ISOLATED BOUNDARY SINGULARITIES OF SEMILINEAR ELLIPTIC EQUATIONS

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Boundary isolated singularities of positive solutions of some non-monotone semilinear elliptic equations

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Résumé

Assume $\Omega \subset \mathbb{R}^N$ is a smooth domain, $x_0 \in \partial\Omega$ and $q \geq (N+1)/(N-1)$. We study the behavior near x_0 of any positive solution of (E) $-\Delta u = u^q$ in Ω which coincides with some $\zeta \in C^2(\partial\Omega)$ on $\partial\Omega \setminus \{x_0\}$. We prove that, if $(N+1)/(N-1) < q < (N+2)/(N-2)$, u satisfies $u(x) \leq C\,|x-x_0|^{-2/(q-1)}$ and we give the limit of $|x-x_0|^{2/(q-1)}\,u(x)$ as $x \to x_0$. In the case $q = (N+1)/(N-1)$ u satisfies $u(x) \leq C\,|x-x_0|^{1-N}\,(\ln(1/|x|))^{(1-N)/2}$ and a corresponding precise asymptotic is obtained. We also study some existence and uniqueness questions for related equations on spherical domains.

1 Introduction

In this article we study the isolated boundary singularities of a positive function u satisfying, for some $q > 1$,

$$-\Delta u = u^q, \tag{1.1}$$

in a smooth domain $\Omega \subset \mathbb{R}^N$ ($N \geq 2$). More precisely we assume that u is continuous in $\overline{\Omega} \setminus \{x_0\}$ where x_0 is some boundary point of Ω and coincides on $\partial\Omega \setminus \{x_0\}$ with a smooth function ζ defined on the whole boundary $\partial\Omega$. This function may develop an isolated singularity at x_0 and the whole question is to describe its behaviour near x_0. When (1.1) is replaced by the *equation with the other sign*, namely

$$-\Delta u + |u|^{q-1}u = 0, \tag{1.2}$$

the problem has been adressed first by Gmira and Véron [14]. Later on it was extended to nonsmooth domains in [15]. One of the key result in [14] is the existence of a critical exponent $q = q_1 := (N+1)/(N-1)$. It is proved that if $q \geq q_1$ any boundary isolated singularity for a solution of (1.2) is removable. On the opposite, if $1 < q < q_1$, there exist solutions with boudary isolated singularity. Furthermore, such singular behaviour have been completely characterized : either if $u \geq 0$, or if $1 + 2/N \leq q < q_1$, or if $N = 2$. For positive solutions, the behaviour of u near x_0 can be of two types :

(i) Either

$$\lim_{\substack{x \to x_0 \\ \frac{x-x_0}{|x-x_0|} \to \sigma}} |x-x_0|^{2/(q-1)} u(x) = \omega(\sigma)$$

locally uniformly in σ where ω satisfies a particular nonlinear elliptic equation on an hemisphere S of the unit sphere S^{N-1},

(ii) or there exists $k \geq 0$ such that

$$u(x) \approx kP(x, x_0) \quad as\ x \to x_0$$

where P is the Poisson kernel in Ω.

Furthermore it is proved that the two types of behaviour truly exist. This result play a fundamental role in the theory of boundary trace of positive solutions of (1.2) which was later on developed by Marcus and Véron [20], [21], [22] using analytic tools, and Le Gall [18] or Dynkin and Kuznetsov [10],[11] with a probabilistic approach.

For equation (1.1), the interior singularity problem was initiated by Lions [19] in the subcritical case $1 < q < N/(N-2)$ and thouroughly analyzed by Gidas and Spruck in a the seminal article [13] in the range $N/(N-2) \leq q < q_2 := (N+2)/(N-2)$. There are several key points in their article, in particular :

(i) Existence of a universal a priori estimate : any positive solution of (1.1) in the ball $B_1(a)\backslash\{a\}$ satisfies

$$u(x) \leq C|x-a|^{-2/(q-1)} \quad \forall x \in B_{1/2}(a) \setminus \{a\}, \tag{1.3}$$

for some $C = C(N,q) > 0$.

(ii) Non-existence of positive smooth solution of (1.1) in $\mathbb{R}^N$.

(iii) Uniqueness of positive solutions of a class of equations on the unit sphere S^{N-1},

$$-\Delta'\omega + \ell w = w^q \tag{1.4}$$

where Δ' is the Laplace-Beltrami operator and $0 < (q-1)\ell \leq N-1$. In this case $q \leq q_3 := (N+1)/(N-3)$.

(iii) Existence of the limit of $|x-a|^{2/(q-1)}u(x)$ when $x \to a$.

The boundary singularity problem associated to (1.1) has been treated by Bidaut-Véron and Vivier [6] in the subcritical range $1 < q < q_1$. Here it is proved that there exists $k \geq 0$ such that

$$u(x) \approx kP(x, x_0) \quad \text{as } x \to x_0.$$

In this article we consider the range of exponent $q_1 \leq q < q_2$ and we prove a series of results which extend Gidas and Spruck's work to the boundary singularity framework. Our main results are the following.

Theorem A *If $1 < q < q_2$, any positive solution u of (1.1) which is continuous in $\overline{\Omega} \setminus \{x_0\}$ and coincide on $\partial\Omega \setminus \{x_0\}$ with some function $\zeta \in C(\partial\Omega)$, satisfies*

$$u(x) \leq C|x-x_0|^{-2/(q-1)} \quad \forall x \in B_{R_0}(x_0) \cap \Omega, \tag{1.5}$$

for some R_0 and $C = C(N, q, \Omega, ||\zeta||_{L^\infty(B_{R_0}(x_0)\cap\partial\Omega\cap\Omega)}) > 0$.

The proof of this result is based upon a topological argument (*the doubling lemma*) and a method introduced by Poláčik, Quittner and Souplet [24].

Theorem B *Let $q_1 \le q \le q_3$, $q \neq q_2$, $\zeta \in C(\partial\Omega)$ and u be as above and satisfies (1.5). If we suppose that the boundary singular point x_0 is 0 and the outward normal unit vector to $\partial\Omega$ at 0 is $-\mathbf{e_N}$, then either u extends to 0 as a continuous function, or*

$$u(x) = \begin{cases} |x|^{-2/(q-1)}\omega_0(x/|x|) + \circ(1)) & \text{if } q > q_1 \\ k_N^* |x|^{1-N} \ln(1/|x|)^{(1-N)/2}(x_N/|x| + \circ(1)) & \text{if } q = q_1, \end{cases} \tag{1.6}$$

where k_N^* is some positive constant depending on N and ω_0 is the unique positive solution of

$$-\Delta'\omega = \frac{2(N - q(N-2))}{(q-1)^2}\omega + w^q \tag{1.7}$$

in $S_+^{N-1} := S^{N-1} \cap \{x : x_N > 0\}$ which vanishes on the equator ∂S_+^{N-1}.

Furthermore, it is proved in [9] that positive solutions with the two above behaviour truly exist. Besides the classical energy approach for this problem, a key point for proving the asymptotic when $q > q_1$ is a uniqueness result for positive solutions of (1.7). This is obtained by proving first that such solutions depend only on the angular variable $\cos^{-1}(x_N/|x|)$ [23] and then that the corresponding ODE admits a unique positive solution. This last point is based on a delicate adaptation of a series of results due to Kwong [16] and Kwong and Li [17]. The type of equation (1.7) is imbedded into a more general problem, namely

$$\begin{cases} -\Delta'\omega = \ell\omega + w^{q-1}\omega & \text{in } S \\ \omega = 0 & \text{on } \partial S; \end{cases} \tag{1.8}$$

where S is a smooth subdomain of S^{N-1}, $N \ge 3$. We prove that ω satisfies a general Pohozaev type identity

$$\left(\frac{(N-3)q}{2(q+1)} - \frac{N+1}{2(q+1)}\right)\int_S |\nabla'\omega|^2 \phi \, d\sigma - \frac{d}{2(q+1)}\left(\ell(q-1) + N - 1\right)\int_S \omega^2\phi \, d\sigma = \frac{1}{2}\int_{\partial S} |\nabla'\omega|^2 \langle \nabla'\phi, \nu\rangle \, d\tau, \tag{1.9}$$

where ϕ is any spherical harmonic of degree 1. As a consequence we derive

Theorem C *Assume $S \subset S_+^{N-1}$.*

(i) If $\ell \ge N - 1$ any nonnegative solution of (1.9) is trivial.

(ii) If $1 < q < q_3$ and $\ell < N - 1$, there exists a positive solution to (1.9).

(iii) Assume S is star-shaped in the sense that there exists a spherical harmonics ϕ positive on S such that $\langle \nabla'\phi, \nu\rangle \le 0$ on ∂S. Then, if $q \ge q_3$, $\ell(q-1) \le 1 - N$, with one of these two last inequalities strict, any solution of (1.9) is trivial. If $q = q_3$, $q = q_3$, $\ell(q-1) = 1 - N$, the same result holds provided $\omega \ge 0$.

Theorems A, B and C have been announced in [3]. Our paper is organised as follows. 1- Introduction. 2- The *a priori* estimate. 3- Characterization of the isolated singularities. 4- Equation on a spherical domain.

2 The *a priori* estimate

In this section Ω is a smooth bounded domain in $\mathbb{R}^N$ and $0 \in \partial\Omega$. We denote $\rho(x) = \text{dist}\,(x, \partial\Omega)$, $\overline{\Omega}^* = \overline{\Omega} \setminus \{0\}$ and $\partial\Omega^* = \partial\Omega \setminus \{0\}$, and define the three exponents $q_1 = (N+1)/(N-1)$, $q_2 = (N+2)/(N-2)$ and $q_3 = (N+1)/(N-3)$. We consider a positive function $u \in C(\overline{\Omega}^*)$ satisfying

$$\begin{cases} -\Delta u = u^q & \text{in } \Omega \\ \quad u = \zeta & \text{in } \partial\Omega^*. \end{cases} \tag{2.1}$$

where $\zeta \in C(\partial\Omega)$. In the sequel, C will denote a positive constant depending upon structural assumptions, but not on the variables, the value of which may change from line to line. We first recall the following standard consequence of the super harmonicity

Theorem 2.1 *For any $q > 0$, $u^q \in L^1(\Omega, \rho dx)$ and the boundary trace of u is a positive Radon with the form $\zeta dS + \gamma\delta_0$ for some $\gamma \geq 0$. Furthermore*

$$u(x) = \int_\Omega G(x,y)u^q(y)dy + \int_{\partial\Omega} P(x,y)\zeta(y)dS(y) + \gamma P(x,0) \quad \forall x \in \Omega \tag{2.2}$$

where G (resp. P) is the Green (resp. Poisson) kernel in Ω. If $q \geq q_1$, then $\gamma = 0$.

Proof. Since u is super harmonic and positive, we can apply Doob's theorem which asserts that $\Delta u \in L^1(\Omega, \rho dx)$, that the boundary trace of u is a positive Radon measure and that u is the sum of the Green potential of $-\Delta$ and the Poisson potential of its boundary trace. In our case the boundary trace may have an atom only at 0 and it coincides with ζ on $\partial\Omega^*$. Thus $u^q \in L^1(\Omega, \rho dx)$ and we obtain (2.2). Clearly $u(x) \geq \gamma P(x,0)$ in Ω. If $q \geq q_1$, $P(.,0) \notin L^{q_1}(\Omega, \rho dx)$, therefore $\gamma = 0$. □

The main result of this section is the a priori estimate.

Theorem 2.2 *Assume $N \geq 2$, $1 < q < q_2$. There exists $C = C(q, \Omega, ||\zeta||_{L^\infty}) > 0$ such that for any positive solution of (3.1) there holds*

$$u(x) \leq C|x|^{-2/(q-1)} \quad \forall x \in \Omega. \tag{2.3}$$

The proof is based on a beautiful method developed by Poláčik, Quittner and Souplet [24]. We recall their *doubling lemma*

Lemma 2.3 *Let (X, d) be a complete metric space, $\Sigma \subset X$ be closed, $D \subset \Sigma$ be non-empty and $\Gamma => \Sigma \setminus D$. Let $M : D \mapsto (0, \infty)$ be a function bounded on the compact subsets of D. Let $k > 0$ be fixed. If there exists $y \in D$ such that*

$$M(y)\text{dist}\,(y, \Gamma) > 2k, \tag{2.4}$$

there exists $x \in D$ such that

$$\begin{aligned} &(i) && M(x)\text{dist}\,(x, \Gamma) > 2k \\ &(ii) && M(x) \geq M(y) \\ &(iii) && 2M(x) \geq M(z) \quad \forall z \in D \cap \overline{B}_{k/M(x)}(x). \end{aligned} \tag{2.5}$$

Lemma 2.4 *Let $N \geq 2$ and $1 < q < q_2$ ($q > 1$ if $N = 2$). There exists $R_0 > 0$ and $C = C(N, q, \delta) > 0$ such that for any $R \leq R_0$, $\delta > 0$, $\zeta \in C^{\partial\Omega}$ such that $\|\zeta\|_{L^\infty} \leq \delta$ and any nonnegative solution u of*

$$\begin{cases} -\Delta u = u^q \quad \text{in } \Omega \cap (B_R \setminus \overline{B}_{R/2}) \\ \quad u = \zeta \quad \text{on } \partial\Omega \cap (B_R \setminus B_{R/2}), \end{cases} \tag{2.6}$$

there holds

$$u(x) \leq C \left(\min\{R - |x|, |x| - R/2\}\right)^{-2/(q-1)} \quad \forall x \in \Omega \cap (B_R \setminus B_{R/2}). \tag{2.7}$$

Proof. We can always assume that $R_0 = 1$ and $R \leq 1$. By setting $u(x) = R^{-2/(q-1)} v(x/R)$ and $z(x/R) = R^{2/(q-1)}\zeta(x)$, we can suppose that $R = 1$. Then Ω is changed into $R^{-1}\Omega$ which has a bounded curvature since $R > 1$ which, for the sake of simplicity, will be still denoted by Ω . If we assume that the result is not true, then for any $k > 0$ there exists a solution v_k of (2.6) with boundary value z_k, with $\Omega \cap (B_R \setminus B_{R/2})$ replaced by $\Omega \cap (B_1 \setminus B_{1/2})$ and $y_k \in \Omega \cap (B_1 \setminus B_{1/2})$ such that

$$M_k(y_k) \geq \frac{2k}{m(y_k)},$$

where we have set $m(y) = \min\{1 - |y|, 1/2 - |y|\}$ and $M_k(y) = v_k^{(q-1)/2}(y)$ for $y \in \Omega \cap (B_1 \setminus B_{1/2})$. We apply Lemma 2.3 with $X = \mathbb{R}^N$, $\Sigma = \overline{\Omega \cap (B_1 \setminus \overline{B}_{1/2})} = \overline{\Omega} \cap (\overline{B}_1 \setminus B_{1/2})$, $D = \left(\Omega \cap (B_1 \setminus \overline{B}_{1/2})\right) \cup \left(\partial\Omega \cap (\overline{B}_1 \setminus B_{1/2})\right)$ and $\Gamma = \Sigma \setminus D = \Omega \cap \{x : |x| = 1 \text{ or } |x| = 1/2\}$. Since the curvature of $\partial\Omega$ is uniformly bounded (even with respect to R), $m(y) \leq \text{dist}\,(y, \Gamma) \leq \theta m(y)$ for some $\theta \geq 1$ and any $y \in D$. Furthermore, by taking the initial R_0 small enough, we can assume after the scaling that $\theta < 2$. By Lemma 2.3 there exists $x_k \in D$ such that

$$\begin{aligned} &(i) \qquad M_k(x_k)\text{dist}\,(x_k, \Gamma) > 2k \\ &(ii) \qquad M_k(x_k) \geq M_k(y_k) \\ &(iii) \qquad 2M_k(x_k) \geq M_k(z) \quad \forall z \in D_k, \end{aligned} \tag{2.8}$$

where $D_k := D \cap \overline{B}_{k/M_k(x_k)}(x_k)$. Set $\tilde{D}_k = \overline{\Omega} \cap \overline{B}_{k/M_k(x_k)}(x_k)$, then $D_k \subset \tilde{D}_k$. Conversely, if $z \in \tilde{D}_k$, then

$$|z| \leq |x_k| + \frac{k}{M_k(x_k)} < |x_k| + \frac{\text{dist}\,(x_k, \Gamma)}{2} < |x_k| + \frac{\theta}{2}(1 - |x_k|) \leq 1. \tag{2.9}$$

Furthermore, since $|z| \geq |x_k| - |x_k - z| \geq |x_k| - kM_k^{-1}(x_k)$, it follows also from (2.8)-(i)

$$|z| > |x_k| - \frac{\text{dist}\,(x_k, \Gamma)}{2} \geq |x_k| - \frac{\theta m(x_k)}{2} \geq |x_k| - \frac{\theta}{2}(|x_k| - \frac{1}{2}) = \frac{\theta}{4} + \left(1 - \frac{\theta}{2}\right)|x_k| \geq \frac{1}{2}. \tag{2.10}$$

It implies $D_k = \tilde{D}_k$. Next we set

$$w_k(\xi) = \frac{1}{M_k^{2/q-1)}(x_k)} v_k\left(x_k + \frac{\xi}{M_k(x_k)}\right) = \frac{1}{v_k(x_k)} v_k\left(x_k + \frac{\xi}{M_k(x_k)}\right).$$

The function w_k is defined for $x_k + M_k^{-1}(x_k)\xi \in D$, that is $1/2 < \left|x_k + M_k^{-1}(x_k)\xi\right| < 1$. If we notice again that, for $|\xi| < k$, we have

$$\left|x_k + M_k^{-1}(x_k)\xi\right| \leq |x_k| + kM_k^{-1}(x_k) \leq 1$$

and

$$\left|x_k + M_k^{-1}(x_k)\xi\right| \geq |x_k| - kM_k^{-1} \geq 2^{-1}$$

from (2.8)–(2.9), thus $x_k + M_k^{-1}(x_k)\xi \in D_k$. Furthermore $w_k(\xi) = \zeta_k(\xi)$ if $x_k + \dfrac{\xi}{M_k(x_k)} \in \partial\Omega$, where

$$\zeta_k(\xi) = \frac{1}{M_k^{2/q-1)}(x_k)} z_k \left(x_k + \frac{\xi}{M_k(x_k)}\right).$$

Clearly w_k satisfies

$$-\Delta w_k = w_k^q \tag{2.11}$$

in and $w_k(0) = 1$, $w_k^{(q-1)/2}(\xi) \leq 2$ for any ξ such that $x_k + M_k^{-1}(x_k)\xi \in D_k$. By the local elliptic equations estimates, interior and up to the boundary, the set of functions $\{w_k\}$ is locally compact for the uniform convergence topology of $\{\xi; |\xi| \leq k, \xi \in M_k(x_k)\,(D_k - x_k)\}$. At end, there exist a sub-sequence $\{k_n\}$ such that $x_{k_n} \to x_0$ for some $x_0 \in \overline{D}$ and $w_{k_n} \to w$ in the local uniform topology of $\mathbb{R}^N$ if $x_0 \in \Omega$, or of some half-space H if $x_0 \in \partial\Omega$. Furthermore $w(0) = 1$, $0 \leq w \leq 2$ and is a solution of (2.11), either in whole $\mathbb{R}^N$, or in H, in which case it vanishes on ∂H. In the two cases it is classical that such a solution cannot exist (see [13] and [8]), which completes the proof. □

Proof of Theorem 2.2. It is an immediate consequence of Lemma 2.4 in taking, for $|x| \leq 4R_0/3$, $R = 4|x|/3$ in inequality (2.7). □

3 Caracterization of the isolated singularities

Assume Ω is a smooth domain in $\mathbb{R}^N$ and $0 \in \partial\Omega$. We shall denote

$$\overline{\Omega}^* = \overline{\Omega} \setminus \{0\} \quad \text{and} \quad \partial\Omega^* = \partial\Omega \setminus \{0\}.$$

We recall that $q_1 = (N+1)/(N-1)$, $q_2 = (N+2)/(N-2)$ and $q_3 = (N+1)/(N-3)$.

Since we are dealing with the local behaviour of u near 0, we can assume that Ω is bounded, otherwile we replace it by a smooth bounded domain the boundary of which coincides with $\partial\Omega$ in a neighborhood of 0. We denote by $(x_1, ...x_N)$ the coordinates of x and by $\mathcal{B} = \{\mathbf{e_1}, ..., \mathbf{e_N}\}$ the canonical orthonormal basis in $\mathbb{R}^N$. The preliminaries and the geometric framework of the study are similar to the ones developed in [14], but, for the sake of completeness, we repeat their presentation. Let u be a positive function satisfying

$$\begin{cases} -\Delta u = u^q & \text{in } \Omega \\ u = \zeta & \text{in } \partial\Omega^*. \end{cases} \tag{3.1}$$

Up to a rotation we can assume that the tangent plane $T_0\Omega$ to $\partial\Omega$ at 0 is the hyperplane $\{x = (x_1, ..., x_N) : x_N = 0\}$ and the outward unit normal vector is $-\mathbf{e_N}$. There exist a neighborhood G of 0 and a C^3 real value function ϕ defined on $G \cap T_0\Omega$ such that

$$G \cap \partial\Omega = \{x = (x', x_N) : x' \in G \cap T_0\Omega, x_N = \phi(x')\}.$$

Furthermore $\phi(0) = \nabla\phi(0) = 0$. Setting $\Phi(x) = y$, with $y_i = x_i$ if $i = 1, ..., N-1$ and $y_N = x_N - \phi(x')$, we can assume that Φ is a C^2 diffeomorphism from G to $\tilde{G} = \Phi(G)$, and $\Phi(\Omega \cap G) = \tilde{G} \cap \mathbb{R}^N_+$. Let $z \in C^3(\overline{\Omega}$ be the harmonic extension of ζ in Ω. We denote

$$u(x) - z(x) = \tilde{u}(y), \quad z(x) = \tilde{z}(y), \quad \zeta(x) = \tilde{\zeta}(y), \tag{3.2}$$

for $x = \Phi^{-1}(y)$ with $y \in \tilde{G} \cap \mathbb{R}^N_+$. Notice that $\tilde{u} \geq 0$ by Theorem 2.1. A lengthy computation yields to

$$\begin{cases} -\Delta\tilde{u} - |\nabla\phi|^2 \tilde{u}_{y_N, y_N} + 2\langle\nabla\phi, \nabla\tilde{u}_{y_N}\rangle + \tilde{u}_{y_N}\Delta\phi = (\tilde{u} + \tilde{z})^q & \text{in } \tilde{G} \cap \mathbb{R}^N_+ \\ \tilde{u} = 0 \quad \text{on } \tilde{G} \cap \mathbb{R}^{N-1} \setminus \{0\}. \end{cases} \tag{3.3}$$

By scaling, we can also assume that $\overline{B_1(0)} \subset \tilde{\Omega}$. Let $(r, \sigma) \in \mathbb{R}_+ \times S^{N-1}$ be the spherical coordinates in $\mathbb{R}^N$, $S^{N-1}_+ = S^{N-1} \cap \mathbb{R}_+$. If $q_1 < q < q_3$, we denote by ω the unique positive solution to

$$\begin{cases} \Delta'\omega + \lambda_{N,q}\omega + |\omega|^{q-1}\omega = 0 & \text{on } S^{N-1}_+ \\ \omega = 0 \quad \text{on } \partial S^{N-1}_+, \end{cases} \tag{3.4}$$

where

$$\lambda_{N,q} = \frac{2}{q-1}\left(\frac{2q}{q-1} - N\right). \tag{3.5}$$

Theorem 3.1 *Assume $q_1 < q < q_2$. Let $\zeta \in C^3(\partial\Omega)$ and $u \in C(\overline{\Omega}^*) \cap C^2(\Omega)$ be a solution of (3.1). Then*

(i) either

$$\lim_{\substack{x \to 0 \\ x/|x| \to \sigma}} |x|^{2/(q-1)} u(x) = \omega(\sigma), \quad \textit{locally uniformly on } S^{N-1}_+, \tag{3.6}$$

(ii) or u can be extended to $\overline{\Omega}$ as a continuous function.

We recall that, in this range of exponents, there exists a positive constant $c = c(\Omega, q, \|\zeta\|_{L^\infty})$ such that

$$u(x) \leq c\,|x|^{-2/(q-1)}, \tag{3.7}$$

by Theorem 2.2. Therefore Theorem 3.3 is an immediate consequence of the next result, were we prove that the convergence of $|x|^{2/(q-1)}$ occurs provided the above estimate holds and $q \in (q_1, q_3) \setminus q_2$.

Proposition 3.2 *Assume $q_1 < q < q_3$, $q \neq q_2$. $\zeta \in C^3(\partial\Omega)$ and $u \in C(\overline{\Omega}^*) \cap C^2(\Omega)$ be a solution of (3.1) such that $|x|^{2/(q-1)}\, u(x)$ is bounded. Then, either (3.6) holds, or u can be extended to $\overline{\Omega}$ as a continuous function.*

Proof. The three first steps of the proof present some similarities with elements of the proof of [14, Th 4.1], so we shall just give a short view of them, adapted to the specific equation (3.3).
Step 1- Reduction to an asymptotically autonomous equation. In spherical coordinates we denote by ∇' the tangential gradient to S^{N-1} identified to the covariant derivative via the imbedding $S^{N-1} \hookrightarrow \mathbb{R}^N$ and by $\mathbf{n} = y/|y|$ the outward normal unit vector to S^{N-1}. Lenghty computations show that (3.3) reads as follows in spherical coordinates,

$$\begin{aligned}
&r^2\tilde{u}_{rr}\left(1 - 2\phi_r\langle \mathbf{n}, \mathbf{e}_N\rangle + |\nabla\phi|^2 \langle \mathbf{n}, \mathbf{e}_N\rangle^2\right)\\
&+r\tilde{u}_r\left(N - 1 - r\langle \mathbf{n}, \mathbf{e}_N\rangle\Delta\phi - 2\langle \nabla'\langle \mathbf{n}, \mathbf{e}_N\rangle, \nabla'\phi\rangle + r\,|\nabla\phi|^2 \langle \nabla'\langle \mathbf{n}, \mathbf{e}_N\rangle, \mathbf{e}_N\rangle\right)\\
&+\langle \nabla'\tilde{u}, \mathbf{e}_N\rangle\left(2\phi_r - |\nabla\phi|^2 \langle \mathbf{n}, \mathbf{e}_N\rangle - r\Delta\phi\right)\\
&+r\langle \nabla'\tilde{u}_r, \mathbf{e}_N\rangle\left(\langle \mathbf{n}, \mathbf{e}_N\rangle\,|\nabla\phi|^2 - 2\phi_r\right) - 2\langle \nabla'\tilde{u}_r, \nabla'\phi\rangle\langle \mathbf{n}, \mathbf{e}_N\rangle\\
&+|\nabla\phi|^2 \langle \nabla'\langle \nabla'\tilde{u}, \mathbf{e}_N\rangle, \mathbf{e}_N\rangle - \frac{2}{r}\langle \nabla'\langle \nabla'\tilde{u}, \mathbf{e}_N\rangle, \nabla'\phi\rangle + \Delta'\tilde{u} + r^2(\tilde{u} + \tilde{z})^q = 0.
\end{aligned} \tag{3.8}$$

Use the transformation

$$t = \ln r, \quad t \leq 0, \quad ,\tilde{u}(r,\sigma) = r^{-2/(q-1)}v(t,\sigma), \quad \tilde{z}(r,\sigma) = r^{-2/(q-1)}\alpha(t,\sigma),$$

we obtain finally that v satisfies

$$\begin{aligned}
(1+\epsilon_1)\, v_{tt} + \left(N - 2\frac{q+1}{q-1} + \epsilon_2\right) v_t + (\lambda_{N,q} + \epsilon_3)\, v + \Delta' v + \langle \nabla' v, \overrightarrow{\epsilon_4}\rangle\\
+ \langle \nabla' v_t, \overrightarrow{\epsilon_5}\rangle + \langle \nabla'\langle \nabla v, \mathbf{e}_N\rangle, \overrightarrow{\epsilon_6}\rangle + (v+\alpha)^q = 0,
\end{aligned} \tag{3.9}$$

on $(-\infty, 0] \times S^{N-1}_+$, where the ϵ_j are functions of t and σ which are linear functions of ϕ, $\nabla\phi$, $\Delta\phi$ and $\nabla\phi^2$. Furthermore $v = 0$ on $(-\infty, 0] \times \partial S^{N-1}_+$.
Step 2- The a priori estimates. By the choice of ϕ, we have

$$|\phi(x)| \leq Cr^2,\ |\nabla\phi(x)| = \sqrt{|\phi_r|^2 + r^{-2}\,|\nabla'\phi|^2} \leq Cr \ \text{ and } \ |\Delta\phi| \leq C.$$

Thus

$$|\epsilon_j(t)| \leq Ce^t \quad j = 1, ..., 6. \tag{3.10}$$

Since ζ is positive and bounded, $0 \leq \alpha(t,\sigma) \leq Ce^{2t/(q-1)}$. Moreover v and $(v+\alpha)^q$ are also bounded, thus we can use the Agmon-Douglis-Nirenberg results on $(-\infty, 0] \times S^{N-1}_+$, to derive $W^{2,p}$ estimates for any $p < \infty$ in any cylinder $[T-1, T+1] \times \overline{S^{N-1}_+}$ ($T \leq -2$). By the Morrey imbedding theorem, for any $0 < \gamma < 1$ there exists $M > 0$ such that

$$\|v\|_{C^{1,\gamma}([T-1,T+1]\times\overline{S^{N-1}_+})} \leq M. \tag{3.11}$$

Next we compute $\epsilon_{j\,t}$ and $\nabla\epsilon_j$. Using the fact that $D^2\phi$ and $D^3\phi$ are bounded (because $\partial\Omega$ is C^3), we derive that, for any $j = 1, ..., 6$, there exists $C_j > 0$ such that

$$\max\{|\epsilon_{j\,t}(\sigma,t)| + \left|\nabla'\epsilon_j(\sigma,t)\right| : \sigma \in S_+^{N-1}\} \leq C_j e^t, \tag{3.12}$$

for $t \leq 0$. Furthermore, the explicit values of the ϵ_j given in [14] show that, for $j = 1, 5$ and 6, there holds

$$\max\{|\epsilon_{j\,tt}(\sigma,t)| + \left|\nabla'\epsilon_{j\,t}(\sigma,t)\right| + \left|D^2\epsilon_{j\,t}(\sigma,t)\right| : \sigma \in S_+^{N-1}\} \leq C_j e^t. \tag{3.13}$$

Actually, for such indices, the ϵ_j depend only on the first derivatives of ϕ. Therefore, it follows from Schauder estimates,

$$\begin{aligned}\|v(t,.)\|_{C^{2,\gamma}(\overline{S_+^{N-1}})} + \|v_t(t,.)\|_{C^{1,\gamma}(\overline{S_+^{N-1}})} + \|v_{tt}(t,.)\|_{C^{\gamma}(\overline{S_+^{N-1}})} \\ + \sup_{|h|\leq 1} |h|^{-\gamma} \|v_{tt}(t+h,.) - v_{tt}(t,.)\|_{C(\overline{S_+^{N-1}})} \leq M.\end{aligned} \tag{3.14}$$

Multiplying (3.9) by v_t and integrating over S_+^{N-1} yields to

$$\begin{aligned}&\int_{S_+^{N-1}} \left(N - 2\frac{q+1}{q-1} + \epsilon_2 + \frac{1}{2}\epsilon_{1\,t}\right) v_t^2 d\sigma - \frac{1}{2}\int_{S_+^{N-1}} \epsilon_{1\,t} v^2 d\sigma \\ &= \frac{1}{2}\frac{d}{dt}\int_{S_+^{N-1}} \left(|\nabla' v|^2 - (1+\epsilon_1)v_t^2 - (\lambda_{N,q} + \epsilon_3)v^2 - \frac{2}{q+1}(v+\alpha)^{q+1}\right) d\sigma \\ &+ \int_{S_+^{N-1}} \left((v+\alpha)^q \alpha_t - v_t\left(\langle\nabla' v, \overrightarrow{\epsilon_4}\rangle - \langle\nabla' v_t, \overrightarrow{\epsilon_5}\right) - \langle\nabla'\langle\nabla' v, \mathbf{e}_N\rangle, \overrightarrow{\epsilon_6}\rangle\right) d\sigma.\end{aligned} \tag{3.15}$$

Notice that $N - 2(q+1)/(q-1) \neq 0$ since $q \neq q_2$, thus, by (3.10), (3.12) and (3.11), there holds

$$\int_{-\infty}^0 \int_{S_+^{N-1}} v_t^2 d\sigma < \infty. \tag{3.16}$$

Differentiating (3.9) with respect to t, multiplying by v_{tt} and using (3.10)-(3.16), we derive

$$\int_{-\infty}^0 \int_{S_+^{N-1}} \left(v_{tt}^2 + \left|\nabla' v_t\right|^2\right) d\sigma < \infty. \tag{3.17}$$

Because $t \mapsto \displaystyle\int_{S_+^{N-1}} v_t^2(t,.)d\sigma$ and $t \mapsto \displaystyle\int_{S_+^{N-1}} v_{tt}^2(t,.)d\sigma$ are uniformly continuous on $(-\infty, 0]$, by (3.14), we obtain finally,

$$\lim_{\tau\to-\infty} \int_{S_+^{N-1}} \left(v_t^2 + v_{tt}^2\right)(\tau,.)d\sigma = 0. \tag{3.18}$$

Step 3- The convergence. Let $\mathcal{T}(v) = \bigcup_{t\leq 0}\{v(t,.)\}$ be the negative trajectory of v. By (3.14) and Ascoli's theorem, $\mathcal{T}(v)$ is relatively compact in in $C^2(\overline{S_+^{N-1}})$. By standard results, the limit set $\Gamma = \bigcap_{\tau<0} \overline{\bigcup_{t\leq\tau}\{v(t,.)\}}$ is a non-empty connected compact set. Let $t \to -\infty$ in (3.9), we

derive from (3.10), (3.12) and (3.18), that any $\omega \in \Gamma$ is a positive solution of (3.4). Therefore, either

$$\lim_{t\to-\infty} v(t,.) = \omega(.) \tag{3.19}$$

in $C^2(\overline{S_+^{N-1}})$, or

$$\lim_{t\to-\infty} v(t,.) = 0 \tag{3.20}$$

in the same topology.

Step 4- Removable singularity. Assume that (3.20) holds. We write (3.9) under the form

$$v_{tt} + \left(N - 2\frac{q+1}{q-1}\right) v_t + \lambda_{N,q} v + \Delta' v + H + (v+\alpha)^q = 0, \tag{3.21}$$

where

$$H = \epsilon_1 v_{tt} + \epsilon_2 v_t + \epsilon_3 v + + \langle \nabla' v, \overrightarrow{\epsilon_4}\rangle + \langle \nabla' v_t, \overrightarrow{\epsilon_5}\rangle + \langle \nabla' \langle \nabla v, \mathbf{e}_N\rangle, \overrightarrow{\epsilon_6}\rangle. \tag{3.22}$$

Thus, by (3.12), (3.13),

$$H^*(t) := \max\{|H(\sigma,t)| + |H_t(\sigma,t)| + \left|\nabla' H(\sigma,t)\right| : \sigma \in S_+^{N-1}\} \le Ce^t. \tag{3.23}$$

Since the first eigenvalue of $-\Delta$ in $W_0^{1,2}(S_+^{N-1})$ is $N-1$,

$$\int_{S_+^{N-1}} v\Delta v \, d\sigma \le (1-N)\int_{S_+^{N-1}} v^2 \, d\sigma.$$

If we set

$$X(t) = \left(\int_{S_+^{N-1}} v^2 \, d\sigma\right)^{1/2}$$

then

$$\int_{S_+^{N-1}} v_t v \, d\sigma = X(t)X'(t) \quad \text{and} \quad \int_{S_+^{N-1}} v_{tt} v \, d\sigma \le X(t)X''(t).$$

Furthermore $(\tilde{v}+\alpha)^q \le 2^{q-1}(\tilde{v}^q + \alpha^q) \le 2^{q-1}\tilde{v}^q + Ce^{2qt/(q-1)}$. Multiplying (3.9) by v, integrating over S_+^{N-1} and using the previous estimates yields to

$$X'' + \left(N - 2\frac{q+1}{q-1}\right) X' - (N - 1 - \lambda_{N,q} - \epsilon(t))\, X + Ce^{2qt/(q-1)} + H^* \ge 0, \tag{3.24}$$

where $\epsilon(t) = \|v(t,.)\|_{L^\infty}^{q-1} \to 0$ as $t \to -\infty$ and C is some positive constant. By assumption $\ell = N - 1 - \lambda_{N,q} > 0$. For $0 < \epsilon < \ell$ there exists $T < 0$ such that $\epsilon(t) < \epsilon$ on $(-\infty, T]$. Because $Ce^{2qt/(q-1)} + H^* \le C_v e^t$, X satisfies

$$X'' + \left(N - 2\frac{q+1}{q-1}\right) X' - (\ell - \epsilon)\, X + C_v e^t \ge 0 \tag{3.25}$$

on $(-\infty, T]$. The linear equation

$$Y'' + \left(N - 2\frac{q+1}{q-1}\right) Y' - (\ell - \epsilon)\, Y = 0, \tag{3.26}$$

has two linearly independent solutions

$$Y_{1,\epsilon}(t) = e^{r_{1,\epsilon}t} \quad \text{and} \quad Y_2(t) = e^{r_{2,\epsilon}t},$$

where $r_{1,\epsilon}$ and $r_{2,\epsilon}$ are respectively the positive and the negative root of the algebraic equation

$$r^2 + \left(N - 2\frac{q+1}{q-1}\right) r - \ell + \epsilon = 0.$$

Furthermore

$$r_{1,\epsilon} \to \frac{q+1}{q-1} \quad \text{and} \quad r_{2,\epsilon} \to \frac{2}{q-1} + 1 - N \quad \text{as} \quad \epsilon \to 0.$$

We find a particular solution $\tilde{Y}$ of the non-homogeneous equation

$$Y'' + \left(N - 2\frac{q+1}{q-1}\right) Y' - (\ell - \epsilon)\, Y = -C_v e^t$$

under the form

$$\tilde{Y}(t) = A_v e^t \quad \text{where} \quad A_v = -C_v \left(\lambda_{N,q} - 2\frac{q+1}{q-1} - \epsilon\right)^{-1}.$$

Because $X(t) = o(Y_2(t))$ at $-\infty$, we can apply maximum principle to (3.25) and get

$$X(t) \le \frac{X(T)}{Y_1(T)} Y_1(t) + \tilde{Y}(t) \le C Y_1(t) + A_v e^t \quad \text{on} \quad (-\infty, T]. \tag{3.27}$$

This proves that the $L^2(S_+^{N-1})$ norm of $v(t,.)$ is exponentially decaying as $t \to -\infty$. If we return to (3.9) and apply the L^2 regularity theory, we derive that the $W^{2,2}(S_+^{N-1})$ norm of $v(t,.)$ satisfies

$$\|v(t,.)\|_{W^{2,2}} \le C(Y_1(t) + e^t) \quad \text{on} \quad (-\infty, T]. \tag{3.28}$$

By the Sobolev imbedding theorems

$$\|v(t,.)\|_{L^p} \le C(Y_1(t) + e^t) \quad \text{on} \quad (-\infty, T], \tag{3.29}$$

where

$$p = \begin{cases} 2(N-1)/(N-5) & \text{if } N \ge 6 \\ \text{any } p < \infty & \text{if } N = 5 \\ \infty & \text{if } N \le 4. \end{cases}$$

Therefore

$$\|v^q(t,.)\|_{L^r} \le C Y_1^q(t) \quad \text{on} \quad (-\infty, T]. \tag{3.30}$$

and

$$r > \begin{cases} \dfrac{2(N-1)}{q_3(N-5)} = r^* > 2 & \text{if } N \ge 6 \\ \text{any } p < \infty & \text{if } N \le 5. \end{cases}$$

Returning to equation (3.9) we derive from (3.29) and (3.30) and the Agmon-Douglis-Nirenberg estimates that

$$\|v(t,.)\|_{W^{2,r}} \leq C(Y_1(t)+e^t) \quad \text{on} \quad (-\infty,T], \tag{3.31}$$

which is an improvement with respect to (3.28). Iterating this process and using Schauder estimates, we finally get

$$\|v(t,.)\|_{C^2} \leq C(Y_1(t)+e^t) \quad \text{on} \quad (-\infty,T]. \tag{3.32}$$

Therefore the function H^* defined in (3.23) satisfies $H^*(t) \leq C_v(Y_1(t)+e^t)e^t$. If we return to (3.24) and follows the same method as above, we derive

$$X(t) \leq \frac{X(T)}{Y_1(T)}Y_1(t) + C(Y_1(t)+e^t)e^t \quad \text{on} \quad (-\infty,T]. \tag{3.33}$$

Iterating this process we end with

$$\|v(t,.)\|_{C^2} \leq CY_1(t) \quad \text{on} \quad (-\infty,T] \quad \text{on} \quad (-\infty,T]. \tag{3.34}$$

If we return to the function u, and use the fact that ϵ can be taken as small as we want, we derive that u remains bounded in Ω. This implies that u is a smooth solution in $\overline{\Omega}$ by using the representation formula (2.2) with $\gamma = 0$. □

In the critical case $q = q_1$ it appears a superposition of the linear and nonlinear effects since their characteristic exponents $2/(q-1)$ and $N-1$ coincide. Similar phenomena have been observed several times in the past [25], [1], [7]. We recall that $x_N\big|_{S^{N-1}_+}$ generates the first eigenspace of $-\Delta'$ in $W^{1,2}_0(S^{N-1}_+)$. We denote by ϕ_1 the positive first eigenfunction normalized by $\|\phi_1\|_{L^2(S^{N-1}_+)} = 1$ and

$$k^*_N = \left(\frac{2\int_{S^{N-1}_+} \phi_1^{2N/(N-1)} d\sigma}{N(N-1)} \right)^{(1-N)/2}. \tag{3.35}$$

Theorem 3.3 *Assume $q = q_1$ and ζ and u are as in Theorem 3.3. Then*
(i) either

$$\lim_{\substack{x \to 0 \\ x/|x| \to \sigma}} |x|^{N-1} (-\ln|x|)^{(N-1)/2} u(x) = k^*_N \phi_1(\sigma), \quad \textit{locally uniformly on } S^{N-1}_+, \tag{3.36}$$

(ii) or u can be extended to $\overline{\Omega}$ as a continuous function.

We consider the following representation of S^{N-1}_+

$$S^{N-1}_+ = \{x = \{x', x_N\} = (\sin\theta\,\sigma', \cos\theta) : \sigma' \in S^{N-2}, \theta \in [0, \pi/2]\}.$$

Then the $SO(N)$- invariant surface measure on S^{N-1} is $d\sigma = \sin^{N-2}\theta\, d\sigma' d\theta$, where $d\sigma'$ is the $SO(N-1)$- invariant surface measure on S^{N-2}. With this representation, $\phi_1(\sigma) = \phi_1(\theta) = \cos\theta$, and the Laplace-Beltrami operator inherits the following representation,

$$\Delta'\psi = \frac{1}{\sin^{N-2}\theta}\left(\sin^{N-2}\theta\psi_\theta\right)_\theta + \frac{1}{\sin^2\theta}\Delta^*\psi,$$

for any $\psi \in C^2(S^{N-1})$, where Δ^* is the Laplace-Beltrami operator on S^{N-2}.

Lemma 3.4 *Under the assumptions of Theorem 3.3 there exists $C > 0$ such that*

$$\int_{S_+^{N-1}} v(t,\sigma)\phi_1(\sigma)d\sigma \le C(-t)^{-1/(q_1-1)} \quad \forall t < 0. \tag{3.37}$$

Proof. Steps 1-3 of the proof of Proposition 3.2 are valid in the sense that v is bounded, from Theorem 2.2, and satisfies

$$v_{tt} - Nv_t + (N-1)v + \Delta' v + H + (v+\alpha)^{q_1} = 0, \tag{3.38}$$

with H defined in (3.22). Uniform bounds and estimates (3.16)-(3.18) hold for v. Therefore the limit set of the trajectory of $\{v(t,.)\}_{t\le 0}$ is reduced to 0 since the set of positive solutions of (3.4) is reduced to zero. Since $(v+\alpha)^{q_1} = v^{q_1} + q_1\theta v^{q_1-1}\alpha$ with $0 \le \theta \le 1$, we write (3.38) as

$$v_{tt} - Nv_t + (N-1)v + \Delta' v + v^{q_1} = -q\theta v^{q_1-1}\alpha - H := H^*, \tag{3.39}$$

and $H^*(\sigma,t)$ satisfies (3.23) as H does. Set $y(t) = \displaystyle\int_{S_+^{N-1}} v(t,\sigma)\phi_1(\sigma)d\sigma$, then

$$y'' - Ny' + \int_{S_+^{N-1}} v^q(t,\sigma)\phi_1(\sigma)d\sigma = \int_{S_+^{N-1}} H^*\phi_1(\sigma)d\sigma. \tag{3.40}$$

The use of convexity and (3.16)-(3.18) transform (3.40) into

$$y'' - Ny' + cy^{q_1} \le Ce^t.$$

Set $\tilde{y}(t) = y(t) + Ae^t$, $A > 0$ to be fixed, then

$$\begin{aligned}\tilde{y}'' - N\tilde{y}' + c\tilde{y}^{q_1} &\le Ce^t - (N-1)Ae^t + c\left(\tilde{y}^{q_1} - y^{q_1}\right)\\ &\le Ce^t - (N-1)Ae^t + cq_1Ay^{q_1-1}e^t.\end{aligned}$$

Since $y(t) \to 0$ as $t \to -\infty$, the right-hand side can be made negative for $t \le T < 0$ by taking$A$ large enough. Because

$$(\tilde{y}^{1-q_1})'' - N(\tilde{y}^{1-q_1})' - c(q-1) \ge 0$$

from the last above inequality, $Z = (\tilde{y}^{1-q_1})'$ satisfies

$$Z' - NZ \ge c(q_1-1) \Longrightarrow Z(t) \le \frac{c(q_1-1)}{N}\left(e^{N(t-T)} - 1\right) + e^{N(t-T)}Z(T) \quad \forall t \le T. \tag{3.41}$$

By taking $t \le T^* < T$, the right-hand side of (3.41) can be made less than $-c/2N(q_1-1) = -B$, thus

$$\tilde{y}^{1-q_1}(T^*) - \tilde{y}^{1-q_1}(t) \le -B(T^*-t) \Longrightarrow \tilde{y}(t) \le \left(\tilde{y}^{1-q_1}(T^*) + B(T^*-t)\right)^{-1/(q_1-1)}, \tag{3.42}$$

which implies (3.37). □

Lemma 3.5 *Under the assumptions of Theorem 3.3, there exists $C > 0$ such that*

$$\|v(t,.)\|_{L^\infty} \leq C(-t)^{-1/q_1}. \tag{3.43}$$

Proof. We proceed by contradiction in assuming that

$$\limsup_{t\to-\infty} (-t)^{1/q_1}\rho(t) = \infty,$$

where we have set $\rho(t) = \|v(t,.)\|_{L^\infty}$. By [7, Appendix A.2], there exists $\eta \in C^\infty((-\infty,0])$ with the following properties

$$\eta > 0,\ \eta' < 0,\ \lim_{t\to-\infty} \eta(t) = 0, \tag{3.44}$$

$$0 < \limsup_{t\to-\infty} \rho(t)/\eta(t) < \infty, \tag{3.45}$$

$$\lim_{t\to-\infty} (-t)^{1/q_1}\eta(t) = \infty, \tag{3.46}$$

$$(\eta'/\eta)',\ (\eta''/\eta)' \in L^1(-\infty,0), \tag{3.47}$$

$$\lim_{t\to-\infty} \eta'(t)/\eta(t) = \lim_{t\to-\infty} \eta''(t)/\eta(t) = 0. \tag{3.48}$$

The function $w^* = v/\eta$ satisfies

$$w^*_{tt} - \left(N - 2\frac{\eta'}{\eta}\right) w^*_t + \left(N - 1 + \frac{\eta''}{\eta} - N\frac{\eta'}{\eta}\right) w^* + \Delta' w^* + \eta^{q_1-1} w^{*\,q_1} - \frac{H^*}{\eta} = 0. \tag{3.49}$$

Since w is uniformly bounded, we deduce from (3.44)–(3.47), the L^p and Schauder estimates,

$$\begin{aligned} &\|w^*(t,.)\|_{C^{2,\gamma}(\overline{S^{N-1}_+})} + \|w^*_t(t,.)\|_{C^{1,\gamma}(\overline{S^{N-1}_+})} + \|w^*_{tt}(t,.)\|_{C^{\gamma}(\overline{S^{N-1}_+})} \\ &\qquad + \sup_{|h|\leq 1} |h|^{-\gamma} \|w^*_{tt}(t+h,.) - w^*_{tt}(t,.)\|_{C(\overline{S^{N-1}_+})}, \leq M. \end{aligned} \tag{3.50}$$

with $\gamma \in (0,1)$, which implies

$$\int_{-\infty}^0 \int_{S^{N-1}_+} \left(w^{*\,2}_{tt} + \left|\nabla' w^*_t\right|^2 \right) d\sigma < \infty. \tag{3.51}$$

The negative trajectory $\mathcal{T}(v) = \bigcup_{t\leq 0}\{w^*(t,.)\}$ of w^* is relatively compact in $C^2(\overline{S^{N-1}_+})$. By (3.48) the limit set of the trajectory $\Gamma = \bigcap_{\tau<0} \overline{\bigcup_{t\leq\tau}\{w^*(t,.)\}}$ is a non-empty connected compact set of $\ker(\Delta' + \lambda_1 I)$, not reduced to zero because of (3.45). Therefore there exists $\{t_n\}_{n\in\mathbb{N}}$ tending to $-\infty$ and $\beta > 0$ such that

$$\lim_{n\to\infty} \frac{v(t_n,\sigma)}{\eta(t_n)} = 2\beta\phi_1(\sigma)$$

in the $C^1(S^{N-1}_+)$-topology. If we return to the function u which is a super solution for the Laplace equation with zero boundary conditions, it implies that, for n large enough,

$$u(x) \geq c_N \eta(t_n)\beta P(x,0) \quad \forall x \in \Omega \setminus B_{e^{t_n}}. \tag{3.52}$$

Thus, using Theorem 2.1,

$$c_N^{q_1}\left(\eta(t_n)\right)^{q_1}\int_{\Omega\setminus B_{e^{t_n}}} P^{q_1}(x,0)\rho(x)dx \le \int_\Omega u^{q_1}\rho(x)dx < \infty.$$

Replacing the second integral in $\Omega\setminus B_{e^{t_n}}$ by an integral in a truncated cone $C = C^S \cap \{x : e^{t_n} \le |x| \le 1\}$ with vertex 0, we derive

$$\int_{\Omega\setminus B_{e^{t_n}}} P^{q_1}(x,0)\rho(x)dx \ge c\int_{e^{t_n}}^1 t^{-1}dt = c(-t_n).$$

Thus, by (3.46), $u^{q_1} \notin L(\Omega, \rho\,dx$, contradiction. □

We recall the following classical result from perturbation theory of linear second order differential equation with constant coefficients [2, Chap 6 §13]

Lemma 3.6 *Let ϵ_1 and ϵ_2 be two continuous functions defined on $(-\infty, 0]$ tending to 0 as $t \to -\infty$ and $a, b > 0$ such that $a^2 - 4b \ge 0$. Then there exist two linearly independent solutions y_1 and y_2 of*

$$y'' - (a + \epsilon_1(t)y'(t) + (b + \epsilon_2(t))y(t) = 0 \quad \text{on } (-\infty, 0) \tag{3.53}$$

such that

$$\lim_{t\to-\infty}\frac{y_1'(t)}{y_1'(t)} = c_1 \text{ and } \lim_{t\to-\infty}\frac{y_2'(t)}{y_2'(t)} = c_2, \tag{3.54}$$

where

$$c_1 = \frac{a+\sqrt{a^2-4b}}{2} \text{ and } c_2 = \frac{a-\sqrt{a^2-4b}}{2}. \tag{3.55}$$

Furthermore, if $f \in C((-\infty, 0])$ is such that $(1-t)^\alpha f(t)$ is bounded for some $\alpha > 0$, then there exists a solution u of

$$y'' - (a + \epsilon_1(t)y'(t) + (b + \epsilon_2(t))y(t) = f(t) \quad \text{on } (-\infty, 0), \tag{3.56}$$

such that $(1-t)^\alpha u(t)$ remains bounded on $(-\infty, 0]$.

In the next lemma we improve (3.43) step by step, up we reach the optimal estimate.

Lemma 3.7 *Under the assumptions of Theorem 3.3, there exists $C > 0$ such that*

$$\|v(t,.)\|_{L^\infty} \le C(-t)^{-1/(q_1-1)}. \tag{3.57}$$

Proof. Step 1- By Lemma 3.5 the function $w_1(t,\sigma) = (-t)^{1/q_1}v(t,\sigma)$ is uniformly bounded on $(-\infty, 1] \times S_+^{N-1}$ where it satisfies

$$w_{1\,tt} - \left(N + \frac{2}{q_1 t}\right)w_{1\,t} + \left(N - 1 + \frac{1+q_1}{q_1^2t^2} + \frac{N}{q_1 t}\right)w_1 + \Delta' w_1 + (-t)^{-1+1/q_1}w_1^{q_1} - (-t)^{1/q_1}H^* = 0. \tag{3.58}$$

Denote respectively by $\tilde{w}_1$ and $\tilde{G}_1$ the projections of w_1 and $(-t)^{-1+1/q_1}w^{q_1} - (-t)^{1/q_1}H^*$ onto the orthogonal space to $\ker(\Delta' + (N-1)I)$ in $W_0^{1,2}(S_+^{N-1})$. Then

$$\tilde{w}_{1\,tt} - \left(N + \frac{2}{q_1 t}\right)\tilde{w}_{1\,t} + \left(N - 1 + \frac{1+q_1}{q_1^2 t^2} + \frac{N}{q_1 t}\right)\tilde{w}_1 + \Delta'\tilde{w}_1 + \tilde{G}_1 = 0 \quad \text{in } (-\infty, 0) \times S_+^{N-1}.$$

Then $||\tilde{G}_1(t,.)||_{L^\infty} \leq c(-t)^{-1+1/q_1}$. Since $\tilde{w}_1$ is uniformly bounded, the standard regularity theory for ellipitic equation implies that $\tilde{w}_{1\,t}$ inherits the same properties. Therefore the function $t \mapsto \tilde{R}_1(t) := ||\tilde{w}_1(t,.)||_{L^2}$ satisfies

$$\tilde{R}_1'' - \left(N + \frac{2}{q_1 t}\right)\tilde{R}_1' + \left(N - 1 + \frac{1+q_1}{q_1^2 t^2} + \frac{N}{q_1 t}\right)\tilde{R}_1 \geq -c(-t)^{-1+1/q_1}.$$

It follows from the maximum principle and Lemma 3.6 that $\tilde{R}_1$ satisfies

$$\tilde{R}_1(t) \leq C(-t)^{-1+1/q_1} \quad \forall t \leq -1.$$

As in the proof of Lemma 3.5, the L^2 estimate implies a $C^{2,\gamma}$ one,

$$\begin{aligned} &\|\tilde{w}_1(t,.)\|_{C^{2,\gamma}(\overline{S_+^{N-1}})} + \|\tilde{w}_{1\,t}(t,.)\|_{C^{1,\gamma}(\overline{S_+^{N-1}})} + \|\tilde{w}_{1\,tt}(t,.)\|_{C^{\gamma}(\overline{S_+^{N-1}})} \\ &\qquad + \sup_{|h|\leq 1} |h|^{-\gamma} \|\tilde{w}_{1\,tt}(t+h,.) - \tilde{w}_{1\,tt}(t,.)\|_{C(\overline{S_+^{N-1}})} \leq M(-t)^{-1+1/q_1}. \end{aligned} \tag{3.59}$$

Combining (3.59) with (3.37) yields to

$$\|v(t,.)\|_{L^\infty} \leq C(-t)^{-\alpha_1} \quad \forall t < 0, \tag{3.60}$$

with $\alpha_1 = \min\{1, 1/(q_1-1)\} = \min\{1, (N-1)/2\}$. If $1 \geq (N-1)/2$ we have proved (3.57).

Step 2- If $1 < (N-1)/2$, we iterate this procedure in setting $w_2(t,\sigma) = (-t)v(t,\sigma)$ and denote by $\tilde{w}_2$ the projection of w_2 onto $(\ker(\Delta' + (N-1)I))^\perp$. *Mutadis mutandis*, it leads to

$$\tilde{w}_{2\,tt} - \left(N + \frac{2}{t}\right)\tilde{w}_{2\,t} + \left(N - 1 + \frac{2}{t^2} + \frac{N}{t}\right)\tilde{w}_2 + \Delta'\tilde{w}_2 + \tilde{G}_2 = 0 \quad \text{in } (-\infty, 0) \times S_+^{N-1},$$

where $||\tilde{G}_2(t,.)||_{L^\infty} \leq c(-t)^{1-q_1}$ and finally,

$$\|v(t,.)\|_{L^\infty} \leq C(-t)^{-\alpha_2} \quad \forall t < 0, \tag{3.61}$$

with $\alpha_2 = \min\{q_1, 1/(q_1-1)\} = \min\{q_1, (N-1)/2\}$. We continue this procedure up we reach

$$\|v(t,.)\|_{L^\infty} \leq C(-t)^{-\alpha_k} \quad \forall t < 0, \tag{3.62}$$

with $\alpha_k = \min\{q_1^{k-1}, 1/(q_1-1)\} = (N-1)/2$. □

Remark. It follows from the last step in the proof of (3.57) that the projection $\tilde{v}$ of v onto $\ker(\Delta' + (N-1)I)^\perp$ satisfies

$$\|\tilde{v}(t,.)\|_{L^\infty} \leq C(-t)^{-(N+1)/2} \quad \forall t < 0, \tag{3.63}$$

Proof of Theorem 3.3. Step 1. Existence of a limit. The function $w(t,.) = (-t)^{(N-1)/2}v(t,.)$ is bounded and satisfies

$$\begin{aligned} w_{tt} - \left(N + \frac{N-1}{t}\right) w_t + \left(N - 1 + \frac{N^2-1}{4t^2}\right) w + \Delta' w \\ - \frac{1}{t}\left(w^{q_1} - \frac{N(N-1)}{2} w\right) - (-t)^{(N-1)/2} H^* = 0, \end{aligned} \tag{3.64}$$

in $(-\infty, 0) \times S_+^{N-1}$. The function w satisfies the same bounds as the ones in (3.50). Therefore multiplying by w_t and integrating over S_+^{N-1} yields to the identity

$$\begin{aligned} &\frac{d}{dt}\int_{S_+^{N-1}} \left[\frac{w_t^2}{2} + \left(N - 1 + \frac{N^2-N}{4t^2}\right)\frac{w^2}{2} - \frac{|\nabla' w|^2}{2} - \frac{1}{t}\left(\frac{w^{q_1+1}}{q_1+1} - \frac{N^2-N}{4} w^2\right)\right] d\sigma \\ &= \left(N + \frac{N-1}{t}\right)\int_{S_+^{N-1}} w_t^2 d\sigma \\ &\qquad + \int_{S_+^{N-1}} \left[(-t)^{(N-1)/2} H^* w_t - \frac{N^2-N}{4t^3} w^2 + \frac{1}{t^2}\left(\frac{w^{q_1+1}}{q_1+1} - \frac{N^2-N}{4} w^2\right)\right] d\sigma. \end{aligned} \tag{3.65}$$

With the help of estimates (3.23), it implies again

$$\lim_{t\to-\infty} ||w_t(t,.)||_{L^2} = \lim_{t\to-\infty} ||w_{tt}(t,.)||_{L^2} = 0, \tag{3.66}$$

and clearly the L^2 norm can be replaced by the L^∞ one. The limit set $\Gamma = \bigcap_{\tau<0} \overline{\bigcup_{t\le\tau}\{w(t,.)\}}$ of the negative trajectory $\mathcal{T}$ of $w(t,.)$ is therefore a non-empty compact connected subset of $\ker(\Delta' + (N-1)I)$ which means an interval of the form $\{\lambda\phi_1 : a \le \lambda \le b\}$ where $a \ge 0$. Let $z(t) = \displaystyle\int_{S_+^{N-1}} w(t,\sigma)\phi_1(\sigma)d\sigma$ and $\tilde{w}$ be the the projection of w onto $\ker(\Delta' + (N-1)I)^\perp$. By (3.63) $\|\tilde{w}(t,.)\|_{L^\infty} \le -C/t$. We can write the equation satisfied by z under the form

$$\begin{aligned} z'' - \left(N + \frac{N-1}{t}\right) z' + \frac{N^2-1}{4t^2} z - \frac{1}{t}\left(\int_{S_+^{N-1}} (\tilde{w} + z\phi_1)^{q_1}\phi_1 d\sigma - \frac{N(N-1)}{2} z\right) \\ - (-t)^{(N-1)/2}\int_{S_+^{N-1}} H^*\phi_1 d\sigma = 0. \end{aligned} \tag{3.67}$$

Set

$$d = \int_{S_+^{N-1}} \phi_1^{q_1+1} d\sigma,$$

then (3.67) becomes

$$z'' - \left(N + \frac{N-1}{t}\right) z' - \frac{1}{t}\left(dz^{q_1} - \frac{N(N-1)}{2} z\right) = \Psi, \tag{3.68}$$

where

$$\Psi = (-t)^{(N-1)/2}\int_{S_+^{N-1}} H^*\phi_1 d\sigma - \frac{N^2-1}{4t^2} z - \frac{1}{t}\int_{S_+^{N-1}} \left((\tilde{w} + z\phi_1)^{q_1} - z^{q_1}\phi_1^{q_1}\right)\phi_1 d\sigma.$$

Because of the estimate on $\|\tilde{w}(t,.)\|_{L^\infty}$ and

$$|(\tilde{w}+z\phi_1)^{q_1}-z^{q_1}\phi_1^{q_1}|\le q_1|\tilde{w}|(z\phi_1)^{q_1-1}, \tag{3.69}$$

it follows that $\|\Psi(t,.)\|_{L^\infty}\le Ct^{-2}$. By a straightforward modification of the end of the proof of [4, Corollary 4.2, p 1061] we obtain that z admits a limit $\ell\ge 0$ when $t\to-\infty$, and

$$d\ell^q-\frac{N(N-1)}{2}\ell=0\Longrightarrow \ell\in\left\{0,\left(\frac{N(N-1)}{2d}\right)^{(N-1)/2}\right\}. \tag{3.70}$$

If $\ell=\left(\frac{N(N-1)}{2d}\right)^{(N-1)/2}$, we have (3.36) with (3.35).

Step 2. We claim that if $\ell=0$, the singularity is removable. Let $X(t)=\|w(t,.)\|_{L^2}$. As in the proof of Proposition 3.2-Step 4, we derive from (3.64) that for any $\epsilon>0$ there exists $t\le t_\epsilon<0$ such that

$$X''-\left(N+\frac{N-1}{t}\right)X'+\frac{1}{t}\left(\frac{N(N-1)-\epsilon}{2}+\frac{N^2-1}{4t}\right)X+C(-t)^{(N-1)/2}e^t\ge 0. \tag{3.71}$$

We perform the following change of unknown, $X(t)=(-t)^{(N-1)/2}e^{Nt/2}Y(t)$, then Y satisfies

$$Y''-\left(\frac{N^2}{4}+\frac{\epsilon}{2t}\right)Y+Ce^{(2-N)t/2}\ge 0. \tag{3.72}$$

By [2, p 126-127] there exist two fundamental solutions of the associated homogeneous linear equation

$$Y''-\left(\frac{N^2}{4}+\frac{\epsilon}{2t}\right)Y=0. \tag{3.73}$$

with the following asymptotic form

$$Y_1(t)=e^{-Nt/2}(-t)^{-\epsilon/4}(1+\circ(1)) \ \text{ and } \ Y_2(t)=e^{Nt/2}(-t)^{\epsilon/4}(1+\circ(1)), \tag{3.74}$$

as $t\to-\infty$, and a super solution of (3.72) $Y_0=Ae^{(2-N)t/2}$ for some $A>0$. Then $Y-Y_0$ is a subsolution of (3.73) for $t\le t_\epsilon^*\le t_\epsilon<0$. Because $(Y-Y_0)_+=\circ(Y_1)$ at $-\infty$, it follows that $Y\le Y_0+CY_2$ for some $C>0$. This implies

$$\left(\int_{S_+^{N-1}}w^2(\sigma,t)d\sigma\right)^{1/2}:=X(t)\le C(-t)^{(N-1)/2}e^t\quad\forall t\le -1.$$

Furthermore the L^2-norm can be replaced by a L^∞ one. Therefore, the solution v of (3.38)-(3.39) satisfies the estimate

$$\|v(t,.)\|_{L^\infty}\le Ce^t\quad\forall t\le 0. \tag{3.75}$$

Iterating this estimate as in the proof of Proposition 3.2-Step 4, we finally obtain

$$\|v(t,.)\|_{L^\infty}\le C(-t)^{\epsilon/4}e^{Nt}\quad\forall t\le 0\Longrightarrow\tilde{u}(x)\le C|x|(\ln(1/|x|)^{\epsilon/4}. \tag{3.76}$$

This implies the claim since ϵ is arbitrary small. □

Remark. The characterization of boundary isolated singularities we have obtained are completed by the existence of singular solutions which have been obtained by Del Pino, Musso and Pacard [9]. In particular they prove the following result :

Theorem 1.1 *There exists a number $p_N > (N+1)/(N-1)$ such that if $(N+1)/(N-1) \leq p_N$, then the following holds : given points $\xi_1, \xi_2, ..., \xi_k \in \partial\Omega$, there exists a very weak solution u to problem (1.1) such that $u \in C^2(\bar{\Omega} \setminus \{\xi_1, \xi_2, ..., \xi_k\}$ and $u(x) \to \infty$ as $x \to \xi_j$ non-tangentially, for all $i = 1, 2, ..., k$.*

Furthermore the solutions they construct have the singular behaviour we have obtained therein. They conjecture that $p_N = (N+2)/(N-2)$.

4 Equation on a spherical domain

4.1 Existence and non-existence

We assume $d \geq 2$, denote by S^d the unit sphere in $\mathbb{R}^{d+1} = \{x = \{x_1, ..., x_{d+1}\}\}$. A C^2 spherical domain S is said to be *starshaped* if there exists a spherical harmonic of degree one ϕ such that $\phi > 0$ on S and $\langle \nabla' \phi, \nu \rangle \leq 0$ on ∂S, where ∇' is the covariant gradient on S^d and ν is the unit outward vector tangent to S on ∂S. In practice we can assume that, up to a rotation, $S \subset S^d_+$ where $S^d_+ := S^d \cap \{x : x_{d+1} > 0\}$ and ϕ the first eigenfunction of the Laplace-Beltrami operator on $-\Delta'$ in $W^{1,2}_0(S^d_+)$. We recall that the corresponding eigenvalue is d. For $\ell \in \mathbb{R}$ and $q > 1$, we consider the problem of finding a positive solution to

$$\begin{cases} \Delta' v + \ell v + |v|^{q-1} v = 0 \quad \text{on } S \\ \qquad\qquad\qquad\quad v = 0 \quad \text{on } \partial S, \end{cases} \tag{4.1}$$

where $S \subset S^d$ is some C^2 domain. We first prove the following spherical Pohozaev type formula.

Proposition 4.1 *Any C^2 solution u of (4.1) satisfies*

$$\begin{aligned} \frac{d-2}{2(q+1)} \left(q - \frac{d+2}{d-2} \right) \int_S |\nabla' v|^2 \, \phi \, d\sigma - \frac{d}{2(q+1)} \left(\ell(q-1) + d \right) \int_S v^2 \phi \, d\sigma \\ = \frac{1}{2} \int_{\partial S} |\nabla' v|^2 \, \langle \nabla' \phi, \nu \rangle \, d\tau. \end{aligned} \tag{4.2}$$

Proof. If we set

$$P = \langle \nabla' \phi, \nabla' v \rangle \nabla' v,$$

then

$$\int_S div \, P \, d\sigma = \int_{\partial S} \langle P, \nu \rangle \, d\tau.$$

Thus

$$div \, P = \langle \nabla' \phi, \nabla' v \rangle \Delta' v + D^2 v(\nabla' \phi, \nabla' v) + D^2 \phi(\nabla' v, \nabla' v). \tag{4.3}$$

where $D^2 v$ is the Hessian operator. Now

$$D^2 v(\nabla'\phi, \nabla' v) = \frac{1}{2}\langle \nabla' \left|\nabla' v\right|^2, \nabla'\phi\rangle. \tag{4.4}$$

Furthermore[1], there holds classicaly

$$D^2\phi + \phi\, g = 0 \tag{4.5}$$

where $g = (g_{i,j})$ is the metric tensor on S^d. Notice that this last relation implies

$$tr(\, D^2\phi) + tr(\, g)\phi = 0,$$

with $tr(\, g) = \sum_i g_{i,i} = d$ and $tr(\, D^2\phi) = \Delta'\phi$, thus we recover the eigenvalue problem. Therefore

$$div\, P = -\langle \nabla'\phi, \nabla' v\rangle \left(\ell v + |v|^{q-1} v\right) + \frac{1}{2}\langle \nabla' \left|\nabla' v\right|^2, \nabla'\phi\rangle - \phi \left|\nabla' v\right|^2. \tag{4.6}$$

Integrating, we derive

$$\begin{aligned}\int_S div\, P\, d\sigma &= -\int_S \langle \nabla'\phi, \nabla' v\rangle \left(\ell v + |v|^{q-1} v\right)\, d\sigma + \frac{1}{2}\int_S \langle \nabla' \left|\nabla' v\right|^2, \nabla'\phi\rangle\, d\sigma - \int_S \phi \left|\nabla' v\right|^2\, d\sigma \\ &= A + B + C.\end{aligned}$$

Then

$$\begin{aligned}A &= -\int_S \langle \nabla'\phi, \nabla' v\rangle \left(\ell v + |v|^{q-1} v\right)\, d\sigma \\ &= -\int_S \langle \nabla'\phi, \nabla' \left(\frac{\ell}{2} v^2 + \frac{1}{q+1}\left|v\right|^{q+1}\right)\rangle\, d\sigma \\ &= \int_S \left(\frac{\ell}{2} v^2 + \frac{1}{q+1}\left|v\right|^{q+1}\right) \Delta'\phi\, d\sigma \\ &= -d\int_S \left(\frac{\ell}{2} v^2 + \frac{1}{q+1}\left|v\right|^{q+1}\right) \phi\, d\sigma,\end{aligned}$$

and

$$\begin{aligned}B &= \frac{1}{2}\int_S \langle \nabla' \left|\nabla' v\right|^2, \nabla'\phi\rangle\, d\sigma \\ &= -\frac{1}{2}\int_S \left|\nabla' v\right|^2 \Delta'\phi\, d\sigma + \frac{1}{2}\int_{\partial S} \left|\nabla' v\right|^2 \langle \nabla'\phi, \nu\rangle\, d\tau \\ &= \frac{d}{2}\int_S \phi \left|\nabla' v\right|^2\, d\sigma + \frac{1}{2}\int_{\partial S} \left|\nabla' v\right|^2 \langle \nabla'\phi, \nu\rangle\, d\tau,\end{aligned}$$

which implies

$$\begin{aligned}&\int_S div\, P\, d\sigma \\ &\qquad = -\int_S \left(\frac{\ell d}{2} v^2 + \frac{d}{q+1}\left|v\right|^{q+1}\right) \phi\, d\sigma + \frac{d-2}{2}\int_S \phi \left|\nabla' v\right|^2\, d\sigma + \frac{1}{2}\int_{\partial S} \left|\nabla' v\right|^2 \langle \nabla'\phi, \nu\rangle\, d\tau.\end{aligned}$$

[1]The authors are grateful to Saïd Ilias for pointing them this identity

By (4.1),

$$\begin{aligned}\int_S \left(|v|^{q+1} + \ell v^2\right) \phi \, d\sigma \, d\sigma &= -\int_S v\phi\Delta' v \, d\sigma \\ &= \int_S \langle \nabla'(v\phi), \nabla' v\rangle \, d\sigma \\ &= \int_S \phi \, |\nabla' v|^2 \, d\sigma + \int_S v\langle \nabla'\phi, \nabla' v\rangle \, d\sigma \\ &= \int_S \phi \, |\nabla' v|^2 \, d\sigma + \frac{d}{2}\int_S v^2\phi \, d\sigma,\end{aligned}$$

because $v\langle \nabla'\phi, \nabla' v\rangle = 2^{-1}\langle \nabla'\phi, \nabla' v^2\rangle$ and $\Delta'\phi = -d\phi$. Therefore

$$\int_S |v|^{q+1} \, \phi \, d\sigma = \int_S \left|\nabla' v\right|^2 \phi \, d\sigma + \left(\frac{d}{2} - \ell\right) \int_S v^2\phi \, d\sigma.$$

The next formula follows

$$\begin{aligned}\int_S div \, P \, d\sigma = \left(\frac{d-2}{2} - \frac{d}{q+1}\right) \int_S |\nabla' v|^2 \, \phi \, d\sigma \\ - \left(\frac{\ell d}{2} + \frac{d(d-2\ell)}{2(q+1)}\right) \int_S v^2\phi \, d\sigma + \frac{1}{2}\int_{\partial S} |\nabla' v|^2 \, \langle \nabla'\phi, \nu\rangle \, d\tau.\end{aligned} \tag{4.7}$$

Because both ϕ and v vanish on ∂S, there holds $|\nabla' v| = |\langle \nabla' v, \nu\rangle|$ and

$$\int_{\partial S} \langle \nabla'\phi, \nabla' v\rangle \langle \nabla' v, \nu\rangle \, d\tau - \frac{1}{2}\int_{\partial S} \left|\nabla' v\right|^2 \langle \nabla'\phi, \nu\rangle \, d\tau = \frac{1}{2}\int_{\partial S} \left|\nabla' v\right|^2 \langle \nabla'\phi, \nu\rangle \, d\tau.$$

The previous identities imply (4.2). □

Proposition 4.2 *Assume $S \subset S^d_+$.*
I- If $\ell \geq d$, any C^2 nonnegative solution to (4.1) is trivial.
II- If $1 < q < (d+2)/(d-2)$ and $\ell < d$, there exists a positive solution to (4.1).
III- If S is starshaped, $q \geq (d+2)/(d-2)$, $\ell(q-1) \leq -d$, with one of the two inequalities strict, then any C^2 solution to (4.1) is trivial. If $q = (d+2)/(d-2)$, $\ell(q-1) = -d$ the same result holds in the class of nonnegative solutions.

Proof. We recall that ϕ is positive on S^d_+. If we multiply by ϕ and integrate, we obtain

$$(-\ell + d)\int_{S^d_+} v\phi \, d\sigma = \int_{S^d_+} v^q\phi \, d\sigma.$$

Therefore, a necessary condition for existence of positive solutions of (4.1) is $\ell < d$.
If we assume $1 < q < (d+2)/(d-2)$ and $\ell < d$, we obtain a positive solution by minimizing

$$\mathcal{J}(w) = \int_{S^d_+} (\left|\nabla' w\right|^2 - \ell w^2) d\sigma$$

over the set of

$$\Sigma = \left\{ w \in W_0^{1,2}(S_+^d) : \int_{S_+^d} |w|^{q+1}\, d\sigma = 1 \right\}.$$

Clearly the methods fails if $q = (d+2)/(d-2)$.
If S is starshaped, the right-hand side of (4.2) is nonpositive, non-identically zero. By Proposition 4.1, if $q \geq (d+2)/(d-2)$ and $\ell(q-1) \geq d$, with one of these inequality being strict, the left-hand side of (4.2) is nonnegative while the right-hand one is nonpositive. If $q = (d+2)/(d-2)$, $\ell(q-1) = d$ and u is positive, then the right-hand side of (4.2) is positive. □

4.2 Uniqueness

The next results deal with uniqueness of positive solutions to (4.1). By a standard adaptation of Gidas-Ni-Nirenberg moving planes method to S^d, any positive solution of (4.1) depends only on the distance to the north pole. If we set $\theta = \cos^{-1}(x_N/|x|)$ and denote $v(x) = v(\theta)$, then

$$\begin{cases} v'' + (d-1)\cot\theta\, v' + \ell v + v^q = 0 \quad \text{in } (0, \pi/2) \\ v'(0) = 0, \quad v(\pi/2) = 0. \end{cases} \tag{4.8}$$

Proposition 4.3 *Assume $d \geq 3$, ℓ is any real number and $1 < q < (d+2)/(d-2)$, or $d = 2$, $\ell \leq 2(3-q)/(q+3)(q-1)$ and $q > 1$. Then problem (4.8) admits at most one positive solution.*

Proof. The methods consiste in adapting to our situation the proofs of Kwong and Li [17] dealing with positive solutions of

$$\begin{cases} u'' + \dfrac{d-1}{r} u' + q(r)u + u^p = 0 \quad \text{in } (0, b) \\ u'(0) = 0\,, \quad u(b) = 0. \end{cases} \tag{4.9}$$

Given $\alpha > 0$ and $w(\theta) = (\sin\theta)^\alpha\, v(\theta)$. Then w satisfies

$$w'' + (d-1-2\alpha)\cot\theta\, w' + \left(\alpha(d-1-\alpha) + \ell + \frac{\alpha(\alpha+2-d)}{\sin^2\theta} \right) w + \frac{w^q}{(\sin\theta)^{\alpha(q-1)}} = 0 \tag{4.10}$$

in $(0, \pi/2)$. Multiplying the equation by $(\sin\theta)^\beta$ for some β to be fixed, we get

$$(\sin\theta)^\beta w'' + (d-1-2\alpha)(\sin\theta)^{\beta-1}\cos\theta w' + (\sin\theta)^{\beta-\alpha(q-1)} w^q + G(\theta) w = 0$$

where

$$G(\theta) = (\sin\theta)^{\beta-2}\left((\alpha(d-1-\alpha)+\ell)\sin^2\theta + \alpha(\alpha+2-d) \right). \tag{4.11}$$

If we take $\alpha = 2(d-1)/(q+3)$ and $\beta = \alpha(q-1) = 2(d-1)(q-1)/(q+3)$ we obtain

$$(\sin\theta)^\beta w'' + \frac{\beta}{2}(\sin\theta)^{\beta-1}\cos\theta w' + w^q + G(\theta) w = 0.$$

Since

$$\frac{d}{d\theta}\left((\sin\theta)^\beta \frac{(w')^2}{2} \right) = \left((\sin\theta)^\beta w'' + \frac{\beta}{2}(\sin\theta)^{\beta-1}\cos\theta w' \right) w'$$

and

$$\frac{d}{d\theta}\left(G(\theta)\frac{(w')^2}{2}\right) = G(\theta)ww' + G'(\theta)\frac{(w')^2}{2}$$

we derive

$$\frac{dE(\theta)}{d\theta} = G'(\theta)\frac{(w')^2}{2} \tag{4.12}$$

with

$$E(\theta) = (\sin\theta)^\beta\frac{(w')^2}{2} + \frac{w^{q+1}}{q+1} + G(\theta)\frac{w^2}{2}. \tag{4.13}$$

Noticing that

$$G'(\theta) = \left[(\alpha(d-1-\alpha)+\ell)\beta\sin^2\theta + \alpha(\alpha+2-d)(\beta-2)\right](\sin\theta)^{\beta-3}\cos\theta,$$

with $\beta-2 = 2[(d-2)q-(d+2)]/(q+3)$, $\alpha+2-d = -[q(d-2)+d-4]$. If $d \geq 3$, the assumption $1 < q < (d+2)/(d-2)$, implies that $\alpha(\alpha+2-d)(\beta-2) > 0$, thus $G' > 0$ for θ small enough. Then we encounter the two possibilities already found in [17] :

(i) either G is increasing on $(0, \pi/2)$,

(ii) or there exists $c \in (0, \pi/2)$ such that G is increasing on $(0, c)$ and decreasing on $(c, \pi/2)$.

If $d = 2$, $\alpha(\alpha+2-d)(\beta-2) = \alpha^2(\beta-2) = -8\alpha^2/(q+3) < 0$. Thus $G' < 0$ for θ small enough, and, if $\ell \leq 2(3-q)/q+3)(q-1)$, we have

(iii) $G' < 0$ on $[0, \pi/2)$,

which is also a situation which is treated by [17]. The situations where G is nondeacreasing on $[0, c]$ and nonincreasing on $[c, \pi/2]$ for some $c \in [0, \pi/2]$ is the analogue of the Λ condition in [17], and it implies uniqueness even if G can tend to $-\infty$ at 0, because in all the cases $\lim_{\theta_0\to 0} G(\theta)(w'(\theta))^2 = 0$. □

In the case $d = 2$ another change of variable leads to a new uniqueness result in the proof of which we adapt some ideas due to Kwong [16] in a singular case, in avoiding his misprints.

Proposition 4.4 *Assume $d = 2$ and $2 < q < 5$ and $\ell \geq -1/4$. Then problem (4.8) admits at most one positive solution.*

Proof. If $w(\theta) = \sqrt{\sin\theta}\, v(\theta)$, then equation (4.10) becomes

$$w'' + \left(\lambda + \frac{1}{4} + \frac{1}{4\sin^2\theta}\right)w + (\sin\theta)^{-(q-1)/2}w^q = 0 \quad \text{in } (0, \pi/2), \tag{4.14}$$

with boundary value $w(0) = w(\pi/2) = 0$. Because of the singularity at 0, the problem is locally well-posed if we look for initial conditions at 0 under the form which corresponds to $v(0) = c > 0$ and $v'(0) = 0$, namely

$$w(\theta) = c\sqrt{\sin\theta}(1 + \circ(1)) \text{ and } \ w'(\theta) = \frac{c}{2\sqrt{\sin\theta}}(1 + \circ(\sin\theta)) \text{ as } \ \theta \to 0. \tag{4.15}$$

We set

$$F(\theta, w) = \left(\lambda + \frac{1}{4\sin^2\theta}\right)w + (\sin\theta)^{-(q-1)/2}w^q.$$

By concavity any positive solution has a unique positive maximum on $(0,\pi/2)$ and is increasing and then decreasing. If it exists two different positive solutions v_1 and v_2 to problem (4.8), they must intersect at least once in $(0,\pi/2)$ because of superlinearity. The same holds for the corresponding solutions w_1 and w_2 of (4.14). Let $d \in (0,\pi/2)$ be the largest of such intersections. It always exists otherwhile it would imply $w_1'(\pi/2) = w_2'(\pi/2)$ and $w_1 = w_2$. We can assume that $w_1'(\pi/2) < w_2'(\pi/2) < 0$ and $w_1 > w_2$ on $(d,\pi/2]$, thus $w_1'(d) > w_2'(d)$. Set $\gamma = w_i(d)$ $(i = 1, 2)$.

Step 1. We claim that w_1 and w_2 cannot be both decreasing on $(d,\pi/2)$. By the inverse function theorem, the functions $\theta \mapsto w_i(\theta)$ are invertible on $(d,\pi/2)$, with inverse functions $\theta_i(w)$ with $w \in (0,\gamma)$; furthermore $\theta_1 > \theta_2$. Set $w_i'(\theta) = W_i(w)$ on $(0,\gamma)$. Then $0 \geq W_1(\gamma) > W_2(\gamma)$ and $0 > W_2(0) > W_1(0)$. Thus there exists $\alpha \in (0,\gamma)$ where $W_2(\alpha) = W_1(\alpha)$. By the chain rule

$$\frac{dW_i}{dw} = w_i''(\theta)\left(\frac{dw_i}{d\theta}\right)^{-1} = -\frac{F(\theta_i(w),w)}{W_i(w)} \Longrightarrow \frac{dW_i^2}{dw} = -2F(\theta_i(w),w). \tag{4.16}$$

Because $\theta_1(w) > \theta_2(w)$, $-F(\theta_1(w),w) \geq -F(\theta_2(w),w)$, which implies

$$\frac{dW_1^2}{dw} > \frac{dW_2^2}{dw} \quad \text{on} \quad (0,\gamma).$$

Integrating on $(0,\gamma)$ yields to $-W_1^2(0) > -W_2^2(0)$, equivalently $(w_2'(\pi/2))^2 > (w_1'(\pi/2))^2$, a contradiction.

The two solutions v_1 and v_2 are such that $v_1(0) = c_1 > v_2(0) = c_2 > 0$. Thus $w_1 > w_2$ near 0. If we assume that $w_1 - w_2$ has more than one zero in $(0,\pi/2)$ we denote by d' the smallest of these zeros and set $\gamma' = w_i(c')$.

Step 2. We claim that w_1 and w_2 cannot be both increasing on $(0,d')$. We use the inverse function theorem on $(0,c')$ define θ_1 and θ_2 as functions of w defined on $(0,\gamma')$ and set $\tilde{W}_i(w) = \sin\theta\, w_i'(\theta)$ on $(0,\gamma')$. Since

$$\sin\theta\, w_i'(\theta) = (\sin\theta)^{3/2} v_i'(\theta) + \frac{\cos\theta}{2\sqrt{\sin\theta}} v_i(\theta), \tag{4.17}$$

we have $\tilde{W}_1(w) > \tilde{W}_2(w)$ near $w = 0$. The relations $0 < v_1'(d') < v_2'(d')$ and $v_1(d') = v_2(d') = \gamma'$ combined with (4.17) implies $0 < \tilde{W}_1(\gamma') < \tilde{W}_2(\gamma')$. Using again the chain rule

$$\frac{d\tilde{W}_i}{dw} = \frac{d}{d\theta}(\sin\theta\, w_i')\left(\frac{dw_i}{d\theta}\right)^{-1} = \cos(\theta_i(w)) - \frac{(\sin^2\theta_i(w))\, F(\theta_i(w),w)}{\tilde{W}_i(w)}. \tag{4.18}$$

Because $\sin^2\theta F(\theta,w) = (\ell\sin^2\theta + 4^{-1})w + (\sin\theta)^{(5-q)/2} w^q$ and $q \leq 5$, there holds

$$\frac{d\tilde{W}_2}{dw} < \cos(\theta_1(w)) - \frac{(\sin^2\theta_1(w))\, F(\theta_1(w),w)}{\tilde{W}_2(w)},$$

which implies

$$\frac{d\tilde{W}_2^2}{dw} - \cos(\theta_1(w))\tilde{W}_2 < \frac{d\tilde{W}_1^2}{dw} - \cos(\theta_1(w))\tilde{W}_1. \tag{4.19}$$

The fact that $\tilde{W}_i(0) = 0$, yields some complications. If we introduce $M(w) = \tilde{W}_2 + \tilde{W}_1$ and $X = \tilde{W}_2^2 - \tilde{W}_1^2$ it follows

$$X'(w) - \cos\theta_1(w) M(w) X(w) \leq 0,$$

from which we derive

$$X(w) \geq e^{\int_w^{\gamma'} \cos\theta_1(s) M(s) ds} X(\gamma') \quad \forall w \in (0, \gamma'].$$

Because $X(\gamma') > 0$, $X(w) > 0$ or equivalently $\tilde{W}_2(w) > \tilde{W}_1(w)$ which contradicts $\tilde{W}_2(w) < \tilde{W}_1(w)$ for w small enough. This proves the claim.

End of the proof. As a consequence of Steps 1 and 2, the two solutions v_1 and v_2 can have only one intersection, say d, and achieve their maximum in the two separate intervals $(0, d)$ and $(d, \pi/2)$. Assume that $0 > w_1'(\pi/2) > w_2'(\pi/2)$. Since there exists only one intersection, $w_1(\theta) > w_2(\theta)$ and $w_1'(\theta) > w_2'(\theta)$ near 0. Let $\theta_i \in (0, \pi/2)$ be the point where w_i achieves its maximum in $(0, \pi/2)$ (clearly $\theta_1 < d < \theta_2$) and $\gamma_i = w_i(\theta_i)$. If $\gamma_1 \geq \gamma_2$, it follows from (4.16), with the sign modification induced by the fact that $\theta_1(w) < \theta_2(w)$,

$$\frac{dW_1^2}{dw} < \frac{dW_2^2}{dw} \quad \text{in } (0, \gamma_2) \Longrightarrow (w_1'(\theta_2))^2 - (w_1'(\pi/2))^2 < -(w_2'(\pi/2))^2,$$

since $w_2'(\theta_2) = 0$. Which is a contradiction. Thus $\gamma_1 < \gamma_2$. In this case we procede as in Step 2 and derive, with the same notations,

$$\tilde{W}_2^2(w) - \tilde{W}_1^2(w) \geq e^{\int_w^{\gamma_1} \cos\theta_1(s) M(s) ds} \tilde{W}_2^2(\gamma_1) \quad \forall w \in (0, \gamma_1],$$

a contradiction since the left-hand side is negative near 0. □

4.3 Applications to Emden-Fowler equations

If $d = N - 1$ ($N \geq 3$) and

$$\ell = \lambda_{N,q} = \frac{2}{q-1}\left(\frac{2q}{q-1} - N\right)$$

we consider, for $q > 1$,

$$\begin{cases} \Delta' v + \lambda_{N,q} v + |v|^{q-1} v = 0 & \text{on } S_+^{N-1} \\ v = 0 & \text{on } \partial S_+^{N-1}. \end{cases} \tag{4.20}$$

Then Proposition 4.3 and Proposition 4.4 imply straithforwardly the next result.

Corollary 4.5 *I- If $1 < q \leq (N+1)/(N-1)$, any C^2 nonnegative solution to (4.20) is trivial.*
II- If $q \geq (N+1)/(N-3)$ any C^2 solution to (4.1) is trivial.
III- If $N \geq 4$ and $(N+1)/(N-1) < q < (N+1)/(N-3)$, or $N = 3$ and $2 < q \leq 5$, there exists a unique positive solution to (4.1).

Remark. In the case $q = (N+1)/(N-3) = (d+2)/(d-2)$, (4.20) takes the form

$$\begin{cases} \Delta' v - \dfrac{d(d-2}{4} v + |v|^{4/(d-2)} v = 0 & \text{on } S_+^{N-1} \\ v = 0 & \text{on } \partial S_+^{N-1} \end{cases} \tag{4.21}$$

The operator

$$v \mapsto \Box v = -\Delta' v + \frac{d(d-2)}{4} v$$

is the conformal Laplacian on S^d ($d(d-2)/4$ is the scalar curvature), the following transformation holds,

$$\begin{cases} \Delta \tilde{v} + |\tilde{v}|^{4/(d-2)} \tilde{v} = 0 \quad \text{on } B_1 \\ \tilde{v} = 0 \quad \text{on } \partial B_1 \end{cases} \tag{4.22}$$

where Π_s is the the south pole stereographic projection from $S^d \setminus \{A\}$ onto $\mathbb{R}^d$, and

$$\tilde{v}(x) = \left(\frac{2}{1+|x|^2} \right)^2 v(\Pi_s^{-1}(x)).$$

Thus (4.21) admits no solution by Pohožaev's identity.

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