(v^q+\alpha^q),$$

we get

$$\int\limits_{S_+^{N-1}} v(v+\alpha)^q\,d\sigma \le 2^q \int\limits_{S_+^{N-1}} \left(v^{q+1}+v\alpha^q\right) d\sigma$$

It follows from Hölder's inequality that

$$\int\limits_{S_+^{N-1}} v(v+\alpha)^q\,d\sigma \le 2^q\Big(X^2\|v(t,\cdot)\|^{q-1}_{L^\infty(S_+^{N-1})} + X\|\alpha(t,\cdot)\|^q_{L^{2q}(S_+^{N-1})}\Big). \tag{7.9}$$

We may assume that u is a nontrivial solution of (1.1). By the strong maximum principle, we have $u>0$ in Ω, thus $X>0$. Combining (7.5), (7.6) and (7.8)–(7.9), one gets

$$X_{tt} - \Big(N-\tfrac{2(q+1)}{q-1}\Big)X_t + \Big(\ell_{N,q}-N+1+2^q\|v(t,\cdot)\|^{q-1}_{L^\infty}\Big)X \ge \\ \ge -\big(\|H(t,\cdot)\|_{L^2}+2^q\|\alpha(t,\cdot)\|^q_{L^{2q}}\big)$$

(to simplify the notation we drop the explicit dependence of the set S_+^{N-1}). From the definition of the function α, there exists $C>0$ such that

$$2^q\|\alpha(t,\cdot)\|^q_{L^{2q}} \le C\mathrm{e}^{-\frac{2qt}{q-1}}.$$

In view of (7.2), given $\varepsilon>0$ there exists $t_0>0$ such that

$$2^q\|v(t,\cdot)\|^{q-1}_{L^\infty} \le \varepsilon \quad \text{on } [t_0,\infty).$$

We deduce that for every $t\ge t_0$ we have

$$X_{tt} - \Big(N-\tfrac{2(q+1)}{q-1}\Big)X_t + (\ell_{N,q}-N+1+\varepsilon)\,X \ge -\|H(t,\cdot)\|_{L^2} - C\mathrm{e}^{-\frac{2qt}{q-1}}.$$

We shall show that

$$X(t) \le C\mathrm{e}^{-\frac{2t}{q-1}} \quad \forall t\ge 0,$$

The conclusion will now follow from a bootstrap argument. Note that the linear equation

$$Z_{tt} - \Big(N-\tfrac{2(q+1)}{q-1}\Big)Z_t + (\ell_{N,q}-N+1)\,Z = 0$$

has two linearly independent solutions:

$$Z_1(t) = \mathrm{e}^{-\frac{q+1}{q-1}t} \quad \text{and} \quad Z_2(t) = \mathrm{e}^{(N-\frac{q+1}{q-1})t}$$

We can then take $\varepsilon>0$ small enough so that the linear equation

$$Z_{tt} - \Big(N-\tfrac{2(q+1)}{q-1}\Big)Z_t + (\ell_{N,q}-N+1+\varepsilon)\,Z = 0$$

has two linearly independent solutions:

$$Z_{1,\varepsilon}(t) = \mathrm{e}^{r_{1,\varepsilon}t} \quad \text{and} \quad Z_{1,\varepsilon}(t) = \mathrm{e}^{r_{2,\varepsilon}t}$$

such that

$$r_{1,\varepsilon} < -\frac{2}{q-1} \quad \text{and} \quad r_{2,\varepsilon}>0.$$

In particular,

$$Z_{2,\varepsilon}(t) \to +\infty \quad \text{as } t\to+\infty.$$

From assumption (7.1), v is bounded. In view of (6.3) and Proposition 6.1 with $p=2$, there exists $C_1>0$ such that

$$\|H(t,\cdot)\|_{L^2}\le C_1\mathrm{e}^{-t}\quad\forall t\ge 0.$$

Thus,

$$X_{tt}-\Big(N-\tfrac{2(q+1)}{q-1}\Big)X_t+(\ell_{N,q}-N+1)\,X\ge -\hat{C}_1\mathrm{e}^{-t}.$$

Since

$$X(t)\to 0\quad\text{as } t\to+\infty,$$

from the maximum principle there exists a constant $\tilde{C}_1>0$ such that

$$X(t)\le \tilde{C}_1(Z_{1,\varepsilon}(t)+\mathrm{e}^{-t}).$$

If $r_{1,\varepsilon}\ge -1$, then

$$X(t)\le 2\tilde{C}_1 Z_{1,\varepsilon}(t).$$

Since $r_{1,\varepsilon}<-\frac{2}{q-1}$, the estimate above implies that u is bounded and thus by standard elliptic estimates u is continuous. Otherwise $r_{1,\varepsilon}<-1$, in which case,

$$X(t)\le 2\tilde{C}_1\mathrm{e}^{-t}.$$

Thus, by Proposition 7.1 for every $T\ge 2$,

$$\|v\|_{W^{2,2}(Q_{T,2})}\le \widetilde{C}_1\mathrm{e}^{-T}.$$

In view of (6.3), there exists $C_2>0$ such that

$$\|H(t,\cdot)\|_{L^2}\le C_2\mathrm{e}^{-2t}\quad\forall t\ge 0.$$

Thus,

$$X_{tt}-\Big(N-\tfrac{2(q+1)}{q-1}\Big)X_t+(\ell_{N,q}-N+1)\,X\ge -\hat{C}_2\mathrm{e}^{-2t}.$$

This implies as before that

$$X(t)\le \tilde{C}_2(Z_{1,\varepsilon}(t)+\mathrm{e}^{-2t}).$$

If $r_{1,\varepsilon}\ge -2$, then

$$X(t)\le 2\tilde{C}_2 Z_{1,\varepsilon}(t)$$

and u is bounded. Otherwise $r_{1,\varepsilon}<-2$, in which case,

$$X(t)\le 2\tilde{C}_2\,\mathrm{e}^{-2t}.$$

We can continue this argument and deduce in finitely many steps that

$$X(t)\le 2\tilde{C}_k Z_{1,\varepsilon}(t).$$

Applying Proposition 6.1 with $p>\frac{N}{2}$, we deduce that for every $T\ge 2$,

$$\|v\|_{W^{2,p}(Q_{T,1})}\le C\Big(Z_{1,\varepsilon}(T)+\mathrm{e}^{-\frac{2T}{q-1}}\Big)\le C\,\mathrm{e}^{-\frac{2T}{q-1}}.$$

Thus, by Morrey's embedding,

$$\|v\|_{L^\infty(Q_{T,1})}\le C\,\mathrm{e}^{-\frac{2T}{q-1}}.$$

This implies that u is bounded and hence continuous in $\overline{\Omega}$. □

The conclusion of Theorem 7.1 is false with the critical exponent $q = q_1$. In fact, combining Theorem 1.3 and the result of del Pino-Musso-Pacard mentioned in the Introduction (Theorem 1.4), when $q = q_1$ there exist solutions of (1.1) such that

$$u(x) \sim x_N |x|^{-N} \big(\log \tfrac{1}{|x|} \big)^{-\frac{N-1}{2}}$$

in a neighborhood of 0. These solutions are necessarily discontinuous at 0 but, since $\frac{2}{q_1 - 1} = N - 1$,

$$\lim_{x \to 0} |x|^{\frac{2}{q_1 - 1}} u(x) = 0.$$

The right statement in this case is the following:

Theorem 7.2. *Let $q = q_1$ and let u be a nonnegative solution of (1.1). If*

$$\lim_{x \to 0} |x|^{N-1} \big(\log \tfrac{1}{|x|} \big)^{\frac{N-1}{2}} u(x) = 0,$$

then u can be continuously extended at 0.

Proof. Let

$$W(t) = t^{\frac{N-1}{2}} \|v(t, \cdot)\|_{L^2(S_+^{N-1})} \quad \forall t \geq 0,$$

where v is the function given by (6.1). By assumption, $W(t) \to 0$ as $t \to +\infty$. As in the proof of Theorem 7.1, for any $\varepsilon > 0$ there exists $t_0 > 0$ such that for every $t \geq t_0$,

$$W_{tt} + \big(N - \tfrac{N-1}{t} \big) W_t + \frac{1}{t} \Big(-\tfrac{N(N-1)}{2} + \varepsilon + \tfrac{N^2 - 1}{4t} \Big) W \geq \\ \geq -t^{\frac{N-1}{2}} \|H(t, \cdot)\|_{L^2} - C\, t^{\frac{N-1}{2}} \mathrm{e}^{-(N+1)t}.$$

The linear equation

$$W_{tt} + \big(N - \tfrac{N-1}{t} \big) W_t + \frac{1}{t} \Big(-\tfrac{N(N-1)}{2} + \tfrac{N^2 - 1}{4t} \Big) W = 0$$

has two linearly independent solutions W_1 and W_2 such that for t sufficiently large (see Lemma A.2 below)

$$W_1(t) = t^{\frac{N-1}{2}} \mathrm{e}^{-Nt} (1 + o(1)) \quad \text{and} \quad W_2(t) = t^{\frac{N-1}{2}} (1 + o(1)).$$

We can then take $\varepsilon > 0$ small enough so that the linear equation

$$W_{tt} + \big(N - \tfrac{N-1}{t} \big) W_t + \frac{1}{t} \Big(\tfrac{-N(N-1)}{2} + \varepsilon + \tfrac{N^2 - 1}{4t} \Big) W = 0$$

has two linearly independent solutions $W_{1,\varepsilon}$ and $W_{2,\varepsilon}$ such that

$$W_{1,\varepsilon}(t) \leq C t^{\frac{N-1}{2}} \mathrm{e}^{-(N-1)t}$$

and

$$W_{2,\varepsilon}(t) \to +\infty \quad \text{as } t \to +\infty.$$

In view of (6.3) and Proposition 6.1 with $p = 2$, there exists $C_1 > 0$ such that

$$\|H(t, \cdot)\|_{L^2} \leq C_1 t^{-\frac{N-1}{2}} \mathrm{e}^{-t} \quad \forall t > 0.$$

Thus,

$$W_{tt} + \big(N - \tfrac{N-1}{t} \big) W_t + \frac{1}{t} \Big(-\tfrac{N(N-1)}{2} + \varepsilon + \tfrac{N^2 - 1}{4t} \Big) W \geq -C \mathrm{e}^{-t}.$$

Since

$$W(t) \to 0 \quad \text{as } t \to +\infty,$$

from the maximum principle there exists a constant $\tilde{C}_1 > 0$ such that

$$W(t) \le \tilde{C}_1(W_{1,\varepsilon}(t) + \mathrm{e}^{-t}).$$

Thus,

$$W(t) \le \hat{C}_1 \mathrm{e}^{-t}.$$

Thus, by Proposition 6.1 with $p = 2$, for every $T \ge 2$,

$$\|v\|_{W^{2,2}(Q_{T,2})} \le \widehat{C}_1 t^{-\frac{N-1}{2}} \mathrm{e}^{-T}.$$

In view of (6.3), there exists $\hat{C}_2 > 0$ such that

$$\|H(t,\cdot)\|_{L^2} \le \hat{C}_2 t^{-\frac{N-1}{2}} \mathrm{e}^{-2t} \quad \forall t \ge 0.$$

We can continue this argument as in the previous theorem and deduce after finitely many steps that

$$W(t) \le \hat{C}_k W_{1,\varepsilon}(t) \le \tilde{C}\, t^{\frac{N-1}{2}} \mathrm{e}^{-(N-1)t},$$

which implies that u is bounded and hence continuous in $\overline{\Omega}$. □

8. Proof of Theorem 1.2

We first establish the following

Proposition 8.1. *Let $q_1 \le q < q_3$, with $q \ne q_2$. If u is a solution of (1.1) such that*

$$|x|^{\frac{2}{q-1}}\, u(x) \quad \textit{is bounded in } \Omega,$$

then for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $x \in \Omega \setminus \{0\}$, $\frac{x}{|x|} \in S^{N-1}_+$ and $|x| < \delta$, then

$$\left| |x|^{\frac{2}{q-1}} u(x) - w\big(\tfrac{x}{|x|}\big) \right| < \varepsilon \tag{8.1}$$

where w is a solution of (1.3).

Proof. Let v be the function given by (6.1). We first rewrite equation (6.2) under the form

$$v_{tt} + \ell_{N,q} v + \Delta' v + (v+\alpha)^q - \Big(N - \tfrac{2(q+1)}{q-1}\Big) v_t = H, \tag{8.2}$$

where H is given by (7.4). Multiplying (8.2) by v_t and integrating over S^{N-1}_+ yields

$$\int\limits_{S^{N-1}_+} v_t v_{tt}\, d\sigma + \ell_{N,q} \int\limits_{S^{N-1}_+} v_t v\, d\sigma + \int\limits_{S^{N-1}_+} v_t\, \Delta' v\, d\sigma + \int\limits_{S^{N-1}_+} v_t (v+\alpha)^q\, d\sigma + \\ - \Big(N - \tfrac{2(q+1)}{q-1}\Big) \int\limits_{S^{N-1}_+} (v_t)^2\, d\sigma = \int\limits_{S^{N-1}_+} v_t H\, d\sigma.$$

Thus,

(8.3)
$$\frac{d}{dt}\int\limits_{S_+^{N-1}}\left[\frac{(v_t)^2}{2}+\frac{\ell_{N,q}v^2}{2}-\frac{|\nabla' v|^2}{2}+\frac{(v+\alpha)^{q+1}}{q+1}\right]d\sigma-\left(N-\tfrac{2(q+1)}{q-1}\right)\int\limits_{S_+^{N-1}}(v_t)^2\,d\sigma=$$
$$=\int\limits_{S_+^{N-1}}\left[v_tH+\alpha_t(v+\alpha)^q\right]d\sigma.$$

From our assumption on u, v is bounded. It follows from (6.4) and the Sobolev imbedding that v, v_t and $\nabla' v$ are uniformly bounded in $S_+^{N-1}\times\mathbb{R}_+$. Integrating (8.3) from 0 to T, for any $T>0$, one deduces that

$$\int\limits_{S_+^{N-1}}\left|\frac{(v_t)^2}{2}+\frac{\ell_{N,q}v^2}{2}-\frac{|\nabla' v|^2}{2}+\frac{(v+\alpha)^{q+1}}{q+1}\right|d\sigma\le C\quad\text{in }\mathbb{R}_+$$

for some constant $C>0$. On the other hand,

$$\int\limits_{S_+^{N-1}}|v_tH|\,d\sigma\le C\mathrm{e}^{-t}.$$

Moreover, since v is bounded and α satisfies (6.5), we have

$$\int\limits_{S_+^{N-1}}|\alpha_t|(v+\alpha)^q\,d\sigma\le C\mathrm{e}^{-\frac{2t}{q-1}}.$$

Thus, integrating (8.3) on $(0,+\infty)$, we obtain

$$\left|N-\tfrac{2(q+1)}{q-1}\right|\int_0^{+\infty}\int\limits_{S_+^{N-1}}v_t^2\,d\sigma<+\infty.$$

Since $q\neq q_2$, $N-\frac{2(q+1)}{q-1}\neq 0$. Hence,

$$\int_0^{+\infty}\int\limits_{S_+^{N-1}}v_t^2\,d\sigma<+\infty.$$

By (6.4) and Morrey's estimates, v_t is uniformly continuous on Q_0. We deduce that

$$v_t(t,\cdot)\to 0\quad\text{uniformly in }S_+^{N-1}\text{ as }t\to+\infty.$$

We now prove that

$$v(t,\cdot)\to w\quad\text{uniformly in }S_+^{N-1}\text{ as }t\to+\infty,$$

where w is a nonnegative solution of (1.3). For this purpose, we study the limit set of the trajectories of v, namely the set

$$\Gamma=\bigcap_{\tau>0}\overline{\bigcup_{t\ge\tau}\{v(t,.)\}},$$

where the closure is computed with respect to the usual norm in $C^0(S_+^{N-1})$. Since Γ is the intersection of a decreasing family of closed connected subsets of $C^0(S_+^{N-1})$,

Γ is closed and connected. In addition, since v is uniformly continuous in Q_0, it follows from the Arzelà-Ascoli theorem that Γ is also compact and nonempty.

We claim that every $w \in \Gamma$ satisfies problem (1.3). Indeed, let (t_k) be a sequence of nonnegative real numbers such that $t_k \to +\infty$ and

$$v(t_k, \cdot) \to w \quad \text{uniformly in } S_+^{N-1}.$$

Clearly, w is nonnegative and $w = 0$ on ∂S_+^{N-1}. For each $k \geq 1$, let

$$V_k : (s, \sigma) \in [0,1] \times S_+^{N-1} \longmapsto v(t_k + s, \sigma).$$

For every $\varphi \in C_0^\infty(S_+^{N-1})$ and for every $\varepsilon \in (0,1)$, from the equation satisfied by v we have

$$\int_0^\varepsilon \int\limits_{S_+^{N-1}} \Big[(V_k)_{tt}\varphi + \ell_{N,q} V_k \varphi + V_k \Delta' \varphi + (V_k + \alpha)^q \varphi - \Big(N - \tfrac{2(q+1)}{q-1}\Big)(V_k)_t \varphi\Big]\, d\sigma\, dt =$$

$$= \int_{t_k}^{t_k+\varepsilon} \int\limits_{S_+^{N-1}} H\varphi\, d\sigma\, dt,$$

As $k \to +\infty$,

$$\int_{t_k}^{t_k+\varepsilon} \int\limits_{S_+^{N-1}} H\varphi\, d\sigma\, dt \to 0.$$

Since $v_t \to 0$ uniformly as $t \to +\infty$, we also have

$$\int_0^\varepsilon \int\limits_{S_+^{N-1}} (V_k)_t \varphi\, d\sigma\, dt \to 0.$$

Note that

$$\int_0^\varepsilon \int\limits_{S_+^{N-1}} (V_k)_{tt} \varphi\, d\sigma\, d\tau = \int\limits_{S_+^{N-1}} \big[v_t(t_k + \varepsilon, \sigma) - v_t(t_k, \sigma)\big]\varphi\, d\sigma \to 0.$$

Since the sequence (V_k) is bounded in C^1, passing to a subsequence if necessary, we may assume that for some continuous function W,

$$V_k \to W \quad \text{uniformly in } [0,1] \times S_+^{N-1}.$$

We conclude that for every $\varepsilon \in (0,1)$,

$$\int_0^\varepsilon \int\limits_{S_+^{N-1}} \Big[\ell_{N,q} W\varphi - W\Delta'\varphi + W^q \varphi\Big]\, d\sigma\, dt = 0.$$

Dividing both sides by ε and letting $\varepsilon \to 0$, we get

$$\int\limits_{S_+^{N-1}} \Big[\ell_{N,q} W(0,\sigma)\varphi - W(0,\sigma)\Delta'\varphi + (W(0,\sigma))^q \varphi\Big]\, d\sigma = 0.$$

Since $w = W(0, \cdot)$, we conclude that w satisfies (1.3). Hence, every element of Γ is a nonnegative solution of (1.3). Since these solutions form a discrete subset of $C^0(S_+^{N-1})$ and Γ is connected (in our case, the set of nonnegative solutions is $\{0, \omega\}$,

where ω is the unique positive solution of (1.3)), Γ contains a single element. In particular,

$$v(t,\cdot) \to w \quad \text{uniformly in } S_+^{N-1} \text{ as } t \to +\infty.$$

The proposition follows from this convergence. □

Proof of Theorem 1.2. Let u be a solution of (1.1). Since $q < q_2$, by Theorem 1.5 there exists $C > 0$ such that for every $x \in \Omega$,

$$0 \le |x|^{\frac{2}{q-1}}\, u(x) \le C.$$

Thus, by Proposition 8.1, there exists a solution w of (1.3) such that (8.1) holds. Either w is the unique positive solution of (1.3) (see Theorem 1.1) or $w = 0$. If $w = 0$, then

$$\lim_{x\to 0} |x|^{\frac{2}{q-1}}\, u(x) = 0.$$

Hence, by Theorem 7.1 u can be continuously extended at 0. □

9. Proof of Theorem 1.3

We first prove an estimate which improves Theorem 1.5 when $q = q_1$, except that we do not know whether the constant C below can be chosen independently of the solution.

Theorem 9.1. *Assume that $q = q_1$. Then, every solution of (1.1) satisfies*

$$u(x) \le C|x|^{-(N-1)} \big(\log \tfrac{1}{|x|}\big)^{-\frac{N-1}{2}} \quad \forall x \in \Omega,$$

for some constant $C > 0$ possibly depending on the solution.

In the proof of this result we need the following lemma:

Lemma 9.1. *Let $a = q_1$ and $E = \ker[\Delta' + (N-1)I]$. Given a solution of (1.1), denote by v the function given by (6.1). If*

$$v = v_1 + v_2$$

is the decomposition of v as the orthogonal projections in $L^2(S_+^{N-1})$ onto E and $E^\perp$, respectively, then

$$\|v_1(t,\cdot)\|_{L^2(S_+^{N-1})} \le C\, t^{-\frac{N-1}{2}} \quad \text{and} \quad \|v_2(t,\cdot)\|_{L^2(S_+^{N-1})} \le C\, \mathrm{e}^{-\frac{t}{2}} \quad \forall t > 0. \tag{9.1}$$

Proof. Denoting by ϕ_1 the first eigenfunction of Δ' with $\|\phi_1\|_{L^1} = 1$, we have

$$v_1(t,\sigma) = y(t)\phi_1(\sigma) \quad \text{where} \quad y(t) = \int\limits_{S_+^{N-1}} v(t,\sigma)\phi_1(\sigma)\, d\sigma.$$

Since $q = q_1$, equation (7.3) becomes

$$v_{tt} + Nv_t + (N-1)v + \Delta' v + (v+\alpha)^{q_1} = H, \tag{9.2}$$

with H defined in (7.4). Since $\alpha \ge 0$, we have $(v+\alpha)^{q_1} \ge v^{q_1}$. Thus,

$$v_{tt} + Nv_t + (N-1)v + \Delta' v + v^{q_1} \le H.$$

By Jensen's inequality,

$$y^{q_1} \le \int\limits_{S_+^{N-1}} v^{q_1}\phi_1\, d\sigma.$$

Multiplying (9.2) by ϕ_1 and integrating over S^{N-1}_+, we get

$$y'' + Ny' + y^{q_1} \leq \int\limits_{S^{N-1}_+} H\phi_1 \, d\sigma.$$

By Theorem 1.5, v is uniformly bounded in $\mathbb{R}_+ \times S^{N-1}_+$. In particular, by (6.3) and Proposition 6.1 with $p = 2$, we have for every $t \geq 0$,

$$\int\limits_{S^{N-1}_+} H\phi_1 \, d\sigma \leq C\, \mathrm{e}^{-t}.$$

Thus,

$$y'' + Ny' + y^{q_1} \leq C\, \mathrm{e}^{-t}.$$

Applying Lemma A.1 we deduce that

$$y(t) \leq Ct^{-\frac{N-1}{2}} \quad \forall t > 0.$$

This concludes the proof of the first estimate in (9.1).

In order to prove the estimate for v_2, let

$$Y(t) = \|v_2(t,\cdot)\|_{L^2(S^{N-1}_+)} \quad \forall t \geq 0.$$

Since $v(t,\sigma) = y(t)\phi_1(\sigma) + v_2(t,\sigma)$, we have

$$v_t = y_t\phi_1 + (v_2)_t \quad \text{and} \quad v_{tt} = y_{tt}\phi_1 + (v_2)_{tt}.$$

Using the orthogonality between ϕ_1 and v_2,

$$YY_t = \int\limits_{S^{N-1}_+} v_2(v_2)_t \, d\sigma = \int\limits_{S^{N-1}_+} v_2\big[y_t\phi_1 + (v_2)_t\big] \, d\sigma = \int\limits_{S^{N-1}_+} v_2 v_t \, d\sigma.$$

From the first equality, we have

$$|Y_t| \leq \|v_2(t,\cdot)\|_{L^2}.$$

One also shows that

$$YY_{tt} \geq \int\limits_{S^{N-1}_+} v_2 v_{tt} \, d\sigma.$$

On the other hand, since the second eigenvalue of the Laplace-Beltrami operator $-\Delta'$ in $W^{1,2}_0(S^{N-1}_+)$ is $2N$,

$$2NY^2 \leq \int\limits_{S^{N-1}_+} |\nabla' v_2|^2 \, d\sigma = -\int\limits_{S^{N-1}_+} v_2\, \Delta' v_2 \, d\sigma = -\int\limits_{S^{N-1}_+} v_2\, \Delta' v \, d\sigma.$$

Multiply (9.2) by v_2 and integrate over S^{N-1}_+. As in the proof of Theorem 7.1, for every $\varepsilon > 0$ there exists $t_1 > 0$ such that for every $t \geq t_1$,

$$Y_{tt} + NY_t - (N + 1 - \varepsilon)Y \geq -C\, \mathrm{e}^{-t}.$$

Note that for $\varepsilon > 0$ small the linear equation

$$Z_{tt} + NZ_t - (N + 1 - \varepsilon)Z = 0$$

has two linearly independent solutions $Z_{1,\varepsilon}$ and $Z_{2,\varepsilon}$ such that

$$Z_{1,\varepsilon}(t) = \mathrm{e}^{r_{1,\varepsilon}t} \quad \text{and} \quad Z_{2,\varepsilon}(t) = \mathrm{e}^{r_{2,\varepsilon}t}$$

with

$$r_{1,\varepsilon} \le -\frac{1}{2} \quad \text{and} \quad r_{2,\varepsilon} > 0.$$

Since $Y(t) \to 0$ as $t \to +\infty$, applying the maximum principle one deduces that

$$Y(t) \le C(Z_{1,\varepsilon}(t) + \mathrm{e}^{-t}).$$

In particular,

$$Y(t) \le C\mathrm{e}^{-\frac{t}{2}}.$$

This gives the estimate for v_2. □

Proof of Theorem 9.1. By Lemma 9.1 above, we have

$$\|v(t,\cdot)\|_{L^2} \le C\, t^{-\frac{N-1}{2}} \quad \forall t > 0.$$

Inserting this estimate into estimate (6.4) for some $p > \frac{N}{2}$ the result follows. □

Proof of Theorem 1.3. By Theorem 9.1, the function $w : [0, +\infty) \to \mathbb{R}$ given by

$$w(t,\sigma) = t^{\frac{N-1}{2}} v(t,\sigma)$$

is bounded. By a straightforward computation, w satisfies

$$w_{tt} + \left(N - \tfrac{N-1}{t}\right) w_t + \left(N - 1 + \tfrac{N^2-1}{4t^2}\right) w + \Delta' w + \\ + \frac{1}{t}\left(w^{q_1} - \tfrac{N(N-1)}{2} w\right) = t^{\frac{N-1}{2}} H, \tag{9.3}$$

where H is given by (7.4). Let $\phi : S_+^{N-1} \to \mathbb{R}$ be the function defined by $\phi(\sigma) = \frac{\sigma_N}{|\sigma|}$; we recall that ϕ is an eigenfunction of $-\Delta'$ in $W_0^{1,2}(S_+^{N-1})$ associated to the first eigenvalue $N-1$. Let

$$z(t) = \fint_{S_+^{N-1}} w(t,\sigma)\phi(\sigma)\, d\sigma \quad \forall t \ge 0.$$

Multiplying (9.3) by ϕ and integrating over S_+^{N-1}, we obtain the following equation satisfied by z:

$$z_{tt} + \left(N - \tfrac{N-1}{t}\right) z_t + \tfrac{N^2-1}{4t^2} z + \frac{1}{t} \fint_{S_+^{N-1}} w^{q_1} \phi\, d\sigma - \tfrac{N(N-1)}{2t} z = t^{\frac{N-1}{2}} \fint_{S_+^{N-1}} H\phi\, d\sigma.$$

Thus,

$$z_{tt} + \left(N - \tfrac{N-1}{t}\right) z_t + \frac{1}{t}\left(\theta z^{q_1} - \tfrac{N(N-1)}{2} z\right) = \Psi,$$

where

$$\theta = \fint_{S_+^{N-1}} \phi^{q_1+1}\, d\sigma$$

and

$$\Psi = t^{\frac{N-1}{2}} \fint_{S_+^{N-1}} H\phi\, d\sigma - \tfrac{N^2-1}{4t^2} z + \frac{1}{t} \fint_{S_+^{N-1}} \left[(z\phi)^{q_1} - w^{q_1}\right]\phi\, d\sigma.$$

By Lemma 9.1, we have

$$\|z(t)\phi - w(t,\cdot)\|_{L^2} \le C\, t^{\frac{N-1}{2}} \mathrm{e}^{-\frac{t}{2}}. \tag{9.4}$$

Since

$$|(z\phi)^{q_1} - w^{q_1}| \le q_1 |z\phi - w| \big[(z\phi)^{q_1-1} + w^{q_1-1}\big],$$

z is bounded in $\mathbb{R}_+$ and w is bounded in $\mathbb{R}_+ \times S^{N-1}_+$,

$$\int\limits_{S^{N-1}_+} \big|(z\phi)^{q_1} - w^{q_1}\big| \phi \, d\sigma \le \|z\phi - w\|_{L^2} \left[z^{q_1-1} \|\phi^{q_1-1}\|_{L^2} + \|w^{q_1-1}\|_{L^2} \right]$$

$$\le C\, t^{\frac{N-1}{2}} \mathrm{e}^{-\frac{t}{2}}.$$

By Proposition 9.1, (6.3) and Proposition 6.1 with $p = 2$,

$$\|H(t,\cdot)\|_{L^2} \le C\, t^{-\frac{N-1}{2}} \mathrm{e}^{-t}.$$

Thus,

$$\|\Psi(t,.)\|_{L^\infty} \le C\Big(\mathrm{e}^{-t} + t^{-2} + t^{\frac{N-3}{2}} \mathrm{e}^{-\frac{t}{2}}\Big) \le \tilde{C}\, t^{-2}.$$

By a straightforward modification of the end of the proof of [5, Corollary 4.2], z admits a limit $\kappa \ge 0$ when $t \to +\infty$, where κ satisfies

$$\theta \kappa^{q_1} - \tfrac{N(N-1)}{2} \kappa = 0.$$

Therefore, either $\kappa = 0$ or $\kappa = \Big(\frac{N(N-1)}{2\theta}\Big)^{\frac{N-1}{2}}$.
By (9.4) we deduce that, as $t \to +\infty$,

$$t^{\frac{N-1}{2}} v(t,\cdot) \to \kappa\phi \quad \text{in } L^2(S^{N-1}_+).$$

By Proposition 6.1 with $p > \frac{N}{2}$ and Morrey's estimates, we conclude that

$$t^{\frac{N-1}{2}} v(t,\cdot) \to \kappa\phi \quad \text{uniformly in } S^{N-1}_+.$$

Rewriting the convergence in terms of u, we conclude that either (1.5) holds or

$$|x|^{N-1} \big(\log \tfrac{1}{|x|}\big)^{\frac{N-1}{2}} u(x) \to 0 \text{as } x \to 0. \tag{9.5}$$

If (9.5) holds, then u must be continuous in view of Theorem 7.2. □

Appendix A. Some ode lemmas

We gather in this section a couple of ODE results which are used in this paper. These results are presumably well-known to specialists:

Lemma A.1. *Given $T > 0$, let $y \in C^2([T, +\infty))$ be a nonnegative function such that*

$$\begin{cases} y_{tt} + a y_t + b y^q \le c\, \mathrm{e}^{-t} & \text{in } (T, +\infty), \\ \lim\limits_{t \to +\infty} y(t) = 0, \end{cases}$$

where $q, a > 1$ and $b, c > 0$. Then, there exists $C > 0$ such that

$$0 \le y(t) \le C\, t^{-\frac{1}{q-1}} \quad \forall t \ge T. \tag{A.1}$$

Proof. Given $A > 0$, let

$$z(t) = y(t) + A\mathrm{e}^{-t} \quad \forall t \ge T.$$

Then, z satisfies

$$z_{tt} + a z_t + b z^q \le \big[c - (a-1)A\big] \mathrm{e}^{-t} + b(z^q - y^q).$$

By convexity of the function $t \in \mathbb{R}_+ \mapsto t^q$,

$$y^q \geq z^q - qz^{q-1}A\mathrm{e}^{-t}.$$

Thus,

$$z_{tt} + az_t + bz^q \leq \big[c - (a - 1 + bqz^{q-1})A\big]\mathrm{e}^{-t}. \tag{A.2}$$

Since $a > 1$ and $z(t) \to 0$ as $t \to \infty$, we can choose $T_1 > T$ and $A > 0$ sufficiently large so that the right-hand side of (A.2) is negative on $[T_1, \infty)$. Thus,

$$z_{tt} + az_t + bz^q \leq 0 \quad \text{in } [T_1, \infty). \tag{A.3}$$

Let $w = z^{1-q}$. By a straightforward computation, we have

$$w_{tt} + aw_t \geq -(q-1)\frac{z_{tt} + az_t}{z^q}. \tag{A.4}$$

Combining (A.3)–(A.4), we deduce that

$$w_{tt} + aw_t \geq b(q-1) \quad \text{in } [T_1, \infty).$$

The function $x = w_t$ satisfies

$$x_t + ax \geq b(q-1) \quad \text{in } [T_1, \infty).$$

Thus, taking $T_2 > T_1$ sufficiently large,

$$x(t) \geq \tfrac{b(q-1)}{a} + c_1\mathrm{e}^{-at} \geq \tfrac{b(q-1)}{2a} \quad \forall t \geq T_2.$$

Since $w_t = x$, choosing $T_3 > T_2$ large enough, we then get

$$w(t) \geq \tfrac{b(q-1)}{4a}\, t \quad \forall t \geq T_3.$$

Therefore,

$$z(t) \leq \Big(\tfrac{4a}{b(q-1)}\, t^{-1}\Big)^{\frac{1}{q-1}} \quad \forall t \geq T_3.$$

We can now enlarge the constant in the right-hand side so that this estimate holds for every $t \geq T$. This immediately implies (A.1). □

Lemma A.2. *Let $a, a_1, b, b_1 \in \mathbb{R}$ with $a \neq 0$. Then, the equation*

$$y_{tt} + \big(a - \tfrac{a_1}{t}\big)y_t + \tfrac{1}{t}\big(b - \tfrac{b_1}{t}\big)y = 0 \quad \text{in } (0, +\infty),$$

has two linearly independent solutions y_1 and y_2 such that

$$y_1(t) = t^{a_1 + \frac{b}{a}}\mathrm{e}^{-at}(1 + o(1)) \quad \text{and} \quad y_2(t) = t^{-\frac{b}{a}}(1 + o(1))$$

for t sufficiently large.

Proof. Let

$$z(t) = \mathrm{e}^{\frac{at}{2}}\, t^{-\frac{a_1}{2}} y(t).$$

Then, z satisfies the equation

$$z_{tt} - \big(\tfrac{a^2}{4} - \tfrac{A_1}{t} + \tfrac{A_2}{t^2}\big)z = 0,$$

where $A_1 = b + \frac{aa_1}{2}$ and $A_2 = b_1 + \frac{a_1}{2} + \frac{a_1^2}{4}$. By [3, pp. 126–127], the equation satisfied by z has two linearly independent solutions with the following asymptotic behaviors as $t \to +\infty$:

$$z_1(t) = \mathrm{e}^{-\frac{at}{2}}\, t^{\frac{A_1}{a}}(1 + o(1)) \quad \text{and} \quad z_2(t) = \mathrm{e}^{\frac{at}{2}}\, t^{-\frac{A_1}{a}}(1 + o(1)).$$

Rewriting these formulas in terms of the function y, the result follows. □

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