A stochastic calculus for multidimensional fractional Brownian motion with arbitrary Hurst index

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A rough path over multidimensional fractional Brownian motion with arbitrary Hurst index by Fourier normal ordering

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Fourier normal ordering [36] is a new algorithm to construct explicit rough paths over arbitrary Hölder-continuous multidimensional paths. We apply in this article the Fourier normal ordering ordering algorithm to the construction of an explicit rough path over multi-dimensional fractional Brownian motion B with arbitrary Hurst index α (in particular, for $\alpha \leq 1/4$, which was till now an open problem) by regularizing the iterated integrals of the analytic approximation of B defined in [34]. The regularization procedure is applied to 'Fourier normal ordered' iterated integrals obtained by permuting the order of integration so that innermost integrals have highest Fourier modes. The algebraic properties of this rough path are best understood using two Hopf algebras: the Hopf algebra of decorated rooted trees [6] for the multiplicative or Chen property, and the shuffle algebra for the geometric or shuffle property. The rough path lives in Gaussian chaos of integer orders and is shown to have finite moments.

As well-known, the construction of a rough path is the key to defining a stochastic calculus and solve stochastic differential equations driven by B.

The article [37] gives a quick overview of the method.

Keywords: fractional Brownian motion, stochastic integrals, rough paths, Hopf algebra of decorated rooted trees

Mathematics Subject Classification (2000): 05C05, 16W30, 60F05, 60G15, 60G18, 60H05

Contents

0 Introduction

The (two-sided) fractional Brownian motion $t \to B_t$, $t \in \mathbb{R}$ (fBm for short) with Hurst exponent α, $\alpha \in (0,1)$, defined as the centered Gaussian process with covariance

$$\mathbb{E}[B_s B_t] = \frac{1}{2}(|s|^{2\alpha} + |t|^{2\alpha} - |t-s|^{2\alpha}), \tag{0.1}$$

is a natural generalization in the class of Gaussian processes of the usual Brownian motion (which is the case $\alpha = \frac{1}{2}$), in the sense that it exhibits two fundamental properties shared with Brownian motion, namely, it has stationary increments, viz. $\mathbb{E}[(B_t - B_s)(B_u - B_v)] = \mathbb{E}[(B_{t+a} - B_{s+a})(B_{u+a} - B_{v+a})]$ for every $a, s, t, u, v \in \mathbb{R}$, and it is self-similar, viz.

$$\forall \lambda > 0, \quad (B_{\lambda t}, t \in \mathbb{R}) \overset{(law)}{=} (\lambda^\alpha B_t, t \in \mathbb{R}). \tag{0.2}$$

One may also define a d-dimensional vector Gaussian process (called: *d-dimensional fractional Brownian motion*) by setting $B_t = (B_t(1), \ldots, B_t(d))$ where $(B_t(i), t \in \mathbb{R})_{i=1,\ldots,d}$ are d independent (scalar) fractional Brownian motions.

Its theoretical interest lies in particular in the fact that it is (up to normalization) the only Gaussian process satisfying these two properties.

A standard application of Kolmogorov's theorem shows that fBm has a version with α^--Hölder continuous (i.e. κ-Hölder continuous for every $\kappa < \alpha$) paths. In particular, fBm with small Hurst parameter α is a natural, simple model for continuous but very irregular processes.

There has been a widespread interest during the past ten years in constructing a stochastic integration theory with respect to fBm and solving stochastic differential equations driven by fBm, see for instance [24, 15, 5, 30, 31]. The multi-dimensional case is very different from the one-dimensional case. When one tries to integrate for instance a stochastic differential equation driven by a two-dimensional fBm $B = (B(1), B(2))$ by using any kind of Picard iteration scheme, one encounters very soon the problem of defining the Lévy area of B which is the antisymmetric part of $\mathcal{A}_{ts} := \int_s^t dB_{t_1}(1) \int_s^{t_1} dB_{t_2}(2)$. This is the simplest occurrence of iterated integrals $\boldsymbol{B}^k_{ts}(i_1, \ldots, i_k) := \int_s^t dB_{t_1}(i_1) \ldots \int_s^{t_{k-1}} dB_{t_k}(i_k)$, $i_1, \ldots, i_k \leq d$ for d-dimensional fBm $B = (B(1), \ldots, B(d))$ which lie at the heart of the rough path theory due to T. Lyons, see [25, 26]. An alternative construction has been given by M. Gubinelli in [16] under the name of 'algebraic rough path theory', which we now propose to describe briefly.

Assume $\Gamma_t = (\Gamma_t(1), \dots, \Gamma_t(d))$ is some non-smooth d-dimensional path which is α-Hölder continuous. Integrals such as $\int f_1(\Gamma_t)d\Gamma_t(1) + \dots + f_d(\Gamma_t)d\Gamma_t(d)$ do not make sense a priori because Γ is not differentiable (Young's integral [23] works for $\alpha > \frac{1}{2}$ but not beyond). In order to define the integration of a differential form along Γ, it is enough to define a *geometric rough path* $(\mathbf{\Gamma}^1, \dots, \mathbf{\Gamma}^{\lfloor 1/\alpha \rfloor})$ lying above Γ, $\lfloor 1/\alpha \rfloor$=entire part of $1/\alpha$, where $\mathbf{\Gamma}^1_{ts} = (\delta\Gamma)_{ts} := \Gamma_t - \Gamma_s$ is the *increment* of Γ between s and t, and each $\mathbf{\Gamma}^k = (\mathbf{\Gamma}^k(i_1, \dots, i_k))_{1 \le i_1, \dots, i_k \le d}$, $k \ge 2$ is a *substitute* for the iterated integrals $\int_s^t d\Gamma_{t_1}(i_1) \int_s^{t_1} d\Gamma_{t_2}(i_2) \dots \int_s^{t_{k-1}} d\Gamma_{t_k}(i_k)$ with the following three properties:

(i) (*Hölder continuity) each component of* $\mathbf{\Gamma}^k$ *is* $k\alpha^-$*-Hölder continuous, that is to say,* $k\kappa$*-Hölder for every* $\kappa < \alpha$*;*

(ii) (multiplicativity) letting $\delta\mathbf{\Gamma}^k_{tus} := \mathbf{\Gamma}^k_{ts} - \mathbf{\Gamma}^k_{tu} - \mathbf{\Gamma}^k_{us}$*, one requires*

$$\delta\mathbf{\Gamma}^k_{tus}(i_1, \dots, i_k) = \sum_{k_1 + k_2 = k} \mathbf{\Gamma}^{k_1}_{tu}(i_1, \dots, i_{k_1}) \mathbf{\Gamma}^{k_2}_{us}(i_{k_1+1}, \dots, i_k); \quad (0.3)$$

(iii) (geometricity)

$$\mathbf{\Gamma}^{n_1}_{ts}(i_1, \dots, i_{n_1}) \mathbf{\Gamma}^{n_2}_{ts}(j_1, \dots, j_{n_2}) = \sum_{\boldsymbol{k} \in \mathrm{Sh}(\boldsymbol{i}, \boldsymbol{j})} \mathbf{\Gamma}^{n_1+n_2}_{ts}(k_1, \dots, k_{n_1+n_2}) \quad (0.4)$$

where $\mathrm{Sh}(\boldsymbol{i}, \boldsymbol{j})$ *is the subset of permutations of* $i_1, \dots, i_{n_1}, j_1, \dots, j_{n_2}$ *which do not change the orderings of* $(i_1, \dots, i_{n_1})$ *and* $(j_1, \dots, j_{n_2})$*.*

The multiplicativity property implies in particular the following identity for the (non anti-symmetrized) Lévy area:

$$\mathcal{A}_{ts} = \mathcal{A}_{tu} + \mathcal{A}_{us} + (B_t(1) - B_u(1))(B_u(2) - B_s(2)) \quad (0.5)$$

while the geometric property implies

$$\begin{aligned} &\int_s^t dB_{t_1}(1) \int_s^{t_1} dB_{t_2}(2) + \int_s^t dB_{t_2}(2) \int_s^{t_2} dB_{t_1}(1) \\ &= \left(\int_s^t dB_{t_1}(1) \right) \left(\int_s^t dB_{t_2}(2) \right) = (B_t(1) - B_s(1))(B_t(2) - B_s(2)). \end{aligned} \quad (0.6)$$

Then there is a standard procedure which allows to define out of these data iterated integrals of any order and to solve differential equations driven by Γ.

The multiplicativity property (0.3) and the geometric property (0.4) are satisfied by smooth paths, as can be checked by direct computation. So the most natural way to construct such a multiplicative functional is to start from some smooth approximation Γ^η, $\eta \stackrel{>}{\to} 0$ of Γ such that each iterated integral $\mathbf{\Gamma}_{ts}^{k,\eta}(i_1,\dots,i_k)$, $k \leq \lfloor 1/\alpha \rfloor$ converges in the $k\kappa$-Hölder norm for every $\kappa < \alpha$.

This general scheme has been applied to fBm in a paper by L. Coutin and Z. Qian [9] and later in a paper by the author [34], using different schemes of approximation of B by B^η with $\eta \to 0$. In both cases, the variance of the Lévy area has been proved to diverge in the limit $\eta \to 0$ when $\alpha \leq 1/4$.

The approach developed in [34] makes use of a complex-analytic process Γ defined on the upper half-plane $\Pi^+ = \{z = x + \mathrm{i}y \mid y > 0\}$, called Γ-*process* or better *analytic fractional Brownian motion* (afBm for short) [33]. Fractional Brownian motion B_t appears as the *real part* of the *boundary value* of Γ_z when $\mathrm{Im}\ z \stackrel{>}{\to} 0$. A natural approximation of B_t is then obtained by considering

$$B_t^\eta := \Gamma_{t+\mathrm{i}\eta} + \overline{\Gamma_{t+\mathrm{i}\eta}} = 2\mathrm{Re}\ \Gamma_{t+\mathrm{i}\eta} \tag{0.7}$$

for $\eta \stackrel{>}{\to} 0$. We show in subsection 3.1 that B^η may be written as a Fourier integral,

$$B_t^\eta = c_\alpha \int_{\mathbb{R}} e^{-\eta|\xi|} |\xi|^{\frac{1}{2}-\alpha} \frac{e^{\mathrm{i}t\xi} - 1}{\mathrm{i}\xi}\ W(d\xi) \tag{0.8}$$

for some constant c_α, where $(W(\xi), \xi \geq 0)$ is a standard complex Brownian motion extended to $\mathbb{R}$ by setting $W(-\xi) = -\bar{W}(\xi)$, $\xi \geq 0$. When $\eta \to 0$, one retrieves the well-known harmonizable representation of B [32].

The so-called *analytic iterated integrals*

$$\int_s^t f_1(z_1) d\Gamma_{z_1}(1) \int_s^{z_1} f_2(z_2) d\Gamma_{z_2}(2) \dots \int_s^{z_{d-1}} f_d(z_d) d\Gamma_{z_d}(d)$$

(where $f_1,\dots,f_d$ are analytic functions), defined a priori for $s,t \in \Pi^+$ by integrating over complex paths wholly contained in Π^+, converge to a finite limit when $\mathrm{Im}\ s, \mathrm{Im}\ t \to 0$ [34], which is the starting point for the construction of a rough path associated to Γ [33]. The main tool for proving this kind of results is analytic continuation.

Computing iterated integrals associated to $B_t = 2\lim_{\eta\to 0} \mathrm{Re}\ \Gamma_{t+\mathrm{i}\eta}$ instead of Γ yields analytic iterated integrals, together with mixed integrals such as for instance $\int_s^t d\Gamma_{z_1}(1) \int_s^{z_1} \overline{d\Gamma_{z_2}(2)}$. For these the analytic continuation method may no longer be applied because Cauchy's formula fails to hold, and the above quantities may be shown to diverge when $\mathrm{Re}\ s, \mathrm{Re}\ t \to 0$, see [34, 35].

Let us explain first how to define a Lévy area for B. Proofs (as well as a sketch of the Fourier normal ordering method for general iterated integrals) may be found in [37]. As mentioned before, the *uncorrected area* $\mathcal{A}^\eta_{ts} := \int_s^t dB^\eta_{u_1}(1) \int_s^{u_1} dB^\eta_{u_2}(2)$ diverges when $\eta \to 0^+$. The idea is now to find some *increment counterterm* $(\delta Z^\eta)_{ts} = Z^\eta_t - Z^\eta_s$ such that the *regularized area* $\mathcal{RA}^\eta_{ts} := \mathcal{A}^\eta_{ts} - (\delta Z^\eta)_{ts}$ converges when $\eta \to 0^+$. Note that the multiplicativity property (0.5) holds for $\mathcal{RA}^\eta$ as well as for $\mathcal{A}^\eta$ since $(\delta Z^\eta)_{ts} = (\delta Z^\eta)_{tu} + (\delta Z^\eta)_{us}$. This counterterm Z^η may be found by using a suitable decomposition of $\mathcal{A}^\eta_{ts}$ into the sum of :

– an *increment term*, $(\delta G^\eta)_{ts}$;

– a *boundary term* denoted by $\mathcal{A}^\eta_{ts}(\partial)$.

The simplest idea one could think of would be to set

$$(\delta G^\eta)_{ts} = \int_s^t dB^\eta_{u_1}(1) B^\eta_{u_1}(2), \tag{0.9}$$

and

$$\mathcal{A}^\eta_{ts}(\partial) = -\int_s^t dB^\eta_{u_1}(1) \ . \ B^\eta_s(2) = -B^\eta_s(2)(B^\eta_t(1) - B^\eta_s(1)). \tag{0.10}$$

Alternatively, rewriting $\mathcal{A}^\eta_{ts}$ as $\int_s^t dB^\eta_{u_2}(2) \int_{u_2}^t dB^\eta_{u_1}(1)$, one may equivalently set

$$(\delta G^\eta)_{ts} = -\int_s^t dB^\eta_{u_2}(2) B^\eta_{u_2}(1) \tag{0.11}$$

and

$$\mathcal{A}^\eta_{ts}(\partial) = \int_s^t dB^\eta_{u_2}(2) \ . \ B^\eta_t(1) = B^\eta_t(1)(B^\eta_t(2) - B^\eta_s(2)). \tag{0.12}$$

Now δG^η diverges when $\eta \to 0^+$, but since it is an increment, it may be discarded (i.e. it might be used as a counterterm). The problem is, $\mathcal{A}^\eta_{ts}(\partial)$ converges when $\eta \to 0^+$ in the κ-Hölder norm for every $\kappa < \alpha$, but not in the 2κ-Hölder norm (which is of course well-known and may be seen as the starting point for rough path theory).

It turns out that a slight adaptation of this poor idea gives the solution. Decompose $\mathcal{A}^\eta_{ts}$ into a double integral in the Fourier coordinates ξ_1, ξ_2 using (0.8). Use the first increment/boundary decomposition (0.9,0.10) for all indices $|\xi_1| \leq |\xi_2|$, and the second one (0.11,0.12) if $|\xi_1| > |\xi_2|$. Then $\mathcal{A}^\eta_{ts}(\partial)$, defined as the sum of two contributions, one coming from (0.10) and the

other from (0.12), *does converge* in the 2κ-Hölder norm when $\eta \to 0^+$, for every $\kappa < \alpha$.

As for the increment term δG^η, defined similarly as the sum of two contributions coming from (0.9) and (0.11), it diverges as soon as $\alpha \leq 1/4$, but may be discarded at will. Actually we use in this article a *minimal regularization scheme*: only the close-to-diagonal (i.e. $\xi_1/\xi_2 \approx -1$) terms in the double integral defining δG^η make it diverge. Summing over an appropriate subset, e.g. $-\xi_1 \notin [\xi_2/2, 2\xi_2]$ yields an increment which converges (for *every* $\alpha \in (0, \frac{1}{2})$) when $\eta \to 0$ in the 2κ-Hölder norm for every $\kappa < \alpha$.

Let $\alpha < 1/4$. As noted in [35], the uncorrected Lévy area $\mathcal{A}^\eta$ of the regularized process B^η converges in law to a Brownian motion when $\eta \to 0^+$ after a rescaling by the factor $\eta^{\frac{1}{2}(1-4\alpha)}$. In the latter article, the following question was raised: is it possible to define a counterterm X^η living on the same probability space as fBm, such that (i) the rescaled process $\eta^{\frac{1}{2}(1-4\alpha)} X^\eta$ converges in law to Brownian motion; (ii) $(B^\eta, \mathcal{A}^\eta - X^\eta)$ is a multiplicative or almost multiplicative functional in the sense of [23], Definition 7.1; (iii) $\mathcal{A}^\eta - X^\eta$ converges in the 2κ-Hölder norm for every $\kappa < \alpha$ when $\eta \to 0$? The counterterm $X^\eta := \mathcal{A}^\eta - \mathcal{R}\mathcal{A}^\eta$ gives a solution to this problem.

The above ideas have a suitable generalization to iterated integrals $\int dB(i_1) \dots \int dB(i_n)$ of order $n \geq 3$. There is one more difficulty though: decomposing $(B^\eta)'_{u_j}(i_j)$ into $c_\alpha \int dW_{\xi_j}(i_j) e^{\mathrm{i}u_j\xi_j} e^{-\eta|\xi_j|} |\xi_j|^{\frac{1}{2}-\alpha}$, an extension of the first increment/boundary decomposition (0.9, 0.10), together with a suitable regularization scheme, yield the correct Hölder estimate *provided* $|\xi_1| \leq \dots \leq |\xi_n|$. What should one do then if $|\xi_{\sigma(1)}| \leq \dots \leq |\xi_{\sigma(n)}|$ for some permutation σ instead ? The idea is to permute the order of integration by using Fubini's theorem, and write $\int_s^t dB^\eta_{u_1}(i_1) \dots \int_s^{u_{n-1}} dB^\eta_{u_n}(i_n)$ as some *iterated tree integral* $\int dB^\eta_{u_1}(i_{\sigma(1)}) \dots \int dB^\eta_{u_n}(i_{\sigma(n)})$. The integration domain, in the general case, becomes a little involved, and necessitates the introduction of combinatorial tools on trees, such as admissible cuts for instance. The underlying structures are those of the Hopf algebra of decorated rooted trees [6, 7] (as already noted in [21] or [17]), and of the Hopf shuffle algebra [27, 28]. The proof of the multiplicative and of the geometric properties for the regularized rough path, as well as the Hopf algebraic reinterpretation, are to be found in [36]. The general idea (see subsection 2.5 for more details) is that the fundamental objects are *skeleton integrals* (a particular type of tree integrals) defined in subsection 2.1, and that *any* regularization of the skeleton integrals (possibly even trivial) yielding finite quantities with the correct Hölder regularity produces a regularized rough path, which implies a large degree of arbitrariness in the definition. The idea of can-

celling singularities by building iteratively counterterms, originated from the Bogolioubov-Hepp-Parasiuk-Zimmermann (BPHZ) procedure for renormalizing Feynmann diagrams in quantum field theory [18], mathematically formalized in terms of Hopf algebras by A. Connes and D. Kreimer, has been applied during the last decade in a variety of contexts ranging from numerical methods to quantum chromodynamics or multi-zeta functions, see for instance [21, 28, 38]. We plan to such a (less arbitrary) construction in the near future (see discussion at the end of subsection 2.5).

The main result of the paper may be stated as follows.

Theorem 0.1 *Let $B = (B(1), \dots, B(d))$ be a d-dimensional fBm of Hurst index $\alpha \in (0,1)$, defined via the harmonizable representation, with the associated family of approximations B^η, $\eta > 0$ living in the same probability space, see eq. (0.8). Then there exists a rough path $(\mathcal{R}\mathbf{B}^{1,\eta} = \delta B^\eta, \dots, \mathcal{R}\mathbf{B}^{\lfloor 1/\alpha \rfloor, \eta})$ over B^η $(\eta > 0)$, living in the chaos of order $1, \dots, \lfloor 1/\alpha \rfloor$ of B, satisfying properties (ii) (multiplicative property) and (iii) (geometric property) of the Introduction, together with the following estimates:*

(uniform Hölder estimate) There exists a constant $C > 0$ such that, for every $s, t \in \mathbb{R}$ and $\eta > 0$,

$$\mathbb{E}|\mathcal{R}\mathbf{B}^{n,\eta}_{ts}(i_1, \dots, i_n)|^2 \leq C|t-s|^{2n\alpha};$$

(rate of convergence) there exists a constant $C > 0$ such that, for every $s, t \in \mathbb{R}$ and $\eta_1, \eta_2 > 0$,

$$\mathbb{E}|\mathcal{R}\mathbf{B}^{n,\eta_1}_{ts}(i_1, \dots, i_n) - \mathcal{R}\mathbf{B}^{n,\eta_2}_{ts}(i_1, \dots, i_n)|^2 \leq C|\eta_1 - \eta_2|^{2\alpha}.$$

These results imply the existence of an explicit rough path $\mathcal{R}\mathbf{B}$ over B, obtained as the limit of $\mathcal{R}\mathbf{B}^\eta$ when $\eta \to 0$.

Here is an outline of the article. We first recall briefly some definitions and preliminary results on algebraic rough path theory in Section 1, which show in particular that Theorem 0.1 implies the convergence of $\mathcal{R}\mathbf{B}^\eta$ to a rough path $\mathcal{R}\mathbf{B}$ over fractional Brownian motion B when $\eta \to 0$. Section 2 is dedicated to tree combinatorics and to the introduction of quite general regularization schemes for the iterated integrals of an arbitrary smooth path Γ. The proof of the multiplicative and geometric properties are to be found

in [36] and are not reproduced here. We apply a suitable regularization scheme to the construction of the regularized rough path $\mathcal{R}\mathbf{B}^\eta$ in section 3, and prove the Hölder and rate of convergence estimates of Theorem 0.1 for the iterated integrals $\mathcal{R}\mathbf{B}^{n,\eta}(i_1,\dots,i_n)$ with distinct indices, $i_1 \neq \dots \neq i_n$. We conclude in Section 4 by showing how to extend these results to coinciding indices, and introducing a new, real-valued, two-dimensional Gaussian process which we call *two-dimensional antisymmetric fractional Brownian motion*, to which the above construction extends naturally.

Notations. The group of permutations of $\{1,\dots,n\}$ will be denoted by Σ_n. The Fourier transform is $\mathcal{F} : f \to \mathcal{F}f(\xi) = \frac{1}{\sqrt{2\pi}} \int f(x) e^{-\mathrm{i}x\xi} dx$. If $|a| \leq C|b|$ for some constant C (a and b depending on some arbitrary set of parameters), then we shall write $|a| \lesssim |b|$.

1 The analysis of rough paths

The present section will be very sketchy since the objects and results needed in this work have alread been presented in great details in [33]. The fundational paper on the subject of algebraic rough path theory is due to M. Gubinelli [16], see also [17] for more details in the case $\alpha < 1/3$. Let us recall briefly the original problem motivating the introduction of rough paths. Let $\Gamma : \mathbb{R} \to \mathbb{R}^d$ be some fixed irregular (i.e. not differentiable) path, say κ-Hölder, and $f : \mathbb{R} \to \mathbb{R}^d$ some function which is also irregular (mainly because one wants to consider functions f obtained as a composition $g \circ \Gamma$ where $g : \mathbb{R}^d \to \mathbb{R}^d$ is regular). Can one define the integral $\int f_x d\Gamma_x$? The answer depends on the Hölder regularity of f and Γ. Assuming f is γ-Hölder with $\kappa + \gamma > 1$, then one may define the so-called *Young integral* [23] $\int_s^t f_x d\Gamma_x$ as the Riemann sum type limit $\lim_{|\Pi|\to 0} \sum_{\{t_j\}\in\Pi} f_{t_i}(\Gamma_{t_{i+1}} - \Gamma_{t_i})$, where $\Pi = \{s = t_0 < \dots < t_n = t\}$ is a partition of $[s,t]$ with mesh $|\Pi|$ going to 0. Then the resulting path $Y_t - Y_s := \int_s^t f_x d\Gamma_x$ has the same regularity as Γ. If $\kappa + \gamma \leq 1$ instead, this is no more possible in general. One way out of this problem, giving at the same time a coherent way to solve differential equations driven by Γ, is to define a class of Γ-*controlled paths* $\mathcal{Q}$, such that the above integration problem may be solved uniquely in this class by a formula generalizing the above Riemann sums, in which formal iterated integrals $\mathbf{\Gamma}^n(i_1,\dots,i_n)$ of Γ appear as in the Introduction.

Definition 1.1 (Hölder spaces) *Let $\kappa \in (0,1)$ and $T > 0$ fixed.*

(i) Let $C_1^\kappa = C_1^\kappa([0,T],\mathbb{C})$ be the space of complex-valued κ-Hölder functions f in one variable with (semi-)norm $||f||_\kappa = \sup_{s,t\in[0,T]} \frac{|f(t)-f(s)|}{|t-s|^\kappa}$.

(ii) Let $C_2^\kappa = C_2^\kappa([0,T],\mathbb{C})$ be the space of complex-valued functions $f = f_{t_1,t_2}$ of two variables vanishing on the diagonal $t_1 = t_2$, such that $||f||_{2,\kappa} < \infty$, where $||\ .\ ||_{2,\kappa}$ is the following norm:

$$||f||_{2,\kappa} = \sup_{s,t\in[0,T]} \frac{|f_{t_1,t_2}|}{|t-s|^\kappa}. \tag{1.1}$$

(iii) Let $C_3^\kappa = C_3^\kappa([0,T],\mathbb{C})$ be the space of complex-valued functions $f = f_{t_1,t_2,t_3}$ of three variables vanishing on the subset $\{t_1 = t_2\} \cup \{t_2 = t_3\} \cup \{t_1 = t_3\}$, such that $||f||_{3,\kappa} < \infty$ for some generalized Hölder semi-norm $||\ .\ ||_{3,\kappa}$ defined for instance in [16], section 2.1.

Definition 1.2 (increments) *(i) Let f be a function of one variable: then the increment of f, denoted by δf, is $(\delta f)_{ts} := f(t) - f(s)$.*

(ii) Let $f = f_{ts}$ be a function of two variables: then we define

$$(\delta f)_{tus} := f_{ts} - f_{tu} - f_{us}. \tag{1.2}$$

Note that $\delta \circ \delta(f) = 0$ if f is a function of one variable.

Let $\Gamma = (\Gamma(1),\dots,\Gamma(d)) : [0,T] \to \mathbb{R}^d$ be a κ-Hölder path, and $(\mathbf{\Gamma}^1_{ts}(i_1) := \Gamma_t(i_1) - \Gamma_s(i_1), \mathbf{\Gamma}^2_{ts}(i_1,i_2),\dots,\mathbf{\Gamma}^{\lfloor 1/\kappa\rfloor}_{ts}(i_1,\dots,i_{\lfloor 1/\kappa\rfloor}))$ be a rough path lying above Γ, satisfying properties (i) (Hölder property), (ii) (multiplicativity property) and (iii) (geometricity property) of the Introduction.

Definition 1.3 (controlled paths) *Let $z = (z(1),\dots,z(d)) \in C_1^\kappa$ for some $\kappa < \alpha$ and $N = \lfloor 1/\kappa\rfloor + 1$. Then z is called a (Γ-)controlled path if its increments can be decomposed into*

$$\delta z(i) = \sum_{n=1}^{N} \sum_{(i_1,\dots,i_n)} \mathbf{\Gamma}^n(i_1,\dots,i_n).f^n(i_1,\dots,i_n;i) + g^0(i) \tag{1.3}$$

for some remainders $g^0(i) \in C_2^{N\kappa}$ and some paths $f^n(i_1,\dots,i_n;i) \in (C_1^\kappa)^n$ such that

$$\delta f^n(i_1,\dots,i_n;i) = \sum_{l=1}^{N-1-n} \sum_{(j_1,\dots,j_l)} \mathbf{\Gamma}^l(j_1,\dots,j_l).f^{l+n}(j_1,\dots,j_l,i_1,\dots,i_n;i) + g^n(i_1,\dots,i_n;i), \quad n = 1,\dots,N \tag{1.4}$$

for some remainder terms $g^n(i_1,\dots,i_n;i) \in C_2^{(N-n)\kappa}$.

We denote by $\mathcal{Q}_\kappa$ *the space of all such paths, and by* $\mathcal{Q}_{\alpha^-}$ *the intersection* $\cap_{\kappa<\alpha}\mathcal{Q}_\kappa$.

We may now state the main result.

Proposition 1.4 (see [17], Theorem 8.5, or [33], Proposition 3.1) *Let* $z \in \mathcal{Q}_{\alpha^-}$. *Then the limit*

$$\int_s^t z_x d\Gamma_x := \lim_{|\Pi|\to 0} \sum_{k=0}^{n}\sum_{i=1}^{d}\left[\delta X_{t_{k+1},t_k}(i) z_{t_k}(i) + \sum_{n=1}^{N-1}\sum_{(i_1,\dots,i_n)} \mathbf{\Gamma}^{n+1}_{t_{k+1},t_k}(i_1,\dots,i_n,i)\zeta^n_{t_k}(i_1,\dots,i_n;i)\right] \tag{1.5}$$

exists in the space $\mathcal{Q}_{\alpha^-}$.

Assume Γ is a centered Gaussian process, and Γ^η a family of Gaussian approximations of Γ living in its first chaos. Then the Proposition below gives very convenient moment conditions for a family of rough paths $(\Gamma^\eta, \mathbf{\Gamma}^{2,\eta},\dots,\mathbf{\Gamma}^{\lfloor 1/\kappa\rfloor,\eta})$ to converge in the right Hölder norms when $\eta \to 0$, thereby defining a rough path above Γ.

Proposition 1.5 *Let* Γ *be a* d*-dimensional centered Gaussian process admitting a version with a.s.* α^-*-Hölder paths. Let* $N = \lfloor 1/\alpha\rfloor$. *Assume:*

1. *there exists a family* Γ^η, $\eta \to 0^+$ *of Gaussian processes living in the first chaos of* Γ *and an overall constant* C *such that*

   (i)
   $$\mathbb{E}|\Gamma^\eta_t - \Gamma^\eta_s|^2 \le C|t-s|^{2\alpha}; \tag{1.6}$$

   (ii)
   $$\mathbb{E}|\Gamma^\eta_t - \Gamma^\varepsilon_t|^2 \le C|\varepsilon-\eta|^{2\alpha}; \tag{1.7}$$

   (iii) $\forall t \in [0,T]$, $\Gamma^\eta_t \overset{L^2}{\to} \Gamma_t$ *when* $\eta \to 0$;

2. *there exists a truncated multiplicative functional* $(\mathbf{\Gamma}^{1,\eta}_{ts} = \Gamma^\eta_t - \Gamma^\eta_s, \mathbf{\Gamma}^{2,\eta}_{ts},\dots,\mathbf{\Gamma}^{N,\eta}_{ts})$ *lying above* Γ^η *and living in the* n*-th chaos of* Γ, $n = 1\dots,N$, *such that, for every* $2 \le k \le N$,

   (i)
   $$\mathbb{E}|\mathbf{\Gamma}^{k,\eta}_{ts}|^2 \le C|t-s|^{2k\alpha}; \tag{1.8}$$

(ii)

$$\mathbb{E}|\mathbf{\Gamma}_{ts}^{k,\varepsilon} - \mathbf{\Gamma}_{ts}^{k,\eta}|^2 \leq C|\varepsilon - \eta|^{2\alpha}. \tag{1.9}$$

Then $(\mathbf{\Gamma}^{1,\eta}, \ldots, \mathbf{\Gamma}^{N,\eta})$ *converges in* $L^2(\Omega; C_2^\kappa([0,T], \mathbb{R}^d) \times C_2^{2\kappa}([0,T], \mathbb{R}^{d^2}) \times \ldots \times C_2^{N\kappa}([0,T], \mathbb{R}^{d^N}))$ *for every* $\kappa < \alpha$ *to a rough path* $(\mathbf{\Gamma}^1, \ldots, \mathbf{\Gamma}^N)$ *lying above* Γ.

Short proof (see [33], Lemma 5.1, Lemma 5.2 and Prop. 5.4). The main ingredient is the Garsia-Rodemich-Rumsey (GRR for short) lemma [14] which states that, if $f \in C_2^\kappa([0,T], \mathbb{C})$,

$$||f||_{2,\kappa} \leq C \left(||\delta f||_{3,\kappa} + \left(\int_0^T \int_0^T \frac{|f_{vw}|^{2p}}{|w-v|^{2\kappa p+2}} \, dv \, dw \right)^{1/2p} \right) \tag{1.10}$$

for every $p \geq 1$.

Then properties (1.6,1.8) imply by using the GRR lemma for p large enough, Jensen's inequality and the equivalence of L^p-norms for processes living in a fixed Gaussian chaos

$$\mathbb{E}||\mathbf{\Gamma}^{k,\eta}||_{2,k\kappa} \lesssim \mathbb{E}||\delta\mathbf{\Gamma}^{k,\eta}||_{3,k\kappa} + C. \tag{1.11}$$

By using the multiplicative property (ii) in the Introduction and induction on k, $\mathbb{E}||\delta\mathbf{\Gamma}^{k,\eta}||_{3,k\kappa}$ may in the same way be proved to be bounded by a constant.

On the other hand, properties (1.6,1.7,1.8,1.9), together with the equivalence of L^p-norms, imply (for every $\kappa < \alpha$)

$$\mathbb{E}|\mathbf{\Gamma}_{ts}^{k,\varepsilon} - \mathbf{\Gamma}_{ts}^{k,\eta}|^2 \lesssim |t-s|^{2k\kappa}|\varepsilon - \eta|^{2(\alpha-\kappa)} \tag{1.12}$$

hence, by the same arguments,

$$\mathbb{E}||\mathbf{\Gamma}^{k,\varepsilon} - \mathbf{\Gamma}^{k,\eta}||_{2,k\kappa} \lesssim |\varepsilon - \eta|^{\alpha-\kappa} \tag{1.13}$$

which shows that $\mathbf{\Gamma}^{k,\varepsilon}$ is a Cauchy sequence in $C_2^{k\kappa}([0,T], \mathbb{R}^{d^k})$. □

2 Tree combinatorics and the Fourier normal ordering method

2.1 From iterated integrals to trees

It was noted already long time ago [3] that iterated integrals could be encoded by trees. This remark has been exploited in connection with the

construction of the rough path solution of (partial, stochastic) differential equations in [17]. The correspondence between trees and itegrated integrals goes simply as follows.

Definition 2.1 *A decorated rooted tree (to be drawn growing* up*) is a finite tree with a distinguished vertex called* root *and edges oriented* downwards *(i.e. directed towards the root), such that every vertex wears an integer label.*

If $\mathbb{T}$ is a decorated rooted tree, we let $V(\mathbb{T})$ be the set of its vertices (including the root), and $\ell : V(\mathbb{T}) \to \mathbb{N}$ be its vertex labeling.

More generally, a decorated rooted forest *is a finite set of decorated rooted trees. If $\mathbb{T} = \{\mathbb{T}_1, \dots, \mathbb{T}_l\}$ is a forest, then we shall write $\mathbb{T}$ as the formal commutative product $\mathbb{T}_1 \dots \mathbb{T}_l$.*

Definition 2.2 *Let $\mathbb{T}$ be a decorated rooted tree.*

- *Letting $v, w \in V(\mathbb{T})$, we say that v* connects directly to *w, and write $v \to w$ or equivalently $w = v^-$, if (v, w) is an edge oriented downwards from v to w. (Note that v^- exists and is unique except if v is the root).*
- *If $v_m \to v_{m-1} \to \dots \to v_1$, then we shall write $v_m \twoheadrightarrow v_1$, and say that v_m* connects to *v_1. By definition, all vertices (except the root) connect to the root.*
- *Let $(v_1, \dots, v_{|V(\mathbb{T})|})$ be an ordering of $V(\mathbb{T})$. Assume that $(v_i \twoheadrightarrow v_j) \Rightarrow (i > j)$ (in particular, v_1 is the root). Then we shall say that the ordering is* compatible *with the* tree partial ordering *defined by $\twoheadrightarrow$.*

Definition 2.3 *(i) Let $\Gamma = (\Gamma(1), \dots, \Gamma(d))$ be a d-dimensional smooth path, and $\mathbb{T}$ a decorated rooted tree such that $\ell : V(\mathbb{T}) \to \{1, \dots, d\}$. Then $I_{\mathbb{T}}(\Gamma) : \mathbb{R}^2 \to \mathbb{R}$ is the iterated integral defined as*

$$[I_{\mathbb{T}}(\Gamma)]_{ts} := \int_s^t d\Gamma_{x_{v_1}}(\ell(v_1)) \int_s^{x_{v_2^-}} d\Gamma_{x_{v_2}}(\ell(v_2)) \dots \int_s^{x_{v_{|V(\mathbb{T})|}^-}} d\Gamma_{x_{v_{|V(\mathbb{T})|}}}(\ell(v_{|V(\mathbb{T})|})) \tag{2.1}$$

where $(v_1, \dots, v_{|V(\mathbb{T})|})$ is any ordering of $V(\mathbb{T})$ compatible with the tree partial ordering.

In particular, if $\mathbb{T}$ is a trunk tree with n vertices (see Fig. 1) – so that the tree ordering is total – we shall write

$$I_{\mathbb{T}}(\Gamma) = I_n^{\ell}(\Gamma), \tag{2.2}$$

where

$$[I_n^\ell(\Gamma)]_{ts} := \int_s^t d\Gamma_{x_1}(\ell(1)) \int_s^{x_1} d\Gamma_{x_2}(\ell(2)) \dots \int_s^{x_{n-1}} d\Gamma_{x_n}(\ell(n)). \quad (2.3)$$

(ii) (generalization) Assume $\mathbb{T}$ is a subtree of $\tilde{\mathbb{T}}$. Let μ be a Borel measure on $\mathbb{R}^{\tilde{\mathbb{T}}}$. Then

$$[I_{\tilde{\mathbb{T}}}(\mu)]_{ts} := \int_s^t \int_s^{x_{v_1^-}} \dots \int_s^{x_{v_{|V(\mathbb{T})|}^-}} \mu(dx_{v_1}, \dots, dx_{v_{|V(\mathbb{T})|}}) \quad (2.4)$$

is a measure on $\mathbb{R}^{\tilde{\mathbb{T}}\setminus\mathbb{T}}$.

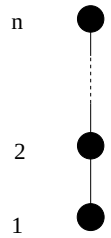

Figure 1: Trunk tree.

Assume $\mathbb{T} = \tilde{\mathbb{T}}$ so $[I_{\tilde{\mathbb{T}}}(\mu)]_{ts}$ is a *number*. Then case (i) may be seen as a particular case of case (ii) with $\mu = d\Gamma(\ell(v_1)) \otimes \dots \otimes d\Gamma(\ell(v_{|V(\mathbb{T})|}))$. Conversely, case (ii) may be seen as a multilinear extension of case (i), and will turn out to be useful later on for the regularization procedure. Note however that (i) uses the labels of $\mathbb{T}$ while (ii) *doesn't*.

The above correspondence extends by (multi)linearity to the *algebra of decorated rooted trees* which we shall now introduce.

Definition 2.4 (algebra of decorated rooted trees) *(i) Let $\mathcal{T}$ be the free commutative algebra over $\mathbb{R}$ generated by decorated rooted trees. If $\mathbb{T}_1, \mathbb{T}_2, \dots \mathbb{T}_l$ are (decorated rooted) trees, then the product $\mathbb{T}_1 \dots \mathbb{T}_l$ is the forest with connected components $\mathbb{T}_1, \dots, \mathbb{T}_l$.*

(ii) Let $\mathbb{T}' = \sum_{l=1}^L m_l \mathbb{T}_l \in \mathcal{T}$, where $m_l \in \mathbb{R}$ and each $\mathbb{T}_l = \mathbb{T}_{l,1} \dots \mathbb{T}_{l,L(l)}$ is a forest with labels in the set $\{1, \dots, d\}$, and Γ be a smooth d-dimensional path as above. Then

$$[I_{\mathbb{T}'}(\Gamma)]_{ts} := \sum_{l=1}^L m_l [I_{\mathbb{T}_{l,1}}(\Gamma)]_{ts} \dots [I_{\mathbb{T}_{l,L(l)}}(\Gamma)]_{ts}. \quad (2.5)$$

Let us now rewrite these iterated integrals by using Fourier transform.

Definition 2.5 (formal integral) *Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth, compactly supported function such that $\mathcal{F}f(0) = 0$. Then the formal integral $\int^t f = -\int_t f$ of f is defined as $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} (\mathcal{F}f)(\xi) \frac{e^{\mathrm{i}t\xi}}{\mathrm{i}\xi}\, d\xi$.*

Formally one may write:

$$\int^t e^{\mathrm{i}x\xi} dx = \int_{\pm \mathrm{i}\infty}^t e^{\mathrm{i}x\xi} dx = \frac{e^{\mathrm{i}t\xi}}{\mathrm{i}\xi} \tag{2.6}$$

(depending on the sign of ξ). The condition $\mathcal{F}f(0) = 0$ prevents possible infra-red divergence when $\xi \to 0$.

The *skeleton integrals* defined below must be understood in a *formal* sense because of the possible infra-red divergences.

Definition 2.6 (skeleton integrals) *(i) Let $\mathbb{T}$ be a tree with $\ell : \mathbb{T} \to \{1, \dots, d\}$ and Γ be a d-dimensional compactly supported, smooth path. Let $(v_1, \dots, v_{|V(\mathbb{T})|})$ be any ordering of $V(\mathbb{T})$ compatible with the tree partial ordering. Then the* skeleton integral *of Γ along $\mathbb{T}$ is by definition*

$$[\mathrm{Sk} I_{\mathbb{T}}(\Gamma)]_t = \int^t d\Gamma_{x_{v_1}}(\ell(v_1)) \int^{x_{v_2^-}} d\Gamma_{x_2}(\ell(v_2)) \dots \int^{x_{v_{|V(\mathbb{T})|}^-}} d\Gamma_{x_{v_{|V(\mathbb{T})|}}}(\ell(v_{|V(\mathbb{T})|})). \tag{2.7}$$

(ii) (multilinear extension, see Definition 2.3) Assume $\mathbb{T}$ is a subtree of $\tilde{\mathbb{T}}$, and μ a compactly supported Borel measure on $\mathbb{R}^{\tilde{\mathbb{T}}}$. Then

$$[\mathrm{Sk} I_{\mathbb{T}}(\mu)]_t = \int^t \int^{x_{v_2^-}} \dots \int^{x_{v_{|V(\mathbb{T})|}^-}} \mu(dx_{v_1}, \dots, dx_{v_{|V(\mathbb{T})|}}) \tag{2.8}$$

is a measure on $\mathbb{R}^{\tilde{\mathbb{T}} \setminus \mathbb{T}}$.

Formally again, $[\mathrm{Sk} I_{\mathbb{T}}(\Gamma)]_t$ may be seen as $[I_{\mathbb{T}}(\Gamma)]_{t,\pm\mathrm{i}\infty}$. Note that (denoting by $\hat{\mu}$ the partial Fourier transform of μ with respect to $(x_v)_{v \in V(\mathbb{T})}$), the following equation holds,

$$[\mathrm{Sk}\, \mathrm{I}_{\mathbb{T}}(\mu)]_t = (2\pi)^{-|V(\mathbb{T})|/2} \langle \hat{\mu}, \left[\mathrm{Sk}\, \mathrm{I}_{\mathbb{T}} \left((x_v)_{v \in V(\mathbb{T})} \to e^{\mathrm{i} \sum_{v \in V(\mathbb{T})} x_v \xi_v} \right) \right]_t \rangle. \tag{2.9}$$

Lemma 2.7 *The following formula holds:*

$$[\mathrm{Sk}I_{\mathbb{T}}(\Gamma)]_t = (\mathrm{i}\sqrt{2\pi})^{-|V(\mathbb{T})|} \int \dots \int_{\mathbb{R}^{\mathbb{T}}} \prod_{v\in V(\mathbb{T})} d\xi_v \, . \, e^{\mathrm{i}t\sum_{v\in V(\mathbb{T})}\xi_v} \frac{\prod_{v\in V(\mathbb{T})} \mathcal{F}(\Gamma'(\ell(v)))(\xi_v)}{\prod_{v\in V(\mathbb{T})}(\xi_v + \sum_{w\twoheadrightarrow v}\xi_w)}. \tag{2.10}$$

Proof. We use induction on $|V(\mathbb{T})|$. After stripping the root of $\mathbb{T}$ (denoted by 0) there remains a forest $\mathbb{T}' = \mathbb{T}'_1 \dots \mathbb{T}'_J$, whose roots are the vertices directly connected to 0. Assume

$$[\mathrm{Sk}I_{\mathbb{T}'_j}(\Gamma)]_{x_0} = \int \dots \int \prod_{v\in V(\mathbb{T}'_j)} d\xi_v \, . \, e^{\mathrm{i}x_0\sum_{v\in V(\mathbb{T}'_j)}\xi_v} F_j((\xi_v)_{v\in\mathbb{T}'_j}). \tag{2.11}$$

Note that

$$\mathcal{F}\left(\prod_{j=1}^{J} \mathrm{Sk}I_{\mathbb{T}'_j}(\Gamma)\right)(\xi) = \int_{\sum_{v\in V(\mathbb{T})\setminus\{0\}}\xi_v=\xi} \prod_{v\in V(\mathbb{T})\setminus\{0\}} d\xi_v \prod_{j=1}^{J} F_j((\xi_v)_{v\in V(\mathbb{T}'_j)}). \tag{2.12}$$

Then

$$\begin{aligned}
[\mathrm{Sk}I_{\mathbb{T}}(\Gamma)]_t &= \int^t d\Gamma_{x_0}(\ell(0)) \prod_{j=1}^{J} [\mathrm{Sk}I_{\mathbb{T}'_j}(\Gamma)]_{x_0} \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{d\xi}{\mathrm{i}\xi} e^{\mathrm{i}t\xi} \mathcal{F}\left(\Gamma'(\ell(0)) \prod_{j=1}^{J} \mathrm{Sk}I_{\mathbb{T}'_j}(\Gamma)\right)(\xi) \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} d\xi_0 \mathcal{F}(\Gamma'(\ell(0)))(\xi_0) e^{\mathrm{i}t\xi_0} \, . \\
&\qquad \int_{-\infty}^{+\infty} \frac{d\xi}{\mathrm{i}\xi} e^{\mathrm{i}t(\xi-\xi_0)} \int_{\sum_{v\in V(\mathbb{T})\setminus\{0\}}\xi_v=\xi-\xi_0} d\xi_v \prod_{j=1}^{J} F_j((\xi_v)_{v\in V(\mathbb{T}'_j)})
\end{aligned} \tag{2.13}$$

hence the result. □

Skeleton integrals are the fundamental objects from which regularized rough paths will be constructed in the next subsections.

2.2 Coproduct structure and increment-boundary decomposition

Consider for an example the trunk tree $\mathbb{T}^{\mathrm{Id}_n}$ (see subsection 2.4 for an explanation of the notation) with vertices $n \to n-1 \to \ldots \to 1$ and labels $\ell : \{1,\ldots,n\} \to \{1,\ldots,d\}$, and the associated iterated integral (assuming $\Gamma = (\Gamma(1),\ldots,\Gamma(d))$ is a smooth path)

$$[I_n^\ell(\Gamma)]_{ts} = [I_{\mathbb{T}^{\mathrm{Id}_n}}(\Gamma)]_{ts} = \int_s^t d\Gamma_{x_1}(\ell(1)) \ldots \int_s^{x_{n-1}} d\Gamma_{x_n}(\ell(n)). \tag{2.14}$$

Cutting $\mathbb{T}^{\mathrm{Id}_n}$ at some vertex $v \in \{2,\ldots,n\}$ produces two trees, $L_v\mathbb{T}^{\mathrm{Id}_n}$ (*left* or rather *bottom* part of $\mathbb{T}^{\mathrm{Id}_n}$) and $R_v\mathbb{T}^{\mathrm{Id}_n}$ (*right* or *top* part), with respective vertex subsets $\{1,\ldots,v-1\}$ and $\{v,\ldots,n\}$. One should actually see the couple $(L_v\mathbb{T}^{\mathrm{Id}_n}, R_v\mathbb{T}^{\mathrm{Id}_n})$ as $L_v\mathbb{T}^{\mathrm{Id}_n} \otimes R_v\mathbb{T}^{\mathrm{Id}_n}$ sitting in the tensor product algebra $\mathcal{T}\otimes\mathcal{T}$. Then multiplicative property (ii) in the Introduction reads

$$[\delta I_{\mathbb{T}^{\mathrm{Id}_n}}(\Gamma)]_{tus} = \sum_{v \in V(\mathbb{T}^{\mathrm{Id}_n})\setminus\{1\}} [I_{L_v\mathbb{T}^{\mathrm{Id}_n}}(\Gamma)]_{tu}[I_{R_v\mathbb{T}^{\mathrm{Id}_n}}(\Gamma)]_{us}. \tag{2.15}$$

On the other hand, one may rewrite $[I_{\mathbb{T}^{\mathrm{Id}_n}}(\Gamma)]_{ts}$ as the sum of the *increment term*

$$\begin{aligned}[\delta G]_{ts} \quad &= \textstyle\int^t d\Gamma_{x_1}(\ell(1)) \int^{x_1} d\Gamma_{x_2}(\ell(2)) \ldots \int^{x_{n-1}} d\Gamma_{x_n}(\ell(n)) \\ &\quad - \textstyle\int^s d\Gamma_{x_1}(\ell(1)) \int^{x_1} d\Gamma_{x_2}(\ell(2)) \ldots \int^{x_{n-1}} d\Gamma_{x_n}(\ell(n))\end{aligned} \tag{2.16}$$

and of the *boundary term*

$$\begin{aligned}[I_{\mathbb{T}^{\mathrm{Id}_n}}(\Gamma)(\partial)]_{ts} = -\sum_{n_1+n_2=n} &\int_s^t d\Gamma_{x_1}(\ell(1)) \ldots \int_s^{x_{n_1-1}} d\Gamma_{x_{n_1}}(\ell(n_1)) \ . \\ . \int_s^s d\Gamma_{x_{n_1+1}}(\ell(n_1+1)) &\int_s^{x_{n_1+1}} d\Gamma_{x_{n_1+2}}(\ell(n_1+2)) \ldots \int_s^{x_{n-1}} d\Gamma_{x_n}(\ell(n)).\end{aligned} \tag{2.17}$$

The above decomposition is fairly obvious for $n = 2$ (see Introduction) and obtained by easy induction for general n. Thus (using tree notation this time)

$$[I_{\mathbb{T}^{\mathrm{Id}_n}}(\Gamma)]_{ts} = [\delta \mathrm{Sk} I_{\mathbb{T}^{\mathrm{Id}_n}}]_{ts} - \sum_{v \in V(\mathbb{T}^{\mathrm{Id}_n})\setminus\{1\}} [I_{L_v\mathbb{T}^{\mathrm{Id}_n}}(\Gamma)]_{ts} \ . \ [\mathrm{Sk} I_{R_v\mathbb{T}^{\mathrm{Id}_n}}(\Gamma)]_s. \tag{2.18}$$

The above considerations extend to arbitrary trees (or also forests) as follows.

Definition 2.8 (admissible cuts) *1. Let $\mathbb{T}$ be a tree, with set of vertices $V(\mathbb{T})$ and root denoted by 0. If $\boldsymbol{v} = (v_1, \ldots, v_J)$, $J \geq 1$ is any totally disconnected subset of $V(\mathbb{T}) \setminus \{0\}$, i.e. $v_i \not\twoheadrightarrow v_j$ for all $i, j = 1, \ldots, J$, then we shall say that $\boldsymbol{v}$ is an* admissible cut *of $\mathbb{T}$, and write $\boldsymbol{v} \models V(\mathbb{T})$. We let $R_{\boldsymbol{v}}\mathbb{T}$ be the sub-forest (or sub-tree if $J = 1$) obtained by keeping only the vertices above $\boldsymbol{v}$, i.e. $V(R_{\boldsymbol{v}}\mathbb{T}) = \boldsymbol{v} \cup \{w \in V(\mathbb{T}) : \ \exists j = 1, \ldots, J, w \twoheadrightarrow v_j\}$, and $L_{\boldsymbol{v}}\mathbb{T}$ be the sub-tree obtained by keeping all other vertices.*

2. Let $\mathbb{T} = \mathbb{T}_1 \ldots \mathbb{T}_l$ be a forest, together with its decomposition into trees. Then an admissible cut of $\mathbb{T}$ is a disjoint union $\boldsymbol{v}_1 \cup \ldots \cup \boldsymbol{v}_l$, $\boldsymbol{v}_i \subset \mathbb{T}_i$, where $\boldsymbol{v}_i$ is either $\emptyset$, $\{0_i\}$ (root of $\mathbb{T}_i$) or an admissible cut of $\mathbb{T}_i$. By definition, we let $L_{\boldsymbol{v}}\mathbb{T} = L_{\boldsymbol{v}_1}\mathbb{T}_1 \ldots L_{\boldsymbol{v}_l}\mathbb{T}_l$, $R_{\boldsymbol{v}}\mathbb{T} = R_{\boldsymbol{v}_1}\mathbb{T}_1 \ldots R_{\boldsymbol{v}_l}\mathbb{T}_l$ (if $\boldsymbol{v}_i = \emptyset$, resp. $\{0_i\}$, then $(L_{\boldsymbol{v}_i}\mathbb{T}_i, R_{\boldsymbol{v}_i}\mathbb{T}_i) := (\mathbb{T}_i, \emptyset)$, resp. $(\emptyset, \mathbb{T}_i)$).

We exclude by convention the two trivial cuts $\emptyset \cup \ldots \cup \emptyset$ and $\{0_1\} \cup \ldots \cup \{0_l\}$.

See Fig. 2 and 3. Defining the co-product operation $\Delta : \mathcal{T} \to \mathcal{T} \otimes \mathcal{T}$, $\mathbb{T} \to e \otimes \mathbb{T} + \mathbb{T} \otimes e + \sum_{\boldsymbol{v} \models V(\mathbb{T})} L_{\boldsymbol{v}}\mathbb{T} \otimes R_{\boldsymbol{v}}\mathbb{T}$ (where e stands for the empty tree, which is the unit of the algebra) yields a coalgebra structure on $\mathcal{T}$ which makes it (once the antipode – which we do not need here – is defined) a Hopf algebra (see articles by A. Connes and D. Kreimer [6, 7, 8]). The convention is usuall to write $\boldsymbol{v} = c$ (cut), $L_{\boldsymbol{v}}\mathbb{T} = R^c(\mathbb{T})$ (root part), $R_{\boldsymbol{v}}\mathbb{T} = P^c(\mathbb{T})$ and $\Delta(\mathbb{T}) = e \otimes \mathbb{T} + \mathbb{T} \otimes e + \sum_c P^c(\mathbb{T}) \otimes R^c(\mathbb{T})$ (note the inversion of the order of the factors in the tensor product).

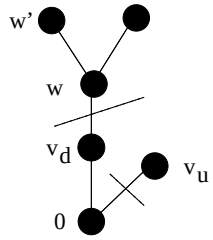

Figure 2: Admissible cut.

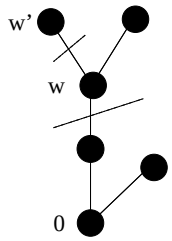

Figure 3: Non-admissible cut.

Eq. (2.15) extends to the general formula (called: *tree multiplicative property*), which one can find in [21] or [17],

$$[\delta I_{\mathbb{T}}(\Gamma)]_{tus} = \sum_{\boldsymbol{v} \models V(\mathbb{T})} [I_{L_{\boldsymbol{v}}\mathbb{T}}(\Gamma)]_{tu} [I_{R_{\boldsymbol{v}}\mathbb{T}}(\Gamma)]_{us}, \tag{2.19}$$

satisfied by any regular path Γ for any tree $\mathbb{T}$.

Letting formally $s = \pm \mathrm{i}\infty$ in eq. (2.19) yields

$$[I_{\mathbb{T}}(\Gamma)]_{tu} = [\delta \mathrm{Sk} I_{\mathbb{T}}]_{tu} - \sum_{v \in V(\mathbb{T}) \setminus \{0\}} [I_{L_v\mathbb{T}}(\Gamma)]_{tu} \, . \, [\mathrm{Sk} I_{R_v\mathbb{T}}(\Gamma)]_u. \tag{2.20}$$

which generalizes eq. (2.18). Conversely, eq. (2.20) implies the tree multiplicative property eq. (2.19), as shown in Lemma 2.10 below.

2.3 Regularization procedure

Definition 2.9 (regularization procedure for skeleton integrals) *Let $\tilde{\mathbb{T}} = \{v_1 < \ldots < v_{|\tilde{T}|}\}$ be a tree, $\mathbb{T} \subset \tilde{\mathbb{T}}$ a subtree, μ a compactly supported Borel measure on $\mathbb{R}^{\tilde{\mathbb{T}}}$ such that* $\mathrm{supp}\hat{\mu} \subset \{(\xi_1, \ldots, \xi_{|V(\tilde{T})|}) \mid |\xi_1| \leq \ldots \leq |\xi_{|V(\tilde{\mathbb{T}})|}\}$*, and $D_{reg} \subset \mathbb{R}^{\mathbb{T}}$ a Borel subset.*

*The (formal) D_{reg}-*regularized skeleton integral $\mathcal{R}\mathrm{Sk}\,\mathrm{I}_{\mathbb{T}}$ *is the linear mapping (see eq. (2.9))*

$$\mu \to [\mathcal{R}\mathrm{Sk}\,\mathrm{I}_{\mathbb{T}}(\mu)]_t = (2\pi)^{-|V(\mathbb{T})|/2} \langle \hat{\mu}, \mathbf{1}_{D_{reg}}(\xi) \, . \, \left[\mathrm{Sk}\,\mathrm{I}_{\mathbb{T}} \left((x_v)_{v \in V(\mathbb{T})} \to e^{\mathrm{i} \sum_{v \in V(\mathbb{T})} x_v \xi_v} \right) \right]_t \rangle \tag{2.21}$$

where $\hat{\mu}$ is the partial Fourier transform of μ with respect to $(x_v)_{v \in V(\mathbb{T})}$.

By assumption we shall only allow $D_{reg} = \mathbb{R}$ if $\mathbb{T}$ is a tree reduced to one vertex.

Lemma 2.10 (regularization) *Let $\mathbb{T} = \mathbb{T}_1 \dots \mathbb{T}_l$ be a forest, together with its tree decomposition. Define by induction on $|V(\mathbb{T})|$ the regularized integration operator $[\mathcal{R}I_{\mathbb{T}}]_{ts}$ by*

$$\prod_{j=1}^{l} \left\{ \left[\delta\mathcal{R}\mathrm{Sk}\,\mathrm{I}_{\mathbb{T}_j}\right]_{ts} - \sum_{\boldsymbol{v}\models V(\mathbb{T}_j)} \left[\mathcal{R}I_{L_{\boldsymbol{v}}\mathbb{T}_j}\right]_{ts} \left[\mathcal{R}\mathrm{Sk}\,\mathrm{I}_{R_{\boldsymbol{v}}\mathbb{T}_j}\right]_{s} \right\} \tag{2.22}$$

Then $[\mathcal{R}I_{\mathbb{T}}]_{ts}$ satisfies the following tree multiplicative property:

$$[\delta\mathcal{R}I_{\mathbb{T}}]_{tus} = \sum_{\boldsymbol{v}\models V(\mathbb{T})} [\mathcal{R}I_{L_{\boldsymbol{v}}\mathbb{T}}]_{tu} \ . \ [\mathcal{R}I_{R_{\boldsymbol{v}}\mathbb{T}}]_{us}\,. \tag{2.23}$$

By analogy with eq. (2.16, 2.17, 2.18), $\left[\delta\mathcal{R}\mathrm{Sk}\,\mathrm{I}_{\mathbb{T}_j}\right]_{ts}$, resp. $[\mathcal{R}I_{\mathbb{T}_j}(\partial)]_{ts} := -\sum_{\boldsymbol{v}\models V(\mathbb{T}_j)} \left[\mathcal{R}I_{L_{\boldsymbol{v}}\mathbb{T}_j}\right]_{ts} \left[\mathcal{R}\mathrm{Sk}\,\mathrm{I}_{R_{\boldsymbol{v}}\mathbb{T}_j}\right]_{s}$ may be called the increment*, resp.* boundary operators *associated to the tree $\mathbb{T}_j$.*

Remark. By Definition 2.9, the condition $[\mathcal{R}I_{\mathbb{T}}]_{ts} = [I_{\mathbb{T}}]_{ts}$ holds for a tree reduced to one vertex. This implies in the end that one has constructed a rough path over the *original* path Γ.

Proof. If the multiplicative property (2.23) holds for trees, then it holds automatically for forests since $[\mathcal{R}I_{\mathbb{T}_1\dots\mathbb{T}_l}]_{ts}$ is the product $\prod_{j=1}^{l}[\mathcal{R}I_{\mathbb{T}_j}]_{ts}$. Hence we may assume that $\mathbb{T}$ is a tree, say, with n vertices. Suppose (by induction) that the above multiplicative property (2.23) holds for all trees with $\leq n-1$ vertices. Then

$$\begin{aligned}
[\delta\mathcal{R}I_{\mathbb{T}}]_{tus} &= \sum_{\boldsymbol{v}\models V(\mathbb{T})} \left(-\left[\delta\mathcal{R}I_{L_{\boldsymbol{v}}\mathbb{T}}\right]_{tus} \left[\mathcal{R}\mathrm{Sk}\,\mathrm{I}_{R_{\boldsymbol{v}}\mathbb{T}}\right]_{s} + \left[\mathcal{R}I_{L_{\boldsymbol{v}}\mathbb{T}}\right]_{tu} \left[\delta\mathcal{R}\mathrm{Sk}\,\mathrm{I}_{R_{\boldsymbol{v}}\mathbb{T}}\right]_{us}\right) \\
&= \sum_{\boldsymbol{v}\models V(\mathbb{T})} \sum_{\boldsymbol{w}\models V(L_{\boldsymbol{v}}\mathbb{T})} \Big(-\left[\mathcal{R}I_{L_{\boldsymbol{w}}\circ L_{\boldsymbol{v}}(\mathbb{T})}\right]_{tu} \left[\mathcal{R}I_{R_{\boldsymbol{w}}\circ L_{\boldsymbol{v}}(\mathbb{T})}\right]_{us} \left[\mathcal{R}\mathrm{Sk}\,\mathrm{I}_{R_{\boldsymbol{v}}\mathbb{T}}\right]_{s} \\
&\qquad + \left[\mathcal{R}I_{L_{\boldsymbol{v}}\mathbb{T}}\right]_{tu} \left[\delta\mathcal{R}\mathrm{Sk}\,\mathrm{I}_{R_{\boldsymbol{v}}\mathbb{T}}\right]_{us}\Big)\,.
\end{aligned} \tag{2.24}$$

Let $\boldsymbol{x} = \boldsymbol{v} \amalg \boldsymbol{w} := \boldsymbol{v} \cup \boldsymbol{w} \setminus \{i \in \boldsymbol{v}\cup\boldsymbol{w} \mid \exists j \in \boldsymbol{v}\cup\boldsymbol{w} \mid i \twoheadrightarrow j\}$. Then one easily proves that $L_{\boldsymbol{w}} \circ L_{\boldsymbol{v}}(\mathbb{T}) = L_{\boldsymbol{x}}(\mathbb{T})$, $R_{\boldsymbol{v}}(\mathbb{T}) = R_{\boldsymbol{v}} \circ R_{\boldsymbol{x}}(\mathbb{T})$ and $R_{\boldsymbol{w}} \circ L_{\boldsymbol{v}}(\mathbb{T}) = L_{\boldsymbol{v}} \circ R_{\boldsymbol{x}}(\mathbb{T})$. Hence

$$\begin{aligned}
[\delta\mathcal{R}I_{\mathbb{T}}]_{tus} &= \sum_{\boldsymbol{x}\models V(\mathbb{T})} [\mathcal{R}I_{L_{\boldsymbol{x}}\mathbb{T}}]_{tu} \left(- \sum_{\boldsymbol{v}\models V(R_{\boldsymbol{x}}\mathbb{T})} [\mathcal{R}I_{L_{\boldsymbol{v}}(R_{\boldsymbol{x}}\mathbb{T})}]_{us} [\mathcal{R}\mathrm{Sk}\,\mathrm{I}_{R_{\boldsymbol{v}}(R_{\boldsymbol{x}}\mathbb{T})}]_{s} + [\delta\mathcal{R}\mathrm{Sk}\,\mathrm{I}_{R_{\boldsymbol{x}}\mathbb{T}}]_{us} \right) \\
&= \sum_{\boldsymbol{x}\models V(\mathbb{T})} [\mathcal{R}I_{L_{\boldsymbol{x}}\mathbb{T}}]_{tu} [\mathcal{R}I_{R_{\boldsymbol{x}}\mathbb{T}}]_{us}.
\end{aligned} \tag{2.25}$$

□

2.4 Permutation graphs

Consider now a permutation $\sigma \in \Sigma_n$. Applying Fubini's theorem yields

$$\begin{aligned} I_n^\ell(\Gamma) &= \int_s^t d\Gamma_{x_1}(\ell(1)) \int_s^{x_1} d\Gamma_{x_2}(\ell(2)) \dots \int_s^{x_{n-1}} d\Gamma_{x_n}(\ell(n)) \\ &= \int_{s_1}^{t_1} d\Gamma_{x_{\sigma(1)}}(\ell(\sigma(1))) \int_{s_2}^{t_2} d\Gamma_{x_{\sigma(2)}}(\ell(\sigma(2))) \dots \int_{s_n}^{t_n} d\Gamma_{x_{\sigma(n)}}(\ell(\sigma(n))), \end{aligned} \tag{2.26}$$

with $s_1 = s$, $t_1 = t$ and $s_j \in \{s\} \cup \{x_{\sigma(i)}, i < j\}$, $t_j \in \{t\} \cup \{x_{\sigma(i)}, i < j\}$ ($j \geq 2$). Now decompose $\int_{s_j}^{t_j} d\Gamma_{x_{\sigma(j)}}(\ell(\sigma(j)))$ into $\left(\int_s^{t_j} - \int_s^{s_j}\right) d\Gamma_{x_{\sigma(j)}}(\ell(\sigma(j)))$ if $s_j \neq s, t_j \neq t$, and $\int_{s_j}^{t} d\Gamma_{x_{\sigma(j)}}(\ell(\sigma(j)))$ into $\left(\int_s^{t} - \int_s^{s_j}\right) d\Gamma_{x_{\sigma(j)}}(\ell(\sigma(j)))$ if $s_j \neq s$. Then $I_n^\ell(\Gamma)$ has been rewritten as a sum of terms of the form

$$\pm \int_s^{\tau_1} d\Gamma_{x_1}(\ell(\sigma(1))) \int_s^{\tau_2} d\Gamma_{x_2}(\ell(\sigma(2))) \dots \int_s^{\tau_n} d\Gamma_{x_n}(\ell(\sigma(n))), \tag{2.27}$$

where $\tau_1 = t$ and $\tau_j \in \{t\} \cup \{x_i, i < j\}$, $j = 2, \dots, n$. Note the renaming of variables and vertices from eq. (2.26) to eq. (2.27). Encoding each of these expressions by the forest $\mathbb{T}$ with set of vertices $V(\mathbb{T}) = \{1, \dots, n\}$, label function $\ell \circ \sigma$, roots $\{j = 1, \dots, n \mid \tau_j = t\}$, and oriented edges $\{(j, j^-) \mid j = 2, \dots, n, \tau_j = x_{j^-}\}$, yields

$$I_n^\ell(\Gamma) = I_{\mathbb{T}^\sigma}(\Gamma) \tag{2.28}$$

for some $\mathbb{T}^\sigma \in \mathcal{T}$ called **permutation graph associated to** σ.

Summarizing:

Lemma 2.11 (permutation graphs) *To every permutation $\sigma \in \Sigma_n$ is associated a permutation graph*

$$\mathbb{T}^\sigma = \sum_{j=1}^{J_\sigma} g(\sigma, j) \mathbb{T}_j^\sigma \in \mathcal{T}, \tag{2.29}$$

$g(\sigma, j) = \pm 1$, each forest $\mathbb{T}_j^\sigma$ being provided by construction with a total ordering compatible with its tree structure, image of the ordering $\{v_1 < \dots < v_n\}$ of the trunk tree $\mathbb{T}^{\mathrm{Id}_n}$ by the permutation σ. The label function of $\mathbb{T}^\sigma$ is $\ell \circ \sigma$, where ℓ is the original label function of $\mathbb{T}^{\mathrm{Id}_n}$.

Example 2.12 *Let* $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$. *Then*

$$\int_s^t d\Gamma_{x_1}(\ell(1)) \int_s^{t_2} d\Gamma_{x_2}(\ell(2)) \int_s^{t_3} d\Gamma_{x_3}(\ell(3)) =$$
$$- \int_s^t d\Gamma_{x_2}(\ell(2)) \int_s^{x_2} d\Gamma_{x_3}(\ell(3)) \int_s^{x_2} d\Gamma_{x_1}(\ell(1))$$
$$+ \int_s^t d\Gamma_{x_2}(\ell(2)) \int_s^{x_2} d\Gamma_{x_3}(\ell(3)) \ . \ \int_s^t d\Gamma_{x_1}(\ell(1)). \quad (2.30)$$

Hence $\mathbb{T}^\sigma = -\mathbb{T}_1^\sigma + \mathbb{T}_2^\sigma$ *is the sum of a tree and of a forest with two components (see Fig. 4).*

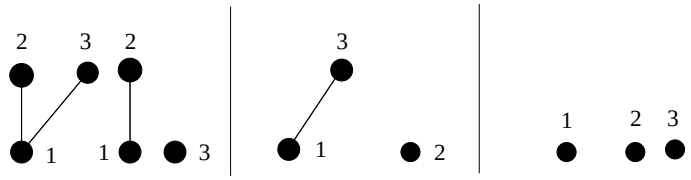

Figure 4: Example 2.12. From left to right: $\mathbb{T}_1^\sigma, \mathbb{T}_2^\sigma$; $L_{\{1\}}\mathbb{T}_1^\sigma \otimes R_{\{1\}}\mathbb{T}_1^\sigma$; $L_{\{1,2\}}\mathbb{T}_1^\sigma \otimes R_{\{1,2\}}\mathbb{T}_1^\sigma$

2.5 Fourier normal ordering algorithm

Let $\Gamma = (\Gamma(1), \ldots, \Gamma(d))$ be a compactly supported, smooth path, and $\mathbf{\Gamma}^n(i_1, \ldots, i_n)$ some iterated integral of Γ. To regularize $\mathbf{\Gamma}^n(i_1, \ldots, i_n)$, we shall apply the following algorithm (a priori formal, since skeleton integrals may be infra-red divergent) :

1. (Fourier projections) Split the measure $\mu = d\Gamma(i_1) \otimes \ldots \otimes d\Gamma(i_n)$ into $\sum_{\sigma \in \Sigma_n} \mathcal{F}^{-1}\left(\mathbf{1}_{D^\sigma}(\xi)\hat{\mu}(\xi)\right)$, where $D^\sigma = \{(\xi_1, \ldots, \xi_n) \in \mathbb{R}^n \mid |\xi_{\sigma(1)}| \leq \ldots \leq |\xi_{\sigma(n)}|\}$, and $\hat{\mu}$ is the Fourier transform of μ. We shall write

$$\mu^\sigma := \mathcal{F}^{-1}\left(\mathbf{1}_{D^\sigma}.\hat{\mu}\right) \circ \sigma = \mathcal{F}^{-1}\left(\mathbf{1}_{D^{\mathrm{Id}_n}}.(\hat{\mu} \circ \sigma)\right); \quad (2.31)$$

2. Rewrite $I_n^\ell\left(\mathcal{F}^{-1}(\mathbf{1}_{D^\sigma}.\hat{\mu})\right)$, where $\ell(j) = i_j$, as $I_{\mathbb{T}^\sigma}(\mu^\sigma) := \sum_{j=1}^{J_\sigma} g(\sigma, j) I_{\mathbb{T}_j^\sigma}(\mu^\sigma)$, where $\mathbb{T}^\sigma$ is the permutation graph defined in subsection 2.4;

3. Replace $I_{\mathbb{T}^\sigma}(\mu^\sigma)$ with some regularized integral as in Definition 2.9 and Lemma 2.10,

$$\mathcal{R}I_{\mathbb{T}^\sigma}(\mu^\sigma) := \sum_{j=1}^{J_\sigma} g(\sigma, j)\mathcal{R}I_{\mathbb{T}_j^\sigma}(\mu^\sigma); \tag{2.32}$$

4. Sum the terms corresponding to all possible permutations, yielding ultimately

$$\mathcal{R}\mathbf{\Gamma}^n(i_1, \ldots, i_n) = \sum_{\sigma \in \Sigma_n} \mathcal{R}I_{\mathbb{T}^\sigma}(\mu^\sigma). \tag{2.33}$$

Explicit formulas for $\Gamma = B^\eta$ may be found in the following section.

Theorem 2.1 *[36]*

$\mathcal{R}\mathbf{\Gamma}$ satisfies the multiplicative (ii) and geometric (iii) properties defined in the Introduction.

The proof given in [36] shows actually that *any* choice of linear maps $[\mathcal{R}\mathrm{Sk}I_{\mathbb{T}}]_t : \mu \to [\mathcal{R}\mathrm{Sk}I_{\mathbb{T}}(\mu)]_t$ such that

(i) $[\mathcal{R}\mathrm{Sk}I_{\mathbb{T}_1.\mathbb{T}_2}(\mu_1 \otimes \mu_2)]_t = [\mathcal{R}\mathrm{Sk}I_{\mathbb{T}_1}(\mu_1)]_t[\mathcal{R}\mathrm{Sk}I_{\mathbb{T}_2}(\mu_2)]_t$ and

(ii) $[\mathcal{R}\mathrm{Sk}I_{\mathbb{T}}(f)]_t = [\mathrm{Sk}I_{\mathbb{T}}(f)]_t = \int^t f(u)\ du$ if $\mathbb{T}$ is the trivial tree with one vertex,

yields a regularized rough path over Γ if Γ is *smooth*. Hence our 'cut' Fourier domain construction is arbitrary if convenient. As already said in the Introduction, it seems natural to look for some more restrictive rules for the regularization; iterated renormalization schemes (such as BPHZ or dimensional regularization) are obvious candidates (work in progress). The question is: is such or such regularization scheme better in any sense ? Contrary to the case of quantum field theory where all renormalization schemes may be implemented by local counterterms, which amount to a change of the value of the (finite number of) parameters in the functional integral (which are experimentally measurable), and give ultimately after resumming the perturbation series one and only one theory, we do not know of any *probabilistically motivated* reason to choose a particular regularization scheme here.

3 Rough path construction for fBm: case of distinct indices

The strategy is now to choose an appropriate regularization procedure, so that regularized skeleton integrals of B^η are finite and satisfy the uniform

Hölder and convergence rate estimates given in Theorem 0.1.

3.1 Analytic approximation of fBm

Recall B may be defined via the harmonizable representation [32]

$$B_t = c_\alpha \int_{\mathbb{R}} |\xi|^{\frac{1}{2}-\alpha} \frac{e^{\mathrm{i}t\xi} - 1}{\mathrm{i}\xi} \, W(d\xi) \tag{3.1}$$

where $(W_\xi, \xi \geq 0)$ is a complex Brownian motion extendeded to $\mathbb{R}$ by setting $W_{-\xi} = -\overline{W}_\xi$ $(\xi \geq 0)$, and $c_\alpha = \frac{1}{2}\sqrt{-\frac{\alpha}{\cos \pi\alpha \Gamma(-2\alpha)}}$.

We shall use the following approximation of B by a family of centered Gaussian processes $(B^\eta, \eta > 0)$ living in the first chaos of B.

Definition 3.1 (approximation B^η) *Let, for $\eta > 0$,*

$$B^\eta_t = c_\alpha \int_{\mathbb{R}} e^{-\eta|\xi|} |\xi|^{\frac{1}{2}-\alpha} \frac{e^{\mathrm{i}t\xi} - 1}{\mathrm{i}\xi} \, W(d\xi). \tag{3.2}$$

The process B^η is easily seen to have a.s. smooth paths. The infinitesimal covariance $\mathbb{E}(B^\eta)'_s(B^\eta)'_t$ may be computed explicitly using the Fourier transform [11]

$$\mathcal{F}K^{',-}_\eta(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} K^{',-}_\eta(x) e^{-\mathrm{i}x\xi} dx = -\frac{\pi\alpha}{2\cos\pi\alpha\Gamma(-2\alpha)} e^{-2\eta|\xi|} |\xi|^{1-2\alpha} \mathbf{1}_{|\xi|>0}, \tag{3.3}$$

where $K^{',-}_\eta(s-t) := \frac{\alpha(1-2\alpha)}{2\cos\pi\alpha}(-\mathrm{i}(s-t) + 2\eta)^{2\alpha-2}$. By taking the real part of these expressions, one finds that B^η has the same law as the analytic approximation of B defined in [34], namely, $B^\eta = \Gamma_{t+\mathrm{i}\eta} + \Gamma_{t-\mathrm{i}\eta} = 2\mathrm{Re}\ \Gamma_{t+\mathrm{i}\eta}$, where Γ is the analytic fractional Brownian motion (see also [33]).

3.2 Choice of the regularization procedure

Let $\sigma \in \Sigma_n$ be a permutation. Recall (see Lemma 2.11) that the permutation graph $\mathbb{T}^\sigma$ may be written as a finite sum $\sum_{j=1}^{J_\sigma} g(\sigma, j)\mathbb{T}^\sigma_j$, where each $\mathbb{T}^\sigma_j$ is a forest which is automatically provided with a total ordering. In the two following subsections, we shall consider regularized tree or skeleton integrals, $\mathcal{R}I_\mathbb{T}$ or $\mathcal{R}\mathrm{Sk}I_\mathbb{T}$, for a forest $\mathbb{T}$ which is one of the $\mathbb{T}^\sigma_j$.

Definition 3.2 *Fix $C_{reg} \in (0,1)$. Let, for $\mathbb{T}$ with set of vertices $V(\mathbb{T}) = \{v_1 < \ldots < v_j\}$,*

$$\mathbb{R}^\mathbb{T}_+ := \big\{(\xi_{v_1}, \ldots, \xi_{v_j}) \in \mathbb{R}^\mathbb{T} \mid |\xi_{v_1}| \leq \ldots \leq |\xi_{v_j}|\big\}, \tag{3.4}$$

$$\mathbb{R}^{\mathbb{T}}_{reg} := \big\{(\xi_{v_1},\dots,\xi_{v_j}) \in \mathbb{R}^{\mathbb{T}}_{+} \mid \forall v \in V(\mathbb{T}), |\xi_v + \sum_{w \twoheadrightarrow v} \xi_w| > C_{reg} \max\{|\xi_w|;\ w \twoheadrightarrow v\}\ \big\}, \tag{3.5}$$

and $\mathcal{R}I_{\mathbb{T}}$, resp. $\mathcal{R}\mathrm{Sk}\,\mathrm{I}_{\mathbb{T}}$ be the corresponding $\mathbb{R}^{\mathbb{T}}_{reg}$-regularized iterated, resp. skeleton integrals as in subsection 2.3.

Condition (3.5) ensures that the denominators in the skeleton integrals are not too small (see Lemma 2.7).

The following Lemma (close to arguments used in the study of random Fourier series [20]) is fundamental for the estimates of the following subsections.

Lemma 3.3 *(i) Let $F(u) = \int_{\mathbb{R}} dW_\xi a(\xi) e^{\mathrm{i}u\xi}$, where $|a(\xi)|^2 \le C|\xi|^{-1-2\beta}$ for some $0 < \beta < 1$: then, for every $u_1, u_2 \in \mathbb{R}$,*

$$\mathbb{E}|F(u_1) - F(u_2)|^2 \le C'|u_1 - u_2|^{2\beta}. \tag{3.6}$$

(ii) Let $\tilde{F}(\eta) = \int_{\mathbb{R}} dW_\xi a(\xi) e^{-\eta|\xi|}$ $(\eta > 0)$, where $|a(\xi)|^2 \le C|\xi|^{-1-2\beta}$ for some $0 < \beta < 1$: then, for every $\eta_1, \eta_2 \in \mathbb{R}_+$,

$$\mathbb{E}|\tilde{F}(\eta_1) - \tilde{F}(\eta_2)|^2 \le C'|\eta_1 - \eta_2|^{2\beta}. \tag{3.7}$$

Proof. Bound $|e^{\mathrm{i}u_1\xi} - e^{\mathrm{i}u_2\xi}|$ by $|u_1 - u_2||\xi|$ for $|\xi| \le \frac{1}{|u_1 - u_2|}$ and by 2 otherwise, and similarly for $|e^{-\eta_1|\xi|} - e^{-\eta_2|\xi|}|$. Note the variance integral is infra-red convergent near $\xi = 0$. □

Remark: Unless $|a(\xi)|^2$ is L^1_{loc} near $\xi = 0$, only the *increments* $F(u_1) - F(u_2)$, $\tilde{F}(\eta_1) - \tilde{F}(\eta_2)$ are well-defined.

3.3 Estimates for the increment term

In this paragraph, as in the next one, we consider regularized tree integrals associated to $\mathcal{R}\mathbf{B}^{n,\eta}(i_1,\dots,i_n)$ where $i_1 \ne \dots \ne i_n$ are distinct indices, so that $B(i_1),\dots,B(i_n)$ are *independent*.

Lemma 3.4 (Hölder estimate and rate of convergence) *Let $\mathbb{T} = \mathbb{T}^\sigma_j$ for some j, and $\alpha < 1/|V(\mathbb{T})|$.*

1. *The skeleton term*

$$[G^{\eta,\sigma}_{\mathbb{T}}(i_1,\dots,i_n)]_u = [\mathcal{R}\mathrm{Sk}\,\mathrm{I}_{\mathbb{T}}\left((dB^\eta(i_1) \otimes \dots \otimes dB^\eta(i_n))^\sigma\right)]_u \tag{3.8}$$

(see eq. (2.31)) writes

$$[G^{\eta,\sigma}_{\mathbb{T}}(i_1,\dots,i_n)]_u = (-\mathrm{i}c_\alpha)^{|V(\mathbb{T})|}\int\dots\int_{(\xi_v)_{v\in V(\mathbb{T})}\in\mathbb{R}^{\mathbb{T}}_{reg}}\prod_{v\in V(\mathbb{T})} dW_{\xi_v}(i_{\sigma(v)})$$
$$e^{\mathrm{i}u\sum_{v\in V(\mathbb{T})}\xi_v}e^{-\eta\sum_{v\in V(\mathbb{T})}|\xi_v|}\frac{\prod_{v\in V(\mathbb{T})}|\xi_v|^{\frac{1}{2}-\alpha}}{\prod_{v\in V(\mathbb{T})}\left[\xi_v+\sum_{w\twoheadrightarrow v}\xi_w\right]}. \quad (3.9)$$

2. *It satisfies the uniform Hölder estimate:*

$$\mathbb{E}\left|[\delta G^{\eta,\sigma}_{\mathbb{T}}(i_1,\dots,i_n)]_{ts}\right|^2 \le C|t-s|^{2\alpha|V(\mathbb{T})|}. \quad (3.10)$$

3. *(rate of convergence) : there exists a constant $C>0$ such that, for every $\eta_1,\eta_2>0$ and $s,t\in\mathbb{R}$,*

$$\mathbb{E}\left|[\delta G^{\eta_1,\sigma}_{\mathbb{T}}(i_1,\dots,i_n)]_{ts}-[\delta G^{\eta_2,\sigma}_{\mathbb{T}}(i_1,\dots,i_n)]_{ts}\right|^2 \le C|\eta_1-\eta_2|^{2\alpha}. \quad (3.11)$$

Proof.

1. Follows from Lemma 2.7 and the definitions of B^η and of regularized integrals in the previous subsections 2.3 and 3.1.

2. (Hölder estimate)

   One may just as well (by multiplying the integral estimates on each tree component) assume $\mathbb{T}$ is a tree, i.e. $\mathbb{T}$ is connected.

   Let $V(\mathbb{T})=\{v_1<\dots<v_{|V(\mathbb{T})|}\}$, so that $|\xi_{v_1}|\le\dots\le|\xi_{v_{|V(\mathbb{T})|}}|$. Since every vertex $v\in V(\mathbb{T})\setminus\{v_1\}$ connects to the root v_1, one has

$$|V(\mathbb{T})|\,.\,|\xi_{v_{|V(\mathbb{T})|}}| \ge |\xi_{v_1}+\dots+\xi_{v_{|V(\mathbb{T})|}}| > C_{reg}|\xi_{v_{|V(\mathbb{T})|}}|, \quad (3.12)$$

   so that $\xi:=\sum_{v\in V(\mathbb{T})}\xi_v$ is comparable to $\xi_{v_{|V(\mathbb{T})|}}$, i.e. belongs to $[C^{-1}\xi_{v_{|V(\mathbb{T})|}},C\xi_{v_{|V(\mathbb{T})|}}]$ if C is some large enough positive constant. Write $[G^{\eta,\sigma}_{\mathbb{T}}(i_1,\dots,i_n)]_u=\int_{\mathbb{R}}e^{\mathrm{i}u\xi}a(\xi)d\xi$.

   Vertices at which 2 or more branches join are called *nodes*, and vertices to which no vertex is connected are called *leaves* (see Fig. 5).

   The set $Br(v_1\twoheadrightarrow v_2)$ of vertices from a leaf or a node v_1 to a node v_2 (or to the root) is called a *branch* if it does not contain any other node. By convention, $Br(v_1\twoheadrightarrow v_2)$ includes v_1 and excludes v_2.

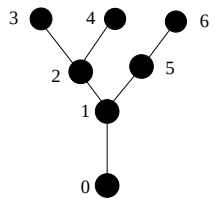

Figure 5: 3,4,6 are leaves; 1, 2 and 5 are nodes, 2 and 5 are uppermost; branches are e.g. $Br(2 \twoheadrightarrow 1)$ or $Br(6 \twoheadrightarrow 1)$.

Consider an uppermost node n, i.e. a node to which no other node is connected, together with the set of leaves $\{w_1 < \ldots < w_J\}$ above n. Let $p_j = |V(Br(w_j \twoheadrightarrow n))|$. Note that $\left(\frac{|\xi_n|^{\frac{1}{2}-\alpha}}{\xi_n + \sum_{w \twoheadrightarrow n} \xi_w}\right)^2 \lesssim |\xi_{w_J}|^{-1-2\alpha}$. Now we proceed to estimate Var $a(\xi)$. On the branch number j from w_j to n,

$$\int \ldots \int_{|\xi_v| \leq |\xi_{w_j}|, v \in Br(w_j \twoheadrightarrow n) \setminus \{w_j\}} \left[\prod_{v \in Br(w_j \twoheadrightarrow n)} \frac{e^{-\eta|\xi_v|} |\xi_v|^{\frac{1}{2}-\alpha}}{\xi_v + \sum_{w \twoheadrightarrow v} \xi_w} \right]^2 \lesssim |\xi_{w_j}|^{-1-2\alpha p_j} \tag{3.13}$$

and (summing over $\xi_{w_1}, \ldots, \xi_{w_{J-1}}$ and over ξ_n)

$$\begin{aligned} &|\xi_{w_J}|^{-1-2\alpha p_J} \int_{|\xi_{w_{J-1}}| \leq |\xi_{w_J}|} d\xi_{w_{J-1}} |\xi_{w_{J-1}}|^{-1-2\alpha p_{J-1}} \\ &\quad \left(\ldots \left(\int_{|\xi_{w_1}| \leq |\xi_{w_2}|} d\xi_{w_1} |\xi_{w_1}|^{-1-2\alpha p_1} \left(\int_{|\xi_n| \leq |\xi_{w_1}|} d\xi_n \frac{|\xi_n|^{1-2\alpha}}{\xi_{w_J}^2} \right) \right) \ldots \right) \\ &\lesssim |\xi_{w_J}|^{-(1+2\alpha p_j) + [2-2\alpha(1+p_1+\ldots+p_{J-1})]-2} = |\xi_{w_J}|^{-1-2\alpha W(n)}, \end{aligned} \tag{3.14}$$

where $W(n) = p_1 + \ldots + p_J + 1 = |\{v : v \twoheadrightarrow n\}| + 1$ is the *weight* of n. One may then consider the reduced tree $\mathbb{T}_n$ obtained by shrinking all vertices above n (including n) to *one* vertex with weight $W(n)$ and perform the same operations on $\mathbb{T}_n$. Repeat this inductively until $\mathbb{T}$ is shrunk to one point. In the end, one gets Var $a(\xi) \lesssim |\xi_{v_{|V(\mathbb{T})|}}|^{-1-2\alpha|V(\mathbb{T})|} \lesssim |\xi|^{-1-2\alpha|V(\mathbb{T})|}$. Now apply Lemma 3.3 (i).

3. (rate of convergence)

Let $X_u^{\eta_1,\eta_2} := [G_{\mathbb{T}}^{\eta_1,\sigma}(i_1,\dots,i_n)]_u - [G_{\mathbb{T}}^{\eta_2,\sigma}(i_1,\dots,i_n)]_u$. Expanding $\prod_{j=1}^{|V(\mathbb{T})|} e^{-\eta_1|\xi_j|} - \prod_{j=1}^{|V(\mathbb{T})|} e^{-\eta_2|\xi_j|}$ as

$$\sum_{j=1}^{|V(\mathbb{T})|} e^{-\eta_2(|\xi_{v_1}|+\dots+|\xi_{v_{j-1}}|)}(e^{-\eta_1|\xi_{v_j}|} - e^{-\eta_2|\xi_{v_j}|})e^{-\eta_1(|\xi_{v_{j+1}}|+\dots+|\xi_{v_{V(\mathbb{T})}}|)}$$

gives $X_u^{\eta_1,\eta_2}$ as a sum, $X_u^{\eta_1,\eta_2} = \sum_{v\in V(\mathbb{T})} X_u^{\eta_1,\eta_2}(v)$, where $X_u^{\eta_1,\eta_2}(v) = \int d\xi_v b_u(\xi_v)(e^{-\eta_1|\xi_v|} - e^{-\eta_2|\xi_v|})$ is obtained from $[G_{\mathbb{T}}^{\eta,\sigma}(i_1,\dots,i_n)]_u$ by replacing $e^{-\eta|\xi_v|}$ with $e^{-\eta_1|\xi_v|} - e^{-\eta_2|\xi_v|}$, and $e^{-\eta|\xi_w|}$, $w \neq v$ either by $e^{-\eta_1|\xi_w|}$ or by $e^{-\eta_2|\xi_w|}$. We want to estimate Var $b_u(\xi_v)$ uniformly in u.

Fix the value of ξ_v in the computations in the above proof for the Hölder estimate. Let w_J be the maximal leaf above v, and $n \twoheadrightarrow v$ be the node just above v if v is not a node, $n = v$ otherwise. Summing over all nodes above n and taking the variance leads to an expression bounded by $|\xi_{w_J}|^{-1-2\alpha W(n)}$, where $W(n) = |\{w \ : \ w \twoheadrightarrow n\}| + 1$ is as before the weight of n. Consider now the corresponding shrunk tree $\mathbb{T}_n$. Let $\mathbb{T}_n(v)$ be the trunk tree defined by $\mathbb{T}_n(v) = \{w \in \mathbb{T}_n : w \twoheadrightarrow v \text{ or } v \twoheadrightarrow w\} \cup \{v\}$; similarly, let $\mathbb{T}(v)$ be the tree defined by $\mathbb{T}(v) = \{w \in \mathbb{T} : w \twoheadrightarrow v \text{ or } v \twoheadrightarrow w\} \cup \{v\}$, so that $\mathbb{T}_n(v)$ is the corresponding shrunk tree. Sum over all vertices $w \in \mathbb{T}_n(v)\setminus\{v\}$. The variance of the coefficient of $e^{-\eta_1|\xi_v|}$ is

$$\begin{aligned} S(\xi_v) &\lesssim \int_{|\xi_n|\geq|\xi_v|} d\xi_n |\xi_n|^{-1-2\alpha W(n)} |\xi_n|^{-1-2\alpha} \int_{|\xi_w|\leq|\xi_n|, w\in\mathbb{T}_n(v)\setminus\{n,v\}} \\ &\qquad \left[\prod_{w\in\mathbb{T}_n(v)\setminus\{n,v\}} d\xi_w \ . \ |\xi_n|^{-(1+2\alpha)}\right] \\ &\lesssim \int_{|\xi_n|\geq|\xi_v|} d\xi_n |\xi_n|^{-2-2\alpha|\mathbb{T}(v)|} \lesssim |\xi_v|^{-1-2\alpha|\mathbb{T}(v)|} \end{aligned} \tag{3.15}$$

if $v \neq n$, and

$$S(\xi_v) \lesssim |\xi_n|^{-1-2\alpha W(n)} \int_{|\xi_w|\leq|\xi_n|, w\in\mathbb{T}_n(v)\setminus\{n\}} \prod_{w\in\mathbb{T}_n(v)\setminus\{n\}} |\xi_n|^{-(1+2\alpha)} \lesssim |\xi_v|^{-1-2\alpha|\mathbb{T}(v)|} \tag{3.16}$$

if $v = n$.

Removing the vertices belonging to $\mathbb{T}(v)$ from $\mathbb{T}$ leads to a forest which gives a finite contribution to the variance. Hence (by Lemma 3.3 (ii)) $\mathbb{E}|X_u^{\eta_1,\eta_2}(v)|^2 \lesssim |\eta_1 - \eta_2|^{2\alpha|\mathbb{T}(v)|}$.

□

The notion of *weight* $W(v)$ *of a vertex* v introduced in this proof will be used again in subsections 3.4 and 4.1.

3.4 Estimates for boundary terms

Let $\mathbb{T} = \mathbb{T}_j^\sigma$ for some $\sigma \in \Sigma_n$, and $i_1 \neq \ldots \neq i_n$ as in the previous subsection. By multiplying the estimates on each tree component, one may just as well assume $\mathbb{T}$ is a tree, i.e. is connected.

We shall now prove estimates for the boundary term $\mathcal{R}I_{\mathbb{T}}\left((dB^\eta(i_1)\otimes\ldots\otimes dB^\eta(i_n))^\sigma\right)(\partial)$ associated to $\mathbb{T}$ (see Lemma 2.10).

Lemma 3.5 *Let* $\mathbb{T} = \mathbb{T}_j^\sigma$ *for some* j *(so that* $n = |V(\mathbb{T})|$*).*

1. *(Hölder estimate) The regularized boundary term* $\mathcal{R}I_{\mathbb{T}}\left((dB^\eta(i_1)\otimes\ldots\otimes dB^\eta(i_n))^\sigma\right)(\partial)$ *satisfies:*

$$\mathbb{E}\left|\left[\mathcal{R}I_{\mathbb{T}}\left((dB^\eta(i_1)\otimes\ldots\otimes dB^\eta(i_n))^\sigma\right)(\partial)\right]_{ts}\right|^2 \leq C|t-s|^{2\alpha|V(\mathbb{T})|} \quad (3.17)$$

   for a certain constant C*.*

2. *(rate of convergence) There exists a positive constant* C *such that, for every* $\eta_1, \eta_2 > 0$*,*

$$\begin{aligned}\mathbb{E}\left|\left[\mathcal{R}I_{\mathbb{T}}\left((dB^{\eta_1}(i_1)\otimes\ldots\otimes dB^{\eta_1}(i_n))^\sigma\right)(\partial)\right]_{ts} - \right.\\ \left. -\left[\mathcal{R}I_{\mathbb{T}}\left((dB^{\eta_2}(i_1)\otimes\ldots\otimes dB^{\eta_2}(i_n))^\sigma\right)(\partial)\right]_{ts}\right|^2 \leq C|\eta_1-\eta_2|^{2\alpha}.\end{aligned} \quad (3.18)$$

Proof.

1. Apply repeatedly Lemma 2.10 to $\mathbb{T}$: in the end, $\left[\mathcal{R}I_{\mathbb{T}}\left((dB^\eta(i_1)\otimes\ldots\otimes dB^\eta(i_n))^\sigma\right)(\partial)\right]_{ts}$ appears as a sum of 'skeleton-type' terms of the form (see Figure 6)

$$\begin{aligned}A_{ts} := [\delta\mathcal{R}\mathrm{Sk}\,\mathrm{I}_{L\mathbb{T}}]_{ts} \,.\, [\mathcal{R}\mathrm{Sk}\,\mathrm{I}_{R_{\boldsymbol{v}_l}\circ L_{\boldsymbol{v}_{l-1}}\circ\ldots\circ L_{\boldsymbol{v}_1}(\mathbb{T})}]_s \ldots [\mathcal{R}\mathrm{Sk}\,\mathrm{I}_{R_{\boldsymbol{v}_2}\circ L_{\boldsymbol{v}_1}(\mathbb{T})}]_s [\mathcal{R}\mathrm{Sk}\,\mathrm{I}_{R_{\boldsymbol{v}_1}\mathbb{T}}]_s \\ \left((dB^\eta(i_1)\otimes\ldots\otimes dB^\eta(i_n))^\sigma\right),\end{aligned} \quad (3.19)$$

   where $\boldsymbol{v}_1 = (v_{1,1} < \ldots < v_{1,J_1}) \models \mathbb{T}$, $\boldsymbol{v}_2 \models L_{\boldsymbol{v}_1}\mathbb{T}$, …, $\boldsymbol{v}_l = (v_{l,1} < \ldots < v_{l,J_l}) \models L_{\boldsymbol{v}_{l-1}} \circ \ldots \circ L_{\boldsymbol{v}_1}(\mathbb{T})$ and $L\mathbb{T} := L_{\boldsymbol{v}_l} \circ \ldots \circ L_{\boldsymbol{v}_1}(\mathbb{T})$. In eq. (3.19) the forest $\mathbb{T}$ has been split into a number of sub-forests,

$L\mathbb{T} \cup \left(\cup_{j=1}^{J} \mathbb{T}_j \right)$; we call this splitting *the splitting associated to* A_{ts} for further reference.

First step.

Let $[B_s^{\boldsymbol{v}_1}[\boldsymbol{\xi}]]_u \prod_{j=1}^{J_1} dW_{\xi_{v_1,j}}(i_{\sigma(v_{1,j})})$ be the contribution to $\mathcal{R}\mathrm{Sk}\, \mathrm{I}_{R_{\boldsymbol{v}_1}\mathbb{T}}$ of all Fourier components such that $\boldsymbol{\xi} = (\xi_{v_{1,1}}, \dots, \xi_{v_{1,J_1}})$, $|\xi_{v_{1,1}}| \le \dots \le |\xi_{v_{1,J_1}}|$ is fixed. For definiteness (see Definition 3.2),

$$\begin{aligned}
&\left[\mathcal{R}\mathrm{Sk}\, \mathrm{I}_{R_{\boldsymbol{v}_1}\mathbb{T}} \left((dB^\eta(i_1) \otimes \dots \otimes dB^\eta(i_n))^\sigma \right) \right]_u ((x_v)_{v \in L_{\boldsymbol{v}_1}\mathbb{T}}) \\
&= \int \dots \int \mathbf{1}_{\mathbb{R}_+^{V(L_{\boldsymbol{v}_1}\mathbb{T}) \cup \boldsymbol{v}_1}} \left((\xi_v)_{v \in V(L_{\boldsymbol{v}_1}\mathbb{T}) \cup \boldsymbol{v}_1} \right) \left[B_s^{\boldsymbol{v}_1}[\boldsymbol{\xi}] \prod_{j=1}^{J_1} dW_{\xi_{v_1,j}}(i_{\sigma(v_{1,j})}) \right] \cdot \\
&\quad \cdot \left[\prod_{v \in L_{\boldsymbol{v}_1}\mathbb{T}} c_\alpha e^{-\eta|\xi_v|} e^{\mathrm{i} x_v \xi_v} |\xi_v|^{\frac{1}{2}-\alpha} dW_{\xi_v}(i_{\sigma(v)}) \right]. \qquad (3.20)
\end{aligned}$$

Then

$$\mathrm{Var}[B_s^{\boldsymbol{v}_1}[\boldsymbol{\xi}]]_s \lesssim \int \dots \int \prod_{v \in \boldsymbol{v}_1} d\xi_v \left[|\xi_v|^{-1-2\alpha} \int \dots \int_{|\xi_w| \ge |\xi_v|, w \in R_v\mathbb{T} \setminus \{v\}} \prod_{w \in R_v\mathbb{T} \setminus \{v\}} |\xi_w|^{-1-2\alpha} \right], \qquad (3.21)$$

hence

$$\mathrm{Var}[B_s^{\boldsymbol{v}_1}[\boldsymbol{\xi}]]_s \lesssim \prod_{v \in \boldsymbol{v}_1} |\xi_v|^{-2|V(R_v\mathbb{T})|\alpha - 1}. \qquad (3.22)$$

Second step.

More generally, let $B_s^{\boldsymbol{v}_1,\dots,\boldsymbol{v}_l}[\boldsymbol{\xi}] \prod_{j=1}^{J_l} dW_{\xi_{v_l,j}}(i_{\sigma(v_{l,j})})$ be the contribution to

$$[\mathcal{R}\mathrm{Sk}\, \mathrm{I}_{R_{\boldsymbol{v}_l} \circ L_{\boldsymbol{v}_{l-1}} \circ \dots \circ L_{\boldsymbol{v}_1}(\mathbb{T})}]_s \dots [\mathcal{R}\mathrm{Sk}\, \mathrm{I}_{R_{\boldsymbol{v}_2} \circ L_{\boldsymbol{v}_1}(\mathbb{T})}]_s [\mathcal{R}\mathrm{Sk}\, \mathrm{I}_{R_{\boldsymbol{v}_1}\mathbb{T}}]_s \left((dB^\eta(i_1) \otimes \dots \otimes dB^\eta(i_n))^\sigma \right) \qquad (3.23)$$

of all Fourier components such that $\boldsymbol{\xi} = (\xi_{v_{l,1}}, \dots, \xi_{v_{l,J_l}})$ is fixed. Then

$$\mathrm{Var}(B_s^{\boldsymbol{v}_1,\dots,\boldsymbol{v}_l}[\boldsymbol{\xi}]) \lesssim \prod_{v \in \boldsymbol{v}_l} |\xi_v|^{-2|V(R_v\mathbb{T})|\alpha - 1} \qquad (3.24)$$

(proof by induction on l).

Third step.

Let $V(L\mathbb{T}) = \{w_1 < \ldots < w_{max}\}$. By definition, $A_{ts} = \int_{\mathbb{R}} a_s(\Xi)(e^{\mathrm{i}\Xi t} - e^{\mathrm{i}\Xi s})d\Xi$, with

$$a_s(\Xi) = \int d\boldsymbol{\xi} \int \ldots \int_{((\xi_w)_{w\in V(L\mathbb{T})})\in D_{\boldsymbol{\xi}}} \prod_{w\in V(L\mathbb{T})} dW_{\xi_w}(i_{\sigma(w)}) \\ \frac{\prod_{w\in V(L\mathbb{T})}(-\mathrm{i}c_\alpha)e^{-\eta|\xi_w|}|\xi_w|^{\frac{1}{2}-\alpha}}{\prod_{w\in V(L\mathbb{T})}(\xi_w + \sum_{w'\twoheadrightarrow w, w'\in V(L\mathbb{T})}\xi_{w'})} B_s^{\boldsymbol{v}_1,\ldots,\boldsymbol{v}_l}[\boldsymbol{\xi}] \tag{3.25}$$

where Fourier components in $D_{\boldsymbol{\xi}}$ satisfy in particular the following conditions:

- $|\xi_w + \sum_{w'\twoheadrightarrow w, w'\in V(L\mathbb{T})}\xi_{w'}| > C_{reg}\max\{|\xi_{w'}| : w' \twoheadrightarrow w, w' \in V(L\mathbb{T})\}$; in particular, $\left(\frac{|\xi_w|^{\frac{1}{2}-\alpha}}{\xi_w+\sum_{w'\twoheadrightarrow w, w'\in V(L\mathbb{T})}\xi_{w'}}\right)^2 \lesssim |\xi_w|^{-1-2\alpha}$;
- $\sum_{w\in V(L\mathbb{T})}\xi_w = \Xi$;
- for every $w \in V(L\mathbb{T})$, $|\xi_w| \leq |\xi_{w_{max}}|$ and $|\xi_w| \leq |\xi_v|$ for every $v \in R(w) := \{v = v_{l,1}, \ldots, v_{l,J_l} \mid v \to w\}$ (note that $R(w)$ may be empty). See Fig. 6.

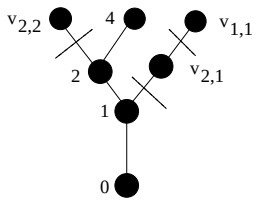

Figure 6: Here $V(L\mathbb{T}) = \{0, 1, 2, 4\}$, $R(0) = R(4) = \emptyset$, $R(1) = \{v_{2,1}\}$, $R(2) = \{v_{2,2}\}$.

Note that $|\Xi| \lesssim |\xi_{w_{max}}| \lesssim |\Xi|$ since every vertex in $V(L\mathbb{T})$ connects to the root (see first lines of the proof of Lemma 3.4 (2)).

If $w \in L\mathbb{T}$, split $R(w)$ into $R(w)_> \cup R(w)_<$, where $R(w)_{\gtrless} := \{v \in R(w) \mid v \gtrless w_{max}\}$. Summing over indices corresponding to vertices in $R\mathbb{T}_> := \{v = v_{l,1}, \ldots, v_{l,J_l} \mid v > w_{max}\} = \cup_{w\in L\mathbb{T}}R(w)_>$, one gets (see again proof of Lemma 3.4 (2))

$$\prod_{v\in R\mathbb{T}_>}\int_{|\xi_v|\geq|\Xi|} d\xi_v|\xi_v|^{-2|V(R_v\mathbb{T})|\alpha-1} \lesssim |\Xi|^{-2\alpha\sum_{v\in R\mathbb{T}_>}|V(R_v\mathbb{T})|}. \tag{3.26}$$

Let $w \in L\mathbb{T} \setminus \{w_{max}\}$ such that $R(w)_< \neq \emptyset$ (note that $R(w_{max})_< = \emptyset$). Let $R(w)_< = \{v_{i_1} < \ldots < v_{i_j}\}$. Then (integrating over $(\xi_v), v \in R(w)_<$)

$$|\xi_w|^{-1-2\alpha} \int_{|\xi_{v_{i_1}}| \geq |\xi_w|} d\xi_{v_{i_1}} \int_{|\xi_{v_{i_2}}| \geq |\xi_{v_{i_1}}|} d\xi_{v_{i_2}} \ldots \int_{|\xi_{v_{i_j}}| \geq |\xi_{v_{i_{j-1}}}|} d\xi_{v_{i_j}}$$
$$|\xi_{v_{i_1}}|^{-2|V(R_{v_{i_1}}\mathbb{T})|\alpha-1} \ldots |\xi_{v_{i_j}}|^{-2|V(R_{v_{i_j}}\mathbb{T})|\alpha-1} \lesssim |\xi_w|^{-1-2\alpha(1+\sum_{v\in R(w)_<}|V(R_v\mathbb{T})|)}. \tag{3.27}$$

In other words, each vertex $w \in L\mathbb{T}$ 'behaves' as if it had a weight $1+\sum_{v\in R(w)_<}|V(R_v\mathbb{T})|$. Hence (by the same method as in the proof of Lemma 3.4 (2)) $\mathrm{Var}(a_s(\xi)) \lesssim |\Xi|^{-1-2\alpha(|V(L\mathbb{T})|+\sum_{v\in R\mathbb{T}_<}|V(R_v\mathbb{T})|)} \cdot |\Xi|^{-2\alpha\sum_{v\in R\mathbb{T}_>}|V(R_v\mathbb{T})|} = |\Xi|^{-1-2\alpha|V(\mathbb{T})|}$. Now apply Lemma 3.3 (i).

2. Similar to the proof of Lemma 3.4 (3). Details are left to the reader.

□

4 End of proof and final remarks

4.1 Estimates: case of coinciding indices

Our previous estimates for $\mathbb{E}|\mathcal{R}\mathbf{B}^{n,\eta}_{ts}(i_1,\ldots,i_n)|^2$ (Hölder estimate) and $\mathbb{E}|\mathcal{R}\mathbf{B}^{n,\eta_1}_{ts}(i_1,\ldots,i_n) - \mathcal{R}\mathbf{B}^{n,\eta_2}_{ts}(i_1,\ldots,i_n)|^2$ (rate of convergence) with $i_1 \neq \ldots \neq i_n$ rest on the independence of the Brownian motions $W(i_1),\ldots,W(i_n)$. We claim that the same estimates also hold true for $\mathbb{E}|\mathcal{R}\mathbf{B}^{n,\eta}(i_1,\ldots,i_n)|^2$ and $\mathbb{E}|\mathcal{R}\mathbf{B}^{n,\eta_1}_{ts}(i_1,\ldots,i_n) - \mathcal{R}\mathbf{B}^{n,\eta_2}_{ts}(i_1,\ldots,i_n)|^2$ if some of the indices $(i_1,\ldots,i_n)$ coincide, with the *same definition* of the regularization procedure $\mathcal{R}$. The key Lemma for the proof is

Lemma 4.1 (Wick's lemma) *(see [22], §5.1.2 and 9.3.4)*

Let $(X_1,\ldots,X_n)$ be a centered Gaussian vector. Denote by $X_{i_1}\diamond\ldots\diamond X_{i_k}$ $(1 \leq i_1,\ldots,i_k \leq n)$ or $:X_{i_1}\ldots X_{i_k}:$ the Wick product of $X_{i_1},\ldots,X_{i_k}$ (also called: normal ordering of the product $X_{i_1}\ldots X_{i_k}$), i.e. the projection of the product $X_{i_1}\ldots X_{i_k}$ onto the k-th chaos of the Gaussian space generated by $X_1,\ldots,X_n$. Then:

1.

$$X_1 \dots X_n = X_1 \diamond \dots \diamond X_n + \sum_{(i_1,i_2)} \mathbb{E}[X_{i_1} X_{i_2}] X_1 \diamond \dots \diamond \check{X}_{i_1} \diamond \dots \diamond \check{X}_{i_2} \diamond \dots \diamond X_n$$
$$+ \dots + \sum_{(i_1,i_2),\dots,(i_{2k+1},i_{2k+2})} \mathbb{E}[X_{i_1} X_{i_2}] \dots \mathbb{E}[X_{i_{2k+1}} X_{i_{2k+2}}]$$
$$X_1 \diamond \dots \diamond \check{X}_{i_1} \diamond \dots \diamond \check{X}_{i_2} \diamond \dots \diamond \check{X}_{i_{2k+1}} \diamond \dots \diamond \check{X}_{i_{2k+2}} \diamond \dots \diamond X_n$$
$$+ \dots, \tag{4.1}$$

where the sum ranges over all partial pairings of indices $(i_1, i_2), \dots, (i_{2k+1}, i_{2k+2})$ $(1 \leq k \leq \lfloor \frac{n}{2} \rfloor - 1)$.

2. For every set of indices $i_1, \dots, i_j, i'_1, \dots, i'_j$,

$$\mathbb{E}\left[(X_{i_1} \diamond \dots \diamond X_{i_j})(X_{i'} \diamond \dots \diamond X_{i'_j}) \right] = \sum_{\sigma \in \Sigma_j} \prod_{m=1}^{j} \mathbb{E}[X_{i_m} X_{i'_{\sigma(m)}}]. \tag{4.2}$$

In our case (considering $\mathcal{RB}^{n,\eta}_{ts}(i_1, \dots, i_n)$) we get a decomposition of the product $dW_{\xi_1}(i_1) \dots dW_{\xi_n}(i_n)$ into $dW_{\xi_1}(i_1) \diamond \dots \diamond dW_{\xi_n}(i_n)$, plus the sum over all possible non-trivial pair contractions, schematically $\langle W'_{\xi_j}(i_j) W'_{\xi_{j'}}(i_{j'}) \rangle = \delta_0(\xi_j + \xi_{j'}) \delta_{i_j, i_{j'}}$.

Consider first the normal ordering of $\mathcal{RB}^{n,\eta}_{ts}(i_1, \dots, i_n)$. As in the proof of Lemma 5.10 in [33], let $\Sigma_{\boldsymbol{i}}$ be the 'index-fixing' subgroup of Σ_n such that : $\sigma' \in \Sigma_{\boldsymbol{i}} \iff \forall j = 1, \dots, n,\ i_{\sigma'(j)} = i_j$. Then (by Wick's lemma and the Cauchy-Schwarz inequality) :

$$\begin{aligned} &\mathrm{Var} : \mathcal{RB}^{n,\eta}_{ts}(i_1, \dots, i_n) := \mathbb{E} \,|: \mathcal{RB}^{n,\eta}_{ts}(i_1, \dots, i_n) :|^2 \\ &= \sum_{\sigma' \in \Sigma_{\boldsymbol{i}}} \mathbb{E}\left[: \mathcal{RB}^{n,\eta}_{ts}(1, \dots, n) : : \mathcal{RB}^{n,\eta}_{ts}(\sigma'(1), \dots, \sigma'(n)) : \right] \\ &\leq |\Sigma_{\boldsymbol{i}}| \,.\, \mathbb{E} |\mathcal{RB}^{n,\eta}(1, \dots, n)|^2, \end{aligned} \tag{4.3}$$

hence the Hölder and rate estimates of section 3 also hold for $: \mathcal{RB}^{n,\eta}(i_1, \dots, i_n) :$.

One must now prove that the estimates of section 3 hold true for all possible contractions of $\mathcal{RB}^{n,\eta}(i_1, \dots, i_n)$. Fixing some non-trivial contraction $(j_1, j_2), \dots, (j_{2l-1}, j_{2l})$, $l \geq 1$, results in an expression $\mathbf{X}^{contr}_{ts}$ belonging to the chaos of order $n - 2l$. By necessity, $i_{j_1} = i_{j_2}, \dots, i_{j_{2l-1}} = i_{j_{2l}}$, but it may well be that there are other index coincidences. The same reasoning

as in the case of : $\mathcal{R}\mathbf{B}^{n,\eta}_{ts}(i_1,\dots,i_n)$: (see eq. (4.3)) shows that one may actually assume $i_m \neq i_{m'}$ if $m \neq m'$ and $\{m,m'\} \neq \{j_1,j_2\},\dots,\{j_{2l-1},j_{2l}\}$. Now (as we shall presently prove) the tree integrals related to the contracted iterated integral $\mathbf{X}^{contr}_{ts}$ may be estimated by considering the tree integrals related to $\check{\mathbf{X}}_{ts} := \mathcal{R}\mathbf{B}^{n-2l,r}_{ts}(i_1,\dots,\check{i_{j_1}},\dots,\check{i_{j_{2l}}},\dots,i_n)$ (which has same law as $\mathcal{R}\mathbf{B}^{n-2l,r}_{ts}(1,\dots,n-2l)$) and (following the idea introduced in the course of the proof of Lemma 3.4) increasing by one the weight W of some other (possibly coinciding) indices $j'_1,\dots,j'_{2l} \neq j_1,\dots,j_{2l}$ – or, in other words, 'inserting' a factor $|\xi_{j'_1}|^{-2\alpha}\dots|\xi_{j'_{2l}}|^{-2\alpha}$ in the variance integrals –. This amounts in the end to increasing the Hölder regularity $(n-2l)\alpha^-$ of $\check{\mathbf{X}}_{ts}$ by $2l\alpha$, which gives the expected regularity.

Fix some permutation $\sigma \in \Sigma_n$, and consider the integral over the Fourier domain $|\xi_{\sigma(1)}| \leq \dots \leq |\xi_{\sigma(n)}|$ as in section 2. Change as before the order of integration and the names of the indices so that $dW_{\xi_{\sigma(j)}}(i_j) \to dW_{\xi_j}(i_{\sigma(j)})$; for convenience, we shall still index the pairing indices as $(j_1,j_2),\dots,(j_{2l-1},j_{2l})$. We may assume that $|j_{2k-1}-j_{2k}| = 1$, $k = 1,\dots,l$ (otherwise $|\xi_m| = |\xi_{j_{2k-1}}| = |\xi_{j_{2k}}|$ for $j_{2k-1} < m < j_{2k}$ or $j_{2k} < m < j_{2k-1}$, which corresponds to a Fourier subdomain of zero Lebesgue measure). In the sequel, we fix $\sigma \in \Sigma_n$ and $(j,j') = (j_{2k-1},j_{2k})$ for some k.

Let $\tilde{\mathbb{T}} = \tilde{\mathbb{T}}_1 \dots \tilde{\mathbb{T}}_L$ be a forest appearing in the decomposition of the permutation graph $\mathbb{T}^\sigma$ as in subsection 2.4. Applying repeatedly Lemma 2.10 to $\tilde{\mathbb{T}}$ leads to a sum of terms obtained from the contraction of $A_{ts} = A_{ts}(1)\dots A_{ts}(L)$, with $A_{ts}(k) = [\delta\mathcal{R}\mathrm{Sk}\,\mathrm{I}_{L\tilde{\mathbb{T}}_k}]_{ts}\prod_j[\mathcal{R}\mathrm{Sk}\,\mathrm{I}_{\mathbb{T}'_{k,j}}]_s\left(\left(\otimes_{v\in V(\tilde{\mathbb{T}}_k)}dB^\eta(i_v)\right)^\sigma\right)$, where $L\tilde{\mathbb{T}}_k, \mathbb{T}'_{k,1},\dots,\mathbb{T}'_{k,j},\dots$ are all subtrees appearing in the splitting associated to $A_{ts}(k)$ (see proof of Lemma 3.5).

Let $\mathbb{T}$ be one of the above trees, either $L\tilde{\mathbb{T}}_k$ or $\mathbb{T}'_{k,j}$. Reconsider the proof of the Hölder estimate or rate of convergence in Lemma 3.4 or Lemma 3.5. The integrals $\left[\mathrm{Sk}\,\mathrm{I}\left((x_v)_{v\in V(\mathbb{T})} \to e^{\mathrm{i}\sum_{v\in V(\mathbb{T})}x_v\xi_v}\right)\right]_u$ appearing in the definition of the regularized skeleton integrals write $i^{-|V(\mathbb{T})|}\frac{e^{\mathrm{i}u\sum_{v\in V(\mathbb{T})}\xi_v}}{\prod_{v\in V(\mathbb{T})}(\xi_v+\sum_{w\to v}\xi_w)}$ (see Lemma 2.7). After the contractions, one must sum over Fourier indices $(\xi_v)_{v\in V(\mathbb{T})}$ such that $(\xi_v)_{v\in V(\mathbb{T})} \in \mathbb{R}^{\mathbb{T}}_{reg}$ and $\xi_{j_{2m-1}} = -\xi_{j_{2m}}$ if both $j_{2m-1},j_{2m} \in V(\mathbb{T})$.

Let $\check{\mathbb{T}}$ be the contracted tree obtained by 'skipping' $\{j_1,\dots,j_{2l}\}\cap V(\mathbb{T})$ while going down the tree $\mathbb{T}$ (see Fig. 7, 8, 9).

The denominator $|\xi_v+\sum_{w\in\mathbb{T},w\twoheadrightarrow v}\xi_w|$ is larger (up to a constant) than the denominator $|\xi_v+\sum_{w\in\check{\mathbb{T}},w\twoheadrightarrow v}\xi_w|$ obtained by considering the same term in the contracted tree integral $\check{\mathbf{X}}_{ts}$ (namely, $|\xi_v+\sum_{w\in\mathbb{T},w\twoheadrightarrow v}\xi_w|$ is of the same

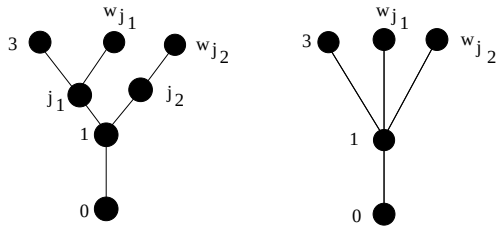

Figure 7: Case (i-a). $\mathbb{T}$ and $\check{\mathbb{T}}$.

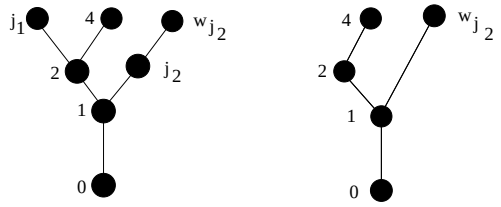

Figure 8: Case (i-b). $\mathbb{T}$ and $\check{\mathbb{T}}$.

order as $\max\{|\xi_w|; w \in \mathbb{T}, w \twoheadrightarrow v\} \geq \max\{|\xi_w|; w \in \check{\mathbb{T}}, w \twoheadrightarrow v\}$). Hence $\mathbb{E}(A_{ts}^{contr})^2$ may be bounded in the same way as $\mathbb{E}A_{ts}^2$ in the proof of Lemma 3.4 or Lemma 3.5, except that each term in the sum over $(\xi_v, v \in V(\mathbb{T}), v \neq j_1, \ldots, j_{2l})$ comes with an extra multiplicative pre-factor $S = S((\xi_v), v \in V(\mathbb{T}), v \neq j_1, \ldots, j_{2l})$ – due to the sum over $(\xi_{j_m})_{m=1,\ldots,2l}$ – which may be seen as an 'insertion'.

Let us estimate this prefactor. We shall assume for the sake of clarity that there is a single contraction $(j_1, j_2) = (j, j')$ (otherwise the prefactor should be evaluated by contracting each tree in several stages, 'skipping' successively $(j_1, j_2), \ldots, (j_{2l-1}, j_{2l})$ by pairs). As already mentioned before, $|j - j'| = 1$ so that j and j' must be successive vertices if they belong to the same branch of the same tree $\mathbb{T}$. Note that, if j and j' are on the same tree, the Fourier index $\Xi := \sum_{v \in V(\mathbb{T})} \xi_v$ (used in the Fourier decomposition of Lemma 3.4 or in the third step of Lemma 3.5) is left unchanged since $\xi_j + \xi_{j'} = 0$.

Case (i): (j, j') belong to unconnected branches of the same tree $\mathbb{T}$. This case splits into three different subcases:

(i-a) neither j nor j' is a leaf. Let w, resp. w' be the leaf above j, resp. j' of maximal index and assume (without loss of generality) that $|\xi_w| \leq$

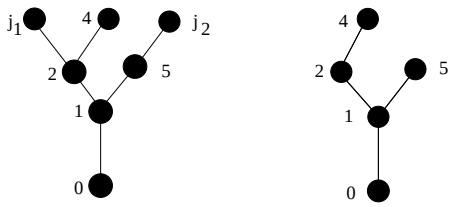

Figure 9: Case (i-c). $\mathbb{T}$ and $\check{\mathbb{T}}$.

$|\xi_{w'}|$. Then

$$S \lesssim \left(\int_{|\xi_j| \le |\xi_w|} d\xi_j \frac{|\xi_j|^{1-2\alpha}}{|\xi_w \xi_{w'}|} \right)^2 \lesssim \left(\int_{|\xi_j| \le |\xi_w|} d\xi_j |\xi_w|^{-1-2\alpha} \right)^2 \lesssim |\xi_w|^{-4\alpha} \tag{4.4}$$

which has the effect of increasing the weight $W(w)$ by 2.

(i-b) j is a leaf, j' is not. Let w' be the leaf of maximal index above j'. Then

$$S \le \left(\int_{|\xi_j| \le |\xi_{w'}|} d\xi_j \frac{|\xi_j|^{1-2\alpha}}{|\xi_j \xi_{w'}|} \right)^2 \lesssim \left(\frac{1}{|\xi_{w'}|} \int_{|\xi_j| \le |\xi_{w'}|} d\xi_j |\xi_j|^{-2\alpha} \right)^2 \lesssim |\xi_{w'}|^{-4\alpha}. \tag{4.5}$$

(i-c) both j and j' are leaves. Let v, resp. v' be the vertex below j, resp. j', i.e. $j \to v$, $j' \to v'$. Then

$$S \lesssim \left(\int_{|\xi_j| \ge \max(|\xi_v|, |\xi_{v'}|)} d\xi_j |\xi_j|^{-1-2\alpha} \right)^2 \lesssim |\xi_v|^{-4\alpha} \tag{4.6}$$

which has the effect of increasing $W(v)$ by 2.

Case (ii): (j, j') are successive vertices on the same branch of the same tree $\mathbb{T}$. Assume (without loss of generality) that $j \to j'$. Then $S = 0$ if j is a leaf (since $\xi_{j'} + \sum_{w \twoheadrightarrow j'} \xi_w = \xi_j + \xi_{j'} = 0$ and such indices fail to meet the condition defining $\mathbb{R}^{\mathbb{T}}_{reg}$), otherwise $S \lesssim |\xi_w|^{-4\alpha}$ if w is the leaf of maximal index above j (by the same argument as in case (i-a)).

Case (iii): (j, j') belong to two different trees, $\mathbb{T}$ and $\mathbb{T}'$.

This case is a variant of case (i). Nothing changes compared to case (i) unless (as in the proof of Lemma 3.4 or in the 3rd step of Lemma 3.5) one

needs to compute the variance of the coefficient $a(\Xi)$ or $a_s(\Xi)$ of $e^{\mathrm{i}u\Xi}$ for Ξ fixed. Assume j belongs to the tree $\mathbb{T} = L\tilde{\mathbb{T}}_k$ while j' is on one of the cut trees $\mathbb{T}'_{k,1}, \dots, \mathbb{T}'_{k,j}, \dots$

Assume first j is not a leaf, and let w be the leaf above j. Then the presence of the extra vertex j modifies the Fourier index Ξ in the Fourier decomposition of $A^{contr}_{ts}(k)$, $A^{contr}_{ts}(k) = \int_{\mathbb{R}} a(\Xi)(e^{\mathrm{i}\Xi t} - e^{\mathrm{i}\Xi s})d\Xi$ or $A^{contr}_{ts}(k) = \int_{\mathbb{R}} a_s(\Xi)(e^{\mathrm{i}\Xi t} - e^{\mathrm{i}\Xi s})d\Xi$, by a factor which is bounded and bounded away from 0, hence $S \lesssim |\xi_w|^{-4\alpha}$ as in case (i-a).

If j is a leaf as in case (i-b) – while w' is as before the leaf of maximal index over j' –, one has: $|\xi_j| \lesssim |\Xi| \lesssim |\xi_j|$. Hence the sum over ξ_j contributes an extra multiplicative pre-factor S to the variance of the coefficient of $a(\Xi)$ or $a_s(\Xi)$ of order

$$S \lesssim \left(\int_{|\Xi|/2 \le |\xi_j| \le 2|\Xi|} d\xi_j \frac{|\xi_j|^{1-2\alpha}}{|\xi_j \xi_{w'}|} \right)^2 \lesssim \left(\int_{|\Xi|/2 \le |\xi_j| \le 2|\Xi|} |\xi_j|^{-1-2\alpha} \right)^2 \lesssim |\Xi|^{-4\alpha}, \tag{4.7}$$

which increases the Hölder index by 2α (see Lemma 3.3).

The case when both j and j' belong to left parts $L\tilde{\mathbb{T}}_k$, $L\tilde{\mathbb{T}}_{k'}$ is similar and left to the reader. □

This concludes at last the proof of Theorem 0.1.

4.2 A remark: about the two-dimensional antisymmetric fBm

Consider a one-dimensional analytic fractional Brownian motion Γ as in [33].

Definition 4.2 *Let* $Z_t = (Z_t(1), Z_t(2)) = (2\mathrm{Re}\ \Gamma_t, 2\mathrm{Im}\ \Gamma_t)$, $t \in \mathbb{R}$. *We call this new centered Gaussian process indexed by* $\mathbb{R}$ *the* two-dimensional antisymmetric fBm.

Its paths are a.s. α^--Hölder. The marginal processes $Z(1)$, $Z(2)$ are usual fractional Brownian motions. The covariance between $Z(1)$ and $Z(2)$ writes (see [33])

$$\mathrm{Cov}(Z_s(1), Z_t(2)) = -\frac{\tan \pi\alpha}{2}[-\mathrm{sgn}(s)|s|^{2\alpha} + \mathrm{sgn}(t)|t|^{2\alpha} - \mathrm{sgn}(t-s)|t-s|^{2\alpha}]. \tag{4.8}$$

Note that we never used any particular linear combination of the analytic/anti-analytic components of B in the estimates of section 3 and 4. Hence these also hold for Z, which gives for free a rough path over Z satisfying Theorem 0.1 of the Introduction.

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