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THE QUASI-INVARIANT LIMIT FOR A KINETIC MODEL OF SOCIOLOGICAL COLLECTIVE BEHAVIOR

LAURENT BOUDIN AND FRANCESCO SALVARANI

ABSTRACT. The paper is devoted to the study of the asymptotic behaviour of a kinetic model proposed to forecast the phenomenon of opinion formation, with both effect of self-thinking and compromise between individuals.

By supposing that the effects of self-thinking and compromise are very weak, we deduce, asymptotically, some simpler models who lose the kinetic structure. We explicitly characterize the asymptotic state of the limiting equation and study the speed of convergence towards equilibrium.

1. INTRODUCTION

The main goal of sociophysics consists in giving a statistical physics modeling of large scale social phenomena, like opinion formation, cultural dissemination or crowd behaviour. This branch of research started in the early eighties with a pioneering paper of Galam, Gefen and Shapir [11].

In the last twenty-five years, a substantial community, which includes mathematicians, physicists and sociologists, has produced many works on the topic: for example, several articles study how to predict the behaviour of voters during an election process, a referendum or some public opinion tendencies [13, 8, 9, 10].

In the literature, different techniques and viewpoints are available.

For instance, several authors base their analysis on Ising models, introduced in social and political sciences by Galam *et al.* [12, 11] (see also, for example, [17, 16]).

Recently, some strategies based on nonequilibrium statistical mechanics have been fruitfully applied [19, 1, 4, 7]. These papers show how the methods of this discipline, originally devoted to the classical field of the kinetic theory of rarefied gases, allow to study the collective behaviour of a large enough number of individuals, where none of which has a dominant role with respect to the others.

In particular, in a previous work [4], we have considered a classical issue of sociophysics, namely the evolution of the opinion about a binary question (for example, the answer to a referendum) in a closed community. Our goal was to make compatible two requirements. First, we wanted to reproduce some sociological collective behavior (being conscious that individuals are not a physical system) and, at the same time, to provide a mathematical

analysis of the equations to go beyond numerical experiments. The model is based on the assumption that the process of opinion formation is obtained through the competition of two opposite effects, described by two operators whose mathematical properties are quite different.

The first one is the binary exchange of ideas between individuals, with a tendency to compromise, described by a collision operator. The second one is the self-thinking process and is modelled as a weighted linear diffusion which vanishes on the boundary of the opinion space. The opinion of individuals is represented by a one-dimensional real variable between (-1) and $(+1)$. The choice of the closed interval $[-1, 1]$ instead of \mathbb{R} means that extreme opinions can actually be reached, and not only asymptotically.

The crucial assumption on the closure of the community means that the total number of individuals is constant and that no phenomenon of birth and death is taken into account. This obviously means that the model provides forecasts on the opinion evolution in a short-time scale. Moreover, here the population dynamics has been neglected.

Hence, from the modelling viewpoint, the time asymptotics problem may not seem relevant, since it means considering immortal individuals with no descendants. In fact, it may be seen as a first step of a later study including a coupling of our model with population dynamics equations.

From a mathematical point of view, the study of the large-time behaviour of the kind of equations proposed in [4] is an active area of research. Indeed, the presence of two phenomena, described by operators with drastically different large-time behaviours, makes the time asymptotics quite interesting: the asymptotic regime, if any, should be driven either by the “dominant” operator or by a state which inherits some properties of both operators as time grows.

In this article, we intend to adopt the mathematical point of view, and consider the time asymptotics of the model proposed in [4]. We shall restrain ourselves, however, to a very particular regime. We shall suppose that both the collision operator and the diffusion term have a small effect in the time evolution of the system. This assumption allows to deduce some approximated equations of the model, then to obtain the stationary state in a closed form and eventually to study the long-time asymptotics of the problem in a simpler way.

We name the resulting equation the quasi-invariant limit of the kinetic-diffusion model defined in [4]. We point out that this point of view is very common, and has been adopted in many articles concerning the study of models for granular gases (see, for example, [2, 3, 15, 18] and the references therein).

This paper is organized as follows. In the next section, we briefly recall the model and its main properties. Section 3 is devoted to the introduction of some quasi-invariant limits, that reflect the relative strength of the operators. Finally, in Section 4, we consider the asymptotic state of the equations

proposed in Section 3, and deduce the rate of decay to the stationary solution for the quasi-invariant approximation. Some numerical experiments concerning the models are also shown to enhance the theoretical study of the equations.

2. ORIGINAL MODEL

In order to make the paper self-consistent, we briefly recall the model proposed in [4] and its main properties. It describes the time evolution of the opinion set of an isolated population about binary questions. Our model is based on only two independent variables: the time $t \in \mathbb{R}_+$ and the opinion variable $x \in \bar{\Omega}$, where Ω denotes the open interval $(-1, 1)$.

The unknown of the model is the density (or distribution function) $f = f(t, x)$, defined on $\mathbb{R}_+ \times \bar{\Omega}$, whose time evolution is described by an integro-differential equation which takes into account two phenomena: self-thinking and binary interactions.

The self-thinking is modelled by a non-homogeneous diffusive term with structure $(\alpha(x)f_x)_x$, where α is a term which forces the diffusive process to respect the bounds of the opinion space $\bar{\Omega}$.

We suppose that the Fourier coefficient α satisfies the assumptions listed below.

Definition 2.1. *Let $\alpha : \bar{\Omega} \rightarrow \mathbb{R}$ be a nonnegative function of class $C^1(\bar{\Omega})$. We say that α is admissible if $\alpha(x) = \alpha(-x)$ for all $x \in \bar{\Omega}$ and $\alpha(-1) = \alpha(1) = 0$.*

The exchange of opinions inside the population is modelled by borrowing the collisional mechanism of a typical interaction in the kinetic theory of gases: whereas in rarefied gas dynamics the particles exchange momentum and energy in such a way that the principles of classical mechanics are satisfied, here the ‘‘collision’’ between individuals allows the exchange of opinions. Let $x, x_* \in \bar{\Omega}$ the opinions of two individuals before an interaction. The opinions after the interaction change according to the following formula:

$$(1) \quad \begin{cases} x' = \frac{x + x_*}{2} + \eta(x) \frac{x - x_*}{2}, \\ x'_* = \frac{x + x_*}{2} + \eta(x_*) \frac{x_* - x}{2}. \end{cases}$$

The function $\eta : \bar{\Omega} \rightarrow \mathbb{R}$, is the *attraction coefficient*: in general, it is a smooth function which describes the degree of attraction of the average opinion with respect to the starting opinion of the agent. Since the interaction must respect the bounds of the interval Ω , we need that $0 \leq \eta < 1$.

Once defined the collision rule (1), the interaction between individuals and the corresponding exchange of opinions is described by a collisional integral of Boltzmann type.

The collisional integral, which will be henceforth denoted as Q , has the classical structure of the dissipative Boltzmann kernels.

Let $\varphi = \varphi(x)$ be a suitably regular test function. We define the weak form of the collision kernel as

$$(2) \quad \langle Q(f, f), \varphi \rangle = \beta \iint_{\Omega^2} f(t, x) f(t, x_*) [\varphi(x') - \varphi(x)] dx_* dx.$$

Note that the particular form of the collision rule (1) only enters through the test function $\varphi(x')$. The cross section $\beta > 0$ is a parameter which governs the probability that an exchange of opinions can occur. In order to keep the description as simple as possible, we suppose that β is purely a positive constant.

The evolution law of the unknown $f = f(t, x)$ is then given by a partial integro-differential equation of second order with respect to x :

$$(3) \quad \int_{\Omega} f_t(t, x) \varphi(x) dx = \int_{\Omega} [\alpha(x) \varphi'(x)]_x f(t, x) dx + \langle Q(f, f), \varphi \rangle$$

posed in $(t, x) \in [0, T] \times \Omega$, $T > 0$, for all $\varphi \in C^2(\bar{\Omega})$, with initial condition

$$(4) \quad f(0, x) = f^{\text{in}}(x) \text{ for all } x \in \bar{\Omega}.$$

For the sake of simplicity, in the whole paper, we shall suppose that $\|f^{\text{in}}\|_{L^1(\Omega)} = 1$.

We point out that Equation (3) translates the presence of two opposite phenomena. The collision term reflects the sociological hypothesis that the individuals, after an exchange of opinion, adjust their own ideas with a tendency to compromise: hence this term gives a concentration effect. On the other hand, the self-thinking introduces a diffusive behavior in the equation, whose properties heavily depend on the functional form of the Fourier coefficient α .

The following results hold [4].

Proposition 2.2. *Let $f = f(t, x)$ be a nonnegative weak solution of (3)–(4), with a nonnegative initial datum $f^{\text{in}} \in L^1(\Omega)$. Then we have*

$$\|f(t, \cdot)\|_{L^1(\Omega)} = \|f^{\text{in}}\|_{L^1(\Omega)} = 1 \quad \text{for a.e. } t \geq 0.$$

Moreover, since $|x| \leq 1$, from the previous result, we immediately deduce that all the moments of f are bounded:

Corollary 2.3. *Let $f = f(t, x)$ be a nonnegative weak solution of problem (3)–(4), with nonnegative initial datum $f^{\text{in}} \in L^1(\Omega)$. Then, for a.e. $t \geq 0$ we have that*

$$\int_{\Omega} x^n f(t, x) dx \leq \|f^{\text{in}}\|_{L^1(\Omega)} = 1$$

for all $n \in \mathbb{N}$.

The following existence theorem guarantees that problem (3)–(4) makes sense.

Theorem 2.4. *Let f^{in} a nonnegative function of class $L^1(\Omega)$. Then there exists a nonnegative weak solution $f \in L^\infty(0, T; L^1(\Omega))$ of (3)–(4), where (3) takes sense in $\mathcal{D}'(-T, T)$.*

3. QUASI-INVARIANT LIMIT

The goal of the paper is the study of the quasi-invariant limit of problem (3)–(4). We shall be interested in situations where only very small modifications of the opinion are allowed by the processes of diffusion and collision.

This means that $\eta(x)$ is very close to 1 for any $x \in \bar{\Omega}$. Since the quasi-invariant limit procedure must be valid in the whole interval Ω , the spatial details of η are not relevant when passing to the limit. Therefore, we can assume that η is a constant, i.e. we write, for a fixed small enough $\varepsilon \in (0, 1/2)$, that

$$\eta_\varepsilon(x) = 1 - 2\varepsilon, \quad \forall x \in \bar{\Omega}.$$

Hence we shall perform the asymptotics $\varepsilon \rightarrow 0^+$.

With this choice, the collision mechanism (1) can be rewritten as

$$(5) \quad \begin{cases} x' = (1 - \varepsilon)x + \varepsilon x_* \\ x'_* = (1 - \varepsilon)x_* + \varepsilon x \end{cases}$$

and its Jacobian is $J(x, x_*) = 1 - 2\varepsilon > 0$.

In order to get the collision term, we write the Taylor expansion of the test function φ up to the second order. For any (x, x') , there exists $\theta \in [0, 1]$ such that

$$\varphi(x') = \varphi(x) + (x' - x)\varphi'(x) + \frac{(x' - x)^2}{2} \varphi''(\theta x' + (1 - \theta)x).$$

Thanks to (5), it is clear that

$$\frac{(x' - x)^2}{2} \varphi''(\theta x' + (1 - \theta)x) = \mathcal{O}(\varepsilon^2).$$

If we denote f_ε the solution of problem (3)–(4) with the collision mechanism (5), whose existence is guaranteed by Theorem 2.4, (2) becomes

$$\begin{aligned} \langle Q(f_\varepsilon, f_\varepsilon), \varphi \rangle &= \beta \iint_{\Omega^2} f_\varepsilon(t, x) f_\varepsilon(t, x_*) \varphi'(x) (x' - x) dx_* dx + \mathcal{O}(\varepsilon^2) \\ &= \varepsilon \beta \iint_{\Omega^2} f_\varepsilon(t, x) f_\varepsilon(t, x_*) \varphi'(x) (x_* - x) dx_* dx + \mathcal{O}(\varepsilon^2), \end{aligned}$$

for any $\varphi \in C^2(\bar{\Omega})$.

The quasi-invariant self-thinking process is modelled by a Fourier coefficient of type

$$(6) \quad \alpha_\varepsilon(x) = \varepsilon^k \alpha(x), \quad x \in \bar{\Omega},$$

such that $k > 0$ and α does not depend on ε . Note that, from now on, α will not depend on ε .

Three categories of choices for k are possible, leading to different kinds of quasi-invariant opinion approximations. In the sequel, we discuss the three situations, which have quite different behaviors.

3.1. The collision-dominated regime. In this subsection, we assume that $k > 1$. Therefore, the effects due to the exchange of opinions between individuals are predominant. If we rescale the time variable as $\tau = \varepsilon t$, the model is reduced to

$$\begin{aligned} \frac{d}{d\tau} \int_{\Omega} f_{\varepsilon} \left(\frac{\tau}{\varepsilon}, x \right) \varphi(x) dx &= \beta \left(\int_{\Omega} f_{\varepsilon} \left(\frac{\tau}{\varepsilon}, x \right) \varphi'(x) dx \right) \left(\int_{\Omega} f_{\varepsilon} \left(\frac{\tau}{\varepsilon}, x \right) x dx \right) \\ &\quad - \beta \int_{\Omega} f_{\varepsilon} \left(\frac{\tau}{\varepsilon}, x \right) x \varphi'(x) dx + \mathcal{O}(\varepsilon^{k-1}). \end{aligned}$$

By letting $\varepsilon \rightarrow 0^+$ in the previous equation, we formally deduce the following equation for the unknown

$$g(\tau, x) = \lim_{\varepsilon \rightarrow 0^+} f_{\varepsilon} \left(\frac{\tau}{\varepsilon}, x \right),$$

posed for $\tau \in [0, +\infty)$:

$$\begin{aligned} \frac{d}{d\tau} \int_{\Omega} g(\tau, x) \varphi(x) dx &= \beta \left(\int_{\Omega} g(\tau, x) \varphi'(x) dx \right) \left(\int_{\Omega} g(\tau, x) x dx \right) \\ (7) \quad &\quad - \beta \int_{\Omega} g(\tau, x) x \varphi'(x) dx \end{aligned}$$

for all $\varphi \in C^2(\bar{\Omega})$, supplemented with the initial condition

$$g(0, x) = f^{\text{in}}(x) \text{ for all } x \in \bar{\Omega}.$$

Equation (7) can be simplified. If we set $\varphi(x) = 1$ and $\varphi(x) = x$ in (7), we respectively deduce that

$$\frac{d}{d\tau} \int_{\Omega} g(\tau, x) dx = \frac{d}{d\tau} \int_{\Omega} x g(\tau, x) dx = 0.$$

Hence the weak form of the quasi-invariant limit for (3)–(4) in the collision-dominated regime has the following structure:

$$\begin{aligned} \frac{d}{d\tau} \int_{\Omega} g(\tau, x) \varphi(x) dx &= \beta m_1(0) \int_{\Omega} g(\tau, x) \varphi'(x) dx \\ (8) \quad &\quad - \beta \int_{\Omega} g(\tau, x) x \varphi'(x) dx \end{aligned}$$

for all $\varphi \in C^2(\bar{\Omega})$, where

$$m_1(0) = \int_{\Omega} f^{\text{in}}(x) x dx,$$

with initial condition

$$(9) \quad g(0, x) = f^{\text{in}}(x) \text{ for all } x \in \bar{\Omega}.$$

This is a linear partial differential equation of first order, with a coefficient depending on the initial datum of the model, more precisely on the first moment of f^{in} .

It is interesting to note that, in this case, all the moments of the solution can be explicitly computed. Indeed, if we consider the test function $\varphi(x) =$

x^n , $n \geq 2$, we obtain from (8) the following evolution equation for the moments:

$$\frac{dm_n}{d\tau} = \beta n [m_1(0)m_{n-1} - m_n], \quad n \geq 2,$$

where

$$m_n(\tau) = \int_{\Omega} x^n g(\tau, x) dx, \quad n \geq 2.$$

By induction, since the first moment is conserved, we can deduce that all the moments are uniformly bounded and that

$$\lim_{\tau \rightarrow +\infty} m_n(\tau) = [m_1(0)]^n$$

for all $n \in \mathbb{N}$.

3.2. The diffusion-dominated regime. The predominance of the self-thinking corresponds to the choice $0 < k < 1$. If we rescale the time as $\tau = \varepsilon^k t$ and disregard all the term of non-zero order with respect to ε , the model is reduced to the non-homogeneous degenerate parabolic equation for the unknown

$$g(\tau, x) = \lim_{\varepsilon \rightarrow 0^+} f_{\varepsilon} \left(\frac{\tau}{\varepsilon}, x \right),$$

posed for $\tau \in [0, +\infty)$:

$$(10) \quad \frac{d}{d\tau} \int_{\Omega} g(\tau, x) \varphi(x) dx = \int_{\Omega} (\alpha \varphi')' (x) g(\tau, x) dx$$

for all $\varphi \in C^2(\bar{\Omega})$, with initial condition

$$(11) \quad g(0, x) = f^{\text{in}}(x) \text{ for all } x \in \bar{\Omega}.$$

This problem is nothing but the weak form of the equation studied in [5].

3.3. The well-balanced regime. In this latter case, we suppose that $k = 1$. If we rescale time as $\tau = \varepsilon t$, the model is reduced to the following form:

$$\begin{aligned} \frac{d}{d\tau} \int_{\Omega} f_{\varepsilon} \left(\frac{\tau}{\varepsilon}, x \right) \varphi(x) dx &= \int_{\Omega} (\alpha \varphi')' (x) f_{\varepsilon} \left(\frac{\tau}{\varepsilon}, x \right) dx \\ &+ \beta \left(\int_{\Omega} f_{\varepsilon} \left(\frac{\tau}{\varepsilon}, x \right) \varphi'(x) dx \right) \left(\int_{\Omega} f_{\varepsilon} \left(\frac{\tau}{\varepsilon}, x \right) x dx \right) \\ &- \beta \int_{\Omega} f_{\varepsilon} \left(\frac{\tau}{\varepsilon}, x \right) x \varphi'(x) dx + \mathcal{O}(\varepsilon). \end{aligned}$$

If $\varepsilon \rightarrow 0^+$, we formally deduce the following partial differential equation (in a weak form) for the unknown

$$g(\tau, x) = \lim_{\varepsilon \rightarrow 0^+} f_{\varepsilon} \left(\frac{\tau}{\varepsilon}, x \right),$$

that is

$$(12) \quad \begin{aligned} \frac{d}{d\tau} \int_{\Omega} g(\tau, x) \varphi(x) dx &= \int_{\Omega} (\alpha\varphi')'(x) g(\tau, x) dx \\ &+ \beta \left(\int_{\Omega} g(\tau, x) \varphi'(x) dx \right) \left(\int_{\Omega} g(\tau, x) x dx \right) \\ &- \beta \int_{\Omega} g(\tau, x) x \varphi'(x) dx, \end{aligned}$$

for all $\varphi \in C^2(\bar{\Omega})$, posed for $\tau \in [0, +\infty)$, with initial condition

$$(13) \quad g(0, x) = f^{\text{in}}(x) \text{ for all } x \in \bar{\Omega}.$$

Equation (12) is a non-homogeneous nonlinear Fokker-Planck-type equation. The nonlinearity comes from the first moment of g , i.e.

$$m_1(\tau) = \int_{\Omega} g(\tau, x) x dx.$$

However, sometimes, this term can be exactly computed. This allows to obtain a further simplification which makes the equation particularly simple. Let us detail two cases.

3.3.1. Even initial datum. The first particular case we consider is obtained when f^{in} is even. With that type of initial datum, problem (12) is invariant by parity. Hence its solution is even for all $t > 0$ and therefore $m_1(\tau) = 0$. Equation (12) can then be simplified into

$$\frac{d}{d\tau} \int_{\Omega} g(\tau, x) \varphi(x) dx = \int_{\Omega} (\alpha\varphi')'(x) g(\tau, x) dx - \beta \int_{\Omega} g(\tau, x) x \varphi'(x) dx$$

for all $\varphi \in C^2(\bar{\Omega})$, posed for $\tau \in [0, +\infty]$. The previous equation is the weak form of a linear Fokker-Planck equation.

3.3.2. Specific form of the Fourier coefficient. When the Fourier coefficient can be written as $\alpha(x) = \kappa(1 - x^2)$ with $\kappa > 0$, we put $\varphi(x) = x$ in Equation (12), and deduce

$$\frac{d}{d\tau} \int_{\Omega} g(\tau, x) x dx = \int_{\Omega} \alpha'(x) g(\tau, x) dx,$$

and hence we get

$$m_1(\tau) = m_1(0) \exp(-2\kappa\tau),$$

where $m_1(0)$ is the first moment of the initial datum

$$m_1(0) = \int_{\Omega} f^{\text{in}}(x) x dx.$$

This leads us to

$$(14) \quad \begin{aligned} \frac{d}{d\tau} \int_{\Omega} g(\tau, x) \varphi(x) dx &= \kappa \int_{\Omega} [(1-x^2)\varphi']'(x) g(\tau, x) dx \\ &- \beta \int_{\Omega} g(\tau, x) x \varphi'(x) dx \\ &+ \beta m_1(0) \exp(-2\kappa\tau) \left(\int_{\Omega} g(\tau, x) \varphi'(x) dx \right), \end{aligned}$$

for all $\varphi \in C^2(\bar{\Omega})$, posed for $\tau \in [0, +\infty)$, with initial condition (13).

Here the equation has a coefficient that depends on the initial datum f^{in} through $m_1(0)$.

3.3.3. *General properties.* In any case, since $x \in \bar{\Omega}$,

$$|m_1(\tau)| \leq \|f^{\text{in}}\|_{L^1(\Omega)}, \quad \text{a.e. } \tau > 0.$$

Since the stationary solution of Equation (12) must be even by parity of the equation, then

$$\lim_{\tau \rightarrow +\infty} m_1(\tau) = 0.$$

4. MATHEMATICAL ASPECTS OF THE QUASI-INVARIANT LIMIT EQUATIONS

4.1. **The collision-dominated regime.** Equation (8) with initial condition (9) is the weak form of a linear partial differential equation of first order. The following result ensures that this problem admits an explicit solution.

Theorem 4.1. *Let $f^{\text{in}} \in H_0^1(\Omega)$ and consider, for any $\tau > 0$, the nonempty open intervals Ω_τ defined by*

$$\Omega_\tau = \left(m_1(0) - (1 + m_1(0))e^{-\beta\tau}, m_1(0) + (1 - m_1(0))e^{-\beta\tau} \right) \subsetneq \Omega,$$

and denote $\Omega_0 = \Omega$. Let us set, for any $\tau \geq 0$,

$$(15) \quad g(\tau, x) = \begin{cases} e^{\beta\tau} f^{\text{in}}((x - m_1(0))e^{\beta\tau} + m_1(0)) & \text{if } x \in \Omega_\tau, \\ 0 & \text{if } x \in \bar{\Omega} \setminus \Omega_\tau. \end{cases}$$

The function g given by (15) is $C^0(\bar{\Omega})$ and is a weak solution to Equation (8) with initial condition (9). Moreover, we have

$$\begin{aligned} \|g(\tau, \cdot)\|_{L^1(\Omega)} &= \|f^{\text{in}}\|_{L^1(\Omega)} \\ |\text{supp } g(\tau, \cdot)| &\leq 2e^{-\beta\tau}. \end{aligned}$$

Proof. Both estimates are immediate. Let us check that g solves (8). We first compute, for $\varphi \in C^1(\bar{\Omega})$,

$$\begin{aligned} \int_{\Omega} g(\tau, x) \varphi(x) dx &= e^{\beta\tau} \int_{\Omega_\tau} f^{\text{in}}((x - m_1(0))e^{\beta\tau} + m_1(0)) \varphi(x) dx \\ &= \int_{\Omega} f^{\text{in}}(y) \varphi((y - m_1(0))e^{-\beta\tau} + m_1(0)) dy, \end{aligned}$$

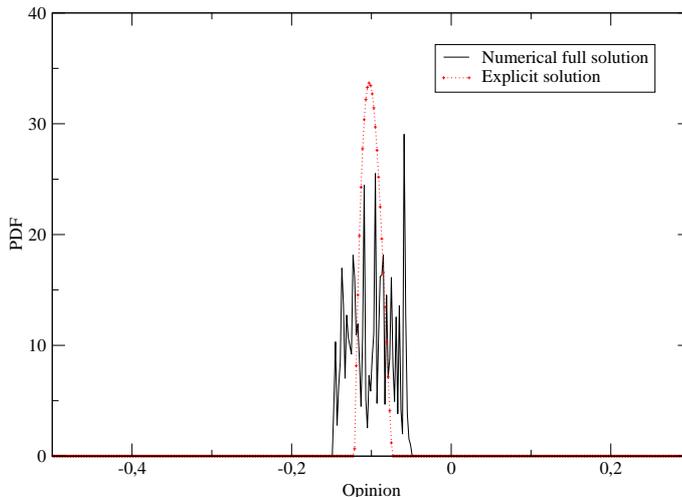


FIGURE 1. Explicit asymptotic and approximate numerical solutions

and then obtain that

$$\frac{d}{d\tau} \int_{\Omega} g(\tau, x) \varphi(x) dx = -\beta e^{-\beta\tau} \int_{\Omega} f^{\text{in}}(y) \varphi'((y - m_1(0))e^{-\beta\tau} + m_1(0)) dy.$$

In the same way, one can compute

$$\int_{\Omega} g(\tau, x) \varphi'(x) dx \quad \text{and} \quad \int_{\Omega} g(\tau, x) x \varphi'(x) dx.$$

It is then easy to prove that g solves (8). \square

Note that, asymptotically in time, Theorem 4.1 ensures that $g(\tau, \cdot)$ converges to the Dirac mass centred at $m_1(0)$ with an exponential rate of decay.

We can numerically recover that behavior. Indeed, we can compare the explicit form of g given by (15) with the numerical solution $f_{\varepsilon}(\cdot/\varepsilon)$ obtained with the code developed in [4]. For the numerical computations, we fix $\varepsilon = 0.01$, $\beta = 50$ as collision frequency, and $k = 4$, $\alpha(x) = (1 - x^2)^{1/3}$ in (6). Moreover, we choose $f^{\text{in}}(x) = 3/8 (1 - x^2)(2 - x)$, so that $m_1(0) = -0.1$. In Figure 1, one can check that both profiles are centred at -0.1 , and have the same concentration effect: indeed, both supports narrowed around -0.1 with respect to τ . It is not surprising that the graphs in Fig. 1 cannot be superimposed, since the diffusive effect is still taken into account for the numerical solution.

4.2. The diffusion-dominated regime. The existence and uniqueness theorem for the approximated equation in the diffusion-dominated case has been proved in [5]. We only add here some results on the convergence speed towards equilibrium which can be explicitly deduced.

If, for example, the even function α satisfies the assumption

$$(16) \quad K := \sup_{x \in (-1,1)} \left[(1-x) \int_{-1}^x \frac{(1+t)^2}{\alpha(t)} dt \right]^{1/2} < +\infty$$

we can use the following theorem on weighted 1-dimensional Poincaré inequalities [6], here written in a simplified form:

Theorem 4.2. *If Condition (16) holds and α is a nonnegative measurable function which is finite a.e., then, for all Lipschitz continuous function f on Ω , we have*

$$\int_{\Omega} \alpha |f'(x)|^2 dx \geq \frac{1}{2K} \int_{\Omega} \left| f(x) - \frac{1}{2} \int_{\Omega} f(v) dv \right|^2 dx.$$

The strong form of Equation (10) with initial condition (9) reads

$$(17) \quad g_{\tau} = (\alpha g_x)_x$$

$$(18) \quad g(0, x) = f^{\text{in}}(x),$$

$$(19) \quad \lim_{x \rightarrow \pm 1} \alpha(x) g_x(\tau, x) = 0.$$

for $x \in \Omega$ and $\tau > 0$.

A standard a-priori estimate for the Neumann problem (17)–(19) can be obtained by differentiating the equation with respect to x , then multiplying each term of the equation by αg_x , and integrating with respect to x in Ω , i.e.

$$(20) \quad \frac{d}{d\tau} \int_{\Omega} \alpha g_x^2 dx = - \int_{\Omega} (\alpha g_x)_x^2 dx \leq 0.$$

Consequently, if $\|\alpha (\partial_x f^{\text{in}})^2\|_{L^1(\Omega)} < +\infty$, then the quantity $\alpha (g_x)^2$ is uniformly bounded in $L^1(\Omega)$.

We now multiply Equation (17) by $(g - 1/2)$ and integrate with respect to $x \in \Omega$. We immediately obtain

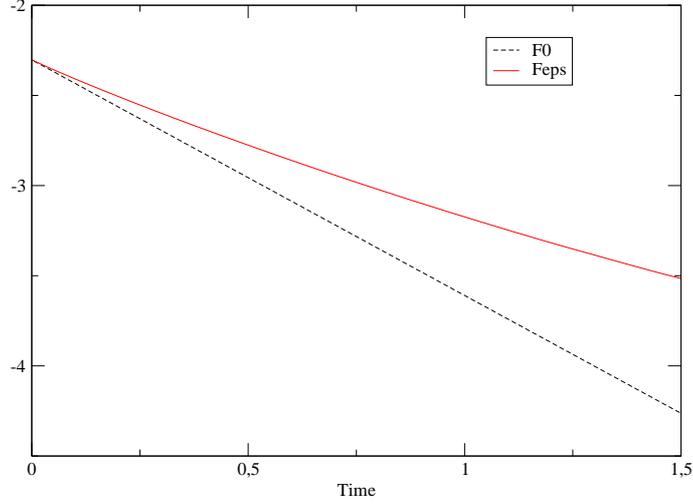
$$\frac{1}{2} \frac{d}{d\tau} \int_{\Omega} \left(g - \frac{1}{2} \right)^2 dx = - \int_{\Omega} \alpha (g_x)^2 dx.$$

The right-hand-side of the above equation is uniformly bounded with respect to t because of (20). Thanks to Theorem 4.2, if α satisfies (16), it comes

$$\frac{d}{d\tau} \int_{\Omega} \left(g - \frac{1}{2} \right)^2 dx = - \frac{1}{K} \int_{\Omega} \left(g - \frac{1}{2} \right)^2 dx,$$

and we get the exponential convergence towards the constant equilibrium solution $1/2$:

$$(21) \quad \left\| g(\tau, \cdot) - \frac{1}{2} \right\|_{L^2(\Omega)} \leq \left\| f^{\text{in}} - \frac{1}{2} \right\|_{L^2(\Omega)} e^{-\tau/2K}.$$

FIGURE 2. Comparison between $\log L^2$ norms

We can also perform a numerical comparison between

$$\tau \mapsto F_\varepsilon(\tau) = 2 \log \left(\left\| f_\varepsilon \left(\frac{\tau}{\varepsilon}, \cdot \right) - \frac{1}{2} \right\|_{L^2(\Omega)} \right)$$

and

$$\tau \mapsto F_0(\tau) = 2 \log \left(\left\| f^{\text{in}} - \frac{1}{2} \right\|_{L^2(\Omega)} \right) - \frac{\tau}{K} \simeq -2.303 - 1.307 \tau.$$

As a matter of fact, the computation of f_ε can be made in the case when $\varepsilon = 0.05$, $\beta = 50$, and $\alpha(x) = (1 - x^2)^{1/3}$, $k = 0.9$ in (6), and $f^{\text{in}}(x) = 3/4 (1 - x^2)$. Note that we can then numerically compute an approximate value of $K \simeq 0.765$ in (16). Figure 2 shows the relative positions of the plots of F_0 and F_ε .

The plot of F_ε is almost a draw line, at least when time is not too large, which suggests for $f_\varepsilon \left(\frac{\tau}{\varepsilon}, \cdot \right)$ an exponential rate of convergence to a stationary solution which is close to the stationary solution of g (i.e. $1/2$). Of course, the two asymptotic states of $f_\varepsilon \left(\frac{\tau}{\varepsilon}, \cdot \right)$ and $g(\tau, \cdot)$ do not coincide, but they differ with an error which is at most of order $\mathcal{O}(\varepsilon^{1-k})$, as shown in the derivation of the quasi-static approximation in Section 3. Since ε^{1-k} is really significant, we cannot be surprised by the fact that estimate (21) is not satisfied by f_ε . Nevertheless, asymptotically, the plot of F_ε should numerically converge to a constant value which gives the error committed, in the L^2 norm, when the asymptotic limit of $f_\varepsilon \left(\frac{\tau}{\varepsilon}, \cdot \right)$ is identified with the asymptotic limit of $g(\tau, x)$.

4.3. The well-balanced regime. This subsection is devoted to prove some important properties of Equation (12). We first characterize their stationary states, as proved in the following proposition:

Proposition 4.3. *There exists a probability density $q \in C^2(\bar{\Omega})$, which is a stationary solution of Equation (12). The stationary solution q has the following explicit form:*

$$q(x) = \xi \exp\left(-\beta \int_0^x \frac{s}{\alpha(s)} ds\right), \quad x \in \bar{\Omega},$$

where $\xi \in \mathbb{R}$ is an arbitrary constant.

Proof. A stationary solution $q(x)$ of the Fokker-Planck equation (12) satisfies the following weak form:

$$(22) \quad \int_{\Omega} (\alpha\varphi')' q dx - \beta \int_{\Omega} x q \varphi' dx = 0$$

for all $\varphi \in C^2(\bar{\Omega})$, by parity (see subsectn. 3.3.3).

If we assume that q is smooth enough, say, for example, $q \in W_{\text{loc}}^{1,1}(\Omega)$, then we can integrate by parts the first term in Equation (22), and deduce that

$$\int_{\Omega} [\alpha q' + \beta x q] \varphi' dx = 0$$

for all $\varphi \in C^2(\bar{\Omega})$. Hence,

$$\alpha q' + \beta x q = 0,$$

and therefore q can be written under the following form:

$$q(x) = \xi \exp\left(-\beta \int_0^x \frac{s}{\alpha(s)} ds\right),$$

where $\xi \in \mathbb{R}$. □

As a next step, we consider the existence theory and the asymptotic decay towards the stationary state for Equation (12). The simultaneous presence of a nonlinearity and of the degeneracy of the second order term makes the study of the general case quite difficult. In this paper, we obtain some results for the case $\alpha(x) = \kappa(1 - x^2)$, with $\kappa > 0$ (see subsectn. 3.3.2). This assumption allows to linearize Equation (12) and to find, for an appropriate choice of the parameter β , an exponential rate of convergence towards equilibrium.

In this framework, the stationary solution is simply of the form $q(x) = (1 - x^2)^{\beta/2}$.

The strategy of the proof is the following. We first consider a lifted version of Equation (14), and then prove by compactness the existence of a

solution of the non-lifted problem. We hence consider the following family of initial-boundary value problems

$$(23) \quad \begin{aligned} \partial_\tau u_\delta &= \partial_x((\alpha + \delta)\partial_x u_\delta) \\ &+ \beta \partial_x(x u_\delta) - \beta e^{-2\kappa\tau} \left(\int_\Omega x f^{\text{in}}(x) dx \right) \partial_x u_\delta, \end{aligned}$$

with initial and boundary conditions

$$(24) \quad u_\delta(0, x) = f^{\text{in}}(x), \quad \lim_{x \rightarrow \pm 1} \partial_x u_\delta(t, x) = 0,$$

where $\delta > 0$ is the *lifting parameter* and the other quantities are the same as in Equation (12).

Standard theory of linear parabolic equations [14] gives the following result:

Proposition 4.4. *Let $f^{\text{in}} \in H^p(\Omega)$, $p \in \mathbb{N}$, and $T > 0$. Then there exists a unique solution $u_\delta \in C([0, T]; H^p(\Omega))$ for the initial-boundary value problem (23)–(24). Moreover, the solution is nonnegative if $f^{\text{in}} \geq 0$ a.e. Finally, there exists a nonnegative constant J only depending on T , β , f^{in} and κ , such that*

$$\|u_\delta(\tau, \cdot)\|_{H^p(\Omega)} \leq J, \quad \forall \delta > 0.$$

Proof. The existence part of the proposition is classical, as well as the non-negativity of the solution [14].

The boundedness of the H^1 norm can be obtained by means of an a priori estimate. We differentiate Equation (23) with respect to x and then multiply it by $\partial_x u_\delta$. After integrating with respect to x in Ω , we deduce

$$\begin{aligned} \frac{1}{2} \frac{d}{d\tau} \int_\Omega (\partial_x u_\delta)^2 dx &= - \int_\Omega (\alpha + \delta) (\partial_{xx} u_\delta)^2 dx \\ &+ 2\kappa \int_\Omega x \partial_x u_\delta \partial_{xx} u_\delta dx + \frac{3}{2} \beta \int_\Omega (\partial_x u_\delta)^2 dx \\ &\leq \left(\frac{3}{2} \beta - \kappa \right) \int_\Omega (\partial_x u_\delta)^2 dx. \end{aligned}$$

We can eventually write

$$(25) \quad \|\partial_x u_\delta\|_{L^2(\Omega)} \leq \|\partial_x f^{\text{in}}\|_{L^2(\Omega)} e^{(3\beta/2 - \kappa)\tau}.$$

The uniform boundedness (with respect to δ) of higher order derivatives is obtained in the same way (note that the equation is linear). Hence, the last part of the proposition follows. \square

Proposition 4.4 allows to prove the following theorem, which guarantees the existence of a weak solution for the well-balanced quasi-invariant limit of (3)–(4), when $\alpha(x) = \kappa(1 - x^2)$.

Theorem 4.5. *Problem (14) posed for $\tau \in [0, +\infty)$, for all $\varphi \in C^2(\bar{\Omega})$, with the nonnegative initial condition $f^{\text{in}} \in H^p(\Omega)$, has a nonnegative weak*

solution in $C([0, T]; H^p(\Omega))$, for any $T > 0$. When $p \geq 2$, the solution is unique in the class of the functions belonging to H^p .

Proof. We consider a family of solution (u_δ) for the initial-boundary value problem (23)–(24), written in the following weak form:

$$\begin{aligned} \frac{d}{d\tau} \int_{\Omega} u_\delta(\tau, x) \varphi(x) dx &= \kappa \int_{\Omega} [(1-x^2)\varphi'(x)]_x u_\delta(\tau, x) dx \\ &\quad - \beta \int_{\Omega} u_\delta(\tau, x) x \varphi'(x) dx \\ &\quad + \beta m_1(0) \exp(-2\kappa\tau) \left(\int_{\Omega} u_\delta(\tau, x) \varphi'(x) dx \right), \end{aligned}$$

for all $\varphi \in C^2(\bar{\Omega})$, posed for $\tau \in [0, +\infty)$, with nonnegative initial datum $f^{\text{in}} \in H^p(\Omega)$.

Since $u_\delta \in C([0, T]; H^p(\Omega))$ for any $\delta > 0$ and since, moreover, the family is uniformly bounded in $H^p(\Omega)$, up to a subsequence, (u_δ) weakly converges in $H^p(\Omega)$, for a.e. τ , let g be its limit. This function g solves the weak form of the initial-boundary value problem described in the theorem. As a matter of fact, the equation itself is linear with respect to the unknown function.

Eventually, we have to check that g satisfies the correct initial condition. Let us integrate the lifted equation with respect to τ in $[0, \theta]$, we obtain that

$$\begin{aligned} &\left| \int_{\Omega} (u_\delta(\theta, x) - f^{\text{in}}(x)) \varphi(x) dx \right| \\ &\leq \kappa \int_0^\theta \int_{\Omega} |[(1-x^2)\varphi'(x)] \partial_x u_\delta(\tau, x)| dx d\tau \\ &\quad + \beta \int_0^\theta \int_{\Omega} |\partial_x (u_\delta(\tau, x) x) \varphi(x)| dx d\tau \\ &\quad + \beta |m_1(0)| \int_0^\theta \int_{\Omega} |\partial_x u_\delta(\tau, x) \varphi(x)| dx d\tau. \end{aligned}$$

Thanks to the upper bound of $\|u_\delta(\tau, \cdot)\|_{H^1(\Omega)}$ given by Proposition 4.3, the right-hand side of the previous inequality vanishes when θ goes to 0. That proves that $g(0, \cdot) = f^{\text{in}}$ in $w\text{-}H^1(\Omega)$.

In order to prove that the solution g is nonnegative, we simply note that weak convergence in H^1 implies strong convergence in L^2 and a.e. Hence, the constructed solution g is nonnegative since the sequence (u_δ) is nonnegative too.

Uniqueness is easily obtained by standard energy estimates for the non-lifted equation in strong form. \square

We note that the previous theorem guarantees uniqueness of the solution (in the class of regular enough functions, say $g(\tau, \cdot) \in H^2(\Omega)$, which is a necessary condition to use the strong formulation) without imposing boundary conditions.

The next step consists in proving that, in fact, the previous result of existence, which is local in time, although with an upper bound time which can be arbitrarily large, is indeed global for a wise choice of parameters κ and β . Moreover, in this case, we deduce exponentially fast convergence towards the stationary solution characterized in Proposition 4.3. The proof is based on a suitable a priori estimate. In order to make the result more readable, we choose $\kappa = 1$.

We start by proving the following property:

Lemma 4.6. *Let g be the solution of Equation (14) posed for $\tau \in [0, +\infty)$, with nonnegative and compactly supported initial condition $f^{\text{in}} \in H_0^p(\Omega)$, with $p \geq 2$. Then $g(t, \pm 1) = 0$ for any $t \in [0, \tau]$.*

Proof. Since $\beta > 0$, the stationary solution q vanishes in $x = \pm 1$. The initial condition f^{in} is compactly supported in Ω . Thanks to a standard Sobolev imbedding, it is also in $L^\infty(\Omega)$. Hence, there exists a suitable stationary solution q^* of Equation (14), such that $q^* \geq f^{\text{in}}$ almost everywhere. Note that the masses of q^* and f^{in} can be different.

We now consider the unique solution of Equation (14) with initial datum $(q^* - f^{\text{in}})$, given by Theorem 4.5. This result ensures that the time evolution of the difference between the solution with initial datum f^{in} and q^* also preserves nonnegativity. Consequently, the trace of the solution g on the boundary is nil. \square

We are almost ready to prove the main result on the initial-value problem for Equation (14) with $\kappa = 1$. We first note that the constant K given by Theorem 4.2 is finite and satisfies $0.76 \leq K \leq 0.77$.

Theorem 4.7. *Let f^{in} be a nonnegative and compactly supported function in $H_0^2(\Omega)$ such that $\|f^{\text{in}}\|_{L^1(\Omega)} = 1$, and $0 < \beta < 1/K$, where the constant K is given by Theorem 4.2. Then the solution g to the well-balanced problem (14) with $\kappa = 1$, posed for $\tau \in [0, +\infty)$, for all $\varphi \in H^1(\Omega)$, with initial datum $f^{\text{in}} \in H^p(\Omega)$, converges exponentially fast to the stationary solution q given by Proposition 4.3, where ξ is chosen such that $\|q\|_{L^1(\Omega)} = 1$.*

Proof. We consider the strong form of Equation (14)

$$g_\tau = (\alpha g_x)_x + \beta(xg)_x - \beta m_1(0)e^{-2\tau} g_x,$$

with initial condition $g(0, x) = f^{\text{in}}(x)$, where $\alpha(x) = 1 - x^2$. Since the initial datum is normalized, it is easy to see that the stationary solution, in this case, is given by

$$q(x) = \xi(1 - x^2)^{\beta/2}, \text{ where } \xi^{-1} = \int_{\Omega} (1 - x^2)^{\beta/2} dx.$$

It is clear that q satisfies

$$(\alpha q')' + \beta(xq)' = 0.$$

Let us subtract the equations respectively satisfied by g_τ and q , multiply the obtained equation by $(g - q)$ and integrate with respect to x in Ω . We easily deduce, after an integration by parts, that

$$\begin{aligned} \frac{1}{2} \frac{d}{d\tau} \int_{\Omega} (g - q)^2 dx &= - \int_{\Omega} \alpha(x) (g - q)_x^2 dx \\ &\quad + \frac{\beta}{2} \int_{\Omega} (g - q)^2 dx - \beta m_1(0) e^{-2\tau} \int_{\Omega} g_x (g - q) dx, \end{aligned}$$

where we have used that both g and q have zero traces on the boundary of the interval Ω . Since g also satisfies (25), using Thm. 4.2, we can write

$$\begin{aligned} \frac{d}{d\tau} \|g - q\|_{L^2}^2 &\leq \left(\beta - \frac{1}{K} \right) \|g - q\|_{L^2}^2 \\ &\quad + \beta m_1(0) \|\partial_x f^{\text{in}}\|_{L^2} \exp \left[\left(\frac{3\beta}{2} - 3 \right) \tau \right] \|g - q\|_{L^2}. \end{aligned}$$

With our assumption on β and the approximate value of K , it is clear that

$$-\gamma_1 := \frac{3\beta}{2} - 3 < 0 \quad \text{and} \quad -\gamma_2 := \frac{1}{2} \left(\beta - \frac{1}{K} \right) < 0.$$

Hence we get

$$\frac{d}{d\tau} \|g - q\|_{L^2} \leq -\gamma_2 \|g - q\|_{L^2} + M e^{-\gamma_1 \tau},$$

where M is a constant. We can safely assume that $\gamma_1 \neq \gamma_2$. Therefore, we can deduce that

$$\begin{aligned} \|g - q\|_{L^2} &\leq \left(\|f^{\text{in}} - q\|_{L^2} + \frac{M}{\gamma_2 - \gamma_1} \right) e^{-\gamma_2 \tau} - \frac{M}{\gamma_2 - \gamma_1} e^{-\gamma_1 \tau} \\ &\leq \|f^{\text{in}} - q\|_{L^2} e^{-\min(\gamma_1, \gamma_2) \tau}, \end{aligned}$$

which means exponential convergence towards equilibrium. \square

Remark 4.8. *Apparently, the exponentially fast convergence of f_ε in τ also holds for at least some values of $\beta \geq 1/K$. As a matter of fact, if we pick $\varepsilon = 10^{-4}$ and $\beta = 30$, it is quite clear, on Fig. 3, that (f_ε) , which is an approximation of g of order $\mathcal{O}(\varepsilon)$, still converges to the stationary solution when the time increases. Nevertheless, the convergence rate is not merely exponential: a piecewise exponential behaviour is indeed shown.*

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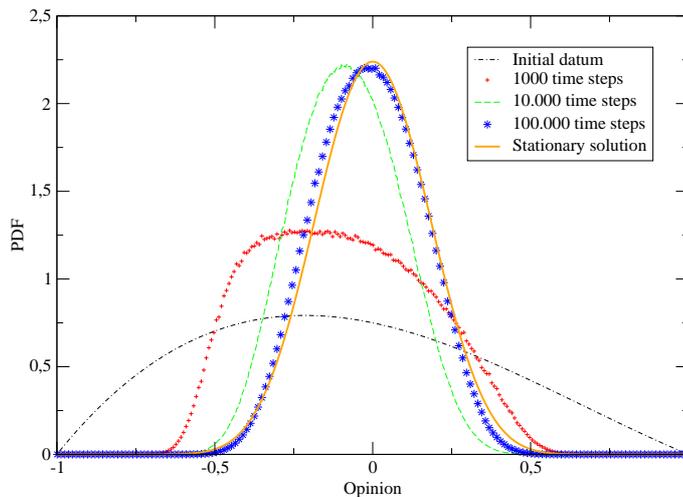


FIGURE 3. Numerical convergence of (f_ε) to the stationary solution

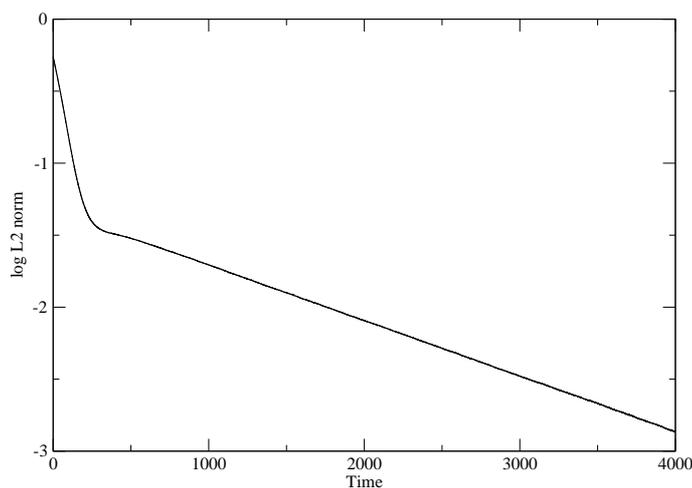


FIGURE 4. Graph of $\log \left(\|f_\varepsilon - q\|_{L^2}^2 \right)$ w.r.t. time

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