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# ISOLATED BOUNDARY SINGULARITIES OF SEMILINEAR ELLIPTIC EQUATIONS

MARIE-FRANÇOISE BIDAUT-VÉRON, AUGUSTO C. PONCE, AND LAURENT VÉRON

ABSTRACT. Given a smooth domain  $\Omega \subset \mathbb{R}^N$  such that  $0 \in \partial\Omega$  and given a nonnegative smooth function  $\zeta$  on  $\partial\Omega$ , we study the behavior near 0 of positive solutions of  $-\Delta u = u^q$  in  $\Omega$  such that  $u = \zeta$  on  $\partial\Omega \setminus \{0\}$ . We prove that if  $\frac{N+1}{N-1} < q < \frac{N+2}{N-2}$ , then  $u(x) \leq C|x|^{-\frac{2}{q-1}}$  and we compute the limit of  $|x|^{\frac{2}{q-1}} u(x)$  as  $x \rightarrow 0$ . We also investigate the case  $q = \frac{N+1}{N-1}$ . The proofs rely on the existence and uniqueness of solutions of related equations on spherical domains.

## CONTENTS

1.	Introduction and main results	1
2.	Pohožaev identity in spherical domains	4
3.	Uniqueness of solutions of a PDE in $S_+^{N-1}$	6
4.	Proof of Theorem 1.1	16
5.	The a priori estimate	17
6.	The geometric and analytic framework	19
7.	Removable singularities at 0	21
8.	Proof of Theorem 1.2	26
9.	Proof of Theorem 1.3	29
	Appendix A. Some ODE lemmas	32
	References	34

## 1. INTRODUCTION AND MAIN RESULTS

Let  $\Omega$  be a smooth open subset of  $\mathbb{R}^N$ , with  $N \geq 2$ , such that  $0 \in \partial\Omega$ . Given  $q > 1$  and  $\zeta \in C^\infty(\partial\Omega)$  with  $\zeta \geq 0$  on  $\partial\Omega$ , consider the problem

$$(1.1) \quad \begin{cases} -\Delta u = u^q & \text{in } \Omega, \\ u \geq 0 & \text{in } \Omega, \\ u = \zeta & \text{on } \partial\Omega \setminus \{0\}. \end{cases}$$

By a solution of (1.1) we mean a function  $u \in C^2(\Omega) \cap C(\overline{\Omega} \setminus \{0\})$  which satisfies (1.1) in the classical sense. A solution may develop an isolated singularity at 0. Our main goal in this paper is to describe the behavior of  $u$  in a neighborhood of this point.

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In the study of boundary singularities of (1.1), one finds three critical exponents; namely,

$$q_1 = \frac{N+1}{N-1}, \quad q_2 = \frac{N+2}{N-2} \quad \text{and} \quad q_3 = \frac{N+1}{N-3}.$$

When  $1 < q < q_1$ , it is proved by Bidaut-Véron-Vivier [6] that for every solution  $u$  of (1.1) there exists  $\alpha \geq 0$  (depending on  $u$ ) such that

$$u(x) = \alpha |x|^{-N} \text{dist}(x, \partial\Omega) (1 + o(1)) \quad \text{as } x \rightarrow 0.$$

In this paper we mainly investigate the case  $q_1 \leq q < q_3$ .

The counterpart of (1.1) for an interior singularity,

$$-\Delta u = u^q \quad \text{in } \Omega \setminus \{x_0\},$$

where  $x_0 \in \Omega$ , was studied by P.-L. Lions [16] in the subcritical case  $1 < q < \frac{N}{N-2}$ , by Aviles [2] when  $q = \frac{N}{N-2}$  and by Gidas-Spruck [11] in the range  $\frac{N}{N-2} < q < \frac{N+2}{N-2}$ . We prove some counterparts of the works of Gidas-Spruck and Aviles in the framework of boundary singularities.

When (1.1) is replaced by an equation with an absorption term,

$$(1.2) \quad -\Delta u + u^q = 0 \quad \text{in } \Omega,$$

the problem has been first addressed by Gmira-Véron [12] (and later to nonsmooth domains in [10]). These results are important in the theory of boundary trace of positive solutions of (1.2) which was developed by Marcus-Véron [17, 18, 19] using analytic tools and by Le Gall [15] and Dynkin-Kuznetsov [8, 9] with a probabilistic approach. We refer the reader to Véron [23] for the case of interior singularities of (1.2).

Let us first consider the case where  $\Omega$  is the upper-half space  $\mathbb{R}_+^N$ , and we look for solutions of (1.1) of the form

$$u(x) = |x|^{-\frac{2}{q-1}} \omega\left(\frac{x}{|x|}\right).$$

By an easy computation,  $\omega$  must satisfy

$$(1.3) \quad \begin{cases} -\Delta' \omega = \ell_{N,q} \omega + \omega^q & \text{in } S_+^{N-1}, \\ \omega \geq 0 & \text{in } S_+^{N-1}, \\ \omega = 0 & \text{on } \partial S_+^{N-1}, \end{cases}$$

where  $\Delta'$  denotes the Laplace-Beltrami operator in the unit sphere  $S^{N-1}$ ,

$$\ell_{N,q} = \frac{2(N-q(N-2))}{(q-1)^2} \quad \text{and} \quad S_+^{N-1} = S^{N-1} \cap \mathbb{R}_+^N.$$

Concerning equation (1.3), we prove

**Theorem 1.1.**

- (i) If  $1 < q \leq q_1$ , then (1.3) admits no positive solution.
- (ii) If  $q_1 < q < q_3$ , then (1.3) admits a unique positive solution.
- (iii) If  $q \geq q_3$ , then (1.3) admits no positive solution.

In Section 3 we study uniqueness of solutions of (1.3) with  $\ell_{N,q}$  replaced by any  $\ell \in \mathbb{R}$ . The proofs are inspired from some interesting ideas taken from Kwong [13] and Kwong-Li [14]. The nonexistence of solutions of (1.3) when  $q \geq q_3$  is based on a Pohožaev identity for spherical domains; see Theorem 2.1 below.

We now consider the case where  $\Omega \subset \mathbb{R}^N$  is a smooth domain such that  $0 \in \partial\Omega$ . Without loss of generality, we may assume that  $-\mathbf{e}_N$  is the outward unit normal vector of  $\partial\Omega$  at 0. We prove the following classification of isolated singularities of solutions of (1.1):

**Theorem 1.2.** *Assume that  $q_1 < q < q_2$ . If  $u$  satisfies (1.1), then either  $u$  can be continuously extended at 0 or for every  $\varepsilon > 0$  there exists  $\delta > 0$  such that if  $x \in \Omega \setminus \{0\}$ ,  $\frac{x}{|x|} \in S_+^{N-1}$  and  $|x| < \delta$ ,*

$$(1.4) \quad \left| |x|^{\frac{2}{q-1}} u(x) - \omega\left(\frac{x}{|x|}\right) \right| < \varepsilon,$$

where  $\omega$  is the unique positive solution of (1.3).

When  $q_2 < q < q_3$ , we have a similar conclusion provided  $u$  satisfies the estimate

$$u(x) \leq C|x|^{-\frac{2}{q-1}} \quad \forall x \in \Omega,$$

for some constant  $C > 0$ ; see Proposition 8.1 below. In the critical case  $q = q_1$  there is a superposition of the linear and nonlinear effects since their characteristic exponents  $\frac{2}{q-1}$  and  $N-1$  coincide. The counterpart of Theorem 1.2 in this case is the following:

**Theorem 1.3.** *Assume that  $q = q_1$ . If  $u$  satisfies (1.1), then either  $u$  can be continuously extended at 0 or for every  $\varepsilon > 0$  there exists  $\delta > 0$  such that if  $x \in \Omega \setminus \{0\}$  and  $|x| < \delta$ ,*

$$(1.5) \quad \left| |x|^{N-1} \left( \log \frac{1}{|x|} \right)^{\frac{N-1}{2}} u(x) - \kappa \frac{x_N}{|x|} \right| < \varepsilon,$$

where  $\kappa$  is a positive constant depending only on the dimension  $N$ .

Our characterization of boundary isolated singularities is complemented by the existence of singular solutions which has been recently obtained by del Pino-Musso-Pacard [21]. We recall their result:

**Theorem 1.4.** *Assume that  $\Omega \subset \mathbb{R}^N$  is a smooth bounded domain. There exists  $p > q_1$  such that for every  $q_1 \leq q < p$  and for every  $\xi_1, \xi_2, \dots, \xi_k \in \partial\Omega$ , there exists a solution of (1.1) with  $\zeta \equiv 0$  such that*

$$u(x) \rightarrow +\infty \quad \text{as } x \rightarrow \xi_j \text{ nontangentially for every } i = 1, 2, \dots, k.$$

In view of Theorems 1.2 and 1.3 any such solution must have the singular behavior we have obtained therein. In [21], the authors conjecture that such solutions exist for every  $q_1 \leq q < q_2$ .

Some of the main ingredients in the proofs of Theorems 1.2 and 1.3 are Theorem 1.1 above concerning existence and uniqueness of positive solutions of (1.3), a removable singularity result (see Theorems 7.1 and 7.2 below) and the following a priori bound of solutions of (1.1):

**Theorem 1.5.** *Assume that  $1 < q < q_2$ . Then, every solution of (1.1) satisfies*

$$(1.6) \quad u(x) \leq C|x|^{-\frac{2}{q-1}} \quad \forall x \in \Omega,$$

for some constant  $C > 0$  independent of the solution.

We establish this estimate using a topological argument, called the *Doubling lemma* (see Lemma 5.1 below), introduced by Poláčik-Quittner-Souplet [22].

Theorems 1.2, 1.3 and 1.5 have been announced in [4].

## 2. POHOŽAEV IDENTITY IN SPHERICAL DOMAINS

We first prove the following Pohožaev identity in spherical domains.

**Theorem 2.1.** *Let  $q > 1$ ,  $\ell \in \mathbb{R}$  and  $S$  be a smooth domain in  $S_+^{N-1}$ . If  $v \in C^2(S) \cap C(\bar{S})$  satisfies*

$$(2.1) \quad \begin{cases} -\Delta' v = \ell v + |v|^{q-1} v & \text{in } S, \\ v = 0 & \text{on } \partial S, \end{cases}$$

then

$$\left(\frac{N-3}{2} - \frac{N-1}{q+1}\right) \int_S |\nabla' v|^2 \phi \, d\sigma - \frac{N-1}{2} \left(\frac{\ell q + N-1}{q+1}\right) \int_S v^2 \phi \, d\sigma = \frac{1}{2} \int_{\partial S} |\nabla' v|^2 \langle \nabla' \phi, \nu \rangle \, d\tau,$$

where  $\nu$  is the outward unit normal vector on  $\partial S$ ,  $\nabla'$  the tangential gradient to  $S^{N-1}$ , and  $\phi$  is a first eigenfunction of the Laplace-Beltrami operator  $-\Delta'$  in  $W_0^{1,2}(S_+^{N-1})$ .

We recall that the first eigenvalue of  $-\Delta'$  in  $W_0^{1,2}(S_+^{N-1})$  is  $N-1$  and the eigenspace associated to this eigenvalue is spanned by the function  $\phi(x) = \frac{x_N}{|x|}$ .

*Proof.* Let

$$P = \langle \nabla' \phi, \nabla' v \rangle \nabla' v.$$

By the Divergence theorem,

$$(2.2) \quad \int_S \operatorname{div} P \, d\sigma = \int_{\partial S} \langle P, \nu \rangle \, d\tau.$$

Note that

$$\operatorname{div} P = \langle \nabla' v, \nabla' \phi \rangle \Delta' v + D^2 v(\nabla' v, \nabla' \phi) + D^2 \phi(\nabla' v, \nabla' v).$$

where  $D^2 v$  is the Hessian operator. Now,

$$D^2 v(\nabla' v, \nabla' \phi) = \frac{1}{2} \langle \nabla' |\nabla' v|^2, \nabla' \phi \rangle.$$

Using the classical identity

$$D^2 \phi + g\phi = 0$$

where  $g = (g_{i,j})$  is the metric tensor on  $S^{N-1}$ , we get

$$D^2 \phi(\nabla' v, \nabla' v) = -g(\nabla' v, \nabla' v)\phi = -|\nabla' v|^2 \phi.$$

We replace these identities in the expression of  $\operatorname{div} P$ ,

$$\operatorname{div} P = -\langle \nabla' v, \nabla' \phi \rangle (\ell v + |v|^{q-1} v) + \frac{1}{2} \langle \nabla' |\nabla' v|^2, \nabla' \phi \rangle - |\nabla' v|^2 \phi.$$

Integrating over  $S$ , we obtain

$$\int_S \operatorname{div} P \, d\sigma = - \int_S \langle \nabla' v, \nabla' \phi \rangle (\ell v + |v|^{q-1} v) \, d\sigma + \frac{1}{2} \int_S \langle \nabla' |\nabla' v|^2, \nabla' \phi \rangle \, d\sigma - \int_S |\nabla' v|^2 \phi \, d\sigma.$$

Note that

$$\begin{aligned} \int_S \langle \nabla' v, \nabla' \phi \rangle (\ell v + |v|^{q-1} v) \, d\sigma &= \int_S \left\langle \nabla' \left( \frac{\ell}{2} v^2 + \frac{1}{q+1} |v|^{q+1} \right), \nabla' \phi \right\rangle \, d\sigma \\ &= - \int_S \left( \frac{\ell}{2} v^2 + \frac{1}{q+1} |v|^{q+1} \right) \Delta' \phi \, d\sigma \\ &= (N-1) \int_S \left( \frac{\ell}{2} v^2 + \frac{1}{q+1} |v|^{q+1} \right) \phi \, d\sigma, \end{aligned}$$

and

$$\begin{aligned} \int_S \langle \nabla' |\nabla' v|^2, \nabla' \phi \rangle d\sigma &= - \int_S |\nabla' v|^2 \Delta' \phi d\sigma + \int_{\partial S} |\nabla' v|^2 \langle \nabla' \phi, \nu \rangle d\tau \\ &= (N-1) \int_S |\nabla' v|^2 \phi d\sigma + \int_{\partial S} |\nabla' v|^2 \langle \nabla' \phi, \nu \rangle d\tau. \end{aligned}$$

These identities imply

$$(2.3) \quad \begin{aligned} \int_S \operatorname{div} P d\sigma &= -\frac{\ell(N-1)}{2} \int_S v^2 \phi d\sigma - \frac{N-1}{q+1} \int_S |v|^{q+1} \phi d\sigma + \frac{N-3}{2} \int_S |\nabla' v|^2 \phi d\sigma + \\ &\quad + \frac{1}{2} \int_{\partial S} |\nabla' v|^2 \langle \nabla' \phi, \nu \rangle d\tau. \end{aligned}$$

On the other hand, since  $v$  satisfies (2.1),

$$\begin{aligned} \int_S (\ell v^2 + |v|^{q+1}) \phi d\sigma &= - \int_S (\Delta' v) v \phi d\sigma \\ &= \int_S \langle \nabla' v, \nabla' (v\phi) \rangle d\sigma = \int_S |\nabla' v|^2 \phi d\sigma + \int_S \langle \nabla' v, \nabla' \phi \rangle v d\sigma. \end{aligned}$$

Since  $v \nabla' v = \frac{1}{2} \nabla' (v^2)$  and  $\Delta' \phi = -(N-1)\phi$ ,

$$\int_S \langle \nabla' v, \nabla' \phi \rangle v d\sigma = \frac{1}{2} \int_S \langle \nabla' (v^2), \nabla' \phi \rangle d\sigma = \frac{N-1}{2} \int_S v^2 \phi d\sigma.$$

Thus,

$$\int_S (\ell v^2 + |v|^{q+1}) \phi d\sigma = \int_S |\nabla' v|^2 \phi d\sigma + \frac{N-1}{2} \int_S v^2 \phi d\sigma.$$

This implies

$$\int_S |v|^{q+1} \phi d\sigma = \int_S |\nabla' v|^2 \phi d\sigma + \left(\frac{N-1}{2} - \ell\right) \int_S v^2 \phi d\sigma.$$

Inserting this identity in (2.3), we obtain

$$(2.4) \quad \begin{aligned} \int_S \operatorname{div} P d\sigma &= \left(\frac{N-3}{2} - \frac{N-1}{q+1}\right) \int_S |\nabla' v|^2 \phi d\sigma - \left(\frac{\ell(N-1)}{2} + \frac{N-1}{q+1} \left(\frac{N-1}{2} - \ell\right)\right) \int_S v^2 \phi d\sigma + \\ &\quad + \frac{1}{2} \int_{\partial S} |\nabla' v|^2 \langle \nabla' \phi, \nu \rangle d\tau. \end{aligned}$$

Since  $v$  vanishes on  $\partial S$ ,  $\nabla' v = \langle \nabla' v, \nu \rangle \nu$  and, in particular,  $|\nabla' v| = |\langle \nabla' v, \nu \rangle|$ . Thus,

$$(2.5) \quad \begin{aligned} \int_{\partial S} \langle P, \nu \rangle d\tau &= \int_{\partial S} \langle \nabla' \phi, \nabla' v \rangle \langle \nabla' v, \nu \rangle d\tau = \int_{\partial S} (\langle \nabla' v, \nu \rangle)^2 \langle \nabla' \phi, \nu \rangle d\tau \\ &= \int_{\partial S} |\nabla' v|^2 \langle \nabla' \phi, \nu \rangle d\tau. \end{aligned}$$

Combining (2.2), (2.4) and (2.5), we get the Pohožaev identity.  $\square$

Using the Pohožaev identity on  $S_+^{N-1}$  we can prove that the Dirichlet problem (2.1) can only have trivial solutions for suitable values of  $q$  and  $\ell$ .

**Corollary 2.1.** *Let  $N \geq 4$ . If  $q \geq q_3$  and  $\ell \leq -\frac{N-1}{q-1}$ , then the function identically zero is the only solution in  $C^2(S^{N-1}) \cap C^2(\overline{S^{N-1}})$  of the Dirichlet problem*

$$\begin{cases} -\Delta'v = \ell v + |v|^{q-1}v & \text{in } S_+^{N-1}, \\ v = 0 & \text{on } \partial S_+^{N-1}. \end{cases}$$

*Proof.* Let  $v$  be a solution of the Dirichlet problem. Applying the Pohožaev identity with  $\phi(x) = \frac{x_N}{|x|}$ , then the left-hand side of the Pohožaev identity is nonnegative, while its right-hand side is nonpositive. Thus, both sides are zero. If at least one of the inequalities  $q \geq q_3$  or  $\ell \leq -\frac{N-1}{q-1}$  is strict, then we immediately deduce that  $v = 0$  in  $S_+^{N-1}$ .

If  $q = q_3$  and  $\ell = -\frac{N-1}{q-1}$ , then

$$\int_{\partial S_+^{N-1}} |\nabla'v|^2 \langle \nabla'\phi, \nu \rangle d\tau = 0.$$

Since  $\langle \nabla'\phi, \nu \rangle < 0$  on  $\partial S_+^{N-1}$ , we conclude that  $\nabla'v = 0$  on  $\partial S_+^{N-1}$ . Define the function  $\tilde{v} : S^{N-1} \rightarrow \mathbb{R}$  by

$$\tilde{v}(x) = \begin{cases} v(x) & \text{if } x \in S_+^{N-1}, \\ 0 & \text{otherwise.} \end{cases}$$

Then,  $\tilde{v}$  satisfies (in the sense of distributions)

$$-\Delta'\tilde{v} = \ell\tilde{v} + |\tilde{v}|^{q-1}\tilde{v} \quad \text{in } S^{N-1}.$$

Since  $\tilde{v}$  vanishes in an open subset of  $S^{N-1}$ , by the unique continuation principle we have  $\tilde{v} = 0$  in  $S^{N-1}$  and the conclusion follows.  $\square$

### 3. UNIQUENESS OF SOLUTIONS OF A PDE IN $S_+^{N-1}$

In this section we address the question of uniqueness of positive solutions of the Dirichlet problem

$$(3.1) \quad \begin{cases} -\Delta'v = \ell v + v^q & \text{in } S_+^{N-1}, \\ v \geq 0 & \text{in } S_+^{N-1}, \\ v = 0 & \text{on } \partial S_+^{N-1}, \end{cases}$$

where  $\ell \in \mathbb{R}$ . A solution of (3.1) is understood in the classical sense.

We shall prove the following results:

**Theorem 3.1.** *Assume that  $N = 2$ . If  $q > 1$ , then for every  $\ell \in \mathbb{R}$  the Dirichlet problem (3.1) has at most one positive solution.*

**Theorem 3.2.** *Assume that  $N \geq 4$ . If  $1 < q < q_3$ , then for every  $\ell \in \mathbb{R}$  the Dirichlet problem (3.1) has at most one positive solution.*

**Theorem 3.3.** *Assume that  $N = 3$ . Then, the Dirichlet problem (3.1) has at most one positive solution under one of the following assumptions:*

- for every  $1 < q \leq 5$  and  $\ell \in \mathbb{R}$ ,
- for every  $q > 5$  and  $\ell \leq \frac{2(3-q)}{(q+3)(q-1)}$ .

**Remark 3.1.** In dimension  $N = 3$  we do not know whether the Dirichlet problem (3.1) has a unique positive solution if  $q > 5$  and  $\ell > \frac{2(3-q)}{(q+3)(q-1)}$ .

We first show that the graphs of two positive solutions of (3.1) must cross.

**Lemma 3.1.** *Assume that  $v_1$  and  $v_2$  are positive solutions of (3.1). If  $v_1 \leq v_2$  in  $S_+^{N-1}$ , then  $v_1 = v_2$ .*

*Proof.* Multiplying by  $v_2$  the equation satisfied by  $v_1$  and integrating by parts, we get

$$\int_{S_+^{N-1}} \langle \nabla v_1, \nabla v_2 \rangle d\sigma = \int_{S_+^{N-1}} (\ell v_1 + (v_1)^q) v_2 d\sigma.$$

Reversing the roles of  $v_1$  and  $v_2$ , we also have

$$\int_{S_+^{N-1}} \langle \nabla v_2, \nabla v_1 \rangle d\sigma = \int_{S_+^{N-1}} (\ell v_2 + (v_2)^q) v_1 d\sigma.$$

Subtracting these identities, we have

$$\int_{S_+^{N-1}} (v_1^{q-1} - v_2^{q-1}) v_1 v_2 d\sigma = 0.$$

Since the integrand is nonnegative we must have  $v_1^{q-1} - v_2^{q-1} = 0$  and the conclusion follows.  $\square$

We now establish the uniqueness result in the case  $N = 2$ .

*Proof of Theorem 3.1.* Denoting by

$$\theta = \arccos \frac{x_2}{|x|},$$

then a solution of (3.1) satisfies

$$\begin{cases} v_{\theta\theta} + \ell v + v^q = 0 & \text{in } (0, \frac{\pi}{2}), \\ v_{\theta}(0) = 0, \quad v(\frac{\pi}{2}) = 0. \end{cases}$$

Moreover, for every  $\theta \in (0, \frac{\pi}{2}]$ ,  $v_{\theta}(\theta) < 0$  (this can be established using for example the moving plane method). Thus,  $v$  is decreasing. Let  $V : [0, v(0)] \rightarrow \mathbb{R}$  be the function defined by

$$(3.2) \quad V(\xi) = v_{\theta}(v^{-1}(\xi)).$$

Then,  $V$  is of class  $C^1$  in  $[0, v(0))$ . Since for every  $\xi \in [0, v(0))$ ,

$$(v^{-1})_{\xi}(\xi) = \frac{1}{v_{\theta}(v^{-1}(\xi))} = \frac{1}{V(\xi)},$$

we deduce that

$$(3.3) \quad (V^2)_{\xi} = 2VV_{\xi} = 2V(v_{\theta\theta} \circ v^{-1})(v^{-1})_{\xi} = 2(v_{\theta\theta} \circ v^{-1}) = -2(\ell\xi + \xi^q).$$

Assume by contradiction that (3.1) has two distinct positive solutions, say  $v_1$  and  $v_2$ . We may assume they are both defined in terms of the variable  $\theta$ . Then, there exists  $c_1 \in (0, \frac{\pi}{2})$  such that  $v_1(c_1) = v_2(c_1)$ . Let  $c_2 \in (c_1, \frac{\pi}{2}]$  be the smallest number



such that  $v_1(c_2) = v_2(c_2)$  (this point  $c_2$  exists since  $v_{1\theta}(c_1) \neq v_{2\theta}(c_1)$ ). Without loss of generality, we may assume that, for every  $\theta \in (c_1, c_2)$ ,

$$v_1(\xi) < v_2(\xi).$$

Let  $V_1$  and  $V_2$  be the functions given by (3.2) corresponding to  $v_1$  and  $v_2$ , respectively. For  $i \in \{1, 2\}$ , let

$$\alpha_i = v_1(c_i) = v_2(c_i).$$

By (3.3), for every  $\xi \in (\alpha_2, \alpha_1)$ ,

$$(V_1^2)_\xi(\xi) = -2(\ell\xi + \xi^q) = (V_2^2)_\xi(\xi).$$

Hence, the function  $V_1^2 - V_2^2$  is constant. On the other hand, since  $v_1 < v_2$  and  $v_1, v_2$  are both decreasing, by uniqueness of the Cauchy problem,

$$v_{1\theta}(c_1) < v_{2\theta}(c_1) < 0 \quad \text{and} \quad v_{2\theta}(c_2) < v_{1\theta}(c_2) < 0.$$

Thus,

$$V_1^2(\alpha_1) - V_2^2(\alpha_1) > 0 \quad \text{and} \quad V_1^2(\alpha_2) - V_2^2(\alpha_2) < 0.$$

This is a contradiction. We conclude that problem (3.1) cannot have more than one positive solution.  $\square$

In order to study (3.1) in the case of higher dimensions, the first step is to rewrite the Dirichlet problem in terms of an ODE. By an adaptation of the moving planes method to  $S^{N-1}$  (see [20]), any positive solution  $v$  of (3.1) depends only on the geodesic distance to the North pole:

$$\theta = \arccos \frac{x_N}{|x|}$$

and  $v$  decreasing with respect to  $\theta$ . Since in this case

$$\Delta'v = \frac{1}{(\sin \theta)^{N-2}} \frac{d}{d\theta} ((\sin \theta)^{N-2} v_\theta),$$

every solution of (3.1) satisfies the following ODE in terms of the variable  $\theta$ :

$$(3.4) \quad \begin{cases} v_{\theta\theta} + (N-2) \cot \theta v_\theta + \ell v + v^q = 0 & \text{in } (0, \frac{\pi}{2}), \\ v_\theta(0) = 0, \quad v(\frac{\pi}{2}) = 0. \end{cases}$$

The heart of the matter is then to apply some ideas from Kwong [13] and Kwong-Li [14], originally dealing with positive solutions of

$$(3.5) \quad \begin{cases} u_{rr} + (N-2) \frac{1}{r} u_r + \ell u + u^q = 0 & \text{in } (0, a), \\ u_r(0) = 0, \quad u(a) = 0. \end{cases}$$

By Lemma 3.1 and the discussion above, the graphs of two positive solutions of (3.4) must intersect in  $(0, \frac{\pi}{2})$ . Of course, the number of intersection points could be arbitrarily large (but always finite in view of the uniqueness of the Cauchy problem). The next lemma allows us to reduce the problem to the case where there could be only one intersection point. The argument is standard and relies on the shooting method; we only give a sketch of the proof.

**Lemma 3.2.** *Assume that (3.4) has two distinct positive solutions. Then, there exist two positive solutions of (3.4) which intersect only once in the interval  $(0, \frac{\pi}{2})$ .*

*Sketch of the proof.* Let  $v_1$  and  $v_2$  be two distinct solutions of (3.4). We may assume that  $v_1(0) > v_2(0)$ . For each  $\alpha \in [0, v_2(0)]$  let  $v^\alpha$  be the (unique) maximal positive solution of

$$\begin{cases} v_{\theta\theta} + (N-2) \cot \theta v_\theta + \ell v + v^q = 0 & \text{in } (0, \frac{\pi}{2}), \\ v_\theta(0) = 0, \quad v(0) = \alpha. \end{cases}$$

Note that  $v^\alpha = v_2$  and  $v^0 = 0$ . As we decrease  $\alpha$  from  $v_2(0)$  to 0,

- the first intersection point  $\sigma_1^\alpha$  between  $v_1$  and  $v^\alpha$  cannot tend to 0 since the derivatives of  $v^\alpha$  in the interval  $[0, \sigma_1^\alpha]$  are uniformly bounded;
- two consecutive intersection points between  $v^\alpha$  and  $v_1$  cannot get arbitrarily close for otherwise there would exist  $\gamma \in [0, v_2(0)]$  and  $\sigma \in [0, \frac{\pi}{2}]$  such that  $v^\gamma(\sigma) = v_1(\sigma)$  and  $v^{\gamma_\theta}(\sigma) = v_{1\theta}(\sigma)$ , but this is impossible.

Combining these facts, we can make the number of intersection points between  $v_1$  and  $v^\alpha$  decrease one by one, until there is left only one intersection point. For the first  $\alpha$  where this happens, we must have  $v^\alpha(\frac{\pi}{2}) = 0$ .  $\square$

The next result is standard but we present a proof for the convenience of the reader.

**Lemma 3.3.** *Assume that  $v_1$  and  $v_2$  are positive solutions of (3.4) whose graphs coincide at a single point of  $(0, \frac{\pi}{2})$ . If  $v_1(0) > v_2(0)$ , then the function*

$$\theta \in (0, \frac{\pi}{2}) \longmapsto \frac{v_2(\theta)}{v_1(\theta)}$$

*is increasing.*

*Proof.* Let  $J : [0, \frac{\pi}{2}] \rightarrow \mathbb{R}$  be the function defined as  $J = v_1 v_{2\theta} - v_2 v_{1\theta}$ . To prove the lemma, it suffices to show that  $J > 0$  in  $(0, \frac{\pi}{2})$ . Using the equations satisfied by  $v_1$  and  $v_2$ , one finds

$$J_\theta = -(N-2) \cot \theta J + (v_1^{q-1} - v_2^{q-1}) v_1 v_2.$$

Thus,

$$\frac{1}{(\sin \theta)^{N-2}} ((\sin \theta)^{N-2} J)_\theta = (v_1^{q-1} - v_2^{q-1}) v_1 v_2.$$

Let  $\sigma \in (0, \frac{\pi}{2})$  be such that  $v_1(\sigma) = v_2(\sigma)$ . Since  $v_{1\theta}(\sigma) \neq v_{2\theta}(\sigma)$ , we have  $v_1 > v_2$  in  $(0, \sigma)$  and  $v_1 < v_2$  in  $(\sigma, \frac{\pi}{2})$ , we conclude that the function

$$\theta \in [0, \frac{\pi}{2}] \longmapsto (\sin \theta)^{N-2} J(\theta)$$

is increasing in  $(0, \sigma)$  and decreasing in  $(\sigma, \frac{\pi}{2})$ . Since it vanishes at 0 and  $\frac{\pi}{2}$ , we have

$$(\sin \theta)^{N-2} J > 0 \quad \text{in } (0, \frac{\pi}{2}).$$

Thus  $J > 0$  in  $(0, \frac{\pi}{2})$  and the conclusion follows.  $\square$

The following identity will be needed in the proofs of Theorems 3.2 and 3.3.

**Lemma 3.4.** *Let  $v$  be a solution of (3.4),  $\alpha = \frac{2(N-2)}{q+3}$  and  $\beta = \frac{2(N-2)(q-1)}{q+3}$ . Set*

$$(3.6) \quad w(\theta) = (\sin \theta)^\alpha v(\theta)$$

*Then,*

$$(3.7) \quad \frac{d}{d\theta} \left( (\sin \theta)^\beta \frac{(w_\theta)^2}{2} + G(\theta) \frac{w^2}{2} + \frac{w^{q+1}}{q+1} \right) = G_\theta(\theta) \frac{w^2}{2},$$

where  $G : (0, \frac{\pi}{2}) \rightarrow \mathbb{R}$  is defined by

$$(3.8) \quad G(\theta) = \left( (\alpha(N-2-\alpha) + \ell)(\sin \theta)^2 + \alpha(\alpha+3-N) \right) (\sin \theta)^{\beta-2}.$$

*Proof.* Let  $w : (0, \frac{\pi}{2}) \rightarrow \mathbb{R}$  be the function defined by (3.6). Then,

$$w_{\theta\theta} + (N-2-2\alpha) \cot \theta w_{\theta} + \left( \alpha(N-2-\alpha) + \ell + \frac{\alpha(\alpha+3-N)}{(\sin \theta)^2} \right) w + \frac{w^q}{(\sin \theta)^{\alpha(q-1)}} = 0.$$

Multiplying this identity by  $(\sin \theta)^{\beta}$ , we get

$$(\sin \theta)^{\beta} w_{\theta\theta} + (N-2-2\alpha)(\sin \theta)^{\beta-1} \cos \theta w_{\theta} + G(\theta) w + (\sin \theta)^{\beta-\alpha(q-1)} w^q = 0$$

where  $G$  is defined by (3.8). We now observe that  $\alpha$  and  $\beta$  satisfy

$$N-2-2\alpha = \frac{\beta}{2} \quad \text{and} \quad \beta - \alpha(q-1) = 0.$$

The identity satisfied by  $w$  becomes

$$(\sin \theta)^{\beta} w_{\theta\theta} + \frac{\beta}{2} (\sin \theta)^{\beta-1} \cos \theta w_{\theta} + G(\theta) w + w^q = 0.$$

Since

$$\frac{d}{d\theta} \left( (\sin \theta)^{\beta} \frac{(w_{\theta})^2}{2} \right) = \left( (\sin \theta)^{\beta} w_{\theta\theta} + \frac{\beta}{2} (\sin \theta)^{\beta-1} \cos \theta w_{\theta} \right) w_{\theta}$$

and

$$\frac{d}{d\theta} \left( G(\theta) \frac{w^2}{2} \right) = G(\theta) w w_{\theta} + G_{\theta}(\theta) \frac{w^2}{2}$$

identity (3.7) follows.  $\square$

The following proof is inspired from Kwong-Li [14].

*Proof of Theorem 3.2.* We use the notation of Lemma 3.4. Let  $E : (0, \frac{\pi}{2}) \rightarrow \mathbb{R}$  be the function defined as

$$E(\theta) = (\sin \theta)^{\beta} \frac{(w_{\theta})^2}{2} + G(\theta) \frac{w^2}{2} + \frac{w^{q+1}}{q+1}.$$

Then, by Lemma 3.4,

$$E_{\theta} = G_{\theta} w^2.$$

We observe that  $E$  can be continuously extended at 0 and  $\frac{\pi}{2}$ . This is clear at  $\frac{\pi}{2}$ , where we take

$$E\left(\frac{\pi}{2}\right) = \frac{(w_{\theta}(\frac{\pi}{2}))^2}{2} = \frac{(v_{\theta}(\frac{\pi}{2}))^2}{2}.$$

To reach the conclusion at 0, it suffices to observe that for every  $\theta \in (0, \frac{\pi}{2})$ ,

$$\begin{aligned} (\sin \theta)^{\beta} (w_{\theta}(\theta))^2 &= (\sin \theta)^{\beta} \left( \alpha (\sin \theta)^{\alpha-1} \cos \theta v(\theta) + (\sin \theta)^{\alpha} v_{\theta}(\theta) \right)^2 \\ &= (\sin \theta)^{2\alpha+\beta-2} \left( \alpha \cos \theta v(\theta) + \sin \theta v_{\theta}(\theta) \right)^2. \end{aligned}$$

Since  $N \geq 4$ ,

$$2\alpha + \beta - 2 = \frac{2(N-3)}{q+3} \left( q + \frac{N-5}{N-3} \right) > 0,$$

the right-hand side of the previous expression converges to 0 as  $\theta \rightarrow 0$ . We can then define

$$E(0) = 0.$$

Notice that

$$G_\theta(\theta) = \left[ (\alpha(N-2-\alpha) + \ell)\beta(\sin\theta)^2 + \alpha(\alpha+3-N)(\beta-2) \right] (\sin\theta)^{\beta-3} \cos\theta.$$

By the choices of  $\alpha$  and  $\beta$ ,

$$\alpha(\alpha+3-N)(\beta-2) = \frac{4(N-2)(N-3)^2}{(q+3)^3} \left( q + \frac{N-5}{N-3} \right) \left( \frac{N+1}{N-3} - q \right).$$

Since  $N \geq 4$  and  $1 < q < \frac{N+1}{N-3}$ , this quantity is positive. Hence, there exists  $\varepsilon > 0$  such that

$$G_\theta(\theta) > 0 \quad \forall \theta \in (0, \varepsilon).$$

In view of the expression of  $G_\theta$ , we have the following possibilities: either

$$(i) \quad G_\theta > 0 \text{ in } (0, \frac{\pi}{2}),$$

or

$$(ii) \quad \text{there exists } c \in (0, \frac{\pi}{2}) \text{ such that } G_\theta > 0 \text{ in } (0, c) \text{ and } G_\theta < 0 \text{ in } (c, \frac{\pi}{2}).$$

Assume by contradiction that (3.1) has more than one solution, hence by Lemma 3.2 problem (3.4) has two positive solutions  $v_1$  and  $v_2$  whose graphs intersect exactly once in the interval  $(0, \frac{\pi}{2})$ . Without loss of generality, we may assume that  $v_1(0) > v_2(0)$ . For  $i \in \{1, 2\}$ , define  $w_i$  and  $E_i$  accordingly.

Assume that  $G$  satisfies property (i) above. Let

$$\gamma = \frac{v_{2\theta}(\frac{\pi}{2})}{v_{1\theta}(\frac{\pi}{2})}.$$

We have

$$(E_2 - \gamma^2 E_1)(0) = 0$$

and, since  $v_2 > v_1$  in a neighborhood of  $\frac{\pi}{2}$ ,

$$(E_2 - \gamma^2 E_1)(0) = \frac{(v_{2\theta}(\frac{\pi}{2}))^2 - \gamma^2 (v_{1\theta}(\frac{\pi}{2}))^2}{2} > 0.$$

Thus, by the Mean value theorem, there exists  $\sigma \in (0, \frac{\pi}{2})$  such that

$$(3.9) \quad (E_2 - \gamma^2 E_1)_\theta(\sigma) > 0.$$

On the other hand, by Lemma 3.3 the function

$$\theta \in (0, \frac{\pi}{2}) \mapsto \frac{v_2(\theta)}{v_1(\theta)}$$

is increasing. In particular, for every  $\theta \in [0, \frac{\pi}{2})$ ,

$$\frac{v_2(\theta)}{v_1(\theta)} < \lim_{\theta \rightarrow \frac{\pi}{2}^-} \frac{v_2(\theta)}{v_1(\theta)} = \frac{v_{2\theta}(\frac{\pi}{2})}{v_{1\theta}(\frac{\pi}{2})} = \gamma.$$

Hence,

$$(w_2)^2 - \gamma^2 (w_1)^2 = (\sin\theta)^{2\alpha} ((v_2)^2 - \gamma^2 (v_1)^2) < 0 \quad \text{in } (0, \frac{\pi}{2}).$$

Thus, by Lemma 3.4 and by assumption (i), we have for every  $\theta \in (0, \frac{\pi}{2})$ ,

$$(E_2 - \gamma^2 E_1)_\theta(\theta) = G_\theta(\theta) ((w_2)^2 - \gamma^2 (w_1)^2) < 0.$$

This contradicts (3.9). Therefore, problem (3.1) cannot have two distinct positive solutions if  $G$  satisfies (i).

We now assume that  $G$  satisfies property (ii) for some point  $c$ . Let

$$\tilde{\gamma} = \frac{v_2(c)}{v_1(c)}.$$

As in the previous case,

$$(E_2 - \tilde{\gamma}^2 E_1)(0) = 0 \quad \text{and} \quad (E_2 - \tilde{\gamma}^2 E_1)\left(\frac{\pi}{2}\right) > 0.$$

Thus, by the Mean value theorem, there exists  $\tilde{\sigma} \in (0, \frac{\pi}{2})$  such that

$$(3.10) \quad (E_2 - \tilde{\gamma}^2 E_1)_\theta(\tilde{\sigma}) > 0.$$

By Lemma 3.3, we have

$$\frac{v_2}{v_1} < \tilde{\gamma} \quad \text{in } (0, c) \quad \text{and} \quad \frac{v_2}{v_1} > \tilde{\gamma} \quad \text{in } (c, \frac{\pi}{2}).$$

Thus, by Lemma 3.4 and by assumption (ii), we have for every  $\theta \in (0, \frac{\pi}{2})$ ,

$$(E_2 - \tilde{\gamma}^2 E_1)_\theta(\theta) = G_\theta(\theta)((w_2)^2 - \tilde{\gamma}^2(w_1)^2) < 0.$$

This contradicts (3.10). Therefore, if  $G$  satisfies (ii), then problem (3.1) has a unique positive solution. The proof of Theorem 3.2 is complete.  $\square$

When  $N = 2$ , the proof of uniqueness of positive solutions of (3.1) is inspired from Kwong-Li [14] (Case 1 below) and Kwong [13] (Case 2 below).

*Proof of Theorem 3.3.* We split the proof in two cases:

*Case 1.*  $q > 1$  and  $\ell \leq \frac{2(3-q)}{(q+3)(q-1)}$ .

Let  $G : (0, \frac{\pi}{2}) \rightarrow \mathbb{R}$  be the function defined by (3.8). Since  $N = 3$ , we have  $\alpha = \frac{2}{q+3}$  and  $\beta = \frac{2(q-1)}{q+3}$ . Thus,

$$\alpha(\alpha + 3 - N)(\beta - 2) = \alpha^2(\beta - 2) = -\frac{32}{(q+3)^3} < 0.$$

Moreover, since by assumption  $\ell \leq \frac{2(3-q)}{(q+3)(q-1)}$ , we have

$$(\alpha(N - 2 - \alpha) + \ell)\beta + \alpha(\alpha + 3 - N)(\beta - 2) = \frac{2(q-1)}{q+3} \left[ \frac{2(q-3)}{(q+3)(q-1)} + \ell \right] \leq 0.$$

Therefore,  $G$  satisfies

(iii)  $G_\theta < 0$  in  $(0, \frac{\pi}{2})$ .

Let  $E : (0, \frac{\pi}{2}) \rightarrow \mathbb{R}$  be the function defined as

$$E(\theta) = (\sin \theta)^\beta \frac{(w_\theta)^2}{2} + G(\theta) \frac{w^2}{2} + \frac{w^{q+1}}{q+1}.$$

Then, by Lemma 3.4,

$$E_\theta = G_\theta w^2.$$

We observe that  $E$  can be continuously extended at  $\frac{\pi}{2}$ , so that

$$E\left(\frac{\pi}{2}\right) = \frac{(v_\theta(\frac{\pi}{2}))^2}{2},$$

but not at 0 since  $E(\theta)$  diverges to  $+\infty$  as  $\theta \rightarrow 0$ .

Assume by contradiction that (3.1) has more than one solution, hence by Lemma 3.2 problem (3.4) has two positive solutions  $v_1$  and  $v_2$  whose graphs intersect exactly once in the interval  $(0, \frac{\pi}{2})$ . Without loss of generality, we may assume that  $v_1(0) > v_2(0)$ . For  $i \in \{1, 2\}$ , define  $w_i$  and  $E_i$  accordingly.

Let

$$\hat{\gamma} = \frac{v_2(0)}{v_1(0)}.$$

By Lemma 3.3 the function

$$\theta \in (0, \frac{\pi}{2}) \mapsto \frac{v_2(\theta)}{v_1(\theta)}$$

is increasing. Thus,

$$(w_2)^2 - \hat{\gamma}^2(w_1)^2 = (\sin \theta)^{2\alpha} ((v_2)^2 - \hat{\gamma}^2(v_1)^2) > 0 \quad \text{in } (0, \frac{\pi}{2}).$$

By Lemma 3.4 and by assumption (iii), we have for every  $\theta \in (0, \frac{\pi}{2})$ ,

$$(3.11) \quad (E_2 - \hat{\gamma}^2 E_1)_\theta(\theta) = G_\theta(\theta) ((w_2)^2 - \hat{\gamma}^2(w_1)^2) < 0.$$

Since  $v_2 > v_1$  in a neighborhood of  $\frac{\pi}{2}$ ,

$$(E_2 - \hat{\gamma}^2 E_1)(\frac{\pi}{2}) = \frac{(v_{2\theta}(\frac{\pi}{2}))^2 - \hat{\gamma}^2(v_{1\theta}(\frac{\pi}{2}))^2}{2} > 0.$$

Although  $E_1$  and  $E_2$  cannot be continuously extended at 0, one checks that

$$\lim_{\theta \rightarrow 0} (E_2(\theta) - \hat{\gamma}^2 E_1(\theta)) = 0.$$

Thus, the function  $E_2 - \hat{\gamma}^2 E_1$  can be continuously extended at 0. By the Mean value theorem, there exists  $\hat{\sigma} \in (0, \frac{\pi}{2})$  such that

$$(E_2 - \hat{\gamma}^2 E_1)_\theta(\hat{\sigma}) > 0.$$

This contradicts (3.11). Therefore, equation (3.1) has at most one positive solution if  $q > 1$  and  $\ell \leq \frac{2(3-q)}{(q+3)(q-1)}$ .

*Case 2.*  $1 < q \leq 5$  and  $\ell > \frac{2(3-q)}{(q+3)(q-1)}$ .

Since  $1 < q \leq 5$ , we have

$$\frac{2(3-q)}{(q+3)(q-1)} + \frac{1}{8} = \frac{(5-q)(9-q)}{(q+3)(q-1)} \geq 0.$$

In particular, under the assumptions on  $q$  and  $\ell$ ,

$$\ell \geq -\frac{1}{4}.$$

(The remaining of the argument only requires  $1 < q \leq 5$  and  $\ell \geq -\frac{1}{4}$ .)

Let  $z : (0, \frac{\pi}{2}) \rightarrow \mathbb{R}$  be the function defined as

$$z(\theta) = (\sin \theta)^{\frac{1}{2}} v(\theta).$$

Then,  $z$  satisfies

$$(3.12) \quad z_{\theta\theta} + \left( \ell + \frac{1}{4} + \frac{1}{4(\sin \theta)^2} \right) z + \frac{z^q}{(\sin \theta)^{\frac{q-1}{2}}} = 0.$$

Assume by contradiction that equation (3.1) has two positive distinct solutions. Denote by  $v_1$  and  $v_2$  two solutions of (3.4) and define  $z_1$  and  $z_2$  accordingly. By Lemma 3.1, the graphs of  $v_1$  and  $v_2$  intersect in  $(0, \frac{\pi}{2})$ . Let  $\sigma_1 < \sigma_2$  be two consecutive points where the graphs of  $z_1$  and  $z_2$  intersect. Without loss of generality, we may assume that

$$z_1 < z_2 \quad \text{in } (\sigma_1, \sigma_2).$$

Let

$$\xi_1 = z_1(\sigma_1) = z_2(\sigma_1) \quad \text{and} \quad \xi_2 = z_1(\sigma_2) = z_2(\sigma_2).$$

We first show that  $z_1$  and  $z_2$  cannot be both decreasing in  $[\sigma_1, \sigma_2]$ . Assume by contradiction that  $z_1$  and  $z_2$  are decreasing in  $[\sigma_1, \sigma_2]$ . In particular,  $\sigma_1 > 0$ . We may consider their inverses  $z_i^{-1} : [\xi_2, \xi_1] \rightarrow [\sigma_1, \sigma_2]$ . For  $i \in \{1, 2\}$ , let  $Z_i : [\xi_2, \xi_1] \rightarrow \mathbb{R}$  be the function given by

$$Z_i(\xi) = z_{i\theta}(z_i^{-1}(\xi))$$

( $Z_i$  is well-defined since  $\sigma_1 > 0$ ). Since

$$z_{1\theta}(\sigma_1) < z_{2\theta}(\sigma_1) < 0 \quad \text{and} \quad z_{2\theta}(\sigma_2) < z_{1\theta}(\sigma_2) < 0,$$

we have

$$(Z_1(\xi_1))^2 > (Z_2(\xi_1))^2 \quad \text{and} \quad (Z_1(\xi_2))^2 < (Z_2(\xi_2))^2.$$

From the Mean value theorem, there exists  $\eta \in (\xi_2, \xi_1)$  such that

$$(3.13) \quad (Z_1^2)_\xi(\eta) > (Z_2^2)_\xi(\eta).$$

On the other hand, for  $i \in \{1, 2\}$  and for every  $\xi \in (\xi_2, \xi_1)$ ,

$$Z_i Z_{i\xi} = z_{i\theta\theta}(z_i^{-1}(\xi)) = -\left(\ell + \frac{1}{4} + \frac{1}{4(\sin z_i^{-1}(\xi))^2}\right)\xi - \frac{\xi^q}{(\sin z_i^{-1}(\xi))^{\frac{q-1}{2}}}.$$

Since  $z_1^{-1} < z_2^{-1}$  in  $(\xi_2, \xi_1)$ , we deduce that

$$(Z_1^2)_\xi = 2Z_1 Z_{1\xi} < 2Z_2 Z_{2\xi} = (Z_2^2)_\xi.$$

This contradicts (3.13). Therefore,  $z_1$  and  $z_2$  cannot be both decreasing in  $(\sigma_1, \sigma_2)$ .

We now show that  $z_1$  and  $z_2$  cannot be both increasing in  $(\sigma_1, \sigma_2)$ . Assume by contradiction that  $z_1$  and  $z_2$  are increasing in  $(\sigma_1, \sigma_2)$ . In particular,  $\sigma_2 < \frac{\pi}{2}$ . We may consider their inverses  $z_i^{-1} : [\xi_1, \xi_2] \rightarrow [\sigma_1, \sigma_2]$ . For  $i \in \{1, 2\}$ , let  $Y_i : [\xi_2, \xi_1] \rightarrow \mathbb{R}$  be the function defined as

$$Y_i(\xi) = z_{i\theta}(z_i^{-1}(\xi))(\sin z_i^{-1}(\xi)).$$

If  $\xi_1 = \sigma_1 = 0$ , this formula is meaningless (since  $z_i$  is not differentiable at 0) but  $Y_i$  can be continuously extended to 0 by taking  $Y_i(0) = 0$ . Since  $z_1 < z_2$  in  $(\sigma_1, \sigma_2)$ , we have

$$(3.14) \quad (Y_1(\xi_1))^2 \leq (Y_2(\xi_1))^2 \quad \text{and} \quad (Y_1(\xi_2))^2 > (Y_2(\xi_2))^2$$

(equality may happen at  $\xi_1$  if  $\xi_1 = 0$ ). On the other hand, for  $i \in \{1, 2\}$ ,

$$\begin{aligned} Y_{i\xi} &= \left( z_{i\theta\theta}(z_i^{-1}(\xi))(\sin z_i^{-1}(\xi)) + z_{i\theta}(z_i^{-1}(\xi))(\cos z_i^{-1}(\xi)) \right) \frac{1}{z_{i\theta}(z_i^{-1}(\xi))} \\ &= \left( z_{i\theta\theta}(z_i^{-1}(\xi))(\sin z_i^{-1}(\xi))^2 \right) \frac{1}{Y_i} + \cos z_i^{-1}(\xi). \end{aligned}$$

Thus,

$$Y_i Y_{i\xi} - Y_i \cos z_i^{-1}(\xi) = -\left( \left( \ell + \frac{1}{4} \right) (\sin z_i^{-1}(\xi))^2 + \frac{1}{4} \right) \xi - (\sin z_i^{-1}(\xi))^{\frac{5-q}{2}} \xi^q.$$

Since  $z_1^{-1} > z_2^{-1}$  in  $(\xi_1, \xi_2)$  and  $\ell \geq -\frac{1}{4}$ ,

$$\left( \ell + \frac{1}{4} \right) (\sin z_1^{-1}(\xi))^2 \geq \left( \ell + \frac{1}{4} \right) (\sin z_2^{-1}(\xi))^2.$$

Since  $q \leq 5$ ,

$$(\sin z_1^{-1}(\xi))^{\frac{5-q}{2}} \geq (\sin z_2^{-1}(\xi))^{\frac{5-q}{2}}.$$

We deduce that

$$Y_1 Y_{1\xi} - Y_1 \cos z_1^{-1}(\xi) \leq Y_2 Y_{2\xi} - Y_2 \cos z_2^{-1}(\xi).$$

Hence,

$$\begin{aligned} ((Y_1)^2 - (Y_2)^2)_\xi &\leq 2(Y_1 \cos z_1^{-1}(\xi) - Y_2 \cos z_2^{-1}(\xi)) \\ &\leq \frac{2 \cos z_2^{-1}(\xi)}{2} (Y_1 - Y_2) \\ &\leq \frac{2 \cos z_2^{-1}(\xi)}{Y_1 + Y_2} ((Y_1)^2 - (Y_2)^2). \end{aligned}$$

Let  $f : (\xi_1, \xi_2) \rightarrow \mathbb{R}$  be the function defined by

$$f(\xi) = \frac{2 \cos z_2^{-1}(\xi)}{Y_1(\xi) + Y_2(\xi)}.$$

Using this notation,

$$((Y_1)^2 - (Y_2)^2)_\xi \leq f(\xi) ((Y_1)^2 - (Y_2)^2).$$

Thus, for every  $\xi \in [\xi_1, \xi_2]$ ,

$$((Y_1)^2 - (Y_2)^2)(\xi) \geq ((Y_1)^2 - (Y_2)^2)(\xi_2) e^{\int_\xi^{\xi_2} f(\tau) d\tau}.$$

Since, by (3.14),

$$((Y_1)^2 - (Y_2)^2)(\xi_1) \leq 0 \quad \text{and} \quad ((Y_1)^2 - (Y_2)^2)(\xi_2) > 0,$$

we have reached a contradiction. Therefore,  $z_1$  and  $z_2$  cannot be both increasing in  $(\sigma_1, \sigma_2)$ .

It follows from equation (3.12) that both  $z_1$  and  $z_2$  are concave. By Lemma 3.1 their graphs must intersect in  $(0, \frac{\pi}{2})$ . Since  $z_1$  and  $z_2$  cannot be simultaneously increasing or decreasing between two intersection points, their graphs intersect exactly once in  $(0, \frac{\pi}{2})$ . Let  $\sigma \in (0, \frac{\pi}{2})$  be such that  $z_1(\sigma) = z_2(\sigma)$ . Without loss of generality, we may assume that

$$z_1 > z_2 \quad \text{in} \quad (0, \sigma) \quad \text{and} \quad z_1 < z_2 \quad \text{in} \quad (\sigma, \frac{\pi}{2}).$$

In particular,

$$z_{1\theta}(\sigma) < z_{2\theta}(\sigma).$$

Since  $z_1$  and  $z_2$  cannot be both increasing in  $(0, \sigma)$  and both are concave, we must have  $z_{1\theta}(\sigma) < 0$ . A similar argument using the interval  $(\sigma, \frac{\pi}{2})$  implies that  $z_{2\theta}(\sigma) > 0$ . Therefore, the maximum of  $z_1$  is achieved in  $(0, \sigma)$  while the maximum of  $z_2$  is achieved in  $(\sigma, \frac{\pi}{2})$ .

Denote the maximum of  $z_i$  by  $m_i$ . We first show that  $m_2 > m_1$ . Indeed, assume by contradiction that  $m_2 \leq m_1$ . Let  $\tilde{\sigma}_2 \in (\sigma, \frac{\pi}{2})$  be such that

$$z_2(\tilde{\sigma}_2) = m_2.$$

Let  $\tilde{\sigma}_1$  be the largest number in  $(0, \frac{\pi}{2})$  such that

$$z_1(\tilde{\sigma}_1) = m_2.$$

The restrictions  $z_i : [\tilde{\sigma}_i, \frac{\pi}{2}] \rightarrow [0, m_2]$  are both decreasing. Let  $\tilde{Z}_i : [0, m_2] \rightarrow \mathbb{R}$  be the function defined as

$$\tilde{Z}_i(\xi) = z_{i\theta}(z_i^{-1}(\xi)).$$

In the interval  $[\tilde{\sigma}_i, \frac{\pi}{2}]$  we have  $z_1^{-1} < z_2^{-1}$ , thus

$$(\tilde{Z}_1^2)_\xi < (\tilde{Z}_2^2)_\xi.$$



Since

$$(\tilde{Z}_1(0))^2 < (\tilde{Z}_2(0))^2 \quad \text{and} \quad (\tilde{Z}_1(m_2))^2 > 0 = (\tilde{Z}_2(m_2))^2,$$

we have a contradiction.

We now show that  $m_1 > m_2$ . Assume by contradiction that  $m_1 \leq m_2$ . Let  $\hat{\sigma}_1 \in (0, \sigma)$  be such that

$$z_1(\hat{\sigma}_1) = m_1.$$

Let  $\hat{\sigma}_2$  be the smallest number in  $(0, \frac{\pi}{2})$  such that

$$z_2(\hat{\sigma}_1) = m_1.$$

The restrictions  $z_i : [0, \hat{\sigma}_i] \rightarrow [0, m_1]$  are both increasing. Let  $\hat{Y}_i : [0, m_1] \rightarrow \mathbb{R}$  be the function defined as

$$\hat{Y}_i(\xi) = z_{i\theta}(z_i^{-1}(\xi))(\sin z_i^{-1}(\xi))$$

if  $\xi \neq 0$  and  $\hat{Y}_i(0) = 0$ . Then,  $\hat{Y}_i$  is continuous. In the interval  $[0, \hat{\sigma}_i]$  we have  $z_1^{-1} < z_2^{-1}$ , thus

$$((\hat{Y}_2)^2 - (\hat{Y}_1)^2)_\xi \leq \frac{2 \cos z_1^{-1}(\xi)}{\hat{Y}_2 + \hat{Y}_1} ((\hat{Y}_2)^2 - (\hat{Y}_1)^2).$$

As before, this contradicts

$$((\hat{Y}_2)^2 - (\hat{Y}_1)^2)(0) \leq 0 \quad \text{and} \quad ((\hat{Y}_2)^2 - (\hat{Y}_1)^2)(m_1) > 0.$$

Since  $m_1 > m_2$  and  $m_1 < m_2$  we have reached a contradiction. Therefore, problem (3.1) can have at most one positive solution.  $\square$

#### 4. PROOF OF THEOREM 1.1

*Proof of (i).* Assume that  $1 < q \leq q_1$ . Let  $\phi$  be a positive eigenfunction of  $-\Delta'$  in  $W_0^{1,2}(S_+^{N-1})$  associated to the first eigenvalue  $N - 1$ , and let  $\omega$  be a solution of (1.3). Using  $\phi$  as test function, we get

$$\int_{S_+^{N-1}} \langle \nabla' \omega, \nabla' \phi \rangle d\sigma = \int_{S_+^{N-1}} (\ell_{N,q} \omega + \omega^q) \phi d\sigma.$$

On the other hand, since  $\phi$  is an eigenfunction of  $-\Delta'$ ,

$$\int_{S_+^{N-1}} \langle \nabla' \omega, \nabla' \phi \rangle d\sigma = (N - 1) \int_{S_+^{N-1}} \omega \phi d\sigma.$$

Thus,

$$(4.1) \quad (N - 1 - \ell_{N,q}) \int_{S_+^{N-1}} \omega \phi d\sigma = \int_{S_+^{N-1}} \omega^q \phi d\sigma.$$

Since  $q \leq q_1$ , we have

$$N - 1 - \ell_{N,q} = \frac{(N-1)(q+1)}{(q-1)^2} \left( q - \frac{N+1}{N-1} \right) \leq 0.$$

Hence, the left-hand side of (4.1) is nonpositive while the right-hand side is non-negative. Thus,

$$\int_{S_+^{N-1}} \omega^q \phi d\sigma = 0$$

We conclude that  $\omega = 0$  in  $S_+^{N-1}$ . Hence, problem (1.3) has no positive solution.  $\square$

*Proof of (ii).* Since  $q > q_1$ ,

$$N - 1 - \ell_{N,q} = \frac{(N-1)(q+1)}{(q-1)^2} \left( q - \frac{N+1}{N-1} \right) > 0.$$

Thus, the functional  $\mathcal{J} : W_0^{1,2}(S_+^{N-1}) \rightarrow \mathbb{R}$  defined by

$$\mathcal{J}(w) = \int_{S_+^{N-1}} (|\nabla' w|^2 - \ell_{N,q} w^2) d\sigma$$

is bounded from below by 0. On the other hand, since  $q < q_3$  we can minimize  $\mathcal{J}$  over the set

$$\left\{ w \in W_0^{1,2}(S_+^{N-1}) ; \int_{S_+^{N-1}} (w^+)^{q+1} d\sigma = 1 \right\}.$$

Let  $w$  be a minimizer. Then,  $w^+$  is also a minimizer, whence  $w = w^+$  and this function satisfies

$$-\Delta' w - \ell_{N,q} w = \lambda w^q \quad \text{in } S_+^{N-1}$$

for some  $\lambda > 0$ . By standard elliptic regularity theory,  $w$  is smooth and vanishes on  $\partial S_+^{N-1}$  in the classical sense. The function  $\lambda^{\frac{1}{q-1}} w$  is therefore a solution of (1.3).  $\square$

*Proof of (iii).* We may assume that  $N \geq 4$ , for otherwise there is nothing to prove. Note that if  $q \geq q_3$ ,

$$\frac{N-1}{q-1} - \ell_{N,q} = -\frac{N-3}{(q-1)^2} \left( q - \frac{N+1}{N-3} \right) \leq 0.$$

Applying Corollary 2.1, we deduce that (1.3) has no positive solution.  $\square$

## 5. THE A PRIORI ESTIMATE

In this section we establish Theorem 1.5 whose proof is based on the following result.

**Proposition 5.1.** *Assume that  $1 < q < q_2$ . Let  $0 < r < \frac{1}{2} \text{diam } \Omega$  and  $\zeta \in C^\infty(\partial\Omega)$  with  $\zeta \geq 0$  on  $\partial\Omega$ . Then, every solution of*

$$(5.1) \quad \begin{cases} -\Delta u = u^q & \text{in } \Omega \cap (B_{2r} \setminus \overline{B}_r), \\ u \geq 0 & \text{in } \Omega \cap (B_{2r} \setminus \overline{B}_r), \\ u = \zeta & \text{on } \partial\Omega \cap (B_{2r} \setminus \overline{B}_r), \end{cases}$$

satisfies

$$(5.2) \quad u(x) \leq C [\text{dist}(x, \Gamma_r)]^{-\frac{2}{q-1}} \quad \forall x \in \Omega \cap (B_{2r} \setminus \overline{B}_r),$$

where  $\Gamma_r = \overline{\Omega} \cap (\partial B_{2r} \cup \partial B_r)$  and  $C > 0$  is a constant independent of  $u$ .

We denote by  $B_r$  the ball of radius  $r$  centered at 0. The proof of this estimate is based on two results: a Liouville theorem for the equation  $-\Delta u = u^q$  in  $\mathbb{R}^N$  or in  $\mathbb{R}_+^N$  (see [7]) and the Doubling lemma of Poláčik-Quittner-Souplet [22] which we recall:

**Lemma 5.1.** *Let  $(X, d)$  be a complete metric space,  $\Gamma \subsetneq X$  and  $\gamma : X \setminus \Gamma \rightarrow (0, +\infty)$ . Assume that  $\gamma$  is bounded on all compact subsets of  $X \setminus \Gamma$ . Given  $k > 0$ , let  $y \in X \setminus \Gamma$  be such that*

$$\gamma(y) \operatorname{dist}(y, \Gamma) > 2k.$$

*Then, there exists  $x \in X \setminus \Gamma$  such that*

- $\gamma(x) \operatorname{dist}(x, \Gamma) > 2k$ ;
- $\gamma(x) \geq \gamma(y)$ ;
- $2\gamma(x) \geq \gamma(z)$ ,  $\forall z \in B_{k/\gamma(x)}(x)$ .

*Proof of Proposition 5.1.* To simplify the notation we may assume that  $\zeta \equiv 0$ . Assume by contradiction that (5.2) is false. Then, for every integer  $k \geq 1$  there exist  $0 < r_k < \frac{1}{2} \operatorname{diam} \Omega$ , a solution  $u_k$  of (5.1) with  $r = r_k$ , and  $y_k \in \Omega \cap (B_{2r_k} \setminus \overline{B}_{r_k})$  such that

$$u_k(y_k) > (2k)^{\frac{2}{q-1}} [\operatorname{dist}(y_k, \Gamma_{r_k})]^{-\frac{2}{q-1}}.$$

Applying the previous lemma with

$$X = \overline{\Omega} \cap (\overline{B}_{2r_k} \setminus B_{r_k}) \quad \text{and} \quad \gamma = u_k^{\frac{q-1}{2}},$$

one finds  $x_k \in X \setminus \Gamma_{r_k}$  such that

- (i)  $u_k(x_k) > (2k)^{\frac{2}{q-1}} [\operatorname{dist}(x_k, \Gamma_{r_k})]^{-\frac{2}{q-1}}$ ;
- (ii)  $u_k(x_k) \geq u_k(y_k)$ ;
- (iii)  $2^{\frac{2}{q-1}} u_k(x_k) \geq u_k(z)$ ,  $\forall z \in B_{R_k}(x_k) \cap \Omega$ , with  $R_k = k[u_k(x_k)]^{-\frac{q-1}{2}}$ .

By (i) we have  $R_k < \frac{1}{2} \operatorname{dist}(x_k, \Gamma_{r_k})$  and thus

$$B_{R_k}(x_k) \cap \Gamma_{r_k} = \emptyset.$$

Since  $\operatorname{dist}(x_k, \Gamma_{r_k}) \leq \frac{1}{2} r_k < \frac{1}{4} \operatorname{diam} \Omega$ , we also deduce from (i) that

$$u_k(x_k) \geq \left( \frac{8k}{\operatorname{diam} \Omega} \right)^{\frac{2}{q-1}}.$$

In particular,

$$u_k(x_k) \rightarrow +\infty \quad \text{as } k \rightarrow +\infty.$$

For every  $k \geq 1$ , let

$$t_k = [u_k(x_k)]^{-\frac{q-1}{2}},$$

$$D_k = \left\{ \xi \in \mathbb{R}^N; |\xi| \leq k \text{ and } x_k + t_k \xi \in \Omega \right\}$$

and

$$v_k(\xi) = \frac{1}{u_k(x_k)} u_k(x_k + t_k \xi) \quad \forall \xi \in D_k.$$

Then,  $v_k$  satisfies

$$-\Delta v_k = v_k^q, \quad 0 \leq v_k \leq 2^{\frac{2}{q-1}} \quad \text{and} \quad v_k(0) = 1.$$

Passing to a subsequence if necessary, we may assume that either

(A) for every  $a > 0$  there exists  $k_0 \geq 1$  such that if  $k \geq k_0$ , then  $B_{at_k}(x_k) \cap \partial\Omega = \emptyset$ ,

or

(B) there exists  $a_0 > 0$  such that for every  $k \geq 1$ ,  $B_{a_0 t_k}(x_k) \cap \partial\Omega \neq \emptyset$ .

Since the sequence  $(v_k)$  is uniformly bounded, it follows that  $(\Delta v_k)$  is also uniformly bounded. In both cases, by elliptic (interior and boundary) estimates, we have for every  $1 < p < +\infty$  and every  $s > 0$ ,

$$\|v_k\|_{W^{2,p}(D_k \cap B_s)} \leq C_{s,p}.$$

If (A) holds, then up to a subsequence  $(v_k)$  converges locally uniformly in  $\mathbb{R}^N$  to some smooth function  $v$  such that

$$-\Delta v = v^q, \quad 0 \leq v \leq 2^{\frac{2}{q-1}} \quad \text{and} \quad v(0) = 1.$$

On the other hand, if (B) holds, then up to a subsequence and a rotation of the domain there exists some smooth function  $v$  defined in  $\mathbb{R}_+^N$  such that  $(v_k)$  converges locally uniformly to  $v$ . Since the sequence  $(v_k)$  is equicontinuous and for every  $k \geq 1$ ,  $v_k(0) = 0$ , we have  $v(0) = 1$ .

In both cases, we deduce that  $v$  is a nontrivial bounded solution of

$$-\Delta v = v^q$$

in  $\mathbb{R}^N$  or in  $\mathbb{R}_+^N$ , which is impossible (see [7]). Therefore, estimate (5.2) must hold.  $\square$

*Proof of Theorem 1.5.* It suffices to establish (1.6) if  $x \in \Omega$  and  $|x| < \frac{3}{4} \text{diam} \Omega$ . For this purpose, we apply Proposition 5.1 with  $r = \frac{2}{3}|x|$ . Since  $\text{dist}(x, \Gamma_r) = \frac{1}{3}r$ , we deduce that

$$u(x) \leq C[\text{dist}(x, \Gamma_r)]^{-\frac{2}{q-1}} = C\left(\frac{r}{3}\right)^{-\frac{2}{q-1}} = \tilde{C}|x|^{-\frac{2}{q-1}}.$$

This establishes the result.  $\square$

## 6. THE GEOMETRIC AND ANALYTIC FRAMEWORK

We recall some of the preliminaries and the geometric framework in [12] which will be used in the remaining of the paper.

We denote by  $(x_1, \dots, x_N)$  the coordinates of  $x \in \mathbb{R}^N$  and by  $\mathcal{B} = \{\mathbf{e}_1, \dots, \mathbf{e}_N\}$  the canonical orthonormal basis in  $\mathbb{R}^N$ . Since we are assuming that the outward unit normal vector is  $-\mathbf{e}_N$ ,  $\partial\Omega$  is the graph of a smooth function in a neighborhood of 0. In other words, there exist a neighborhood  $G$  of 0 and a smooth function  $\phi : G \cap T_0\Omega \rightarrow \mathbb{R}$  such that

$$G \cap \partial\Omega = \left\{ (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} ; x' \in G \cap T_0\Omega \text{ and } x_N = \phi(x') \right\}.$$

Furthermore,

$$\phi(0) = 0 \quad \text{and} \quad \nabla\phi(0) = 0.$$

Setting  $\Phi(x) = y$ , with  $y_i = x_i$  if  $i = 1, \dots, N-1$  and  $y_N = x_N - \phi(x')$ , we can assume that  $\Phi$  is a  $C^\infty$  diffeomorphism from  $G$  to  $\tilde{G} = \Phi(G)$ , and  $\Phi(\Omega \cap G) = \tilde{G} \cap \mathbb{R}_+^N$ . To avoid introducing some additional notation, we will assume that

$$\tilde{G} = B_1.$$

Given  $\zeta \in C^\infty(\partial\Omega)$ , let  $z$  be the harmonic extension of  $\zeta$  in  $\Omega$ . For every solution  $u$  of (1.1), we denote

$$u(x) - z(x) = \tilde{u}(y), \quad z(x) = \tilde{z}(y) \quad \text{and} \quad \zeta(x) = \tilde{\zeta}(y),$$

for every  $x = \Phi^{-1}(y)$  with  $y \in \tilde{G} \cap \mathbb{R}_+^N$ . Since  $u$  is superharmonic and  $u = z$  on  $\partial\Omega$ , we have  $\tilde{u} \geq 0$ . On the other hand, a straightforward computation yields

$$\Delta u = \Delta \tilde{u} + |\nabla \phi|^2 \tilde{u}_{y_N, y_N} - 2\langle \nabla \phi, \nabla \tilde{u}_{y_N} \rangle - \tilde{u}_{y_N} \Delta \phi$$

Thus,  $\tilde{u}$  satisfies the equation

$$-\Delta \tilde{u} - |\nabla \phi|^2 \tilde{u}_{y_N, y_N} + 2\langle \nabla \phi, \nabla \tilde{u}_{y_N} \rangle + \tilde{u}_{y_N} \Delta \phi = (\tilde{u} + \tilde{z})^q.$$

Rewriting this equation in terms of spherical coordinates, one obtains

$$\begin{aligned} (1 + \eta_1) \tilde{u}_{rr} + \frac{1}{r^2} \Delta' \tilde{u} + \frac{N-1 + \eta_2}{r} \tilde{u}_r + (\tilde{u} + \tilde{z})^q &= \\ &= \frac{1}{r^2} \langle \nabla' \tilde{u}, \vec{\eta}_3 \rangle + \frac{1}{r} \langle \nabla' \tilde{u}_r, \vec{\eta}_4 \rangle + \frac{1}{r^2} \langle \nabla' \langle \nabla' \tilde{u}, \mathbf{e}_N \rangle, \vec{\eta}_5 \rangle. \end{aligned}$$

where

$$\begin{aligned} \eta_1 &= -2\phi_r \langle \mathbf{n}, \mathbf{e}_N \rangle + |\nabla \phi|^2 \langle \mathbf{n}, \mathbf{e}_N \rangle^2, \\ \eta_2 &= -r \langle \mathbf{n}, \mathbf{e}_N \rangle \Delta \phi - 2\langle \nabla' \langle \mathbf{n}, \mathbf{e}_N \rangle, \nabla' \phi \rangle + r |\nabla \phi|^2 \langle \nabla' \langle \mathbf{n}, \mathbf{e}_N \rangle, \mathbf{e}_N \rangle, \\ \vec{\eta}_3 &= -\left(2\phi_r - |\nabla \phi|^2 \langle \mathbf{n}, \mathbf{e}_N \rangle - r \Delta \phi\right) \mathbf{e}_N, \\ \vec{\eta}_4 &= -\left(|\nabla \phi|^2 \langle \mathbf{n}, \mathbf{e}_N \rangle - 2\phi_r\right) \mathbf{e}_N + \frac{2}{r} \langle \mathbf{n}, \mathbf{e}_N \rangle \nabla' \phi, \\ \vec{\eta}_5 &= -|\nabla \phi|^2 \mathbf{e}_N + \frac{2}{r} \nabla' \phi. \end{aligned}$$

Taking into account the fact that  $\phi(0) = 0$  and  $\nabla \phi(0) = 0$ ,

$$|\phi(x)| \leq Cr^2, \quad |D\phi(x)| \leq Cr \quad \text{and} \quad |D^2\phi| \leq C.$$

Thus, for every  $j = 1, \dots, 5$ ,

$$\|\eta_j(r, \cdot)\|_{L^\infty} \leq Cr \quad \forall r \in (0, 1).$$

**Lemma 6.1.** *Let*

$$(6.1) \quad t = \log \frac{1}{r}, \quad v(t, \sigma) = r^{\frac{2}{q-1}} \tilde{u}(r, \sigma) \quad \text{and} \quad \alpha(t, \sigma) = r^{\frac{2}{q-1}} \tilde{z}(r, \sigma).$$

*Then,  $v$  satisfies*

$$(6.2) \quad (1 + \epsilon_1) v_{tt} + \Delta' v - \left(N - \frac{2(q+1)}{q-1} + \epsilon_2\right) v_t + (\ell_{N,q} + \epsilon_3) v + (v + \alpha)^q = \\ = \langle \nabla' v, \vec{\epsilon}_4 \rangle + \langle \nabla' v_t, \vec{\epsilon}_5 \rangle + \langle \nabla' \langle \nabla' v, \mathbf{e}_N \rangle, \vec{\epsilon}_6 \rangle,$$

*where  $\epsilon_j$  are functions defined in  $(0, +\infty) \times S_+^{N-1}$  satisfying the estimates*

$$(6.3) \quad \|\epsilon_j(t, \cdot)\|_{L^\infty} \leq Ce^{-t} \quad \forall t \geq 0,$$

*for every  $j = 1, \dots, 6$ .*

We refer the reader to [12] for the proof of Lemma 6.1 and for the explicit expressions of the functions  $\epsilon_j$ .

For every  $T \geq 0$  and  $\delta > 0$ , let

$$Q_T = (T, +\infty) \times S_+^{N-1} \quad \text{and} \quad Q_{T,\delta} = (T - \delta, T + \delta) \times S_+^{N-1}.$$

We have the following  $W^{2,p}$ -estimates satisfied by  $v$ :

**Proposition 6.1.** *Let  $v$  be defined as in Lemma 6.1. If  $v$  is uniformly bounded in  $Q_0$ , then for every  $1 < p < +\infty$ ,*

$$(6.4) \quad \|v\|_{W^{2,p}(Q_{T,1})} \leq C \left( \|v\|_{L^2(Q_{T,2})} + e^{-\frac{2T}{q-1}} \right) \quad \forall T \geq 2,$$

for some positive constant depending on  $\|v\|_{L^\infty}$  and on  $p$ .

*Proof.* Since  $\Delta'$  is uniformly elliptic and  $\Phi$  is a diffeomorphism, the operator  $L$  given by

$$L(v) = (1 + \epsilon_1) v_{tt} + \Delta' v - \left( N - \frac{2(q+1)}{q-1} + \epsilon_2 \right) v_t + \\ - \langle \nabla' v, \vec{\epsilon}_4 \rangle - \langle \nabla' v_t, \vec{\epsilon}_5 \rangle - \langle \nabla' \langle \nabla' v, \mathbf{e}_N \rangle, \vec{\epsilon}_6 \rangle$$

is uniformly elliptic. Let  $\delta > 0$ . By the Agmon-Douglis-Nirenberg estimates (see [1]) applied to the restriction of  $v$  on the set  $Q_{T,1+\delta}$ ,

$$\|v\|_{W^{2,p}(Q_{T,1+\delta})} \leq C \left( \|v\|_{L^p(Q_{T,1+2\delta})} + \|(\alpha + v)^q\|_{L^p(Q_{T,1+2\delta})} \right).$$

Since  $\alpha$  and  $v$  are uniformly bounded in  $Q_0$ , for every  $s \in (1, 2)$  we have

$$\|(\alpha + v)^q\|_{L^p(Q_{T,s})} \leq \|\alpha + v\|_{L^\infty(Q_{T,s})}^{q-1} \|\alpha + v\|_{L^p(Q_{T,s})} \\ \leq C \left( \|\alpha\|_{L^p(Q_{T,s})} + \|v\|_{L^p(Q_{T,s})} \right)$$

Since  $\tilde{z}$  is uniformly bounded in  $\Omega$ ,

$$\|\alpha\|_{L^p(Q_{T,s})} \leq C e^{-\frac{2T}{q-1}} \|\tilde{z}\|_{L^\infty(\Omega)} \leq C e^{-\frac{2T}{q-1}}.$$

Thus,

$$(6.5) \quad \|v\|_{W^{2,p}(Q_{T,1+\delta})} \leq C \left( \|v\|_{L^p(Q_{T,1+2\delta})} + e^{-\frac{2T}{q-1}} \right).$$

In particular,

$$\|v\|_{W^{2,p}(Q_{T,1})} \leq C \left( \|v\|_{L^p(Q_{T,\frac{3}{2}})} + e^{-\frac{2T}{q-1}} \right).$$

By a bootstrap argument based on the estimate (6.5) above and the Sobolev imbedding, we also have

$$\|v\|_{L^p(Q_{T,\frac{3}{2}})} \leq C \left( \|v\|_{L^2(Q_{T,2})} + e^{-\frac{2T}{q-1}} \right).$$

Combining these inequalities, the estimate follows.  $\square$

## 7. REMOVABLE SINGULARITIES AT 0

The goal of this section is to show that solutions of (1.1) which are not too large in a neighborhood of 0 must be continuous at 0.

**Theorem 7.1.** *Let  $q > q_1$  and let  $u$  be a solution of (1.1). If*

$$(7.1) \quad \lim_{x \rightarrow 0} |x|^{\frac{2}{q-1}} u(x) = 0,$$

then  $u$  can be continuously extended at 0.

*Proof.* Let  $v$  be the function given by (6.1). By assumption (7.1), we have

$$(7.2) \quad \lim_{t \rightarrow +\infty} v(t, \cdot) = 0 \quad \text{uniformly in } S_+^{N-1}.$$

We now rewrite (6.2) under the form

$$(7.3) \quad v_{tt} - \left( N - \frac{2(q+1)}{q-1} \right) v_t + \ell_{N,q} v + \Delta' v + (v + \alpha)^q = H,$$

where  $H$  is given by

$$(7.4) \quad H = -\epsilon_1 v_{tt} + \epsilon_2 v_t - \epsilon_3 v + \langle \nabla' v, \vec{\epsilon}_4 \rangle + \langle \nabla' v_t, \vec{\epsilon}_5 \rangle + \langle \nabla' \langle \nabla' v, \mathbf{e}_N \rangle, \vec{\epsilon}_6 \rangle.$$

Thus,

$$(7.5) \quad \int_{S_+^{N-1}} v v_{tt} d\sigma - \left( N - \frac{2(q+1)}{q-1} \right) \int_{S_+^{N-1}} v v_t d\sigma + \ell_{N,q} \int_{S_+^{N-1}} v^2 d\sigma + \int_{S_+^{N-1}} v \Delta' v d\sigma + \int_{S_+^{N-1}} v(v + \alpha)^q d\sigma = \int_{S_+^{N-1}} v H d\sigma.$$

Let

$$X(t) = \|v(t, \cdot)\|_{L^2(S_+^{N-1})} \quad \forall t \geq 0.$$

Note that for every  $t > 0$ ,

$$(7.6) \quad X X_t = \int_{S_+^{N-1}} v v_t d\sigma.$$

Using Hölder's inequality we have

$$|X X_t| \leq \|v(t, \cdot)\|_{L^2(S_+^{N-1})} \|v_t(t, \cdot)\|_{L^2(S_+^{N-1})}.$$

Thus,

$$(7.7) \quad |X_t| \leq \|v_t(t, \cdot)\|_{L^2(S_+^{N-1})}.$$

Computing the derivative with respect to  $t$  on both sides of identity (7.6), we get

$$(X_t)^2 + X X_{tt} = \int_{S_+^{N-1}} (v_t)^2 d\sigma + \int_{S_+^{N-1}} v v_{tt} d\sigma = \|v_t(t, \cdot)\|_{L^2(S_+^{N-1})}^2 + \int_{S_+^{N-1}} v v_{tt} d\sigma.$$

From this identity and estimate (7.7), we deduce that

$$(7.8) \quad X X_{tt} \geq \int_{S_+^{N-1}} v v_{tt} d\sigma.$$

On the other hand, since the first eigenvalue of the Laplace-Beltrami operator  $-\Delta'$  in  $W_0^{1,2}(S_+^{N-1})$  is  $N - 1$ ,

$$(N - 1) X^2 \leq \int_{S_+^{N-1}} |\nabla' v|^2 d\sigma = - \int_{S_+^{N-1}} v \Delta' v d\sigma.$$

By Hölder's inequality,

$$\int_{S_+^{N-1}} v H d\sigma \leq X \|H(t, \cdot)\|_{L^2(S_+^{N-1})}.$$

From the elementary inequality

$$(v + \alpha)^q \leq 2^q(v^q + \alpha^q),$$

we get

$$\int_{S_+^{N-1}} v(v + \alpha)^q d\sigma \leq 2^q \int_{S_+^{N-1}} (v^{q+1} + v\alpha^q) d\sigma$$

It follows from Hölder's inequality that

$$(7.9) \quad \int_{S_+^{N-1}} v(v + \alpha)^q d\sigma \leq 2^q \left( X^2 \|v(t, \cdot)\|_{L^\infty(S_+^{N-1})}^{q-1} + X \|\alpha(t, \cdot)\|_{L^{2q}(S_+^{N-1})}^q \right).$$

We may assume that  $u$  is a nontrivial solution of (1.1). By the strong maximum principle, we have  $u > 0$  in  $\Omega$ , thus  $X > 0$ . Combining (7.5), (7.6) and (7.8)–(7.9), one gets

$$\begin{aligned} X_{tt} - \left( N - \frac{2(q+1)}{q-1} \right) X_t + \left( \ell_{N,q} - N + 1 + 2^q \|v(t, \cdot)\|_{L^\infty}^{q-1} \right) X &\geq \\ &\geq - \left( \|H(t, \cdot)\|_{L^2} + 2^q \|\alpha(t, \cdot)\|_{L^{2q}}^q \right) \end{aligned}$$

(to simplify the notation we drop the explicit dependence of the set  $S_+^{N-1}$ ). From the definition of the function  $\alpha$ , there exists  $C > 0$  such that

$$2^q \|\alpha(t, \cdot)\|_{L^{2q}}^q \leq C e^{-\frac{2qt}{q-1}}.$$

In view of (7.2), given  $\varepsilon > 0$  there exists  $t_0 > 0$  such that

$$2^q \|v(t, \cdot)\|_{L^\infty}^{q-1} \leq \varepsilon \quad \text{on } [t_0, \infty).$$

We deduce that for every  $t \geq t_0$  we have

$$X_{tt} - \left( N - \frac{2(q+1)}{q-1} \right) X_t + (\ell_{N,q} - N + 1 + \varepsilon) X \geq -\|H(t, \cdot)\|_{L^2} - C e^{-\frac{2qt}{q-1}}.$$

We shall show that

$$X(t) \leq C e^{-\frac{2t}{q-1}} \quad \forall t \geq 0,$$

The conclusion will now follow from a bootstrap argument. Note that the linear equation

$$Z_{tt} - \left( N - \frac{2(q+1)}{q-1} \right) Z_t + (\ell_{N,q} - N + 1) Z = 0$$

has two linearly independent solutions:

$$Z_1(t) = e^{-\frac{q+1}{q-1}t} \quad \text{and} \quad Z_2(t) = e^{(N - \frac{q+1}{q-1})t}$$

We can then take  $\varepsilon > 0$  small enough so that the linear equation

$$Z_{tt} - \left( N - \frac{2(q+1)}{q-1} \right) Z_t + (\ell_{N,q} - N + 1 + \varepsilon) Z = 0$$

has two linearly independent solutions:

$$Z_{1,\varepsilon}(t) = e^{r_{1,\varepsilon}t} \quad \text{and} \quad Z_{2,\varepsilon}(t) = e^{r_{2,\varepsilon}t}$$

such that

$$r_{1,\varepsilon} < -\frac{2}{q-1} \quad \text{and} \quad r_{2,\varepsilon} > 0.$$

In particular,

$$Z_{2,\varepsilon}(t) \rightarrow +\infty \quad \text{as } t \rightarrow +\infty.$$



From assumption (7.1),  $v$  is bounded. In view of (6.3) and Proposition 6.1 with  $p = 2$ , there exists  $C_1 > 0$  such that

$$\|H(t, \cdot)\|_{L^2} \leq C_1 e^{-t} \quad \forall t \geq 0.$$

Thus,

$$X_{tt} - \left(N - \frac{2(q+1)}{q-1}\right) X_t + (\ell_{N,q} - N + 1) X \geq -\hat{C}_1 e^{-t}.$$

Since

$$X(t) \rightarrow 0 \quad \text{as } t \rightarrow +\infty,$$

from the maximum principle there exists a constant  $\tilde{C}_1 > 0$  such that

$$X(t) \leq \tilde{C}_1 (Z_{1,\varepsilon}(t) + e^{-t}).$$

If  $r_{1,\varepsilon} \geq -1$ , then

$$X(t) \leq 2\tilde{C}_1 Z_{1,\varepsilon}(t).$$

Since  $r_{1,\varepsilon} < -\frac{2}{q-1}$ , the estimate above implies that  $u$  is bounded and thus by standard elliptic estimates  $u$  is continuous. Otherwise  $r_{1,\varepsilon} < -1$ , in which case,

$$X(t) \leq 2\tilde{C}_1 e^{-t}.$$

Thus, by Proposition 7.1 for every  $T \geq 2$ ,

$$\|v\|_{W^{2,2}(Q_{T,2})} \leq \tilde{C}_1 e^{-T}.$$

In view of (6.3), there exists  $C_2 > 0$  such that

$$\|H(t, \cdot)\|_{L^2} \leq C_2 e^{-2t} \quad \forall t \geq 0.$$

Thus,

$$X_{tt} - \left(N - \frac{2(q+1)}{q-1}\right) X_t + (\ell_{N,q} - N + 1) X \geq -\hat{C}_2 e^{-2t}.$$

This implies as before that

$$X(t) \leq \tilde{C}_2 (Z_{1,\varepsilon}(t) + e^{-2t}).$$

If  $r_{1,\varepsilon} \geq -2$ , then

$$X(t) \leq 2\tilde{C}_2 Z_{1,\varepsilon}(t)$$

and  $u$  is bounded. Otherwise  $r_{1,\varepsilon} < -2$ , in which case,

$$X(t) \leq 2\tilde{C}_2 e^{-2t}.$$

We can continue this argument and deduce in finitely many steps that

$$X(t) \leq 2\tilde{C}_k Z_{1,\varepsilon}(t).$$

Applying Proposition 6.1 with  $p > \frac{N}{2}$ , we deduce that for every  $T \geq 2$ ,

$$\|v\|_{W^{2,p}(Q_{T,1})} \leq C \left( Z_{1,\varepsilon}(T) + e^{-\frac{2T}{q-1}} \right) \leq C e^{-\frac{2T}{q-1}}.$$

Thus, by Morrey's embedding,

$$\|v\|_{L^\infty(Q_{T,1})} \leq C e^{-\frac{2T}{q-1}}.$$

This implies that  $u$  is bounded and hence continuous in  $\bar{\Omega}$ . □

The conclusion of Theorem 7.1 is false with the critical exponent  $q = q_1$ . In fact, combining Theorem 1.3 and the result of del Pino-Musso-Pacard mentioned in the Introduction (Theorem 1.4), when  $q = q_1$  there exist solutions of (1.1) such that

$$u(x) \sim x_N |x|^{-N} \left( \log \frac{1}{|x|} \right)^{-\frac{N-1}{2}}$$

in a neighborhood of 0. These solutions are necessarily discontinuous at 0 but, since  $\frac{2}{q_1-1} = N-1$ ,

$$\lim_{x \rightarrow 0} |x|^{\frac{2}{q_1-1}} u(x) = 0.$$

The right statement in this case is the following:

**Theorem 7.2.** *Let  $q = q_1$  and let  $u$  be a nonnegative solution of (1.1). If*

$$\lim_{x \rightarrow 0} |x|^{N-1} \left( \log \frac{1}{|x|} \right)^{\frac{N-1}{2}} u(x) = 0,$$

*then  $u$  can be continuously extended at 0.*

*Proof.* Let

$$W(t) = t^{\frac{N-1}{2}} \|v(t, \cdot)\|_{L^2(S_+^{N-1})} \quad \forall t \geq 0,$$

where  $v$  is the function given by (6.1). By assumption,  $W(t) \rightarrow 0$  as  $t \rightarrow +\infty$ . As in the proof of Theorem 7.1, for any  $\varepsilon > 0$  there exists  $t_0 > 0$  such that for every  $t \geq t_0$ ,

$$\begin{aligned} W_{tt} + \left(N - \frac{N-1}{t}\right) W_t + \frac{1}{t} \left(-\frac{N(N-1)}{2} + \varepsilon + \frac{N^2-1}{4t}\right) W &\geq \\ &\geq -t^{\frac{N-1}{2}} \|H(t, \cdot)\|_{L^2} - Ct^{\frac{N-1}{2}} e^{-(N+1)t}. \end{aligned}$$

The linear equation

$$W_{tt} + \left(N - \frac{N-1}{t}\right) W_t + \frac{1}{t} \left(-\frac{N(N-1)}{2} + \frac{N^2-1}{4t}\right) W = 0$$

has two linearly independent solutions  $W_1$  and  $W_2$  such that for  $t$  sufficiently large (see Lemma A.2 below)

$$W_1(t) = t^{\frac{N-1}{2}} e^{-Nt} (1 + o(1)) \quad \text{and} \quad W_2(t) = t^{\frac{N-1}{2}} (1 + o(1)).$$

We can then take  $\varepsilon > 0$  small enough so that the linear equation

$$W_{tt} + \left(N - \frac{N-1}{t}\right) W_t + \frac{1}{t} \left(-\frac{N(N-1)}{2} + \varepsilon + \frac{N^2-1}{4t}\right) W = 0$$

has two linearly independent solutions  $W_{1,\varepsilon}$  and  $W_{2,\varepsilon}$  such that

$$W_{1,\varepsilon}(t) \leq Ct^{\frac{N-1}{2}} e^{-(N-1)t}$$

and

$$W_{2,\varepsilon}(t) \rightarrow +\infty \quad \text{as } t \rightarrow +\infty.$$

In view of (6.3) and Proposition 6.1 with  $p = 2$ , there exists  $C_1 > 0$  such that

$$\|H(t, \cdot)\|_{L^2} \leq C_1 t^{-\frac{N-1}{2}} e^{-t} \quad \forall t > 0.$$

Thus,

$$W_{tt} + \left(N - \frac{N-1}{t}\right) W_t + \frac{1}{t} \left(-\frac{N(N-1)}{2} + \varepsilon + \frac{N^2-1}{4t}\right) W \geq -Ce^{-t}.$$

Since

$$W(t) \rightarrow 0 \quad \text{as } t \rightarrow +\infty,$$

from the maximum principle there exists a constant  $\tilde{C}_1 > 0$  such that

$$W(t) \leq \tilde{C}_1(W_{1,\varepsilon}(t) + e^{-t}).$$

Thus,

$$W(t) \leq \hat{C}_1 e^{-t}.$$

Thus, by Proposition 6.1 with  $p = 2$ , for every  $T \geq 2$ ,

$$\|v\|_{W^{2,2}(Q_{T,2})} \leq \hat{C}_1 t^{-\frac{N-1}{2}} e^{-T}.$$

In view of (6.3), there exists  $\hat{C}_2 > 0$  such that

$$\|H(t, \cdot)\|_{L^2} \leq \hat{C}_2 t^{-\frac{N-1}{2}} e^{-2t} \quad \forall t \geq 0.$$

We can continue this argument as in the previous theorem and deduce after finitely many steps that

$$W(t) \leq \hat{C}_k W_{1,\varepsilon}(t) \leq \tilde{C} t^{\frac{N-1}{2}} e^{-(N-1)t},$$

which implies that  $u$  is bounded and hence continuous in  $\bar{\Omega}$ .  $\square$

## 8. PROOF OF THEOREM 1.2

We first establish the following

**Proposition 8.1.** *Let  $q_1 \leq q < q_3$ , with  $q \neq q_2$ . If  $u$  is a solution of (1.1) such that*

$$|x|^{\frac{2}{q-1}} u(x) \text{ is bounded in } \Omega,$$

*then for every  $\varepsilon > 0$  there exists  $\delta > 0$  such that if  $x \in \Omega \setminus \{0\}$ ,  $\frac{x}{|x|} \in S_+^{N-1}$  and  $|x| < \delta$ , then*

$$(8.1) \quad \left| |x|^{\frac{2}{q-1}} u(x) - w\left(\frac{x}{|x|}\right) \right| < \varepsilon$$

*where  $w$  is a solution of (1.3).*

*Proof.* Let  $v$  be the function given by (6.1). We first rewrite equation (6.2) under the form

$$(8.2) \quad v_{tt} + \ell_{N,q} v + \Delta' v + (v + \alpha)^q - \left(N - \frac{2(q+1)}{q-1}\right) v_t = H,$$

where  $H$  is given by (7.4). Multiplying (8.2) by  $v_t$  and integrating over  $S_+^{N-1}$  yields

$$\begin{aligned} \int_{S_+^{N-1}} v_t v_{tt} d\sigma + \ell_{N,q} \int_{S_+^{N-1}} v_t v d\sigma + \int_{S_+^{N-1}} v_t \Delta' v d\sigma + \int_{S_+^{N-1}} v_t (v + \alpha)^q d\sigma + \\ - \left(N - \frac{2(q+1)}{q-1}\right) \int_{S_+^{N-1}} (v_t)^2 d\sigma = \int_{S_+^{N-1}} v_t H d\sigma. \end{aligned}$$

Thus,

$$(8.3) \quad \begin{aligned} \frac{d}{dt} \int_{S_+^{N-1}} \left[ \frac{(v_t)^2}{2} + \frac{\ell_{N,q} v^2}{2} - \frac{|\nabla' v|^2}{2} + \frac{(v+\alpha)^{q+1}}{q+1} \right] d\sigma - \left( N - \frac{2(q+1)}{q-1} \right) \int_{S_+^{N-1}} (v_t)^2 d\sigma = \\ = \int_{S_+^{N-1}} \left[ v_t H + \alpha_t (v+\alpha)^q \right] d\sigma. \end{aligned}$$

From our assumption on  $u$ ,  $v$  is bounded. It follows from (6.4) and the Sobolev imbedding that  $v$ ,  $v_t$  and  $\nabla' v$  are uniformly bounded in  $S_+^{N-1} \times \mathbb{R}_+$ . Integrating (8.3) from 0 to  $T$ , for any  $T > 0$ , one deduces that

$$\int_{S_+^{N-1}} \left| \frac{(v_t)^2}{2} + \frac{\ell_{N,q} v^2}{2} - \frac{|\nabla' v|^2}{2} + \frac{(v+\alpha)^{q+1}}{q+1} \right| d\sigma \leq C \quad \text{in } \mathbb{R}_+$$

for some constant  $C > 0$ . On the other hand,

$$\int_{S_+^{N-1}} |v_t H| d\sigma \leq C e^{-t}.$$

Moreover, since  $v$  is bounded and  $\alpha$  satisfies (6.5), we have

$$\int_{S_+^{N-1}} |\alpha_t| (v+\alpha)^q d\sigma \leq C e^{-\frac{2t}{q-1}}.$$

Thus, integrating (8.3) on  $(0, +\infty)$ , we obtain

$$\left| N - \frac{2(q+1)}{q-1} \right| \int_0^{+\infty} \int_{S_+^{N-1}} v_t^2 d\sigma < +\infty.$$

Since  $q \neq q_2$ ,  $N - \frac{2(q+1)}{q-1} \neq 0$ . Hence,

$$\int_0^{+\infty} \int_{S_+^{N-1}} v_t^2 d\sigma < +\infty.$$

By (6.4) and Morrey's estimates,  $v_t$  is uniformly continuous on  $Q_0$ . We deduce that

$$v_t(t, \cdot) \rightarrow 0 \quad \text{uniformly in } S_+^{N-1} \text{ as } t \rightarrow +\infty.$$

We now prove that

$$v(t, \cdot) \rightarrow w \quad \text{uniformly in } S_+^{N-1} \text{ as } t \rightarrow +\infty,$$

where  $w$  is a nonnegative solution of (1.3). For this purpose, we study the limit set of the trajectories of  $v$ , namely the set

$$\Gamma = \bigcap_{\tau > 0} \overline{\bigcup_{t \geq \tau} \{v(t, \cdot)\}},$$

where the closure is computed with respect to the usual norm in  $C^0(S_+^{N-1})$ . Since  $\Gamma$  is the intersection of a decreasing family of closed connected subsets of  $C^0(S_+^{N-1})$ ,

$\Gamma$  is closed and connected. In addition, since  $v$  is uniformly continuous in  $Q_0$ , it follows from the Arzelà-Ascoli theorem that  $\Gamma$  is also compact and nonempty.

We claim that every  $w \in \Gamma$  satisfies problem (1.3). Indeed, let  $(t_k)$  be a sequence of nonnegative real numbers such that  $t_k \rightarrow +\infty$  and

$$v(t_k, \cdot) \rightarrow w \quad \text{uniformly in } S_+^{N-1}.$$

Clearly,  $w$  is nonnegative and  $w = 0$  on  $\partial S_+^{N-1}$ . For each  $k \geq 1$ , let

$$V_k : (s, \sigma) \in [0, 1] \times S_+^{N-1} \mapsto v(t_k + s, \sigma).$$

For every  $\varphi \in C_0^\infty(S_+^{N-1})$  and for every  $\varepsilon \in (0, 1)$ , from the equation satisfied by  $v$  we have

$$\begin{aligned} \int_0^\varepsilon \int_{S_+^{N-1}} \left[ (V_k)_{tt} \varphi + \ell_{N,q} V_k \varphi + V_k \Delta' \varphi + (V_k + \alpha)^q \varphi - \left( N - \frac{2(q+1)}{q-1} \right) (V_k)_t \varphi \right] d\sigma dt = \\ = \int_{t_k}^{t_k+\varepsilon} \int_{S_+^{N-1}} H \varphi d\sigma dt, \end{aligned}$$

As  $k \rightarrow +\infty$ ,

$$\int_{t_k}^{t_k+\varepsilon} \int_{S_+^{N-1}} H \varphi d\sigma dt \rightarrow 0.$$

Since  $v_t \rightarrow 0$  uniformly as  $t \rightarrow +\infty$ , we also have

$$\int_0^\varepsilon \int_{S_+^{N-1}} (V_k)_t \varphi d\sigma dt \rightarrow 0.$$

Note that

$$\int_0^\varepsilon \int_{S_+^{N-1}} (V_k)_{tt} \varphi d\sigma d\tau = \int_{S_+^{N-1}} [v_t(t_k + \varepsilon, \sigma) - v_t(t_k, \sigma)] \varphi d\sigma \rightarrow 0.$$

Since the sequence  $(V_k)$  is bounded in  $C^1$ , passing to a subsequence if necessary, we may assume that for some continuous function  $W$ ,

$$V_k \rightarrow W \quad \text{uniformly in } [0, 1] \times S_+^{N-1}.$$

We conclude that for every  $\varepsilon \in (0, 1)$ ,

$$\int_0^\varepsilon \int_{S_+^{N-1}} \left[ \ell_{N,q} W \varphi - W \Delta' \varphi + W^q \varphi \right] d\sigma dt = 0.$$

Dividing both sides by  $\varepsilon$  and letting  $\varepsilon \rightarrow 0$ , we get

$$\int_{S_+^{N-1}} \left[ \ell_{N,q} W(0, \sigma) \varphi - W(0, \sigma) \Delta' \varphi + (W(0, \sigma))^q \varphi \right] d\sigma = 0.$$

Since  $w = W(0, \cdot)$ , we conclude that  $w$  satisfies (1.3). Hence, every element of  $\Gamma$  is a nonnegative solution of (1.3). Since these solutions form a discrete subset of  $C^0(S_+^{N-1})$  and  $\Gamma$  is connected (in our case, the set of nonnegative solutions is  $\{0, \omega\}$ ),

where  $\omega$  is the unique positive solution of (1.3)),  $\Gamma$  contains a single element. In particular,

$$v(t, \cdot) \rightarrow w \quad \text{uniformly in } S_+^{N-1} \text{ as } t \rightarrow +\infty.$$

The proposition follows from this convergence.  $\square$

*Proof of Theorem 1.2.* Let  $u$  be a solution of (1.1). Since  $q < q_2$ , by Theorem 1.5 there exists  $C > 0$  such that for every  $x \in \Omega$ ,

$$0 \leq |x|^{\frac{2}{q-1}} u(x) \leq C.$$

Thus, by Proposition 8.1, there exists a solution  $w$  of (1.3) such that (8.1) holds. Either  $w$  is the unique positive solution of (1.3) (see Theorem 1.1) or  $w = 0$ . If  $w = 0$ , then

$$\lim_{x \rightarrow 0} |x|^{\frac{2}{q-1}} u(x) = 0.$$

Hence, by Theorem 7.1  $u$  can be continuously extended at 0.  $\square$

### 9. PROOF OF THEOREM 1.3

We first prove an estimate which improves Theorem 1.5 when  $q = q_1$ , except that we do not know whether the constant  $C$  below can be chosen independently of the solution.

**Theorem 9.1.** *Assume that  $q = q_1$ . Then, every solution of (1.1) satisfies*

$$u(x) \leq C|x|^{-(N-1)} \left( \log \frac{1}{|x|} \right)^{-\frac{N-1}{2}} \quad \forall x \in \Omega,$$

for some constant  $C > 0$  possibly depending on the solution.

In the proof of this result we need the following lemma:

**Lemma 9.1.** *Let  $a = q_1$  and  $E = \ker[\Delta' + (N-1)I]$ . Given a solution of (1.1), denote by  $v$  the function given by (6.1). If*

$$v = v_1 + v_2$$

is the decomposition of  $v$  as the orthogonal projections in  $L^2(S_+^{N-1})$  onto  $E$  and  $E^\perp$ , respectively, then

$$(9.1) \quad \|v_1(t, \cdot)\|_{L^2(S_+^{N-1})} \leq C t^{-\frac{N-1}{2}} \quad \text{and} \quad \|v_2(t, \cdot)\|_{L^2(S_+^{N-1})} \leq C e^{-\frac{t}{2}} \quad \forall t > 0.$$

*Proof.* Denoting by  $\phi_1$  the first eigenfunction of  $\Delta'$  with  $\|\phi_1\|_{L^1} = 1$ , we have

$$v_1(t, \sigma) = y(t)\phi_1(\sigma) \quad \text{where} \quad y(t) = \int_{S_+^{N-1}} v(t, \sigma)\phi_1(\sigma) d\sigma.$$

Since  $q = q_1$ , equation (7.3) becomes

$$(9.2) \quad v_{tt} + Nv_t + (N-1)v + \Delta'v + (v + \alpha)^{q_1} = H,$$

with  $H$  defined in (7.4). Since  $\alpha \geq 0$ , we have  $(v + \alpha)^{q_1} \geq v^{q_1}$ . Thus,

$$v_{tt} + Nv_t + (N-1)v + \Delta'v + v^{q_1} \leq H.$$

By Jensen's inequality,

$$y^{q_1} \leq \int_{S_+^{N-1}} v^{q_1} \phi_1 d\sigma.$$

Multiplying (9.2) by  $\phi_1$  and integrating over  $S_+^{N-1}$ , we get

$$y'' + Ny' + y^{q_1} \leq \int_{S_+^{N-1}} H\phi_1 d\sigma.$$

By Theorem 1.5,  $v$  is uniformly bounded in  $\mathbb{R}_+ \times S_+^{N-1}$ . In particular, by (6.3) and Proposition 6.1 with  $p = 2$ , we have for every  $t \geq 0$ ,

$$\int_{S_+^{N-1}} H\phi_1 d\sigma \leq C e^{-t}.$$

Thus,

$$y'' + Ny' + y^{q_1} \leq C e^{-t}.$$

Applying Lemma A.1 we deduce that

$$y(t) \leq C t^{-\frac{N-1}{2}} \quad \forall t > 0.$$

This concludes the proof of the first estimate in (9.1).

In order to prove the estimate for  $v_2$ , let

$$Y(t) = \|v_2(t, \cdot)\|_{L^2(S_+^{N-1})} \quad \forall t \geq 0.$$

Since  $v(t, \sigma) = y(t)\phi_1(\sigma) + v_2(t, \sigma)$ , we have

$$v_t = y_t\phi_1 + (v_2)_t \quad \text{and} \quad v_{tt} = y_{tt}\phi_1 + (v_2)_{tt}.$$

Using the orthogonality between  $\phi_1$  and  $v_2$ ,

$$YY_t = \int_{S_+^{N-1}} v_2(v_2)_t d\sigma = \int_{S_+^{N-1}} v_2[y_t\phi_1 + (v_2)_t] d\sigma = \int_{S_+^{N-1}} v_2 v_t d\sigma.$$

From the first equality, we have

$$|Y_t| \leq \|v_2(t, \cdot)\|_{L^2}.$$

One also shows that

$$YY_{tt} \geq \int_{S_+^{N-1}} v_2 v_{tt} d\sigma.$$

On the other hand, since the second eigenvalue of the Laplace-Beltrami operator  $-\Delta'$  in  $W_0^{1,2}(S_+^{N-1})$  is  $2N$ ,

$$2NY^2 \leq \int_{S_+^{N-1}} |\nabla' v_2|^2 d\sigma = - \int_{S_+^{N-1}} v_2 \Delta' v_2 d\sigma = - \int_{S_+^{N-1}} v_2 \Delta' v d\sigma.$$

Multiply (9.2) by  $v_2$  and integrate over  $S_+^{N-1}$ . As in the proof of Theorem 7.1, for every  $\varepsilon > 0$  there exists  $t_1 > 0$  such that for every  $t \geq t_1$ ,

$$Y_{tt} + NY_t - (N + 1 - \varepsilon)Y \geq -C e^{-t}.$$

Note that for  $\varepsilon > 0$  small the linear equation

$$Z_{tt} + NZ_t - (N + 1 - \varepsilon)Z = 0$$

has two linearly independent solutions  $Z_{1,\varepsilon}$  and  $Z_{2,\varepsilon}$  such that

$$Z_{1,\varepsilon}(t) = e^{r_{1,\varepsilon}t} \quad \text{and} \quad Z_{2,\varepsilon}(t) = e^{r_{2,\varepsilon}t}$$

with

$$r_{1,\varepsilon} \leq -\frac{1}{2} \quad \text{and} \quad r_{2,\varepsilon} > 0.$$

Since  $Y(t) \rightarrow 0$  as  $t \rightarrow +\infty$ , applying the maximum principle one deduces that

$$Y(t) \leq C(Z_{1,\varepsilon}(t) + e^{-t}).$$

In particular,

$$Y(t) \leq Ce^{-\frac{t}{2}}.$$

This gives the estimate for  $v_2$ .  $\square$

*Proof of Theorem 9.1.* By Lemma 9.1 above, we have

$$\|v(t, \cdot)\|_{L^2} \leq Ct^{-\frac{N-1}{2}} \quad \forall t > 0.$$

Inserting this estimate into estimate (6.4) for some  $p > \frac{N}{2}$  the result follows.  $\square$

*Proof of Theorem 1.3.* By Theorem 9.1, the function  $w : [0, +\infty) \rightarrow \mathbb{R}$  given by

$$w(t, \sigma) = t^{\frac{N-1}{2}} v(t, \sigma)$$

is bounded. By a straightforward computation,  $w$  satisfies

$$(9.3) \quad w_{tt} + \left(N - \frac{N-1}{t}\right) w_t + \left(N - 1 + \frac{N^2-1}{4t^2}\right) w + \Delta' w + \frac{1}{t} \left(w^{q_1} - \frac{N(N-1)}{2} w\right) = t^{\frac{N-1}{2}} H,$$

where  $H$  is given by (7.4). Let  $\phi : S_+^{N-1} \rightarrow \mathbb{R}$  be the function defined by  $\phi(\sigma) = \frac{\sigma_N}{|\sigma|}$ ; we recall that  $\phi$  is an eigenfunction of  $-\Delta'$  in  $W_0^{1,2}(S_+^{N-1})$  associated to the first eigenvalue  $N - 1$ . Let

$$z(t) = \int_{S_+^{N-1}} w(t, \sigma) \phi(\sigma) d\sigma \quad \forall t \geq 0.$$

Multiplying (9.3) by  $\phi$  and integrating over  $S_+^{N-1}$ , we obtain the following equation satisfied by  $z$ :

$$z_{tt} + \left(N - \frac{N-1}{t}\right) z_t + \frac{N^2-1}{4t^2} z + \frac{1}{t} \int_{S_+^{N-1}} w^{q_1} \phi d\sigma - \frac{N(N-1)}{2t} z = t^{\frac{N-1}{2}} \int_{S_+^{N-1}} H \phi d\sigma.$$

Thus,

$$z_{tt} + \left(N - \frac{N-1}{t}\right) z_t + \frac{1}{t} \left(\theta z^{q_1} - \frac{N(N-1)}{2} z\right) = \Psi,$$

where

$$\theta = \int_{S_+^{N-1}} \phi^{q_1+1} d\sigma$$

and

$$\Psi = t^{\frac{N-1}{2}} \int_{S_+^{N-1}} H \phi d\sigma - \frac{N^2-1}{4t^2} z + \frac{1}{t} \int_{S_+^{N-1}} [(z\phi)^{q_1} - w^{q_1}] \phi d\sigma.$$

By Lemma 9.1, we have

$$(9.4) \quad \|z(t)\phi - w(t, \cdot)\|_{L^2} \leq Ct^{\frac{N-1}{2}} e^{-\frac{t}{2}}.$$



Since

$$|(z\phi)^{q_1} - w^{q_1}| \leq q_1 |z\phi - w| [(z\phi)^{q_1-1} + w^{q_1-1}],$$

$z$  is bounded in  $\mathbb{R}_+$  and  $w$  is bounded in  $\mathbb{R}_+ \times S_+^{N-1}$ ,

$$\begin{aligned} \int_{S_+^{N-1}} |(z\phi)^{q_1} - w^{q_1}| \phi \, d\sigma &\leq \|z\phi - w\|_{L^2} \left[ z^{q_1-1} \|\phi^{q_1-1}\|_{L^2} + \|w^{q_1-1}\|_{L^2} \right] \\ &\leq C t^{\frac{N-1}{2}} e^{-\frac{t}{2}}. \end{aligned}$$

By Proposition 9.1, (6.3) and Proposition 6.1 with  $p = 2$ ,

$$\|H(t, \cdot)\|_{L^2} \leq C t^{-\frac{N-1}{2}} e^{-t}.$$

Thus,

$$\|\Psi(t, \cdot)\|_{L^\infty} \leq C \left( e^{-t} + t^{-2} + t^{\frac{N-3}{2}} e^{-\frac{t}{2}} \right) \leq \tilde{C} t^{-2}.$$

By a straightforward modification of the end of the proof of [5, Corollary 4.2],  $z$  admits a limit  $\kappa \geq 0$  when  $t \rightarrow +\infty$ , where  $\kappa$  satisfies

$$\theta \kappa^{q_1} - \frac{N(N-1)}{2} \kappa = 0.$$

Therefore, either  $\kappa = 0$  or  $\kappa = \left( \frac{N(N-1)}{2\theta} \right)^{\frac{N-1}{q_1}}$ .

By (9.4) we deduce that, as  $t \rightarrow +\infty$ ,

$$t^{\frac{N-1}{2}} v(t, \cdot) \rightarrow \kappa \phi \quad \text{in } L^2(S_+^{N-1}).$$

By Proposition 6.1 with  $p > \frac{N}{2}$  and Morrey's estimates, we conclude that

$$t^{\frac{N-1}{2}} v(t, \cdot) \rightarrow \kappa \phi \quad \text{uniformly in } S_+^{N-1}.$$

Rewriting the convergence in terms of  $u$ , we conclude that either (1.5) holds or

$$(9.5) \quad |x|^{N-1} \left( \log \frac{1}{|x|} \right)^{\frac{N-1}{2}} u(x) \rightarrow 0 \text{ as } x \rightarrow 0.$$

If (9.5) holds, then  $u$  must be continuous in view of Theorem 7.2.  $\square$

#### APPENDIX A. SOME ODE LEMMAS

We gather in this section a couple of ODE results which are used in this paper. These results are presumably well-known to specialists:

**Lemma A.1.** *Given  $T > 0$ , let  $y \in C^2([T, +\infty))$  be a nonnegative function such that*

$$\begin{cases} y_{tt} + ay_t + by^q \leq c e^{-t} & \text{in } (T, +\infty), \\ \lim_{t \rightarrow +\infty} y(t) = 0, \end{cases}$$

where  $q, a > 1$  and  $b, c > 0$ . Then, there exists  $C > 0$  such that

$$(A.1) \quad 0 \leq y(t) \leq C t^{-\frac{1}{q-1}} \quad \forall t \geq T.$$

*Proof.* Given  $A > 0$ , let

$$z(t) = y(t) + A e^{-t} \quad \forall t \geq T.$$

Then,  $z$  satisfies

$$z_{tt} + az_t + bz^q \leq [c - (a-1)A] e^{-t} + b(z^q - y^q).$$

By convexity of the function  $t \in \mathbb{R}_+ \mapsto t^q$ ,

$$y^q \geq z^q - qz^{q-1}Ae^{-t}.$$

Thus,

$$(A.2) \quad z_{tt} + az_t + bz^q \leq [c - (a - 1 + bqz^{q-1})A]e^{-t}.$$

Since  $a > 1$  and  $z(t) \rightarrow 0$  as  $t \rightarrow \infty$ , we can choose  $T_1 > T$  and  $A > 0$  sufficiently large so that the right-hand side of (A.2) is negative on  $[T_1, \infty)$ . Thus,

$$(A.3) \quad z_{tt} + az_t + bz^q \leq 0 \quad \text{in } [T_1, \infty).$$

Let  $w = z^{1-q}$ . By a straightforward computation, we have

$$(A.4) \quad w_{tt} + aw_t \geq -(q-1)\frac{z_{tt} + az_t}{z^q}.$$

Combining (A.3)–(A.4), we deduce that

$$w_{tt} + aw_t \geq b(q-1) \quad \text{in } [T_1, \infty).$$

The function  $x = w_t$  satisfies

$$x_t + ax \geq b(q-1) \quad \text{in } [T_1, \infty).$$

Thus, taking  $T_2 > T_1$  sufficiently large,

$$x(t) \geq \frac{b(q-1)}{a} + c_1e^{-at} \geq \frac{b(q-1)}{2a} \quad \forall t \geq T_2.$$

Since  $w_t = x$ , choosing  $T_3 > T_2$  large enough, we then get

$$w(t) \geq \frac{b(q-1)}{4a}t \quad \forall t \geq T_3.$$

Therefore,

$$z(t) \leq \left(\frac{4a}{b(q-1)}t^{-1}\right)^{\frac{1}{q-1}} \quad \forall t \geq T_3.$$

We can now enlarge the constant in the right-hand side so that this estimate holds for every  $t \geq T$ . This immediately implies (A.1).  $\square$

**Lemma A.2.** *Let  $a, a_1, b, b_1 \in \mathbb{R}$  with  $a \neq 0$ . Then, the equation*

$$y_{tt} + \left(a - \frac{a_1}{t}\right)y_t + \frac{1}{t}\left(b - \frac{b_1}{t}\right)y = 0 \quad \text{in } (0, +\infty),$$

*has two linearly independent solutions  $y_1$  and  $y_2$  such that*

$$y_1(t) = t^{a_1 + \frac{b}{a}}e^{-at}(1 + o(1)) \quad \text{and} \quad y_2(t) = t^{-\frac{b}{a}}(1 + o(1))$$

*for  $t$  sufficiently large.*

*Proof.* Let

$$z(t) = e^{\frac{at}{2}}t^{-\frac{a_1}{2}}y(t).$$

Then,  $z$  satisfies the equation

$$z_{tt} - \left(\frac{a^2}{4} - \frac{A_1}{t} + \frac{A_2}{t^2}\right)z = 0,$$

where  $A_1 = b + \frac{aa_1}{2}$  and  $A_2 = b_1 + \frac{a_1}{2} + \frac{a^2}{4}$ . By [3, pp. 126–127], the equation satisfied by  $z$  has two linearly independent solutions with the following asymptotic behaviors as  $t \rightarrow +\infty$ :

$$z_1(t) = e^{-\frac{at}{2}}t^{\frac{A_1}{a}}(1 + o(1)) \quad \text{and} \quad z_2(t) = e^{\frac{at}{2}}t^{-\frac{A_1}{a}}(1 + o(1)).$$

Rewriting these formulas in terms of the function  $y$ , the result follows.  $\square$

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