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A stochastic calculus for multidimensional fractional Brownian motion with arbitrary Hurst index

J er mie Unterberger

We construct in this article an explicit rough path over a multi-dimensional fractional Brownian motion B with arbitrary Hurst index H (in particular, for $H < 1/4$) by regularizing an associated random Fourier series defined in [32]. The regularization procedure is applied to 'Fourier normal ordered' iterated integrals obtained by permuting the order of integration so that innermost integrals have highest Fourier modes. The algebraic properties of this rough path are best understood using the Hopf algebra structure of the algebra of decorated rooted trees. Rough path theory gives then a general procedure to define a stochastic calculus and solve stochastic differential equations driven by this very irregular process. A variant of our regularization scheme is also expected to apply to arbitrary deterministic H older paths.

The last section is also dedicated to the definition of a related two-dimensional Gaussian process, called *antisymmetric two-dimensional fractional Brownian motion*, with the same regularity as B but with dependent components, to which the above construction extends naturally.

Keywords: fractional Brownian motion, stochastic integrals, rough paths, Hopf algebra of decorated rooted trees

Mathematics Subject Classification (2000): 05C05, 16W30, 60F05, 60G15, 60G18, 60H05

0 Introduction

The (two-sided) fractional Brownian motion $t \rightarrow B_t$, $t \in \mathbb{R}$ (fBm for short) with Hurst exponent α , $\alpha \in (0, 1)$, defined as the centered Gaussian process with covariance

$$\mathbb{E}[B_s B_t] = \frac{1}{2}(|s|^{2\alpha} + |t|^{2\alpha} - |t - s|^{2\alpha}), \quad (0.1)$$

is a natural generalization in the class of Gaussian processes of the usual Brownian motion (which is the case $\alpha = \frac{1}{2}$), in the sense that it exhibits two fundamental properties shared with Brownian motion, namely, it has stationary increments, viz. $\mathbb{E}[(B_t - B_s)(B_u - B_v)] = \mathbb{E}[(B_{t+a} - B_{s+a})(B_{u+a} - B_{v+a})]$ for every $a, s, t, u, v \in \mathbb{R}$, and it is self-similar, viz.

$$\forall \lambda > 0, \quad (B_{\lambda t}, t \in \mathbb{R}) \stackrel{(law)}{=} (\lambda^\alpha B_t, t \in \mathbb{R}). \quad (0.2)$$

One may also define a d -dimensional vector Gaussian process (called: *d-dimensional fractional Brownian motion*) by setting $B_t = (B_t(1), \dots, B_t(d))$ where $(B_t(i), t \in \mathbb{R})_{i=1, \dots, d}$ are d independent (scalar) fractional Brownian motions.

Its theoretical interest lies in particular in the fact that it is (up to normalization) the only Gaussian process satisfying these two properties.

A standard application of Kolmogorov's theorem shows that fBm has a version with α^- -Hölder continuous (i.e. κ -Hölder continuous for every $\kappa < \alpha$) paths. In particular, fBm with small Hurst parameter α is a natural, simple model for continuous but very irregular processes.

There has been a widespread interest during the past ten years in constructing a stochastic integration theory with respect to fBm and solving stochastic differential equations driven by fBm, see for instance [23, 15, 5, 28, 29]. The multi-dimensional case is very different from the one-dimensional case. When one tries to integrate for instance a stochastic differential equation driven by a two-dimensional fBm $B = (B(1), B(2))$ by using any kind of Picard iteration scheme, one encounters very soon the problem of defining the Lévy area of B which is the antisymmetric part of $\mathcal{A}_{ts} := \int_s^t dB_{t_1}(1) \int_s^{t_1} dB_{t_2}(2)$. This is the simplest occurrence of iterated integrals $\mathbf{B}_{ts}^k(i_1, \dots, i_k) := \int_s^t dB_{t_1}(i_1) \dots \int_s^{t_{k-1}} dB_{t_k}(i_k)$, $i_1, \dots, i_k \leq d$ for d -dimensional fBm $B = (B(1), \dots, B(d))$ which lie at the heart of the rough path theory due to T. Lyons, see [24, 25]. An alternative construction has been given in [16] under the name of 'algebraic rough path theory', which we now propose to describe briefly.

Assume $\Gamma_t = (\Gamma_t(1), \dots, \Gamma_t(d))$ is some non-smooth d -dimensional path which is α -Hölder continuous. Integrals such as $\int f_1(\Gamma_t) d\Gamma_t(1) + \dots + \int f_d(\Gamma_t) d\Gamma_t(d)$ do not make sense a priori because Γ is not differentiable (Young's integral [22] works for $\alpha > \frac{1}{2}$ but not beyond). In order to define the integration of a differential form along Γ , it is enough to define a *truncated multiplicative functional* or *rough path* $(\mathbf{\Gamma}^1, \dots, \mathbf{\Gamma}^{\lfloor 1/\alpha \rfloor})$ lying above Γ , $\lfloor 1/\alpha \rfloor =$ entire part of $1/\alpha$, where $\mathbf{\Gamma}_{ts}^1 = (\delta\Gamma)_{ts} := \Gamma_t - \Gamma_s$ is the *increment* of Γ between s and t , and each $\mathbf{\Gamma}^k = (\mathbf{\Gamma}^k(i_1, \dots, i_k))_{1 \leq i_1, \dots, i_k \leq d}$, $k \geq 2$ is a *substitute* for the iterated integrals $\int_s^t d\Gamma_{t_1}(i_1) \int_s^{t_1} d\Gamma_{t_2}(i_2) \dots \int_s^{t_{k-1}} d\Gamma_{t_k}(i_k)$ with the following two properties:

(i) (*Hölder continuity*) each component of $\mathbf{\Gamma}^k$ is $k\alpha^-$ -Hölder continuous, that is to say, $k\kappa$ -Hölder for every $\kappa < \alpha$;

(ii) (*multiplicativity*) letting $\delta\mathbf{\Gamma}_{tus}^k := \mathbf{\Gamma}_{ts}^k - \mathbf{\Gamma}_{tu}^k - \mathbf{\Gamma}_{us}^k$, one requires

$$\delta\mathbf{\Gamma}_{tus}^k(i_1, \dots, i_k) = \sum_{k_1+k_2=k} \mathbf{\Gamma}_{tu}^{k_1}(i_1, \dots, i_{k_1}) \mathbf{\Gamma}_{us}^{k_2}(i_{k_1+1}, \dots, i_k). \quad (0.3)$$

If furthermore the following property holds

(iii) (*geometricity*)

$$\mathbf{\Gamma}_{ts}^{n_1}(i_1, \dots, i_{n_1}) \mathbf{\Gamma}_{ts}^{n_2}(j_1, \dots, j_{n_2}) = \sum_{\mathbf{k} \in \text{Sh}(\mathbf{i}, \mathbf{j})} \mathbf{\Gamma}_{ts}^{n_1+n_2}(k_1, \dots, k_{n_1+n_2}) \quad (0.4)$$

where $\text{Sh}(\mathbf{i}, \mathbf{j})$ is the subset of permutations of $i_1, \dots, i_{n_1}, j_1, \dots, j_{n_2}$ which do not change the orderings of (i_1, \dots, i_{n_1}) and (j_1, \dots, j_{n_2}) ,

then $\mathbf{\Gamma}$ is called a *geometric rough path*.

The multiplicativity property implies in particular the following identity for the (non anti-symmetrized) Lévy area:

$$\mathcal{A}_{ts} = \mathcal{A}_{tu} + \mathcal{A}_{us} + (B_t(1) - B_u(1))(B_u(2) - B_s(2)) \quad (0.5)$$

while the geometric property implies

$$\begin{aligned} & \int_s^t dB_{t_1}(1) \int_s^{t_1} dB_{t_2}(2) + \int_s^t dB_{t_2}(2) \int_s^{t_2} dB_{t_1}(1) \\ &= \left(\int_s^t dB_{t_1}(1) \right) \left(\int_s^t dB_{t_2}(2) \right) = (B_t(1) - B_s(1))(B_t(2) - B_s(2)). \end{aligned} \quad (0.6)$$

Then there is a standard procedure which allows to define out of these data iterated integrals of any order and to solve differential equations driven by Γ .

The multiplicativity property (0.3) and the geometric property (0.4) are satisfied by smooth paths, as can be checked by direct computation. So the most natural way to construct such a multiplicative functional is to start from some smooth approximation $\Gamma(\eta)$, $\eta \xrightarrow{>} 0$ of Γ such that each iterated integral $\Gamma_{ts}^k(\eta)(i_1, \dots, i_k)$, $k \leq [q]$ converges in the $k\kappa$ -Hölder norm for every $\kappa < \alpha$.

This general scheme has been applied to fBm in a paper by L. Coutin and Z. Qian [9] and later in a paper by the author [32], using different schemes of approximation of B by B^η with $\eta \rightarrow 0$. In both cases, the variance of the Lévy area has been proved to diverge in the limit $\eta \rightarrow 0$ when $\alpha \leq 1/4$.

The approach developed in [32] makes use of a complex-analytic process Γ defined on the upper half-plane $\Pi^+ = \{z = x + iy \mid y > 0\}$, called Γ -process or better *analytic fractional Brownian motion* (afBm for short) [31]. Fractional Brownian motion B_t appears as the *real part* of the *boundary value* of Γ_z when $\text{Im } z \xrightarrow{>} 0$. A natural approximation of B_t is then obtained by considering

$$B_t^\eta := \Gamma_{t+i\eta} + \overline{\Gamma_{t+i\eta}} = 2\text{Re } \Gamma_{t+i\eta} \quad (0.7)$$

for $\eta \xrightarrow{>} 0$.

The so-called *analytic iterated integrals*

$$\int_s^t f_1(z_1) d\Gamma_{z_1}(1) \int_s^{z_1} f_2(z_2) d\Gamma_{z_2}(2) \dots \int_s^{z_{d-1}} f_d(z_d) d\Gamma_{z_d}(d)$$

(where f_1, \dots, f_d are analytic functions), defined a priori for $s, t \in \Pi^+$ by integrating over complex paths wholly contained in Π^+ , converge to a finite limit when $\text{Im } s, \text{Im } t \rightarrow 0$ [32], which is the starting point for the construction of a rough path associated to Γ [31]. The main tool for proving this kind of results is analytic continuation.

Computing iterated integrals associated to $B_t = 2 \lim_{\eta \rightarrow 0} \text{Re } \Gamma_{t+i\eta}$ instead of Γ yields analytic iterated integrals, together with mixed integrals such as for instance $\int_s^t d\Gamma_{z_1}(1) \int_s^{z_1} \overline{d\Gamma_{z_2}(2)}$. For these the analytic continuation method may no longer be applied because Cauchy's formula fails to hold, and the above quantities may be shown to diverge when $\text{Re } s, \text{Re } t \rightarrow 0$, see [32, 33].

We shall use here instead (essentially for convenience) a *series decomposition* of Γ also introduced in [32]. Let $\mathcal{D} := \{z \in \mathbb{C} \mid |z| < 1\}$ be the unit

disk. We denote by $\Phi : \Pi^+ \rightarrow \mathcal{D}$, $z \mapsto \zeta := \Phi(z) = \frac{z-i}{z+i}$ the Cayley transform (which is a biholomorphism). Letting

$$\tilde{\Gamma}'_{\zeta} := (1 - \zeta)^{2\alpha} \Gamma'_{\Phi^{-1}(\zeta)} \quad (0.8)$$

(essentially a trivial time-change), we obtain a new process $\tilde{\Gamma}'$, called *unit-disk Γ -process* or *unit-disk analytic fractional Brownian motion*, which lives on the open unit disk. This process may be written as a random entire series as follows:

$$\tilde{\Gamma}'_{\zeta} := \sum_{k \geq 0} a_{k+1} \zeta^k \xi_{k+1}, \quad (0.9)$$

where:

– $(\xi_k)_{k \geq 1}$ is a sequence of independent standard complex Gaussian variables, i.e. $\mathbb{E}[\xi_j \xi_k] = 0$, $\mathbb{E}[\xi_j \bar{\xi}_k] = \delta_{j,k}$;

– the coefficients $a_{k+1} = c_{\alpha} \sqrt{\frac{(2-2\alpha)_k}{k!}}$ (with $c_{\alpha} = 2^{\alpha-1} \sqrt{\frac{\alpha(1-2\alpha)}{2 \cos \pi \alpha}}$) scale like $k^{\frac{1}{2}-\alpha}$ when $k \rightarrow \infty$ (by definition, $(2-2\alpha)_k = (2-2\alpha)(3-2\alpha) \dots (1-2\alpha+k) \sim_{k \rightarrow \infty} k^{1-2\alpha}$. $k!$ is Pochhammer's symbol).

Then the integrated series

$$\tilde{\Gamma}_{\zeta} := \sum_{k \geq 1} a_k \frac{\zeta^k}{k} \xi_k, \quad \zeta = r e^{i\theta} \quad (0.10)$$

may be shown to converge in the κ -Hölder norm (for every $\kappa < \alpha$) to a κ -Hölder process on the unit circle \mathcal{C} when $r \rightarrow 1^-$, which is simply the associated random *Fourier series*

$$\tilde{\Gamma}_{e^{i\theta}} = \sum_{k \geq 1} \frac{a_k}{k} e^{ik\theta} \xi_k. \quad (0.11)$$

Such random Fourier series have been extensively studied, see for instance J.-P. Kahane's book [20], where it is shown in particular that $\sum_{k \in \mathbb{Z}} b_k e^{ik\theta} \xi_k$ converges a.s. to an α^- -Hölder continuous random function provided $|b_k|^2 = O(k^{-1-2\alpha})$ when $k \rightarrow \infty$ (which is the case here).

We then let

$$\tilde{B}_{e^{i\theta}} = 2\text{Re } \tilde{\Gamma}_{e^{i\theta}}; \quad (0.12)$$

this α^- -Hölder process is naturally approximated by $\tilde{B}_{e^{i\theta}}^r = 2\text{Re } \tilde{\Gamma}_{r e^{i\theta}}$. We thus obtain a variant of fBm, \tilde{B} , together with a family of real-analytic approximations \tilde{B}^r , $r \rightarrow 1^-$ of \tilde{B} which is very closely related to fBm B and

its family of approximations B^η ; note however the *change of origin*: $\tilde{B}_0 = 0$ but $\tilde{B}_{\Phi(0)} = \tilde{B}_{-1} \neq 0$ whereas $B_0 = 0$ by definition.

Let us explain first how to define a Lévy area for \tilde{B} . As mentioned before, the *uncorrected area* $\tilde{\mathcal{A}}_{\tau\sigma}^r := \int_\sigma^\tau d\tilde{B}_{\zeta_1}^r(1) \int_\sigma^{\zeta_1} d\tilde{B}_{\zeta_2}^r(2)$ diverges when $r \rightarrow 1^-$. The idea is now to find some *increment counterterm* $(\delta Z^r)_{\tau\sigma} = Z_\tau^r - Z_\sigma^r$ such that the *regularized area* $\mathcal{R}\tilde{\mathcal{A}}_{\tau\sigma}^r := \tilde{\mathcal{A}}_{\tau\sigma}^r - (\delta Z^r)_{\tau\sigma}$ converges when $r \rightarrow 1^-$. Note that the multiplicativity property (0.5) holds for $\mathcal{R}\tilde{\mathcal{A}}^r$ as well as for $\tilde{\mathcal{A}}^r$ since $(\delta Z^r)_{\tau\sigma} = (\delta Z^r)_{\tau u} + (\delta Z^r)_{u\sigma}$. This counterterm X may be found by using a suitable decomposition of $\tilde{\mathcal{A}}_{\tau\sigma}^r$ into the sum of :

- an *increment term*, $(\delta G)_{\tau\sigma}$;
- a *boundary term* denoted by $\tilde{\mathcal{A}}_{\tau\sigma}^r(\partial)$.

The simplest idea one could think of would be to set

$$(\delta G)_{\tau\sigma} = \int_\sigma^\tau d\tilde{B}_{\zeta_1}^r(1) \tilde{B}_{\zeta_1}^r(2), \quad (0.13)$$

and

$$\tilde{\mathcal{A}}_{\tau\sigma}^r(\partial) = - \int_\sigma^\tau d\tilde{B}_{\zeta_1}^r(1) \cdot \tilde{B}_\sigma^r(2) = -\tilde{B}_\sigma^r(2)(\tilde{B}_\tau^r(1) - \tilde{B}_\sigma^r(1)). \quad (0.14)$$

Alternatively, rewriting $\tilde{\mathcal{A}}_{\tau\sigma}^r$ as $\int_\sigma^\tau d\tilde{B}_{\zeta_2}^r(2) \int_{\zeta_2}^\tau d\tilde{B}_{\zeta_1}^r(1)$, one may equivalently set

$$(\delta G)_{\tau\sigma} = - \int_\sigma^\tau d\tilde{B}_{\zeta_2}^r(2) \tilde{B}_{\zeta_2}^r(1) \quad (0.15)$$

and

$$\tilde{\mathcal{A}}_{\tau\sigma}^r(\partial) = \int_\sigma^\tau d\tilde{B}_{\zeta_2}^r(2) \cdot \tilde{B}_\tau^r(1) = \tilde{B}_\tau^r(1)(\tilde{B}_\tau^r(2) - \tilde{B}_\sigma^r(2)). \quad (0.16)$$

Now δG diverges when $r \rightarrow 1^-$, but since it is an increment, it may be discarded (i.e. it might be used as a counterterm). The problem is, $\tilde{\mathcal{A}}_{\tau\sigma}^r(\partial)$ converges when $r \rightarrow 1^-$ in the κ -Hölder norm for every $\kappa < \alpha$, but not in the 2κ -Hölder norm (which is of course well-known and may be seen as the starting point for rough path theory).

It turns out that a slight adaptation of this poor idea gives the solution. Decompose $\tilde{\mathcal{A}}_{\tau\sigma}^r$ into a double series in k_1, k_2 using (0.9). Use the first increment/boundary decomposition (0.13,0.14) for all indices $k_1 \leq k_2$, and the second one (0.15,0.16) if $k_1 > k_2$. Then $\tilde{\mathcal{A}}_{\tau\sigma}^r(\partial)$, defined as the sum as two contributions, one coming from (0.14) and the other from (0.16), *does converge* in the 2κ -Hölder norm when $r \rightarrow 1^-$, for every $\kappa < \alpha$.

As for the increment term δG defined similarly as the sum of two contributions coming from (0.13) and (0.15), it diverges, but may be discarded at will. Actually we use in this article a *minimal regularization scheme*: only the close-to-diagonal (i.e. $k_1/k_2 \approx 1$) terms in the double series defining δG make it diverge. Summing over an appropriate subset, e.g. $k_1 \notin [k_2/2, 2k_2]$ yields an increment which converges in the 2κ -Hölder norm for every $\kappa < \alpha$. More precisely still, only the mixed term coming from the iterated integral $\int d\tilde{\Gamma}(1) \int d(\tilde{\Gamma}(2))$ is problematic, the analytic integrals $\int d\tilde{\Gamma}(1) \int d\tilde{\Gamma}(2)$ needing no regularization whatsoever.

Let $\alpha < 1/4$. As noted in [33], the uncorrected Lévy area \mathcal{A}^η of the regularized process B^η converges in law to a Brownian motion when $\eta \rightarrow 0^+$ after a rescaling by the factor $\eta^{\frac{1}{2}(1-4\alpha)}$. In the latter article, the following question was raised: is it possible to define a counterterm X^η living on the same probability space as fBm, such that (i) the rescaled process $\eta^{\frac{1}{2}(1-4\alpha)} X^\eta$ converges in law to Brownian motion; (ii) $(B^\eta, \mathcal{A}^\eta - X^\eta)$ is a multiplicative or almost multiplicative functional in the sense of [22], Definition 7.1; (iii) $\mathcal{A}^\eta - X^\eta$ converges in the 2κ -Hölder norm for every $\kappa < \alpha$ when $\eta \rightarrow 0$? A suitable time-change of the above defined counterterm δZ^r gives a solution to this problem.

The above ideas have a suitable generalization to iterated integrals $\int d\tilde{B}(i_1) \dots \int d\tilde{B}(i_n)$ of order $n \geq 3$. There is one more difficulty though: decomposing $(\tilde{B}^r)'_{\zeta_j}(i_j)$ into $2\text{Re} \sum_{k_j \geq 0} a_{k_j+1} \zeta_j^{k_j} \xi_{k_j+1}(i_j)$, a straightforward extension of the first increment/boundary decomposition (0.13,0.14) yields the correct Hölder estimate *provided* $k_1 \leq \dots \leq k_n$. What should one do then if $k_{\varepsilon(1)} \leq \dots \leq k_{\varepsilon(n)}$ for a suitable permutation ε instead? The idea is to permute the order of integration and write $\int_\sigma^\tau d\tilde{B}_{\zeta_1}(i_1) \dots \int_\sigma^{\zeta_{n-1}} d\tilde{B}_{\zeta_n}(i_n)$ as some iterated integral $\int d\tilde{B}_{\zeta_1}(i_{\varepsilon(1)}) \dots \int d\tilde{B}_{\zeta_n}(i_{\varepsilon(n)})$. The integration domain, in the general case, becomes a little involved, and necessitates the introduction of combinatorial tools on graphs and trees, such as admissible cuts, skeletons, etc. Our terminology is inspired from the renormalization of Feynmann graphs. The underlying structure (as already noted in [17]) is that of the Hopf algebra of decorated rooted trees, see for instance [12] or [19] or references in section 6).

We show in the last section that the above rough path construction extends naturally to the two-dimensional process $Z_t = (Z_t(1), Z_t(2)) := (2\text{Re} \Gamma_t, 2\text{Im} \Gamma_t)$, $t \in \mathbb{R}$, where Γ is a one-dimensional analytic fractional Brownian motion of Hurst index α . Both components of Z are fBm, but

they are correlated in a non-trivial way. Since $\text{Cov}(\Gamma, \Gamma) = 0$, the covariance matrix $\text{Cov}(Z(1), Z(2))$ (which is the imaginary part of the Hermitian covariance kernel of Γ) is antisymmetric, hence we suggest to call Z the *antisymmetric two-dimensional fBm*. This 'new' process is, so we believe, as fundamental as the usual two-dimensional fBm.

The main result of the paper may be stated as follows. Recall $\mathcal{C} = \{\zeta \in \mathbb{C} \mid |\zeta| = 1\}$ is the unit circle.

Theorem 0.1 *Let $\tilde{\Gamma} = (\tilde{\Gamma}(1), \dots, \tilde{\Gamma}(d))$ be a d -dimensional unit-disk analytic fBm of Hurst index α , with real part $\tilde{B} = (\tilde{B}(1), \dots, \tilde{B}(d))$. Then there exists a rough path $(\mathcal{R}\tilde{\mathbf{B}}^{1,r} = \delta\tilde{B}^r, \dots, \mathcal{R}\tilde{\mathbf{B}}^{\lfloor 1/\alpha \rfloor, r})$ over \tilde{B}^r ($r < 1$), living in the chaos of order $1, \dots, \lfloor 1/\alpha \rfloor$ of $\tilde{\Gamma}$, satisfying properties (ii) (multiplicative property) and (iii) (geometric property) of the Introduction, together with the following estimates:*

(Hölder estimate) *There exists a constant $C > 0$ such that, for every $\sigma, \tau \in \mathcal{C}$ and $0 < r < 1$,*

$$\mathbb{E}|\mathcal{R}\tilde{\mathbf{B}}_{\tau\sigma}^{n,r}(i_1, \dots, i_n)|^2 \leq Cd(\sigma, \tau)^{2n\alpha};$$

(rate of convergence) *there exists a constant $C > 0$ such that, for every $\sigma, \tau \in \mathcal{C}$ and $\frac{1}{2} < r_1, r_2 < 1$,*

$$\mathbb{E}|\mathcal{R}\tilde{\mathbf{B}}_{\tau\sigma}^{n,r_1}(i_1, \dots, i_n) - \mathcal{R}\tilde{\mathbf{B}}_{\tau\sigma}^{n,r_2}(i_1, \dots, i_n)|^2 \leq C|r_1 - r_2|^{2\alpha}.$$

These results imply the existence of an explicit rough path $\mathcal{R}\tilde{\mathbf{B}}$ over \tilde{B} , obtained as the limit of $\mathcal{R}\tilde{\mathbf{B}}^r$ when $r \rightarrow 1$.

The general idea underlying our construction could be expressed roughly as follows: take the Fourier decomposition of $\tilde{\Gamma}(i_1), \dots, \tilde{\Gamma}(i_n)$ and move the largest Fourier modes to the right, so that innermost integrals have highest Fourier frequencies (adepts of conformal field theory may like to call this 'Fourier normal ordering', [10]). Hence we expect it to imply (using the ordinary Fourier transform this time) an explicit construction of a rough path above any α -Hölder d -dimensional path. Using Fourier integrals instead of Fourier series for fBm (and working directly with B instead of \tilde{B}) involves some (superficial) technical complications that can easily be overcome (see subsection 7.3). A general construction in the deterministic case will be given shortly in a forthcoming article.

Here is an outline of the article. Sections 1, 2 and 3 are preliminary; we first recall briefly some prerequisites on algebraic rough path theory, and also the definition of the analytic fBm process Γ with its first properties. The regularized Lévy area is constructed in section 4. Sections 5 and 6 are dedicated to the construction in the general multidimensional case; the algebraic stuff needed to deal with the case when the above permutation ε is not trivial is introduced in section 6. We conclude the proof, show how to work directly with the usual fBm instead of the unit-disk fBm and introduce the antisymmetric two-dimensional fBm in section 7.

Notations. Throughout the article, we shall use the following notations: $\mathcal{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ will be the (open) unit disk, with closure $\overline{\mathcal{D}} = \{z \in \mathbb{C} \mid |z| \leq 1\}$. The unit circle $\{z \in \mathbb{C} \mid |z| = 1\}$ will be denoted by \mathcal{C} . If $\sigma, \tau \in \mathcal{C}$, the distance $d(\sigma, \tau)$ between σ and τ will be measured along the circle (i.e. $d(\sigma, \tau)$ is a minimal measure of the angle between its two arguments). The group of permutations of $\{0, \dots, n-1\}$ is denoted by Σ_n . Finally, if $|a| \leq C|b|$ for some constant C (a and b depending on some arbitrary set of parameters), then we shall write $|a| \lesssim |b|$.

1 The analysis of rough paths

The present section will be very sketchy since the objects and results needed in this work have already been presented in great details in [31]. The foundational paper on the subject of algebraic rough path theory is due to M. Gubinelli [16], see also [17] for more details in the case $\alpha < 1/3$. Let us recall briefly the original problem motivating the introduction of rough paths. Let $\Gamma : \mathbb{R} \rightarrow \mathbb{R}^d$ be some fixed irregular (i.e. not derivable) path, say κ -Hölder, and $f : \mathbb{R} \rightarrow \mathbb{R}^d$ some function which is also irregular (mainly because one wants to consider functions f obtained as a composition $g \circ \Gamma$ where $g : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is regular). Can one define the integral $\int f_x d\Gamma_x$? The answer depends on the Hölder regularity of f and Γ . Assuming f is γ -Hölder with $\kappa + \gamma > 1$, then one may define the so-called *Young integral* [22] $\int_s^t f_x d\Gamma_x$ as the Riemann sum type limit $\lim_{|\Pi| \rightarrow 0} \sum_{\{t_j\} \in \Pi} f_{t_i}(\Gamma_{t_{i+1}} - \Gamma_{t_i})$, where $\Pi = \{s = t_0 < \dots < t_n = t\}$ is a partition of $[s, t]$ with mesh $|\Pi|$ going to 0. Then the resulting path $Y_t - Y_s := \int_s^t f_x d\Gamma_x$ has the same regularity as Γ . If $\kappa + \gamma \leq 1$ instead, this is no more possible in general. One way out of this problem, giving at the same time a coherent way to solve differential equations driven by Γ , is to define a class of Γ -controlled paths \mathcal{Q} , such that the above integration problem may be solved uniquely in this

class by a formula generalizing the above Riemann sums, in which formal iterated integrals $\mathbf{\Gamma}^n(i_1, \dots, i_n)$ of Γ appear as in the Introduction.

Definition 1.1 (Hölder spaces) *Let $\kappa \in (0, 1)$ and $T > 0$ fixed.*

- (i) *Let $C_1^\kappa = C_1^\kappa([0, T], \mathbb{C})$ be the space of complex-valued κ -Hölder functions f in one variable with (semi-)norm $\|f\|_\kappa = \sup_{s, t \in [0, T]} \frac{|f(t) - f(s)|}{|t - s|^\kappa}$.*
- (ii) *Let $C_2^\kappa = C_2^\kappa([0, T], \mathbb{C})$ be the space of complex-valued functions $f = f_{t_1, t_2}$ of two variables vanishing on the diagonal $t_1 = t_2$, such that $\|f\|_{2, \kappa} < \infty$, where $\|\cdot\|_{2, \kappa}$ is the following norm:*

$$\|f\|_{2, \kappa} = \sup_{s, t \in [0, T]} \frac{|f_{t_1, t_2}|}{|t - s|^\kappa}. \quad (1.1)$$

- (iii) *Let $C_3^\kappa = C_3^\kappa([0, T], \mathbb{C})$ be the space of complex-valued functions $f = f_{t_1, t_2, t_3}$ of three variables vanishing on the subset $\{t_1 = t_2\} \cup \{t_2 = t_3\} \cup \{t_1 = t_3\}$, such that $\|f\|_{3, \kappa} < \infty$ for some generalized Hölder semi-norm $\|\cdot\|_{3, \kappa}$ defined for instance in [16], section 2.1.*

Definition 1.2 (increments) (i) *Let f be a function of one variable: then the increment of f , denoted by δf , is $(\delta f)_{ts} := f(t) - f(s)$.*

- (ii) *Let $f = f_{ts}$ be a function of two variables: then we define*

$$(\delta f)_{tus} := (\delta f)_{ts} - (\delta f)_{tu} - (\delta f)_{us}. \quad (1.2)$$

Note that $\delta \circ \delta(f) = 0$ if f is a function of one variable.

Let $\Gamma = (\Gamma(1), \dots, \Gamma(d)) : [0, T] \rightarrow \mathbb{R}^d$ be a κ -Hölder path, and $(\mathbf{\Gamma}_{ts}^1(i_1) := \Gamma_t(i_1) - \Gamma_s(i_1), \mathbf{\Gamma}_{ts}^2(i_1, i_2), \dots, \mathbf{\Gamma}_{ts}^{\lfloor 1/\kappa \rfloor}(i_1, \dots, i_{\lfloor 1/\kappa \rfloor}))$ be a rough path lying above Γ , satisfying properties (i) (Hölder property), (ii) (multiplicativity property) and (iii) (geometricity property) of the Introduction.

Definition 1.3 (controlled paths) *Let $z = (z(1), \dots, z(d)) \in C_1^\kappa$ for some $\kappa < \alpha$ and $N = \lfloor 1/\kappa \rfloor + 1$. Then z is called a (Γ) -controlled path if its increments can be decomposed into*

$$\delta z(i) = \sum_{n=1}^N \sum_{(i_1, \dots, i_n)} \mathbf{\Gamma}^n(i_1, \dots, i_n) \cdot f^n(i_1, \dots, i_n; i) + g^0(i) \quad (1.3)$$

for some remainders $g^0(i) \in C_2^{N\kappa}$ and some paths $f^n(i_1, \dots, i_n; i) \in (C_1^\kappa)^n$ such that

$$\begin{aligned} \delta f^n(i_1, \dots, i_n; i) = \\ \sum_{l=1}^{N-1-n} \sum_{(j_1, \dots, j_l)} \mathbf{\Gamma}^l(j_1, \dots, j_l) \cdot f^{l+n}(j_1, \dots, j_l, i_1, \dots, i_n; i) + g^n(i_1, \dots, i_n; i), \quad n = 1, \dots, N \end{aligned} \quad (1.4)$$

for some remainder terms $g^n(i_1, \dots, i_n; i) \in C_2^{(N-n)\kappa}$.

We denote by \mathcal{Q}_κ the space of all such paths, and by \mathcal{Q}_{α^-} the intersection $\bigcap_{\kappa < \alpha} \mathcal{Q}_\kappa$.

We may now state the main result.

Proposition 1.4 (see [17], Theorem 8.5, or [31], Proposition 3.1) *Let $z \in \mathcal{Q}_{\alpha^-}$. Then the limit*

$$\int_s^t z_x d\Gamma_x := \lim_{|\Pi| \rightarrow 0} \sum_{k=0}^n \sum_{i=1}^d \left[\delta X_{t_{k+1}, t_k}(i) z_{t_k}(i) + \sum_{n=1}^{N-1} \sum_{(i_1, \dots, i_n)} \mathbf{\Gamma}_{t_{k+1}, t_k}^{n+1}(i_1, \dots, i_n, i) \zeta_{t_k}^n(i_1, \dots, i_n; i) \right] \quad (1.5)$$

exists in the space \mathcal{Q}_{α^-} .

The Proposition below gives very convenient moment conditions on a family of multiplicative functionals $(\Gamma^\eta, \mathbf{\Gamma}^{2,\eta}, \dots, \mathbf{\Gamma}^{\lfloor 1/\kappa \rfloor, \eta})$ to converge in the right Hölder norms when $\eta \rightarrow 0$ and to define a rough path lying above a centered Gaussian process Γ .

Proposition 1.5 *Let Γ be a d -dimensional centered Gaussian process admitting a version with a.s. α^- -Hölder paths. Let $N = \lfloor 1/\alpha^- \rfloor$. Assume:*

1. *there exists a family Γ^η , $\eta \rightarrow 0^+$ of Gaussian processes living in the first chaos of Γ such that*

$$(i) \quad \mathbb{E} |\Gamma_t^\eta - \Gamma_s^\eta|^2 \leq C |t - s|^{2\alpha}; \quad (1.6)$$

$$(ii) \quad \mathbb{E} |\Gamma_t^\eta - \Gamma_t^\varepsilon|^2 \leq C |\varepsilon - \eta|^{2\alpha}; \quad (1.7)$$

$$(iii) \quad \forall t \in [0, T], \Gamma_t^\eta \xrightarrow{L^2} \Gamma_t \text{ when } \eta \rightarrow 0;$$

2. there exists a truncated multiplicative functional $(\mathbf{\Gamma}_{ts}^{1,\eta} = \Gamma_t^\eta - \Gamma_s^\eta, \mathbf{\Gamma}_{ts}^{2,\eta}, \dots, \mathbf{\Gamma}_{ts}^{N,\eta})$ lying above Γ^η and living in the n -th chaos of Γ , $n = 1 \dots, N$, such that, for every $2 \leq k \leq N$,

$$(i) \quad \mathbb{E}|\mathbf{\Gamma}_{ts}^{k,\eta}|^2 \leq C|t-s|^{2k\alpha}; \quad (1.8)$$

$$(ii) \quad \mathbb{E}|\mathbf{\Gamma}_{ts}^{k,\varepsilon} - \mathbf{\Gamma}_{ts}^{k,\eta}|^2 \leq C|\varepsilon - \eta|^{2\alpha}. \quad (1.9)$$

Then $(\mathbf{\Gamma}^{1,\eta}, \dots, \mathbf{\Gamma}^{N,\eta})$ converges in $L^2(\Omega; C_2^\kappa([0, T], \mathbb{R}^d) \times C_2^{2\kappa}([0, T], \mathbb{R}^{d^2}) \times \dots \times C_2^{N\kappa}([0, T], \mathbb{R}^{d^N}))$ for every $\kappa < \alpha$ to a rough path $(\mathbf{\Gamma}^1, \dots, \mathbf{\Gamma}^N)$ lying above Γ .

Short proof (see [31], Lemma 5.1, Lemma 5.2 and Prop. 5.4). The main ingredient is the Garsia-Rodemich-Rumsey (GRR for short) lemma [14] which states that, if $f \in C_2^\kappa([0, T], \mathbb{C})$,

$$\|f\|_{2,\kappa} \leq C \left(\|\delta f\|_{3,\kappa} + \left(\int_0^T \int_0^T \frac{|f_{vw}|^{2p}}{|w-v|^{2\kappa p+2}} dv dw \right)^{1/2p} \right) \quad (1.10)$$

for every $p \geq 1$.

Then properties (1.6,1.8) imply by using the GRR lemma for p large enough, Jensen's inequality and the equivalence of L^p -norms for processes living in a fixed Gaussian chaos

$$\mathbb{E}\|\mathbf{\Gamma}^{k,\eta}\|_{2,k\kappa} \lesssim \mathbb{E}\|\delta\mathbf{\Gamma}^{k,\eta}\|_{3,k\kappa} + C. \quad (1.11)$$

By using the multiplicative property (ii) in the Introduction and induction on k , $\mathbb{E}\|\delta\mathbf{\Gamma}^{k,\eta}\|_{3,k\kappa}$ may in the same way be proved to be bounded by a constant.

On the other hand, properties (1.6,1.7,1.8,1.9), together with the equivalence of L^p -norms, imply (for every $\kappa < \alpha$)

$$\mathbb{E}|\mathbf{\Gamma}_{ts}^{k,\varepsilon} - \mathbf{\Gamma}_{ts}^{k,\eta}|^2 \lesssim |t-s|^{2k\kappa}|\varepsilon - \eta|^{2(\alpha-\kappa)} \quad (1.12)$$

hence, by the same arguments,

$$\mathbb{E}\|\mathbf{\Gamma}^{k,\varepsilon} - \mathbf{\Gamma}^{k,\eta}\|_{2,k\kappa} \lesssim |\varepsilon - \eta|^{\alpha-\kappa} \quad (1.13)$$

which shows that $\mathbf{\Gamma}^{k,\varepsilon}$ is a Cauchy sequence in $C_2^{k\kappa}([0, T], \mathbb{R}^{d^k})$. \square

Let us conclude with the following remark, which reduces the construction of a rough path for Γ to that of a rough path for $\tilde{\Gamma}$ defined by eq. (0.9), The following Proposition is intuitively true (an explicit reference is missing for that).

Proposition 1.6 *Let $\tilde{\Gamma}$ be an α^- -Hölder path such that*

$$(\delta\Gamma)_{ts} = \int_{\phi^{-1}(s)}^{\phi^{-1}(t)} \phi'(x)(f \circ \phi)(x) d\tilde{\Gamma}_x. \quad (1.14)$$

for some C^∞ -function f and some C^∞ -diffeomorphism ϕ . Assume $\tilde{\Gamma}$ is a rough path lying above $\tilde{\Gamma}$. Then there is an explicit method to construct a rough path $\mathbf{\Gamma}$ above Γ derived from $\tilde{\Gamma}$. The method is functorial and stable under limits, so that in particular, if $\tilde{\Gamma} = \lim_{\eta \rightarrow 0} \tilde{\Gamma}^\eta$, then $\mathbf{\Gamma} = \lim_{\eta \rightarrow 0} \mathbf{\Gamma}^\eta$.

'Proof'. Fix $s = t_0$. Eq. (1.14) writes after an integration by parts

$$\Gamma_t - \Gamma_{t_0} = \left(\phi'(\phi^{-1}(t))f(t)\tilde{\Gamma}_{\phi^{-1}(t)} - \phi'(\phi^{-1}(t_0))f(t_0)\tilde{\Gamma}_{\phi^{-1}(t_0)} \right) - \int_{\phi^{-1}(t_0)}^{\phi^{-1}(t)} (\phi' \cdot f \circ \phi)'_x \tilde{\Gamma}_x dx. \quad (1.15)$$

In other words, Γ is essentially obtained from $\tilde{\Gamma}$ by a regular time-change and the addition of a C^1 -path. Note that (formally or if Γ is regular) eq. (1.14) writes simply $\Gamma'_u = f(u)\tilde{\Gamma}'_{\phi^{-1}(u)}$, which is eq. (0.8) with $\phi = \Phi^{-1}$ and $f(u) = (1 - \Phi(u))^{-2\alpha}$. \square

2 Preliminary results on random Fourier series

Let $f(z) = \sum_{n \in \mathbb{Z}} z^n X_n$ ($z \in \mathcal{C}$) be a random Fourier series constructed out of a sequence of independent centered complex-valued random variables $(X_n)_{n \in \mathbb{Z}}$. The function f may be seen as the boundary value on \mathcal{C} of the sum of an (a priori formal) analytic series $f^+(z) := \sum_{n \geq 0} z^n X_n$ and of an anti-analytic series $f^-(\bar{z}) := \sum_{n \geq 1} \bar{z}^n X_{-n}$, both living on the unit disk. Under very general growth conditions on the sequence $(\text{Var} X_n)_{n \in \mathbb{Z}}$, f^+ , resp. f^- may be shown to be analytic, resp. anti-analytic on the open unit disk. Hence

$$f^r(z) := f^+(rz) + f^-(r\bar{z}) \quad (z \in \mathcal{C}) \quad (2.1)$$

is real-analytic on \mathcal{C} and may be considered as the extension of f to the circle $\mathcal{C}^r := \{z \in \mathbb{C} \mid |z| = r\}$ with radius $r < 1$.

A general question could be to prove Hölder regularity properties of f (or, equivalently, of f^+ and f^-) under appropriate hypotheses on the 'growth'

of the sequence (X_n) , as measured for instance by their variances. 'Almost' optimal hypotheses for *symmetric* variables (X_n) is given in J.-P. Kahane's book [20]; roughly speaking, f is a.s. α^- -Hölder if and only if

$$\sum_{2^j \leq |n| < 2^{j+1}} \text{Var} X_n = O(2^{-2j\alpha}) \quad (2.2)$$

(see [20], section 7.4 for a correct statement, and also the references at the end of chapter 7). In particular, if $\text{Var} X_n = O(n^{-1-2\alpha})$, then f is a.s. α^- -Hölder. The statements in [20] are much more refined; for instance, the modulus of continuity of f , $\omega_f(h) := \sup_{d(z,z') \leq h} |f(z) - f(z')|$, may be proved (under condition (2.2)) to be a $O((|\log h|)^\gamma h^\alpha)$ for an appropriate γ , with $\gamma = \frac{1}{2}$ if the (X_n) are (sub-)Gaussian variables.

It turns out to be much easier in our setting (where f belongs to the n -th chaos of a fixed Gaussian process) to show L^2 -estimates such as $\mathbb{E}|f(z) - f(z')|^2 \leq C|z - z'|^{2n\alpha}$, from which f may be proved to be a.s. $n\kappa$ -Hölder continuous for every $\kappa < \alpha$, by using Kolmogorov's lemma together with the equivalence of L^p -norms.

We shall prove elementary uniform L^2 -estimates for the function f extended continuously to the closed unit disk \bar{D} as

$$f(z) := f^+(z) + f^-(\bar{z}), \quad (2.3)$$

and also for (non necessarily converging) truncated series or the remainder of the series (in the converging case).

In the sequel, we shall always consider random Fourier series $f(z) := \sum_{n \in \mathbb{Z}} z^n X_n$, $z \in \mathcal{C}$, constructed out of a sequence $(X_n)_{n \in \mathbb{Z}}$ of independent, L^2 , centered random variables, together with their above extension $f(z) := f^+(z) + f^-(\bar{z})$ to \bar{D} . For the sake of brevity, we shall simply say that f is a centered random Fourier series.

Notation.

Let $f(z) := \sum_{n \in \mathbb{Z}} a_n z^n$ be a Fourier series. Then ${}^N f(z) := \sum_{|n| \leq N} a_n z^n$ is the truncation to order N of f , and (if f converges) ${}_N f(z) := \sum_{|n| > N} a_n z^n$ is the remainder.

Lemma 2.1 (modulus of continuity for symmetric series) *Let $f(z) := \sum_{n \in \mathbb{Z}} z^n X_n$ be a centered random Fourier series. Assume there is some $\alpha \in (0, 1)$ such that*

$$\left(\sum_{2^j \leq |n| < 2^{j+1}} \text{Var} X_n \right)^{1/2} = O(2^{-j\alpha}) \quad (2.4)$$

(This holds true in particular if $\text{Var}X_n = O(|n|^{-1-2\alpha})$). Then there exists a constant C such that

$$\sup_{z \in \overline{\mathcal{D}}} \sup_{N \geq 0} \mathbb{E}|^N f(z)|^2 \leq C \quad (2.5)$$

and

$$\sup_{z, z' \in \overline{\mathcal{C}}} \sup_{N \geq 0} \mathbb{E}|^N f(z) - ^N f(z')|^2 \leq C|z - z'|^{2\alpha}. \quad (2.6)$$

Proof.

The convergence of the series $\mathbb{E}|^N f(z)|^2$ when $N \rightarrow \infty$ is obvious. Concerning the second inequality, one has:

$$\begin{aligned} \mathbb{E}|^N f(z) - ^N f(z')|^2 &\leq \mathbb{E}|f(z) - f(z')|^2 \leq \sum_{n \in \mathbb{N}} |z^n - (z')^n|^2 (\text{Var}X_n + \text{Var}X_{-n}) \\ &\lesssim |z - z'|^2 \sum_{|n| \leq 1/|z-z'|} n^2 \text{Var}X_n + \sum_{|n| > 1/|z-z'|} \text{Var}X_n \end{aligned} \quad (2.7)$$

by Taylor's formula. Now (grouping together the $(\text{Var}X_n)$ for $2^j \leq |n| < 2^{j+1}$) one gets the result. \square

Let us now consider successively estimates for truncated centered Fourier series and for the remainder of convergent centered Fourier series.

Lemma 2.2 (estimates for truncated centered Fourier series) *Let $f(z) = \sum_{n \in \mathbb{Z}} z^n X_n$ be a centered random Fourier series. Assume, as in Lemma 2.1, that there is some $\alpha \in (0, 1)$ such that*

$$\left(\sum_{2^j \leq |n| < 2^{j+1}} \text{Var}X_n \right)^{1/2} = O(2^{-j\alpha}) \quad (2.8)$$

(in particular if $\text{Var}X_n \lesssim |n|^{-1-2\alpha}$). Then

$$\mathbb{E}|^N f(z) - ^N f(z')|^2 \lesssim (|z - z'|N^{1-\alpha})^2. \quad (2.9)$$

The same bound also holds true if $\alpha \leq 0$ (case of a divergent series) provided $N \lesssim \frac{1}{|z-z'|}$.

Proof.

Let $\alpha < 1$. As in the proof of Lemma 2.1,

$$\mathbb{E}|^N f(z) - ^N f(z')|^2 \lesssim |z - z'|^2 \sum_{|n| \leq N} n^2 \text{Var} X_n \lesssim (|z - z'| N^{1-\alpha})^2 \quad (2.10)$$

if $N \leq \frac{1}{|z-z'|}$. Otherwise (by Lemma 2.1), if $\alpha > 0$ (so the series converges),

$$\mathbb{E}|^N f(z) - ^N f(z')|^2 \lesssim |z - z'|^{2\alpha} \lesssim (|z - z'| N^{1-\alpha})^2. \quad (2.11)$$

□

Lemma 2.3 (remainder estimate for convergent Fourier series) *Let $f(z) = \sum_{n \in \mathbb{Z}} z^n X_n$ be a centered random Fourier series. Assume there is some $\alpha \in (0, 1)$ such that*

$$\left(\sum_{2^j \leq |n| < 2^{j+1}} \text{Var} X_n \right)^{1/2} = O(2^{-j\alpha}) \quad (2.12)$$

(in particular if $\text{Var} X_n = O(|n|^{-1-2\alpha})$).

Then

$$\mathbb{E}|_N f(z)|^2 \leq CN^{-2\alpha}. \quad (2.13)$$

Proof. Elementary. □

3 Series decomposition and approximation of fractional Brownian motion

3.1 Series decomposition of analytic fractional Brownian motion

The easiest way to define $\Gamma = \{\Gamma_z; z \in \Pi^+\}$ makes use of a series expansion involving the analytic functions $\{f_k; k \geq 0\}$, defined on Π^+ by:

$$f_k(z) = a_{k+1} \left[\frac{z+i}{2i} \right]^{2\alpha-2} \left[\frac{z-i}{z+i} \right]^k. \quad (3.1)$$

where

$$a_{k+1} = 2^{\alpha-1} \left[\frac{\alpha(1-2\alpha)(2-2\alpha)_k}{2 \cos(\pi\alpha) k!} \right]^{1/2}. \quad (3.2)$$

The Pochhammer symbol $(x)_k$ is defined by:

$$(x)_k = \prod_{j=0}^{k-1} (x+j) = \frac{\Gamma(x+k)}{\Gamma(x)},$$

where Γ is the usual Gamma function.

Note that

$$a_k \sim_{k \rightarrow \infty} 2^{\alpha-1} \left[\frac{\alpha(1-2\alpha)}{2 \cos \pi \alpha} \right]^{\frac{1}{2}} k^{\frac{1}{2}-\alpha}. \quad (3.3)$$

It is shown in [32] that the series $\sum_{k \geq 0} f_k(z) \overline{f_k(w)}$ converges in absolute value for $z, w \in \Pi^+$, and that the following identity holds true:

$$\sum_{k \geq 0} f_k(z) \overline{f_k(w)} = \frac{\alpha(1-2\alpha)}{2 \cos(\pi\alpha)} (-i(z-\bar{w}))^{2\alpha-2}. \quad (3.4)$$

This fact allows to define the process Γ in the following way:

Proposition 3.1 (see [32] or [31]) *Let $\{\xi_k^1, \xi_k^2; k \geq 1\}$ be two families of independent standard Gaussian random variables, defined on a complete probability space, and for $k \geq 0$, set $\xi_k = \xi_k^1 + i\xi_k^2$. Consider the process Γ' defined for $z \in \Pi^+$ by $\Gamma'_z = \sum_{k \geq 0} f_k(z) \xi_{k+1}$. Then:*

1. Γ' is a well-defined analytic process on Π^+ .
2. Let $\Gamma : (0, 1) \rightarrow \Pi^+$ be any continuous path with endpoints $\Gamma(0) = 0$ and $\Gamma(1) = z$, and set $\Gamma_z = \int_{\Gamma} \Gamma'_u du$. Then Γ is an analytic process on Π^+ . Furthermore, as z runs along any path in Π^+ going to $t \in \mathbb{R}$, the random variables Γ_z converge almost surely to a random variable called again Γ_t .
3. The family $\{\Gamma_t; t \in \mathbb{R}\}$ defines a Gaussian centered complex-valued process, whose covariance function is given by:

$$\mathbb{E}[\Gamma_s \Gamma_t] = 0, \quad \mathbb{E}[\Gamma_s \bar{\Gamma}_t] = \frac{e^{-i\pi\alpha \operatorname{sgn}(s)} |s|^{2\alpha} + e^{i\pi\alpha \operatorname{sgn}(t)} |t|^{2\alpha} - e^{i\pi\alpha \operatorname{sgn}(t-s)} |s-t|^{2\alpha}}{4 \cos(\pi\alpha)}.$$

The paths of this process are almost surely κ -Hölder for any $\kappa < \alpha$.

4. The real part of $\{\Gamma_t; t \in \mathbb{R}\}$ is a fractional Brownian motion indexed by \mathbb{R} .

Lemma 3.2 Let $U := \{z = t + i\eta \in \mathbb{C} \mid 0 \leq t, \eta \leq \frac{1}{2}\}$. Then:

1. There exists a constant C such that:

$$\forall z, z' \in U, \mathbb{E}|\Gamma(z) - \Gamma(z')|^2 \leq C|z - z'|^{2\alpha}. \quad (3.5)$$

2. There exists a.s. a constant $C > 0$ such that

$$\sup_{z, z' \in U} \frac{|\Gamma(z) - \Gamma(z')|}{|z - z'|^\alpha \log^{\frac{1}{2}}(1/|z - z'|)} \leq C. \quad (3.6)$$

Proof.

1. See [32], Lemma 1.5.

2. Let $\Delta_z := \int_0^z \left(\frac{u+i}{2i}\right)^{-2\alpha} \Gamma'_u du$. By the above Proposition,

$$\Delta_z - \Delta_w = 4 \int_w^z \left[\sum_{k \geq 0} a_{k+1} \xi_{k+1} \left(\frac{u-i}{u+i}\right)^k \right] \left(\frac{u+i}{i}\right)^{-2} du.$$

Now use the Cayley transform $\Phi : \Pi^+ \rightarrow D$, $z \rightarrow \zeta := \frac{z-i}{z+i}$, with inverse $z = \Phi^{-1}(\zeta) = i\frac{1+\zeta}{1-\zeta}$: then

$$\Delta_z - \Delta_w = 2i \sum_{k \geq 0} a_{k+1} \xi_{k+1} \int_{\Phi(w)}^{\Phi(z)} \zeta^k d\zeta = 2i \sum_{k \geq 1} \frac{a_k}{k} \xi_k (\Phi(z)^k - \Phi(w)^k).$$

Consider the time-changed process $\tilde{\Gamma} := \Delta \circ \Phi^{-1}$. Note that

$$\tilde{\Gamma}'_\zeta = (1 - \zeta)^{2\alpha} \Gamma'_{\Phi^{-1}(\zeta)} \quad (3.7)$$

and conversely,

$$\Gamma'_z = \left(\frac{z+i}{2i}\right)^{2\alpha} \tilde{\Gamma}'_{\Phi(z)}. \quad (3.8)$$

Then $\tilde{\Gamma}$ lives on the unit disk \mathcal{D} , which allows to use standard results on random Gaussian series on the unit circle by an immediate extension to the closed unit disk. We define the modulus of continuity of $\tilde{\Gamma}$ to be the function

$$\omega(h) := \sup_{\zeta, \zeta' \in \overline{\mathcal{D}}, |\zeta - \zeta'| \leq h} |\tilde{\Gamma}_\zeta - \tilde{\Gamma}_{\zeta'}|.$$

Recall $a_k \sim_{k \rightarrow \infty} c_\alpha k^{\frac{1}{2} - \alpha}$ for some positive constant c_α . By Theorem 7.3.2 in [20] (see preamble to section 2) there exists a.s. a constant C such that

$$\sup_{\zeta, \zeta' \in \overline{\mathcal{D}}, |\zeta - \zeta'| \leq 1/2} \frac{|\tilde{\Gamma}_\zeta - \tilde{\Gamma}_{\zeta'}|}{|\zeta - \zeta'|^\alpha \log^{\frac{1}{2}}(1/|\zeta - \zeta'|)} \leq C. \quad (3.9)$$

Now

$$\begin{aligned} \Gamma_{z'} - \Gamma_z &= \int_z^{z'} \left(\frac{u+i}{2i} \right)^{2\alpha} \tilde{\Gamma}'_{\Phi(u)} du \\ &= \left(\frac{z'+i}{2i} \right)^{2\alpha} (\tilde{\Gamma}_{\Phi(z')} - \tilde{\Gamma}_{\Phi(z)}) + i\alpha \int_z^{z'} \left(\frac{u+i}{2i} \right)^{2\alpha-1} (\tilde{\Gamma}_{\Phi(u)} - \tilde{\Gamma}_{\Phi(z)}) du. \end{aligned} \quad (3.10)$$

Since the prefactors $\left(\frac{z'+i}{2i} \right)^{2\alpha}$, $\left(\frac{u+i}{2i} \right)^{2\alpha-1}$ are bounded (and bounded away from zero) in U , and $\Phi : U \rightarrow \Phi(U)$ is a diffeomorphism, the same bound (up to a constant) holds for $\Gamma_z - \Gamma_{z'}$.

□

3.2 Approximation of fractional Brownian motion

A natural way to get a regular approximation of B_t is the following.

Definition 3.3 *Let, for $\eta > 0$,*

$$B_t^\eta := 2\operatorname{Re} \int_{i\eta}^{t+i\eta} \Gamma'_u du = 2\operatorname{Re} (\Gamma_{t+i\eta} - \Gamma_{i\eta}). \quad (3.11)$$

Note that $t \rightarrow B_t^\eta$ has a.s. real-analytic paths for every fixed $\eta > 0$.

Proposition 3.4 *(see [32])*

Let $T > 0$. The process B^η converges in $L^2(\Omega; C_1^\kappa([0, T]))$ to B for every $\kappa < \alpha$.

3.3 A variant of fBm: the unit-disk fractional Brownian motion

We shall now introduce a process \tilde{B} living on the unit disk, called *unit-disk fractional Brownian motion*, which is more or less equivalent (via the Cayley transform) to fBm.

Definition 3.5 (unit-disk Γ -process) *Let $\tilde{\Gamma}'$ be the random series defined on the open unit disk by*

$$\tilde{\Gamma}'_{\zeta} := \sum_{k \geq 0} a_{k+1} \zeta^k \xi_{k+1}. \quad (3.12)$$

Its covariance kernel writes:

$$\mathbb{E} \tilde{\Gamma}'_{\zeta_1} \overline{\tilde{\Gamma}'_{\zeta_2}} = c_{\alpha}^2 (1 - \zeta_1 \bar{\zeta}_2)^{2\alpha-2}. \quad (3.13)$$

By standard arguments, see for instance [32], one may prove that the above series defines an analytic process on the open unit disk \mathcal{D} . Set

$$\tilde{\Gamma}_{\zeta} := \int_{\gamma} \tilde{\Gamma}'_{\zeta'} d\zeta' = \sum_{k \geq 1} a_k \frac{\zeta^k}{k} \xi_k, \quad \zeta \in \mathcal{D} \quad (3.14)$$

where $\gamma : [0, 1] \rightarrow \mathcal{D}$ is any path in \mathcal{D} such that $\gamma(0) = 0$ and $\gamma(1) = \zeta$. Then (the proof is exactly the same as for the original Γ -process) $\tilde{\Gamma}$ may be extended into an α^- -Hölder process on the closed unit disk $\overline{\mathcal{D}}$.

Definition 3.6 (unit-disk fBm) *Let $\tilde{B}_{\tau}^r = 2\operatorname{Re} \int_0^{r\tau} \tilde{\Gamma}'_{\zeta} d\zeta = 2\operatorname{Re} \tilde{\Gamma}_{r\tau}$ ($r \leq 1$) and $\tilde{B}_{\tau} := \tilde{B}_{\tau}^1 = 2\operatorname{Re} \int_0^{\tau} \tilde{\Gamma}'_{\zeta} d\zeta$.*

The Proposition below may be proved in exactly the same way as Proposition 3.4.

Proposition 3.7 *When $r \rightarrow 1^-$, the process \tilde{B}^r converges in $L^2(\Omega; C_1^{\kappa}(\mathcal{C}))$ to \tilde{B} for every $\kappa < \alpha$.*

The unit-disk Γ -process was originally introduced in the proof of Lemma 3.2. As noted there,

$$\Gamma'_z = \left(\frac{z+i}{2i} \right)^{2\alpha} \tilde{\Gamma}'_{\Phi(z)}.$$

The extra prefactor $\left(\frac{z+1}{2i}\right)^{2\alpha}$ is C^∞ (actually, analytic) for $z \in \Pi^+$, so $\tilde{\Gamma}$ is essentially obtained from Γ by a C^∞ time-change. Hence (see Proposition 1.6), the construction of a rough-path for B follows in a straightforward way from the construction of a rough-path for \tilde{B} . Working with \tilde{B} instead of B will make it possible to use random Fourier series techniques as in section 2.

Note once again the 'change of origin' ($\tilde{\Gamma}$ vanishes at 0 and not at $-1 = \Phi(0)$). This is a priori not important since we shall always consider *increments* of Γ . However, this change of origin will prove to be essential to get Hölder regularity estimates in the sequel (see next section).

4 Definition of a Lévy area for the unit-disk fBm

4.1 Definitions and notations.

Let $\tilde{\Gamma} := (\tilde{\Gamma}(1), \tilde{\Gamma}(2))$ be a two-component unit-disk Γ -process (with Hurst exponent $\alpha < \frac{1}{2}$) living on the closed unit disk $\bar{\mathcal{D}}$, with series decomposition

$$\tilde{\Gamma}'_\zeta(j) = \sum_{k \geq 0} a_{k+1} \zeta^k \xi_{k+1}(j), \quad \zeta \in \mathcal{D}, \quad j = 1, 2 \quad (4.1)$$

and $\tilde{B}_\tau^r := (2\operatorname{Re} \int_0^{r\tau} \tilde{\Gamma}'_\zeta(1) d\zeta, 2\operatorname{Re} \int_0^{r\tau} \tilde{\Gamma}'_\zeta(2) d\zeta)$ ($r \leq 1$) be the corresponding family of approximations of the unit-disk fBm. It proves convenient to write $\tilde{\Gamma}_\zeta(\bar{j})$ instead of $\overline{\tilde{\Gamma}_\zeta(j)}$ and similarly, $\xi_k(\bar{j})$ instead of $-\bar{\xi}_k(j)$ (note the coefficients (a_k) are *real*). We shall use this notation in the whole section. Using the real-valued angle measure $-i \frac{d\zeta}{\zeta} = d\theta$ instead of $d\zeta$, we shall write

$$\int_{r\sigma}^{r\tau} \tilde{\Gamma}'_\zeta(j) d\zeta = \int_{r\sigma}^{r\tau} \left(\sum_{k \geq 1} a_k \zeta^k \xi_k(j) \right) \frac{d\zeta}{\zeta} = \int_\sigma^\tau \left(\sum_{k \geq 1} r^k a_k \zeta^k \xi_k(j) \right) \frac{d\zeta}{\zeta}, \quad (4.2)$$

with complex conjugate $\int_\sigma^\tau \left(\sum_{k \geq 1} r^k a_k \zeta^{-k} \xi_k(\bar{j}) \right) \frac{d\zeta}{\zeta}$.

We want to define a two-time process $\tilde{\mathbf{B}}_{\tau\sigma}^{2,r}(1, 2)$ which is a Lévy area for \tilde{B}^r , i.e. such that $\delta \tilde{\mathbf{B}}_{\tau u \sigma}^{2,r}(1, 2) = \tilde{B}_{\tau u}^r(1) \tilde{B}_{u\sigma}^r(2)$, and converges when $r \rightarrow 1^-$ in $L^2(\Omega; C_1^{2\kappa}(\mathcal{C}))$ to some limiting process $\tilde{\mathbf{B}}_{\tau\sigma}^2(1, 2)$ for every $\kappa < \alpha$, to be interpreted as the Lévy area of \tilde{B} (see section 1).

The natural candidate for $\tilde{\mathbf{B}}^{2,r}(1, 2)$ is simply the usual, uncorrected area process.

Definition 4.1 (uncorrected area process) For $r < 1$ and $\sigma, \tau \in \mathcal{C}$, let $\tilde{\mathcal{A}}_{\tau\sigma}^r$ denote the following second-order iterated integral of the unit-disk fBm \tilde{B} :

$$\tilde{\mathcal{A}}_{\tau\sigma}^r := 2\text{Re} \int_{\sigma}^{\tau} \tilde{\Gamma}'_{r\zeta_1}(1) d\zeta_1 \left(\int_{\sigma}^{\zeta_1} \tilde{\Gamma}'_{r\zeta_2}(2) d\zeta_2 + \text{c.c.} \right) \quad (4.3)$$

where *c.c.* stands for 'complex conjugate'.

It is closely related to the Lévy area defined in [32] and hence diverges when $r \rightarrow 1^-$.

More precisely, decompose $\tilde{\mathcal{A}}_{\tau\sigma}^r$ into

$$\tilde{\mathcal{A}}_{\tau\sigma}^r := \tilde{\Gamma}_{\tau\sigma}^{2,r}(1, 2) + \tilde{\Gamma}_{\tau\sigma}^{2,r}(1, \bar{2}) + \tilde{\Gamma}_{\tau\sigma}^{2,r}(\bar{1}, 2) + \tilde{\Gamma}_{\tau\sigma}^{2,r}(\bar{1}, \bar{2}), \quad (4.4)$$

where $\tilde{\Gamma}_{\tau\sigma}^{2,r}(1, 2)$, resp. $\tilde{\Gamma}_{\tau\sigma}^r(\bar{1}, \bar{2})$ is the usual Lévy area of $\tilde{\Gamma}^r$, resp. $\overline{(\tilde{\Gamma}^r)}$, and

$$\tilde{\Gamma}_{\tau\sigma}^{2,r}(\bar{1}, 2) = \int_{\sigma}^{\tau} \tilde{\Gamma}'_{r\zeta_1}(\bar{1}) d\zeta_1 \int_{\sigma}^{\zeta_1} \tilde{\Gamma}'_{r\zeta_2}(2) d\zeta_2, \quad \tilde{\Gamma}_{\tau\sigma}^{2,r}(1, \bar{2}) = \overline{\left(\tilde{\Gamma}_{\tau\sigma}^{2,r}(\bar{1}, 2) \right)}. \quad (4.5)$$

It is a consequence of [32] (see also [31, 33] for more precise statements) that $\tilde{\Gamma}_{\tau\sigma}^r(1, 2)$ converges in $L^2(\Omega; C_1^{2\kappa}(\mathcal{C}))$ while $\tilde{\Gamma}_{\tau\sigma}^r(\bar{1}, 2)$ diverges; in particular, $\mathbb{E} \left| \tilde{\Gamma}_{\tau\sigma}^r(\bar{1}, 2) \right|^2 \rightarrow_{r \rightarrow 1^-} \infty$.

The idea is now to *subtract* to $\tilde{\Gamma}_{\tau\sigma}^r(\bar{1}, 2)$ some *increment counterterm* $\delta Z_{\tau\sigma}^r(\bar{1}, 2) := Z_{\tau}^r(\bar{1}, 2) - Z_{\sigma}^r(\bar{1}, 2)$, so that:

- the multiplicative property (ii) in the Introduction still holds, i.e. $\delta(\tilde{\Gamma}^{2,r}(\bar{1}, 2) - \delta Z^r(\bar{1}, 2)) = \delta\tilde{\Gamma}^{2,r}(\bar{1}, 2)$ since $\delta \circ \delta = 0$ (see section 1);
- the *regularized Lévy area* $\mathcal{R}\tilde{\Gamma}^{2,r}(\bar{1}, 2) := \tilde{\Gamma}^{2,r}(\bar{1}, 2) - \delta Z^r$ converges in $L^2(\Omega; C_1^{2\kappa}(\mathcal{C}))$.

For that purpose, let us use the series decomposition of $\tilde{\Gamma}$:

Lemma 4.2 *Let, for $k_1, k_2 \geq 1$,*

$$F_{\bar{k}_1, k_2}(\sigma, \tau) := \int_{\sigma}^{\tau} a_{k_1} \zeta_1^{-k_1} \frac{d\zeta_1}{\zeta_1} \int_{\sigma}^{\zeta_1} a_{k_2} \zeta_2^{k_2} \frac{d\zeta_2}{\zeta_2} \quad (4.6)$$

so that

$$\tilde{\Gamma}_{\tau\sigma}^{2,r}(\bar{1}, 2) = \sum_{k_1 \geq 1} \sum_{k_2 \geq 1} r^{k_1 + k_2} F_{\bar{k}_1, k_2}(\sigma, \tau) \xi_{k_1}(\bar{1}) \xi_{k_2}(2). \quad (4.7)$$

Then

$$F_{\bar{k}_1, k_2}(\sigma, \tau) = F_{\bar{k}_1, k_2}^\partial(\sigma, \tau) + (\delta G_{\bar{k}_1, k_2})_{\tau\sigma}, \quad (4.8)$$

where

$$F_{\bar{k}_1, k_2}^\partial(\sigma, \tau) = -a_{k_2} \frac{\sigma^{k_2}}{k_2} \int_\sigma^\tau a_{k_1} \zeta_1^{-k_1} \frac{d\zeta_1}{\zeta_1} \quad (k_1 \leq k_2), \quad -a_{k_1} \frac{\tau^{-k_1}}{k_1} \int_\sigma^\tau a_{k_2} \zeta_2^{k_2} \frac{d\zeta_2}{\zeta_2} \quad (k_1 > k_2) \quad (4.9)$$

and

$$G_{\bar{k}_1, k_2}(\zeta) = a_{k_1} \frac{a_{k_2}}{k_2} \frac{\zeta^{k_2-k_1}}{k_2-k_1} \quad (k_1 < k_2), \quad a_{k_2} \frac{a_{k_1}}{k_1} \frac{\zeta^{k_2-k_1}}{k_2-k_1} \quad (k_1 > k_2), \\ \frac{a_{k_1}^2}{k_1} \ln \zeta \quad (k_1 = k_2). \quad (4.10)$$

Proof. The formulas for $k_1 \leq k_2$ are straightforward (replace $\int_\sigma^{\zeta_1} d\zeta_2$ with $-\int_0^\sigma d\zeta_2 + \int_0^{\zeta_1} d\zeta_2$, using once again as origin 0 instead of $-1 = \Phi(0)$). To obtain the formulas for $k_2 < k_1$, rewrite $F_{\bar{k}_1, k_2}(\sigma, \tau)$ as $\int_\sigma^\tau a_{k_2} \zeta_2^{k_2} \frac{d\zeta_2}{\zeta_2} \int_{\zeta_2}^\tau a_{k_1} \zeta_1^{-k_1} \frac{d\zeta_1}{\zeta_1}$. Note that the two decompositions of F into $F^\partial + \delta G$ coincide when $k_1 = k_2$. \square

Definition 4.3 (regularized unit-disk Lévy area) *Let*

$$Z_\zeta^r(1, 2) := 0, \quad Z_\zeta^r(\bar{1}, 2) := \sum_{k_1 \geq 1} \sum_{k_2 = k_1/2}^{2k_1} r^{k_1+k_2} G_{\bar{k}_1, k_2}(\zeta) \xi_{k_1}(\bar{1}) \xi_{k_2}(2), \quad (4.11)$$

$$Z_\zeta^r(\bar{1}, \bar{2}) = \overline{Z_\zeta^r(1, 2)} = 0, \quad Z_\zeta^r(1, \bar{2}) = \overline{Z_\zeta^r(\bar{1}, 2)}; \quad (4.12)$$

$$\mathcal{R}\tilde{\Gamma}_{\tau\sigma}^{2,r}(i, j) := \tilde{\Gamma}_{\tau\sigma}^{2,r}(i, j) - (\delta Z^r(i, j))_{\tau\sigma} \quad (4.13)$$

with $i = 1$ or $\bar{1}$, $j = 2$ or $\bar{2}$, and finally,

$$\mathcal{R}\tilde{\mathcal{A}}_{\tau\sigma}^r := \mathcal{R}\tilde{\Gamma}_{\tau\sigma}^{2,r}(1, 2) + \mathcal{R}\tilde{\Gamma}_{\tau\sigma}^{2,r}(1, \bar{2}) + \mathcal{R}\tilde{\Gamma}_{\tau\sigma}^{2,r}(\bar{1}, 2) + \mathcal{R}\tilde{\Gamma}_{\tau\sigma}^{2,r}(\bar{1}, \bar{2}). \quad (4.14)$$

Note that the *geometricity property* (iii) in the Introduction holds for $\mathcal{R}\tilde{\mathcal{A}}^r$, which is equivalent to saying that

$$\mathcal{R}\tilde{\Gamma}^{2,r}(\bar{1}, 2) + \mathcal{R}\tilde{\Gamma}^{2,r}(2, \bar{1}) = \tilde{\Gamma}^{2,r}(\bar{1}, 2) + \tilde{\Gamma}^{2,r}(2, \bar{1}), \quad (4.15)$$

or simply, $\delta Z^r(\bar{1}, 2) + \overline{\delta Z^r(2, 1)} = 0$. The last equality is an immediate consequence of the *antisymmetry property* $G_{\bar{k}_1, k_2}(\zeta) + \overline{G_{k_2, k_1}(\zeta)} = 0$.

Recall $d(\sigma, \tau)$ ($\sigma, \tau \in \mathcal{C}$) is the distance along the unit circle.

We shall prove in this section the following two results.

Theorem 4.1 (Hölder estimate) *Let $\alpha < \frac{1}{2}$. There exists a constant C such that, for every $r < 1$ and $\sigma, \tau \in \mathcal{C}$,*

$$\mathbb{E} \left| \mathcal{R}\tilde{\mathcal{A}}_{\tau\sigma}^r \right|^2 \leq C d(\sigma, \tau)^{4\alpha}. \quad (4.16)$$

Theorem 4.2 (rate of convergence) *Let $\alpha < \frac{1}{2}$. There exists a constant C such that, for every $\frac{1}{2} < r_1, r_2 < 1$ and $\sigma, \tau \in \mathcal{C}$,*

$$\mathbb{E} \left| \mathcal{R}\tilde{\mathcal{A}}_{\tau\sigma}^{r_1} - \mathcal{R}\tilde{\mathcal{A}}_{\tau\sigma}^{r_2} \right|^2 \leq C |r_1 - r_2|^{2\alpha}. \quad (4.17)$$

4.2 Uniform Hölder estimate for the regularized unit-disk Lévy area

We shall prove in this subsection Theorem 4.1.

The regularized iterated integral $\mathcal{R}\tilde{\Gamma}^{2,r}(\bar{1}, 2)$ decomposes further as

$$\mathcal{R}\tilde{\Gamma}^{2,r}(\bar{1}, 2) = \delta\mathcal{R}\tilde{\mathbf{G}}^{2,r}(\bar{1}, 2) + \mathcal{R}\tilde{\Gamma}^{2,r}(\bar{1}, 2)(\partial), \quad (4.18)$$

(called respectively the *increment term* and the *boundary term*), where

$$\mathcal{R}\tilde{\mathbf{G}}_{\zeta}^{2,r}(\bar{1}, 2) = \sum_{k_1, k_2 \geq 1: k_2 \notin [k_1/2, 2k_1]} \sum_{k_1, k_2 \geq 1} r^{k_1+k_2} G_{\bar{k}_1, k_2}(\zeta) \xi_{k_1}(\bar{1}) \xi_{k_2}(2) \quad (4.19)$$

and

$$\mathcal{R}\tilde{\Gamma}_{\tau\sigma}^{2,r}(\bar{1}, 2)(\partial) = \sum_{k_1, k_2 \geq 1} \sum_{k_1, k_2 \geq 1} r^{k_1+k_2} F_{\bar{k}_1, k_2}^{\partial}(\sigma, \tau) \xi_{k_1}(\bar{1}) \xi_{k_2}(2). \quad (4.20)$$

By the overall symmetry $k_1 \leftrightarrow k_2$, $\xi(1) \leftrightarrow \xi(2)$, $\sigma \leftrightarrow \tau$, one may restrict the study to the terms in the double series with indices (k_1, k_2) such that $k_1 \leq k_2$. We write $\mathcal{R}\tilde{\mathbf{G}}^{2,r,+}(\bar{1}, 2)$, resp. $\mathcal{R}\tilde{\mathbf{G}}^{2,r,-}(\bar{1}, 2)$ and $\mathcal{R}\tilde{\Gamma}^{2,r,+}(\bar{1}, 2)(\partial)$, resp. $\mathcal{R}\tilde{\Gamma}^{2,r,-}(\bar{1}, 2)(\partial)$ for the sums over the restricted index set $k_1 \leq k_2$, resp. $k_1 > k_2$ and study henceforth only the terms with index '+' (note that this corresponds to the decomposition of $\tilde{\mathbf{G}}^{2,r}$ into its analytic and its anti-analytic part).

4.2.1 Study of the regularized increment term

One has

$$\begin{aligned} \mathcal{R}\tilde{\mathbf{G}}_z^{2,r,+}(\bar{\mathbf{1}}, 2) &= \\ \sum_{k_1, k_2 \geq 1: k_1 < k_2/2} \sum_{k_1 < k_2} r^{k_1+k_2} G_{\bar{k}_1, k_2}^-(z) \xi_{k_1}(\bar{\mathbf{1}}) \xi_{k_2}(2) &= \sum_{m \geq 1} r^m \frac{z^m}{m} \mathcal{R}X_m^{r,+} \end{aligned} \quad (4.21)$$

where (setting $m = k_2 - k_1$)

$$\mathcal{R}X_m^{r,+} := \sum_{1 \leq k_1 < m} r^{2k_1} \frac{a_{k_1} a_{m+k_1}}{m+k_1} \xi_{k_1}(\bar{\mathbf{1}}) \xi_{m+k_1}(2). \quad (4.22)$$

Now (recall $a_k \sim_{k \rightarrow \infty} c_\alpha k^{\frac{1}{2}-\alpha}$)

$$\text{Var} \mathcal{R}X_m^{r,+} \leq \sum_{1 \leq k_1 < m} \left(\frac{a_{m+k_1}}{m+k_1} a_{k_1} \right)^2 \lesssim \left(\frac{a_m}{m} \right)^2 \sum_{1 \leq k_1 < m} a_{k_1}^2 \lesssim m^{1-4\alpha}. \quad (4.23)$$

Hence (by Lemma 2.1)

$$\mathbb{E} \left[|\mathcal{R}\tilde{\mathbf{G}}_z^{2,r,+}(\bar{\mathbf{1}}, 2) - \mathcal{R}\tilde{\mathbf{G}}_{z'}^{2,r,+}(\bar{\mathbf{1}}, 2)|^2 \right] \leq Cd(z, z')^{4\alpha}. \quad (4.24)$$

Remark. Suppose one removes the restriction $k_1 < k_2/2$. Then the increment term (as expected) diverges when $r \rightarrow 1^-$ for $\alpha \leq \frac{1}{4}$: namely, the variance of $X_m^{1,+} := \sum_{k_1=1}^{\infty} \frac{a_{k_1} a_{m+k_1}}{m+k_1} \xi_{k_1}(\bar{\mathbf{1}}) \xi_{m+k_1}(2)$ converges if and only if $\sum_{k_2=1}^{\infty} k_1^{-4\alpha} < \infty$, i.e. if and only if $\alpha > \frac{1}{4}$.

4.2.2 Study of the boundary term

Cut the boundary term $\mathcal{R}\tilde{\mathbf{I}}^{2,r,+}(\bar{\mathbf{1}}, 2)(\partial)$ into two parts: the first part consists of indices $k_1, k_2 \geq 1$ such that $0 \leq k_1 \leq k_2 \leq \frac{1}{d(\sigma, \tau)}$; the second part consists of the remaining indices, for which by necessity $k_2 > \frac{1}{d(\sigma, \tau)}$. Then the first part writes

$$\begin{aligned} X_{\tau\sigma}^\partial &:= \sum_{1 \leq k_1 \leq k_2 \leq 1/d(\sigma, \tau)} \sum_{k_1 < k_2} r^{k_1+k_2} F_{\bar{k}_1, k_2}^\partial(\sigma, \tau) \xi_{k_1}(\bar{\mathbf{1}}) \xi_{k_2}(2) \\ &= - \sum_{k_2=1}^{1/d(\sigma, \tau)} r^{k_2} a_{k_2} \frac{\sigma^{k_2}}{k_2} \xi_{k_2}(2) \left(\delta^{k_2} \tilde{\mathbf{I}}^r(\bar{\mathbf{1}}) \right)_{\tau\sigma} \end{aligned} \quad (4.25)$$

where ${}^{k_2}\tilde{\Gamma}^r(1)_z := \sum_{k_1=1}^{k_2} r^{k_1} \frac{a_{k_1}}{k_1} z^{k_1}$ (see notation in section 2) is the $\tilde{\Gamma}$ -process truncated to order k_2 .

By Lemma 2.2, there exists a constant C such that

$$\mathbb{E} \left| \left(\delta^{k_2} \tilde{\Gamma}^r(1) \right)_{\tau\sigma} \right|^2 \leq C (d(\sigma, \tau) k_2^{1-\alpha})^2.$$

Since $\frac{a_{k_2}}{k_2} k_2^{1-\alpha} \lesssim k_2^{\frac{1}{2}-2\alpha}$, one gets (recall $\alpha < \frac{1}{2}$):

$$\mathbb{E} |X_{\tau\sigma}^\partial|^2 \lesssim d(\sigma, \tau)^2 \sum_{k_2=1}^{1/d(\sigma, \tau)} k_2^{1-4\alpha} \lesssim d(\sigma, \tau)^{4\alpha}. \quad (4.26)$$

The second part writes

$$\begin{aligned} Y_{\tau\sigma}^\partial &:= \sum_{k_2 > \frac{1}{d(\sigma, \tau)}} \sum_{k_1 \leq k_2} r^{k_1+k_2} F_{k_1, k_2}^\partial(\sigma, \tau) \xi_{k_1}(\bar{1}) \xi_{k_2}(2) \\ &= - \sum_{k_2 > \frac{1}{d(\sigma, \tau)}} r^{k_2} a_{k_2} \frac{\sigma^{k_2}}{k_2} \xi_{k_2}(2) \left(\delta^{k_2} \tilde{\Gamma}^r(\bar{1}) \right)_{\tau\sigma}. \end{aligned} \quad (4.27)$$

By Lemma 2.1,

$$\mathbb{E} \left| \left(\delta^{k_2} \tilde{\Gamma}^r(1) \right)_{\tau\sigma} \right|^2 \lesssim d(\sigma, \tau)^{2\alpha}, \quad (4.28)$$

and by Lemma 2.3, one gets

$$\mathbb{E} |Y_{\tau\sigma}^\partial|^2 \lesssim d(\sigma, \tau)^{4\alpha}. \quad (4.29)$$

□

4.3 Rate of convergence of the regularized unit-disk Lévy area

We shall now prove Theorem 4.2. Let $\frac{1}{2} < r_1, r_2 < 1$. We want to estimate the difference $\mathcal{R}\tilde{\Gamma}_{\tau\sigma}^{2, r_1, +}(\bar{1}, 2) - \mathcal{R}\tilde{\Gamma}_{\tau\sigma}^{2, r_2, +}(\bar{1}, 2)$.

The analytic part of the regularized increment term, $\delta\mathcal{R}X^{r_1, r_2, +} := \delta\mathcal{R}\mathbf{G}^{2, r_1, +}(\bar{1}, 2) - \delta\mathcal{R}\mathbf{G}^{2, r_2, +}(\bar{1}, 2)$ writes $\delta\left(\sum_{m \geq 1} \frac{z^m}{m} \mathcal{R}X_m^{r_1, r_2, +}\right)_{\tau\sigma}$, where

$$\begin{aligned} \mathcal{R}X_m^{r_1, r_2, +} &= \sum_{1 \leq k_1 < m} \left[r_1^{m+2k_1} - r_2^{m+2k_1} \right] \frac{a_{m+k_1}}{m+k_1} a_{k_1} \xi_{k_1}(\bar{1}) \xi_{m+k_1}(2) \\ &= r_1^m \sum_{1 \leq k_1 < m} \left((r_1^{2k_1} - r_2^{2k_1}) \frac{a_{m+k_1}}{m+k_1} a_{k_1} \xi_{k_1}(\bar{1}) \xi_{m+k_1}(2) \right) \\ &\quad + (r_1^m - r_2^m) \sum_{1 \leq k_1 < m} r_2^{2k_1} \frac{a_{m+k_1}}{m+k_1} a_{k_1} \xi_{k_1}(\bar{1}) \xi_{m+k_1}(2). \end{aligned} \quad (4.30)$$

Hence

$$\mathcal{R}X^{r_1, r_2, +}(z) = \sum_{m \geq 1} r_1^m \frac{z^m}{m} (\delta Y_m^1)_{r_1^2, r_2^2} + \delta \left(\sum_{m \geq 1} \frac{w^m}{m} Y_m^2 \right)_{zr_1, zr_2}, \quad (4.31)$$

where

$$Y_m^1(z) := \sum_{1 \leq k_1 < m} \frac{a_{m+k_1}}{m+k_1} a_{k_1} z^{k_1} \xi_{k_1}(\bar{1}) \xi_{m+k_1}(2) \quad (4.32)$$

and

$$Y_m^2 := \sum_{1 \leq k_1 < m} r_2^{2k_1} \frac{a_{m+k_1}}{m+k_1} a_{k_1} \xi_{k_1}(\bar{1}) \xi_{m+k_1}(2). \quad (4.33)$$

The sum with the (Y_m^2) is similar to that treated in subsection 4.2.1, hence contributes a term with variance uniformly bounded by $C|r_2 - r_1|^{4\alpha}$.

As for the sum with the Y_m^1 , by Lemma 2.2,

$$\mathbb{E} \left| (\delta Y_m^1)_{r_1^2, r_2^2} \right|^2 \lesssim (r_2 - r_1)^2 m^{3-4\alpha}, \quad m \lesssim \frac{1}{|r_2 - r_1|} \quad (4.34)$$

on the one hand, and

$$\mathbb{E} \left| (\delta Y_m^1)_{r_1^2, r_2^2} \right|^2 \leq 4 \sup_{z \in \bar{\mathcal{D}}} \mathbb{E} |Y_m^1(z)|^2 \lesssim m^{-1-2\alpha} \sum_{k_1=0}^m k_1^{1-2\alpha} \lesssim m^{1-4\alpha} \quad (4.35)$$

on the other hand. We use the first bound for $m \leq \frac{1}{|r_2 - r_1|}$ and the second one for $m > \frac{1}{|r_2 - r_1|}$. Hence

$$\mathbb{E} \left| \sum_{m \leq 1/|r_2 - r_1|} r_1^m \frac{z^m}{m} (\delta Y_m^1)_{r_1^2, r_2^2} \right|^2 \lesssim (r_2 - r_1)^2 \sum_{m \leq 1/|r_2 - r_1|} m^{1-4\alpha} \lesssim C|r_2 - r_1|^{4\alpha}, \quad (4.36)$$

while $\mathbb{E} \left| \sum_{m>1/|r_2-r_1|} r_1^m \frac{z^m}{m} (\delta Y_m^1)_{r_1^2, r_2^2} \right|$ is bounded (by Lemma 2.3) by a constant times $|r_2 - r_1|^{4\alpha}$.

All together one has proved that

$$\mathbb{E} \left| \mathcal{R}\tilde{\mathbf{G}}_{\tau\sigma}^{2,r_1,+}(\bar{1}, 2) - \mathcal{R}\tilde{\mathbf{G}}_{\tau\sigma}^{2,r_2,+}(\bar{1}, 2) \right|^2 \leq C|r_2 - r_1|^{4\alpha}. \quad (4.37)$$

Consider now the difference of the boundary terms, $X_{\tau\sigma}^{r_1, r_2, +}(\partial) := \mathcal{R}\tilde{\mathbf{\Gamma}}^{2,r_1,+}(\bar{1}, 2)(\partial) - \mathcal{R}\tilde{\mathbf{\Gamma}}^{2,r_2,+}(\bar{1}, 2)(\partial)$: one gets

$$\begin{aligned} X_{\tau\sigma}^{r_1, r_2, +}(\partial) &= - \sum_{k_2=1}^{\infty} (r_1^{k_2} - r_2^{k_2}) a_{k_2} \frac{\sigma^{k_2}}{k_2} \xi_{k_2}(2) \left(\delta^{k_2} \tilde{\Gamma}^{r_1}(\bar{1}) \right)_{\tau\sigma} \\ &\quad - \sum_{k_2=1}^{\infty} r_2^{k_2} a_{k_2} \frac{\sigma^{k_2}}{k_2} \xi_{k_2}(2) \left(\delta^{k_2} \tilde{\Gamma}^{r_1}(\bar{1}) - {}^{k_2} \tilde{\Gamma}^{r_2}(\bar{1}) \right)_{\tau\sigma} \end{aligned} \quad (4.38)$$

The variance of the first term is bounded by a constant times

$$\sup_{z \in \mathcal{C}} \left(\mathbb{E} |\tilde{\Gamma}_z^{r_1}(1)|^2 \right) \cdot \mathbb{E} \left| \left(\delta \tilde{\Gamma}(2) \right)_{r_1\sigma, r_2\sigma} \right|^2 \lesssim |r_2 - r_1|^{2\alpha}. \quad (4.39)$$

As for the second term, one rewrites $\delta^{k_2} \tilde{\Gamma}^{r_1}(\bar{1}) - {}^{k_2} \tilde{\Gamma}^{r_2}(\bar{1})$ as $({}^{k_2} \tilde{\Gamma}_{r_1\tau}(\bar{1}) - {}^{k_2} \tilde{\Gamma}_{r_2\tau}(\bar{1})) - ({}^{k_2} \tilde{\Gamma}_{r_1\sigma}(\bar{1}) - {}^{k_2} \tilde{\Gamma}_{r_2\sigma}(\bar{1}))$ and bounds its variance by $4 \sup_{z \in \mathcal{C}} |\tilde{\Gamma}_{r_1z}(\bar{1}) - \tilde{\Gamma}_{r_2z}(\bar{1})|^2 \lesssim |r_1 - r_2|^{2\alpha}$. Hence the variance of the second term is bounded by $C|r_1 - r_2|^{2\alpha} \cdot \sum_{k_2 \geq 1} k_2^{-1-2\alpha} = C'|r_1 - r_2|^{2\alpha}$. \square

5 Iterated integrals of superior order: easy case

5.1 General outline

Consider a d -dimensional unit-disk analytic fractional Brownian motion $\tilde{\Gamma} = (\tilde{\Gamma}(1), \dots, \tilde{\Gamma}(d))$ with independent coordinates, together with its boundary value $\tilde{B}_\zeta := 2\text{Re } \tilde{\Gamma}_\zeta$. Our purpose is to prove the two following results (see Theorem 0.1), which together (using the theoretical results of section 1) entail the existence of a rough path lying above \tilde{B} .

Theorem 5.1 (Hölder estimate) *Let $1 \leq i_1, \dots, i_n \leq d$ ($n \geq 3$) and $\alpha < 1/n$. Then, for every $\sigma, \tau \in \mathcal{C}$, there exists a constant C such that*

$$\mathbb{E} \left| \mathcal{R}\tilde{\mathbf{B}}_{\tau\sigma}^{n,r}(i_1, \dots, i_n) \right|^2 \leq Cd(\sigma, \tau)^{2n\alpha}. \quad (5.1)$$

Theorem 5.2 (rate of convergence) *Let $1 \leq i_1, \dots, i_n \leq d$ ($n \geq 3$) and $\alpha < 1/n$. Then there exists a constant C such that, for every $\frac{1}{2} < r_1, r_2 < 1$ and $\sigma, \tau \in \mathcal{C}$,*

$$\mathbb{E} \left| \mathcal{R}\tilde{\mathbf{B}}_{\tau\sigma}^{n,r_1}(i_1, \dots, i_n) - \mathcal{R}\tilde{\mathbf{B}}_{\tau\sigma}^{n,r_2}(i_1, \dots, i_n) \right|^2 \leq C|r_1 - r_2|^{2\alpha}. \quad (5.2)$$

The proof is a generalization of the estimates of the previous section. Since it is rather long, it will be divided into several steps and take up essentially the rest of the paper. We restrict in this section to the case $i_1 \neq \dots \neq i_n$, so that (by renumbering the components) we shall assume $i_1 = 1, \dots, i_n = n$.

Consider an iterated integral of order n ,

$$\tilde{\mathbf{B}}_{\tau\sigma}^{n,r}(1, \dots, n) = \int_{\sigma}^{\tau} d\tilde{B}_{\zeta_1}^r(1) \int_{\sigma}^{\zeta_1} d\tilde{B}_{\zeta_2}^r(2) \dots \int_{\sigma}^{\zeta_{n-1}} d\tilde{B}_{\zeta_n}^r(n). \quad (5.3)$$

Decompose each \tilde{B} into series as in section 4. For brevity, we rewrite

$$\tilde{\Gamma}'_z(i) + \tilde{\Gamma}'_z(\bar{i}) = 2\operatorname{Re} \tilde{\Gamma}'_z(i) = \sum_{k \in \mathbb{Z}} a_k z^{k-1} \tilde{\xi}_k(i) \quad (5.4)$$

with $\tilde{\xi}_k(i) = \xi_k(i)$ ($k \geq 1$), and

$$a_{-k} = a_{k+2}, \quad \tilde{\xi}_{-k}(i) = -\bar{\xi}_{k+2}(i) = \xi_{k+2}(\bar{i}) \quad (k \geq 0). \quad (5.5)$$

The main problem (compared to the case of the Lévy area) is to define properly the decomposition of $\tilde{\mathbf{B}}^{n,r}(1, \dots, n)$ into the sum of an *increment term* $\delta\tilde{\mathbf{G}}$ and a *boundary term* $\tilde{\mathbf{B}}(\partial)$.

As we noted in the Introduction, the 'natural' boundary term for the Lévy area, namely, $-\delta\tilde{B}_{\tau\sigma}^r(1)\tilde{B}_{\sigma}^r(2)$, was *not* $2\alpha^-$ -Hölder, but only α^- -Hölder. It is only after *restriction* to the set of indices $\{k_1, k_2 \geq 0 \mid k_1 \leq k_2\}$ that this boundary term becomes $2\alpha^-$ -Hölder. The particular feature about the two-dimensional case is that there is a simple way to rewrite the Lévy area $\int_{\sigma}^{\tau} d\tilde{B}_{\zeta_1}^r(1) \int_{\sigma}^{\zeta_1} d\tilde{B}_{\zeta_2}^r(2)$ by moving $\tilde{B}(1)$ to the right of $\tilde{B}(2)$ as $\int_{\sigma}^{\tau} d\tilde{B}_{\zeta_2}^r(2) \int_{\zeta_2}^{\tau} d\tilde{B}_{\zeta_1}^r(1)$, by using, so to speak, the time-reversal symmetry. Then the correct definition of the boundary term requires to split the Lévy area into a sum over each of the two sets of indices ($k_1 < k_2$), ($k_1 > k_2$), and is given by similar but different expressions on each. Note that diagonal indices (i.e. such that $k_1 = k_2$) play no particular role and may be grouped together with either subset.

The idea in the multi-dimensional setting is to split the set of indices \mathbb{Z}^n into $\cup_{\varepsilon \in \Sigma'_n} \mathbb{Z}_\varepsilon^n$, where Σ'_n is the group of permutations of $\{1, \dots, n\}$, and $\mathbb{Z}_\varepsilon^n = \{(k_1, \dots, k_n) \in \mathbb{Z}^n \mid |k_{\varepsilon(1)}| \leq \dots \leq |k_{\varepsilon(n)}|\}$. Then the correct definition of the boundary term over each set of indices \mathbb{Z}_ε^n is obtained by *permuting* the order of integration of the multiple integral $\tilde{\mathbf{B}}^{n,r}(1, \dots, n)$ according to the permutation ε . Now (except if $\varepsilon = \text{Id}$ is the trivial permutation, or corresponds to time-reversal, i.e. $\varepsilon(i) = n - i + 1$, $i = 1, \dots, n$) the resulting expression is much more complicated, which requires some extra combinatorial considerations (see section 6).

In this section, we shall present our estimates in the easier case when ε is the trivial permutation.

Definition 5.1 1. If $A(z) = \sum_{k_1, \dots, k_n \in \mathbb{Z}} f_{k_1, \dots, k_n}(z) \tilde{\xi}_{k_1}(i_1) \dots \tilde{\xi}_{k_n}(i_n)$, then we let

$$\mathcal{P}^+ A(z) = \sum_{|k_1| \leq \dots \leq |k_n|} f_{k_1, \dots, k_n}(z) \tilde{\xi}_{k_1}(i_1) \dots \tilde{\xi}_{k_n}(i_n) \quad (5.6)$$

be the orthogonal projection of $A(z)$ onto the subspace of the n -th Gaussian chaos of $\tilde{\Gamma}(i_1), \dots, \tilde{\Gamma}(i_n)$ generated by $\{\tilde{\xi}_{k_1}(i_1) \dots \tilde{\xi}_{k_n}(i_n), |k_1| \leq \dots \leq |k_n|\}$.

2. Let $\mathbb{Z}^{n,+} := \{(k_1, \dots, k_n) \in \mathbb{Z}^n \mid |k_1| \leq \dots \leq |k_n|\}$ and $\tilde{\mathbf{B}}^{n,r,+}(1, \dots, n) = \mathcal{P}^+ \tilde{\mathbf{B}}^{n,r}(1, \dots, n)$ be the restriction of the sum defining $\tilde{\mathbf{B}}^{n,r}(1, \dots, n)$ to the set of indices $\mathbb{Z}^{n,+}$.

Generally speaking, all the quantities constructed below which are sums over some subset of $\mathbb{Z}^{n,+}$ bear an extra upper index $'+$ '.

Elementary computations yield an increment/boundary decomposition of the iterated integral,

$$\tilde{\mathbf{B}}_{\tau\sigma}^{n,r,+}(1, \dots, n) = \left(\delta \tilde{\mathbf{G}}^{n,r,+}(1, \dots, n) \right)_{\tau\sigma} + \tilde{\mathbf{B}}_{\tau\sigma}^{n,r,+}(1, \dots, n)(\partial) \quad (5.7)$$

where $\tilde{\mathbf{G}}$ and $\tilde{\mathbf{B}}(\delta)$ will be defined presently.

Define first the increment term,

$$\begin{aligned} \tilde{\mathbf{G}}_z^{n,r,+}(1, \dots, n) &= \sum_{(k_1, \dots, k_n) \in \mathbb{Z}^{n,+}} r^{|k_1| + \dots + |k_n|} a_{k_1} \dots a_{k_n} \tilde{\xi}_{k_1}(1) \dots \tilde{\xi}_{k_n}(n) \\ &\int^z \zeta_1^{k_1} \frac{d\zeta_1}{\zeta_1} \int^{\zeta_1} \zeta_2^{k_2} \frac{d\zeta_2}{\zeta_2} \dots \int^{\zeta_{n-1}} \zeta_n^{k_n} \frac{d\zeta_n}{\zeta_n} = \quad (5.8) \\ &\sum_{(k_1, \dots, k_n) \in \mathbb{Z}_*^n \cap \mathbb{Z}^{n,+}} r^{|k_1| + \dots + |k_n|} \frac{a_{k_1} \dots a_{k_n} \tilde{\xi}_{k_1}(1) \dots \tilde{\xi}_{k_n}(n) z^{k_1 + \dots + k_n}}{k_n(k_{n-1} + k_n) \dots (k_1 + \dots + k_n)} + R, \quad (5.9) \end{aligned}$$

where by definition $\int^\tau \zeta^k \frac{d\zeta}{\zeta} = i \int^\theta e^{ik\theta} d\theta := \frac{\tau^k}{k}$ ($k \neq 0$) is the usual integral of monomials, and $\mathbb{Z}_*^n = \{(k_1, \dots, k_n) \in \mathbb{Z}^n \mid k_n \neq 0, k_{n-1} + k_n \neq 0, \dots, k_1 + \dots + k_n \neq 0\}$. The term R is a sum of logarithmic terms which will not bother us since the regularization process will remove the indices (k_1, \dots, k_n) which do not belong to \mathbb{Z}_*^n . Note also that this formal integral is self-conjugate, i.e. $-i \int^\theta e^{-ik\theta} d\theta = -\int^\tau \zeta^{-k} d\zeta = \overline{\left(\frac{\tau^k}{k}\right)}$, so $\tilde{\mathbf{G}}$ is real-valued.

Now, the boundary term $\tilde{\mathbf{B}}_{\tau\sigma}^{n,r,+}(1, \dots, n)(\partial)$ writes:

$$\begin{aligned} \tilde{\mathbf{B}}_{\tau\sigma}^{n,r,+}(1, \dots, n)(\partial) &= \mathcal{P}^+ \left[\sum_{n_1+n_2=n} \tilde{\mathbf{B}}_{\tau\sigma}^{n_1,r,+}(1, \dots, n_1) \cdot \right. \\ &\left. \int_\sigma d\tilde{B}_{\zeta_{n_1+1}}^r(n_1+1) \int^{\zeta_{n_1+1}} d\tilde{B}_{\zeta_{n_1+2}}^r(n_1+2) \dots \int^{\zeta_{n-1}} d\tilde{B}_{\zeta_n}^r(n) \right] \\ &= -\mathcal{P}^+ \sum_{n_1+n_2=n} \tilde{\mathbf{B}}_{\tau\sigma}^{n_1,r,+}(1, \dots, n_1) \tilde{\mathbf{G}}_\sigma^{n_2,r,+}(n_1+1, \dots, n). \quad (5.10) \end{aligned}$$

Consider any regularization procedure $\tilde{\mathbf{G}}^{n,r,+} \rightarrow \mathcal{R}\tilde{\mathbf{G}}^{n,r,+}$ commuting with the projections onto the Fourier components (k_1, \dots, k_n) such that $\delta\mathcal{R}\tilde{\mathbf{G}}^{1,r,+} = \delta\tilde{\mathbf{G}}^{1,r,+} = \delta\tilde{B}^r$ (so that $\mathcal{R}\tilde{\mathbf{B}}_{\tau\sigma}^{1,r} = (\delta\tilde{B}^r)_{\tau\sigma}$, i.e. the rough path we are constructing lies above \tilde{B}^r). Then the regularization scheme for $\tilde{\mathbf{B}}^r$ is defined as follows.

Definition 5.2 (regularization for $\varepsilon = \text{Id}$) Let $i_1 \neq \dots \neq i_n$. We define by induction on n

$$\mathcal{R}\tilde{\mathbf{B}}_{\tau\sigma}^{n,r,+}(i_1, \dots, i_n)(\partial) := -\mathcal{P}^+ \sum_{n_1+n_2=n} \mathcal{R}\tilde{\mathbf{B}}_{\tau\sigma}^{n_1,r,+}(i_1, \dots, i_{n_1}) \mathcal{R}\tilde{\mathbf{G}}_\sigma^{n_2,r,+}(i_{n_1+1}, \dots, i_n) \quad (5.11)$$

and

$$\mathcal{R}\tilde{\mathbf{B}}_{\tau\sigma}^{n,r,+}(i_1, \dots, i_n) := \left(\delta\mathcal{R}\tilde{\mathbf{G}}^{n,r,+}(i_1, \dots, i_n) \right)_{\tau\sigma} + \mathcal{R}\tilde{\mathbf{B}}_{\tau\sigma}^{n,r,+}(i_1, \dots, i_n)(\partial), \quad (5.12)$$

where $\mathcal{R}\tilde{\mathbf{G}}^{n,r,+}$ is an arbitrary regularization of $\tilde{\mathbf{G}}^{n,r,+}$ – commuting with the projections onto the Fourier components (k_1, \dots, k_n) – such that $\delta\mathcal{R}\tilde{\mathbf{G}}^{1,r,+} = \delta\tilde{\mathbf{B}}^r$.

Then (by a straightforward computation, see also [16], eq. (6) with a different sign convention for δ)

$$\begin{aligned} & \left(\delta\mathcal{R}\tilde{\mathbf{B}}^{n,r,+}(i_1, \dots, i_n) \right)_{\tau u\sigma} \\ &= -\mathcal{P}^+ \sum_{n_1+n_2=n} \left\{ \left(\delta\mathcal{R}\tilde{\mathbf{B}}^{n_1,r,+}(i_1, \dots, i_{n_1}) \right)_{\tau u\sigma} \mathcal{R}\tilde{\mathbf{G}}_{\sigma}^{n_2,r,+}(i_{n_1+1}, \dots, i_n) \right. \\ & \quad \left. - \mathcal{R}\tilde{\mathbf{B}}_{\tau u}^{n_1,r,+}(i_1, \dots, i_{n_1}) \left(\delta\mathcal{R}\tilde{\mathbf{G}}^{n_2,r,+}(i_{n_1+1}, \dots, i_n) \right)_{u\sigma} \right\}. \end{aligned} \quad (5.13)$$

Let us prove that the multiplicative property

$$\left(\delta\mathcal{R}\tilde{\mathbf{B}}^{n,r,+}(i_1, \dots, i_n) \right)_{\tau u\sigma} = \mathcal{P}^+ \sum_{n_1+n_2=n} \mathcal{R}\tilde{\mathbf{B}}_{\tau u}^{n_1,r,+}(i_1, \dots, i_{n_1}) \mathcal{R}\tilde{\mathbf{B}}_{u\sigma}^{n_2,r,+}(i_{n_1+1}, \dots, i_n) \quad (5.14)$$

holds for every $n \geq 1$. Assuming it is true for $n = 1, \dots, N-1$, then

$$\begin{aligned} & \mathcal{P}^+ \sum_{n_1+n_2=N} \mathcal{R}\tilde{\mathbf{B}}_{\tau u}^{n_1,r,+}(i_1, \dots, i_{n_1}) \mathcal{R}\tilde{\mathbf{B}}_{u\sigma}^{n_2,r,+}(i_{n_1+1}, \dots, i_N) \\ &= \mathcal{P}^+ \sum_{n_1+n_2=N} \mathcal{R}\tilde{\mathbf{B}}_{\tau u}^{n_1,r,+}(i_1, \dots, i_{n_1}) \left\{ \left(\delta\mathcal{R}\tilde{\mathbf{G}}^{n_2,r,+}(i_{n_1+1}, \dots, i_N) \right)_{u\sigma} - \right. \\ & \quad \left. \sum_{p_1+p_2=n_2} \mathcal{R}\tilde{\mathbf{B}}_{u\sigma}^{p_1,r,+}(i_{n_1+1}, \dots, i_{n_1+p_1}) \mathcal{R}\tilde{\mathbf{G}}_{\sigma}^{p_2,r,+}(i_{n_1+p_1+1}, \dots, i_N) \right\} \\ &= \mathcal{P}^+ \sum_{n_1+n_2=N} \mathcal{R}\tilde{\mathbf{B}}_{\tau u}^{n_1,r,+}(i_1, \dots, i_{n_1}) \left(\delta\mathcal{R}\tilde{\mathbf{G}}^{n_2,r,+}(i_{n_1+1}, \dots, i_N) \right)_{u\sigma} \\ & \quad - \mathcal{P}^+ \sum_{q_1+q_2=N} \left(\delta\mathcal{R}\tilde{\mathbf{B}}^{q_1,r,+}(i_1, \dots, i_{q_1}) \right)_{\tau u\sigma} \mathcal{R}\tilde{\mathbf{G}}_{\sigma}^{q_2,r,+}(i_{q_1+1}, \dots, i_N) \\ &= \left(\delta\mathcal{R}\tilde{\mathbf{B}}^{N,r,+}(i_1, \dots, i_N) \right)_{\tau u\sigma} \end{aligned} \quad (5.15)$$

hence the multiplicative property holds for every $N \geq 1$.

5.2 Definition and estimates of the regularized increment term

The regularized increment term (as in the case of the Lévy area) will be obtained by restricting the sum defining the increment term to an appropriate subset.

Definition 5.3 (regularized increment term for $\varepsilon = 1$) *Let*

$$\mathbb{Z}_{reg}^{n,+} := \{(k_1, \dots, k_n) \in \mathbb{Z}^{n,+} \mid \forall j = 1, \dots, n-1, |k_j + \dots + k_n| > C_{reg}|k_n|\} \quad (5.16)$$

for some constant $C_{reg} \in (0, 1)$. Then we define the regularized increment term $\mathcal{R}\tilde{\mathbf{G}}^{n,r}(1, \dots, n)$ to be

$$\begin{aligned} \mathcal{R}\tilde{\mathbf{G}}_z^{n,r,+}(1, \dots, n) := & \sum_{(k_1, \dots, k_n) \in \mathbb{Z}_{reg}^{n,+}} r^{|k_1| + \dots + |k_n|} \\ & a_{k_1} \dots a_{k_n} \tilde{\xi}_{k_1}(1) \dots \tilde{\xi}_{k_n}(n) \int^z \zeta_1^{k_1} \frac{d\zeta_1}{\zeta_1} \int^{\zeta_1} \zeta_2^{k_2} \frac{d\zeta_2}{\zeta_2} \dots \int^{\zeta_{n-1}} \zeta_n^{k_n} \frac{d\zeta_n}{\zeta_n}. \end{aligned} \quad (5.17)$$

If $k \geq 1$, we denote by $\mathcal{R}^k \tilde{\mathbf{G}}_z^{n,r,+}(1, \dots, n)$, resp. $\mathcal{R}_k \tilde{\mathbf{G}}_z^{n,r,+}(1, \dots, n)$ the corresponding truncated series, resp. remainder as in section 2.

Note that the exponent of z in the series (5.17) is $k_1 + \dots + k_n$ and that condition (5.16) entails the absence of any logarithmic term in the series.

Remark. One may also set simply $\mathcal{R}\tilde{\mathbf{G}}^{n,r,+}(1, \dots, n) \equiv 0$ for every $n \geq 2$ (or, formally, $C_{reg} = \infty$). Our regularization procedure is *minimal* in the sense that we only discard in the sum (5.17) (as in section 4) an adequate subset of indices $\mathbb{Z}^{n,+} \setminus \mathbb{Z}_{reg}^{n,+}$ which makes the series divergent. Note also that $(\mathbb{Z}^{n,+} \setminus \mathbb{Z}_{reg}^{n,+}) \cap \mathbb{N}^n = \emptyset$ if $C_{reg} \in (0, 1)$, so no regularization is required to define iterated integrals of the Γ -process (corresponding to positive indices k_1, \dots, k_n): hence our construction coincides with that of [31] in the case of the analytic fractional Brownian motion Γ .

Lemma 5.4 (Hölder estimate for the regularized increment term)

Let $\alpha < 1/n$. There exists a constant C such that

$$\sup_{\sigma, \tau \in \mathcal{C}} \mathbb{E} \left| \left(\delta \mathcal{R}\tilde{\mathbf{G}}^{n,r,+} \right)_{\tau\sigma}(1, \dots, n) \right|^2 \leq Cd(\sigma, \tau)^{2n\alpha}. \quad (5.18)$$

Also, considering the series truncated to order $k > 0$ instead,

$$\sup_{\sigma, \tau \in \mathcal{C}} \mathbb{E} \left| \left(\delta^k \mathcal{R} \tilde{\mathbf{G}}^{n, r, +} \right)_{\tau \sigma} (1, \dots, n) \right|^2 \leq C' (d(\sigma, \tau) k^{1-2n\alpha})^2. \quad (5.19)$$

Proof. Consider the restricted sum in the above formula (5.9). By definition of $\mathbb{Z}_{reg}^{n, +}$, the denominator is larger than a constant times $|k_n|^n$, while $a_{k_1} \dots a_{k_n} \lesssim |k_n|^{n(1/2-\alpha)}$. Note also that the exponent of z , namely, $k_1 + \dots + k_n$, differs from k_n by a multiplicative factor which is bounded and bounded away from 0 (namely, $C_{reg}|k_n| < |k_1 + \dots + k_n| \leq n|k_n|$). Hence

$$\mathcal{R} \tilde{\mathbf{G}}_z^{n, r, +}(1, \dots, n) = \sum_{K \in \mathbb{Z}} C_K z^K, \quad (5.20)$$

where

$$\text{Var} C_K \lesssim \text{Card} \{ (k_1, \dots, k_n) \in \mathbb{Z}_{reg}^{n, +} \mid k_1 + \dots + k_n = K \} \cdot K^{-n(2\alpha+1)} \lesssim K^{-1-2n\alpha}. \quad (5.21)$$

Hence (by Lemma 2.1 and Lemma 2.2) the conclusion of the Lemma holds.

□

Lemma 5.5 (Rate of convergence for the regularized increment term)

Let $\alpha < 1/n$. Then there exists a constant C such that

$$\mathbb{E} \left| \mathcal{R} \tilde{\mathbf{G}}_z^{n, r_1, +}(1, \dots, n) - \mathcal{R} \tilde{\mathbf{G}}_z^{n, r_2, +}(1, \dots, n) \right|^2 \leq C |r_1 - r_2|^{2n\alpha}. \quad (5.22)$$

Proof.

Let $X_z^{n, r_1, r_2, +} := \mathcal{R} \tilde{\mathbf{G}}_z^{n, r_1, +}(1, \dots, n) - \mathcal{R} \tilde{\mathbf{G}}_z^{n, r_2, +}(1, \dots, n)$: then

$$\begin{aligned} X_z^{n, r_1, r_2, +} &= \sum_{i=1}^n \sum_{(k_1, \dots, k_n) \in \mathbb{Z}_{reg}^{n, +}} \frac{r_2^{|k_1| + \dots + |k_{i-1}|} r_1^{|k_{i+1}| + \dots + |k_n|} \prod_{j \neq i} [a_{k_j} \tilde{\xi}_{k_j}(j)]}{\prod_{j \neq i} ((k_j + 1) + \dots + k_n)} \\ &\quad z^{\sum_{j \neq i} k_j} \frac{a_{k_i}}{k_i + \dots + k_n} \tilde{\xi}_{k_i}(i) \left\{ r_1^{|k_i|} z^{k_i} - r_2^{|k_i|} z^{k_i} \right\}. \end{aligned} \quad (5.23)$$

The variance of the coefficient of $r_1^{|k_i|}$ is

$$s_{k_i} \lesssim \sum_{|k_n| \geq |k_i|} \sum_{|k_j| \leq |k_n|, j=1, \dots, n-1, j \neq i} (|k_n|^{-1-2n\alpha})^n \lesssim \sum_{|k_n| \geq |k_i|} |k_n|^{-2-2n\alpha} \lesssim |k_i|^{-1-2n\alpha} \quad (5.24)$$

if $i \neq n$, and

$$s_{k_n} \lesssim \sum_{|k_1|, \dots, |k_{n-1}| \leq |k_n|} (|k_n|^{-1-2\alpha})^n \lesssim |k_n|^{-1-2n\alpha}. \quad (5.25)$$

Hence (by Lemma 2.1) $\mathbb{E}|X_z^{n, r_1, r_2, +}|^2 \leq C|r_1 z - r_2 z|^{2n\alpha} = C|r_1 - r_2|^{2n\alpha}$.

□

5.3 Estimates for the boundary terms

Lemma 5.6 *Let $\alpha < 1/n$. The regularized boundary term $\mathcal{R}\tilde{\mathbf{B}}_{\tau\sigma}^{n, r}(1, \dots, n)(\partial)$ satisfies:*

$$\mathbb{E} \left| \mathcal{R}\tilde{\mathbf{B}}_{\tau\sigma}^{n, r}(1, \dots, n)(\partial) \right|^2 \leq Cd(\sigma, \tau)^{2n\alpha} \quad (5.26)$$

for a certain constant C .

Proof.

First step.

Take distinct indices $i_0 \neq \dots \neq i_n$ and let $k \in \mathbb{Z}$. Let $\mathcal{R}\tilde{\mathbf{H}}_z^{n, r, +}[k](i_0; i_1, \dots, i_n)$ be the sum of the terms in the series $\mathcal{R}_k \tilde{\mathbf{G}}_z^{n+1, r, +}(i_0, i_1, \dots, i_n)$ such that $k_0 = k$. Then

$$\begin{aligned} & \mathcal{R}\tilde{\mathbf{H}}_z^{n, r, +}[k](i_0; i_1, \dots, i_n) \\ &= r^{|k|} a_k \tilde{\xi}_k(i_0) \sum_{\substack{(k_1, \dots, k_n) \in \mathbb{Z}_{reg}^{n+1}; \\ |k_1| \geq |k|, |k+k_1+\dots+k_n| \geq C_{reg}|k_n|}} r^{|k_1|+\dots+|k_n|} \\ & \quad a_{k_1} \dots a_{k_n} \tilde{\xi}_{k_1}(i_1) \dots \tilde{\xi}_{k_n}(i_n) \int^z \zeta_0^k \frac{d\zeta_0}{\zeta_0} \int^{\zeta_0} \zeta_1^{k_1} \frac{d\zeta_1}{\zeta_1} \dots \int^{\zeta_{n-1}} \zeta_n^{k_n} \frac{d\zeta_n}{\zeta_n}. \end{aligned} \quad (5.27)$$

Then the variance of $\mathcal{R}\tilde{\mathbf{H}}_z^{n, r, +}[k](i_0; i_1, \dots, i_n)$ is bounded uniformly in z (up to a constant) by

$$|k|^{-1-2\alpha} \prod_{j=1}^n \sum_{|k_j| \geq |k|} |k_j|^{-1-2\alpha} \lesssim |k|^{-1-2\alpha(n+1)}. \quad (5.28)$$

Second step.

Apply repeatedly Definition 5.2: $\mathcal{R}\tilde{\mathbf{B}}_{\tau\sigma}^{n, r, +}(1, \dots, n)(\partial)$ may be written as a sum of terms of the form $A_{\tau\sigma} := \mathcal{P}^+(\delta \mathcal{R}\tilde{\mathbf{G}}^{n_1, r, +}(1, \dots, n_1))_{\tau\sigma} B_\sigma$, where

$$\begin{aligned} B_\sigma &:= \mathcal{P}^+ \mathcal{R}\tilde{\mathbf{G}}_\sigma^{n_2-n_1, r, +}(n_1+1, \dots, n_2) \dots \mathcal{R}\tilde{\mathbf{G}}_\sigma^{n_p-n_{p-1}, r, +}(n_{p-1}+1, \dots, n_p) \\ & \quad \cdot \mathcal{R}\tilde{\mathbf{G}}_\sigma^{n-p, r, +}(n_p+1, \dots, n). \end{aligned} \quad (5.29)$$

Write $A_{\tau\sigma} = \sum_{K \in \mathbb{Z}} C'_K(\sigma)(\tau^K - \sigma^K)$ by expanding $\mathcal{R}\tilde{\mathbf{G}}^{n_1, r_1, +}(1, \dots, n_1)$ into series. By definition of $\mathbb{Z}_{reg}^{n_1, +}$, $|K| \lesssim |k_{n_1}| \lesssim |K|$. Let $B_\sigma[k]$ be the expression obtained by keeping all terms in the series decomposition of B_σ such that $k_{n_1+1} = k$: a simple generalization of Step 1 (by induction on p) leads to $\text{Var } B_\sigma[k] \lesssim |k|^{-1-2\alpha(n-n_1)}$. Then

$$\text{Var} C'_K(\sigma) \lesssim \text{Var} C_K \sum_{|k| \geq |K|} \text{Var } B_\sigma[k] \quad (5.30)$$

where C_K is the coefficient of z^K in the series $\mathcal{R}\tilde{\mathbf{G}}^{n_1, r_1, +}(1, \dots, n_1)$. By the proof of lemma 5.4, $\text{Var} C_K \lesssim |K|^{-1-2n_1\alpha}$, hence $\text{Var} C'_K(\sigma) \lesssim |K|^{-1-2n\alpha}$ uniformly in σ . This implies $\mathbb{E}|A_{\tau\sigma}|^2 \lesssim d(\sigma, \tau)^{2n\alpha}$ by Lemma 2.1. \square

Lemma 5.7 (rate of convergence) *Let $\alpha < 1/n$. There exists a constant C such that, for every $\frac{1}{2} < r_1, r_2 < 1$ and $\sigma, \tau \in \mathcal{C}$,*

$$\mathbb{E} \left| \mathcal{R}\tilde{\mathbf{B}}_{\tau\sigma}^{n, r_1, +}(\partial)(1, \dots, n) - \mathcal{R}\tilde{\mathbf{B}}_{\tau\sigma}^{n, r_2, +}(\partial)(1, \dots, n) \right|^2 \leq C|r_1 - r_2|^{2\alpha}. \quad (5.31)$$

Proof. The difference $\mathbf{X}_{\tau\sigma}^{n, r_1, r_2} := \mathcal{R}\tilde{\mathbf{B}}_{\tau\sigma}^{n, r_1, +}(1, \dots, n)(\partial) - \mathcal{R}\tilde{\mathbf{B}}_{\tau\sigma}^{n, r_2, +}(1, \dots, n)(\partial)$ writes (see Definition 5.2)

$$\begin{aligned} & -\mathcal{P}^+ \sum_{n_1+n_2=n} \left(\mathcal{R}\tilde{\mathbf{B}}_{\tau\sigma}^{n_1, r_1, +}(1, \dots, n_1) - \mathcal{R}\tilde{\mathbf{B}}_{\tau\sigma}^{n_1, r_2, +}(1, \dots, n_1) \right) \mathcal{R}\tilde{\mathbf{G}}_{\sigma}^{n_2, r_1, +}(n_1+1, \dots, n) \\ & -\mathcal{P}^+ \sum_{n_1+n_2=n} \mathcal{R}\tilde{\mathbf{B}}_{\tau\sigma}^{n_1, r_2, +}(1, \dots, n_1) \left(\mathcal{R}\tilde{\mathbf{G}}_{\sigma}^{n_2, r_1, +}(n_1+1, \dots, n) - \mathcal{R}\tilde{\mathbf{G}}_{\sigma}^{n_2, r_2, +}(n_1+1, \dots, n) \right). \end{aligned} \quad (5.32)$$

The result follows now from the estimates of subsection 5.2 by induction on n . \square

6 Iterated integrals of superior order: case of distinct indices

6.1 General setting

We keep in this section (with an index shift which will prove later on to be convenient) the hypothesis $i_0 \neq \dots \neq i_{n-1}$ and set $i_0 = 0, \dots, i_{n-1} = n-1$,

but consider this time indices (k_0, \dots, k_{n-1}) in the domain

$$\mathbb{Z}_\varepsilon^n := \{(k_0, \dots, k_{n-1}) \in \mathbb{Z}^n \mid |k_{\varepsilon(0)}| \leq \dots \leq |k_{\varepsilon(n-1)}|\} \quad (6.1)$$

for some fixed (non-trivial) permutation $\varepsilon : \{0, \dots, n-1\} \rightarrow \{0, \dots, n-1\}$. We propose to introduce an appropriate regularization of the part of $\tilde{\mathbf{B}}_{\tau\sigma}^{n,r}(0, \dots, n-1) = \int_\sigma^\tau d\tilde{B}_{\zeta_1}^r(0) \int_\sigma^{\zeta_1} d\tilde{B}_{\zeta_1}^r(1) \dots \int_\sigma^{\zeta_{n-1}} d\tilde{B}_{\zeta_{n-1}}^r(n-1)$ corresponding to this subset of indices.

Permuting the order of integration according to the permutation ε , an arbitrary iterated integral

$$\int_\sigma^\tau \frac{d\zeta_0}{\zeta_0} \int_\sigma^{\zeta_0} \frac{d\zeta_1}{\zeta_1} \dots \int_\sigma^{\zeta_{n-1}} \frac{d\zeta_{n-1}}{\zeta_{n-1}} f(\zeta_0, \dots, \zeta_{n-1})$$

writes

$$\int_{\sigma_0}^{\tau_0} \frac{d\zeta_{\varepsilon(0)}}{\zeta_{\varepsilon(0)}} \dots \int_{\sigma_{n-1}}^{\tau_{n-1}} \frac{d\zeta_{\varepsilon(n-1)}}{\zeta_{\varepsilon(n-1)}} f(\zeta_{\varepsilon(0)}, \dots, \zeta_{\varepsilon(n-1)})$$

for some intervals $[\sigma_0, \tau_0] = [\sigma, \tau], [\sigma_1, \tau_1], \dots, [\sigma_{n-1}, \tau_{n-1}]$. Writing $\begin{pmatrix} \tau_j \\ \sigma_j \end{pmatrix}$ as the j -th row of a $2 \times n$ -matrix, and choosing as labels of the rows the component indices $i_j = j$, leads more generally to the next definition.

Definition 6.1 1. Let $\mathbb{T} = (\mathbb{T}(i, j))_{1 \leq i \leq 2, 0 \leq j \leq n-1}$ be a matrix with 2 lines and n rows, with coefficients in the set $V := \{0, \dots, n-1\} \cup \{\sigma, \tau\}$, and $\ell(\mathbb{T}, j) \in \mathbb{N}$, $j = 0, \dots, n-1$ be some integer labels. It may be seen as an oriented graph with set of vertices V and set of edges $E = \{(v, w) \in \{0, \dots, n-1\} \times V \mid v \rightarrow w\}$, where by definition $i \rightarrow w$ ($i \in \{0, \dots, n-1\}$, $w \in V$), read: i connects directly to w , if $w = \mathbb{T}(1, i)$ or $\mathbb{T}(2, i)$. Write $\mathbb{T}.j$ for the row $\begin{pmatrix} \mathbb{T}(1, j) \\ \mathbb{T}(2, j) \end{pmatrix}$. We call \mathbb{T} an integration graph if:

- (i) $(i, j) \in E$, $i, j \in \{0, \dots, n-1\} \Rightarrow j < i$;
- (ii) $\mathbb{T}.0 = \begin{pmatrix} \tau \\ \sigma \end{pmatrix}$ or $\begin{pmatrix} \sigma \\ \tau \end{pmatrix}$.

We identify two integration graphs \mathbb{T} and \mathbb{T}' if some permutation of the vertices $\varepsilon : \{0, \dots, n-1\} \rightarrow \{0, \dots, n-1\}$ such that $(i, j) \in E(\mathbb{T}) \Leftrightarrow (\varepsilon(i), \varepsilon(j)) \in E(\mathbb{T}')$ and $\ell(\mathbb{T}', \varepsilon(j)) = \ell(\mathbb{T}, j)$ transforms \mathbb{T} into \mathbb{T}' . This defines equivalence classes of integration graphs.

2. Let \mathbb{T} be an integration graph. We call \mathbb{T} an integration tree (or simply a tree) if

$$\mathbb{T} = \begin{pmatrix} \tau & i_1 & i_2 & \dots & i_{n-1} \\ \sigma & \sigma & \sigma & \dots & \sigma \end{pmatrix} \quad (6.2)$$

with $i_j \in \{0, \dots, j-1\}$. It may be seen as a decorated rooted tree with vertices $j = 0$ (root), $1, \dots, n-1$ wearing the label $\ell(\mathbb{T}, j)$.

More generally, if $i_{j_1}, \dots, i_{j_J} = \tau$ for some $j_1, \dots, j_J = 1, \dots, n-1$, then \mathbb{T} is called an integration forest (or simply a forest) and may be seen as a decorated forest (i.e. a disjoint union of decorated rooted trees) with roots $0, j_1, \dots, j_J$.

3. Let $\mathbb{T} = (\mathbb{T}(i, j))_{1 \leq i \leq 2, 0 \leq j \leq n-1}$ be a matrix with coefficients in the set $V \cup \emptyset$. Then \mathbb{T} is called a truncated integration graph if condition (i) is satisfied and $\mathbb{T}.j \neq \begin{pmatrix} \emptyset \\ \emptyset \end{pmatrix}$ for every j .
4. If $i_1 \rightarrow i_2 \rightarrow \dots \rightarrow i_{j-1} \rightarrow w$ for some indices $i_1, \dots, i_{j-1} \in \{0, \dots, n-1\}$ and $w \in V$, then we shall write $i_1 \rightarrow w$, and say that i_1 connects to w .
5. If a, b are some symbols (in practice, $a, b \in \{\sigma, \tau, u\}$ only), then $[\mathbb{T}]_{ba}$ is the graph obtained by replacing all occurrences of the symbol σ , resp. τ , by a , resp. b .

Trees \mathbb{T} grow up starting from their root, but (by convention) arrows go down, so that v is 'below' w if $w \rightarrow v$ (see figures in subsection 6.2 below). The 'boundary vertices' σ, τ and the arrows leading to them are never considered when looking at the tree structure, which is really $\mathbb{T} \setminus \{\sigma, \tau\}$. By a slight abuse of notation, we shall write \mathbb{T} both for the underlying graph structure and for the underlying tree structure.

Definition 6.2 (operations on integration graphs) 1. Assume $\mathbb{T} = (\mathbb{T}(i, j))_{1 \leq i \leq 2, 0 \leq j \leq n-1}$ is some (possibly truncated) integration graph. Then $-\mathbb{T}$ is the graph obtained from \mathbb{T} by permuting the entries of the row number j for some $j = 0, \dots, n-1$.

2. Let $\mathbb{T}_1, \mathbb{T}_2$ be some (possibly truncated) integration graphs of dimension n with the same labels, such that the rows of \mathbb{T}_1 and \mathbb{T}_2 are identical, save for row number j for some $j = 0, \dots, n-1$, and furthermore,

$\mathbb{T}_1(1, j) = \mathbb{T}_2(2, j)$, resp. $\mathbb{T}_1(2, j) = \mathbb{T}_2(1, j)$. Then $\mathbb{T} := \mathbb{T}_1 + \mathbb{T}_2$ is the integration graph of dimension n defined by

$$\begin{aligned} \mathbb{T}.k &= \mathbb{T}_1.k = \mathbb{T}_2.k \quad (k \neq j), \\ \begin{pmatrix} \mathbb{T}(1, j) \\ \mathbb{T}(2, j) \end{pmatrix} &= \begin{pmatrix} \mathbb{T}_2(1, j) \\ \mathbb{T}_1(2, j) \end{pmatrix}, \quad \text{resp.} \quad \begin{pmatrix} \mathbb{T}_1(1, j) \\ \mathbb{T}_2(2, j) \end{pmatrix}. \end{aligned} \quad (6.3)$$

Operations 1 and 2 may be represented simply as operations on the row number j , namely, $\begin{pmatrix} a \\ c \end{pmatrix} + \begin{pmatrix} c \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$.

3. Let \mathbb{T}_1 , resp. \mathbb{T}_2 be a (possibly truncated) integration graph of dimension n_1 , resp. n_2 . Then $\mathbb{T}_1.\mathbb{T}_2 = (\mathbb{T}_1 \mid \mathbb{T}_2)$ is the integration graph of dimension $n_1 + n_2$ obtained by keeping the same labels and moving \mathbb{T}_2 to the right of \mathbb{T}_1 , i.e. such that

$$(\mathbb{T}_1.\mathbb{T}_2).j = \mathbb{T}_1.j \quad (0 \leq j \leq n_1 - 1) \quad (6.4)$$

and $(\mathbb{T}_1.\mathbb{T}_2)(i, n_1 + j) = \mathbb{T}_2(i, j)$ if $\mathbb{T}_2(i, j) = \sigma$ or τ , $\mathbb{T}_2(i, j) + n_1$ otherwise ($0 \leq j \leq n_2 - 1$).

Conversely, if \mathbb{T} is some (possibly truncated) integration graph, then 'cutting' it along every row of the form $\begin{pmatrix} \tau \\ \sigma \end{pmatrix}$ yields a decomposition of \mathbb{T} into a product of indecomposable graphs. In particular, if \mathbb{T} is a forest, then \mathbb{T} decomposes into a product of trees.

The objects we consider live actually in the free commutative algebra over \mathbb{Z} generated by truncated integration graphs, quotiented out by the above rules. Restricting to (non-truncated) forests, the relevant structure is that of the Hopf algebra \mathcal{T} of decorated rooted trees, generated by trees (or rather forests) and with coproduct structure given in subsection 6.2 below. By Lemma 6.7, integration graphs may be seen as linear combinations of forests, so we are really working in \mathcal{T} . The relation between labeled rooted trees and iterated integrals has been known for a long time, see for instance [3, 4]. It has recently received some interest because of the introduction of the Hopf algebra of decorated rooted trees by A. Connes, D. Kreimer [6, 7, 8], see also the papers by C. Brouder, A. Frabetti and L. Foissy [1, 2, 12] in the context of Feynmann graphs in quantum field theory. The notion of integration graphs and the representation by matrices is only a by-product of this, but it is also convenient. The algebraic structure is best seen on trees, but matrices are sometimes more visual and also useful when studying iterated integrals with varying boundaries σ, τ, u .

Definition 6.3 (integration along a tree) Let \mathbb{T} be a (possibly truncated) integration graph with distinct labels $\{\ell_0 := \ell(\mathbb{T}, 0), \dots, \ell_{n-1} := \ell(\mathbb{T}, n-1)\} := V_\ell$, and $f : \mathcal{C}^{V_\ell} := \{(\zeta_v)_{v \in V_\ell}\} \rightarrow \mathbb{C}$ be a Laurent series, $f(\zeta_{\ell_0}, \dots, \zeta_{\ell_{n-1}}) = \sum f_{k_{\ell_0}, \dots, k_{\ell_{n-1}}} \zeta_{\ell_0}^{k_{\ell_0}} \dots \zeta_{\ell_{n-1}}^{k_{\ell_{n-1}}}$. Then we let (assuming the expression below converges)

$$[I_{\mathbb{T}}(f)]_{zw} := \int_{\mathbb{T}(2,0)}^{\mathbb{T}(1,0)} \frac{d\zeta_{\ell_0}}{\zeta_{\ell_0}} \int_{\mathbb{T}(2,1)}^{\mathbb{T}(1,1)} \frac{d\zeta_{\ell_1}}{\zeta_{\ell_1}} \dots \int_{\mathbb{T}(2,n-1)}^{\mathbb{T}(1,n-1)} \frac{d\zeta_{\ell_{n-1}}}{\zeta_{\ell_{n-1}}} f(\zeta_{\ell_0}, \dots, \zeta_{\ell_{n-1}}), \quad (6.5)$$

where all occurrences of τ , resp. σ have been replaced by z , resp. w .

If τ does not occur in \mathbb{T} , we shall simply write $[I_{\mathbb{T}}f]_w$, meaning that all occurrences of σ should be replaced by w .

This definition extends immediately by linearity to formal linear combinations of graphs.

Remark. If \mathbb{T} is a truncated graph, then $\int^\tau \zeta^k \frac{d\zeta}{\zeta}$, resp. $\int_\sigma \zeta^k \frac{d\zeta}{\zeta}$ (as in subsection 5.1) is defined as $\frac{\tau^k}{k}$ ($k \neq 0$), resp. $-\frac{\sigma^k}{k}$. For general truncated graphs, some extra conditions on f (which are always satisfied in our computations) are needed to ensure that no logarithmic terms appear.

Lemma 6.4 Let \mathbb{T} be a (possibly truncated) integration graph with distinct labels $\{\ell_0 := \ell(\mathbb{T}, 0), \dots, \ell_{n-1} := \ell(\mathbb{T}, n-1)\} := V_\ell$. Assume $\mathbb{T} = \mathbb{T}_1 \cdot \mathbb{T}_2$.

1. Let $f = f_{\ell_0} \otimes \dots \otimes f_{\ell_{n-1}} : \mathcal{C}^{V_\ell} \rightarrow \mathbb{C}$, then

$$[I_{\mathbb{T}}(f)]_{\tau\sigma} = [I_{\mathbb{T}_1}(f|_{\ell(V(\mathbb{T}_1))})]_{\tau\sigma} [I_{\mathbb{T}_2}(f|_{\ell(V(\mathbb{T}_2))})]_{\tau\sigma}, \quad (6.6)$$

where, given a subset $W = \{i_1, \dots, i_j\} \subset \ell(V(\mathbb{T}))$, $f|_W := f_{i_1} \otimes \dots \otimes f_{i_j}$.

2. (multilinear extension of 1.) Let $f = \sum f_{k_{\ell_0}, \dots, k_{\ell_{n-1}}} \zeta_{\ell_0}^{k_{\ell_0}} \otimes \dots \otimes \zeta_{\ell_{n-1}}^{k_{\ell_{n-1}}}$ be a converging Laurent series, then

$$[I_{\mathbb{T}}(f)]_{\tau\sigma} = [I_{\mathbb{T}_1}]_{\tau\sigma} \cdot [I_{\mathbb{T}_2}]_{\tau\sigma}(f), \quad (6.7)$$

where by definition

$$[I_{\mathbb{T}_1}]_{\tau\sigma} \cdot [I_{\mathbb{T}_2}]_{\tau\sigma}(f) = \sum f_{k_{\ell_0}, \dots, k_{\ell_{n-1}}} \left[I_{\mathbb{T}_1} \left(\otimes_{l \in \ell(V(\mathbb{T}_1))} \zeta_l^{k_l} \right) \right]_{\tau\sigma} \left[I_{\mathbb{T}_2} \left(\otimes_{l \in \ell(V(\mathbb{T}_2))} \zeta_l^{k_l} \right) \right]_{\tau\sigma}. \quad (6.8)$$

Proof. Immediate. □

Recall $\varepsilon : \{0, \dots, n-1\} \rightarrow \{0, \dots, n-1\}$ is an arbitrary permutation.

Definition 6.5 (permutation integration graphs) Let \mathbb{T}^ε be the integration graph given by

$$\varepsilon(\mathbb{T}(1, j)) = \max\{i' = \varepsilon(1), \dots, \varepsilon(j-1) \mid i' < \varepsilon(j)\}, \quad (6.9)$$

resp. τ if this set is empty, and

$$\varepsilon(\mathbb{T}(2, j)) = \min\{i' = \varepsilon(1), \dots, \varepsilon(j-1) \mid i' > \varepsilon(j)\}, \quad (6.10)$$

resp. σ if this set is empty, with labels $\ell(\mathbb{T}^\varepsilon, j) = j$.

Note our choice of labels, which is identical for all \mathbb{T}^ε , $\varepsilon \in \Sigma_n$.

In particular, $\mathbb{T}^{\text{Id}_n} = \begin{pmatrix} \tau & 0 & 1 & \dots & n-2 \\ \sigma & \sigma & \sigma & \sigma & \sigma \end{pmatrix}$.

It is easy to check that an integration graph \mathbb{T} is a permutation graph (up to the sign, see Definition 6.2) if and only if, for every j , $\mathbb{T}(1, j) \rightarrow \mathbb{T}(2, j)$ or $\mathbb{T}(2, j) \rightarrow \mathbb{T}(1, j)$. For if this is the case, then ε may be retrieved step by step by inserting $\varepsilon(j)$ inside the set $\{\tau, \varepsilon(1), \dots, \varepsilon(j-1), \sigma\}$ between the neighbouring positions $\varepsilon(\mathbb{T}(1, j))$ and $\varepsilon(\mathbb{T}(2, j))$.

The motivation for introducing these permutation graphs comes from the following.

Lemma 6.6 Let, for any Laurent series $f : \mathbb{C}^n \rightarrow \mathbb{C}$ such that the integrals converge,

$$I_{\tau\sigma}^n(f) := \int_\sigma^\tau \frac{d\zeta_0}{\zeta_0} \int_\sigma^{\zeta_0} \frac{d\zeta_1}{\zeta_1} \dots \int_\sigma^{\zeta_{n-2}} \frac{d\zeta_{n-1}}{\zeta_{n-1}} f(\zeta_0, \dots, \zeta_{n-1}). \quad (6.11)$$

Then, for every permutation ε ,

$$I_{\tau\sigma}^n(f) = I_{\mathbb{T}^\varepsilon}(f \circ \varepsilon) \quad (6.12)$$

where by definition, $f \circ \varepsilon(\zeta_0, \dots, \zeta_{n-1}) := f(\zeta_{\varepsilon(0)}, \dots, \zeta_{\varepsilon(n-1)})$.

Proof. Elementary. □

Lemma 6.7 (decomposition of an integration graph) *Let \mathbb{T} be a (non-truncated) integration graph. Split each row of index j , $j \geq 1$, in the following way:*

$$\begin{pmatrix} i_1 \\ i_2 \end{pmatrix} \rightarrow \begin{pmatrix} i_1 \\ \sigma \end{pmatrix} - \begin{pmatrix} i_2 \\ \sigma \end{pmatrix}, \quad \begin{pmatrix} \tau \\ i \end{pmatrix} \rightarrow \begin{pmatrix} \tau \\ \sigma \end{pmatrix} - \begin{pmatrix} i \\ \sigma \end{pmatrix} \quad (6.13)$$

and decompose each forest thus obtained into a product of trees. Then \mathbb{T} writes

$$\mathbb{T} = \sum_l \pm \mathbb{T}_{l,1} \dots \mathbb{T}_{l,L(l)} \quad (6.14)$$

where each $\mathbb{T}_{l,m}$ is an integration tree.

Proof. Elementary. □

We shall be interested for application to fBm in the following function:

Definition 6.8 (defining function for iterated integrals) 1. *Let, for any multi-index $\mathbf{i} = \{i_0, \dots, i_{n-1}\}$,*

$$\begin{aligned} f_{\mathbf{i}}^{n,r,\varepsilon}(\zeta_0, \dots, \zeta_{n-1}) &:= \sum_{|k_{\varepsilon(0)}| \leq \dots \leq |k_{\varepsilon(n-1)}|} \rho(k_0, \dots, k_{n-1}) \\ & r^{|k_0| + \dots + |k_{n-1}|} a_{k_0} \dots a_{k_{n-1}} \tilde{\xi}_{k_0}(i_0) \dots \tilde{\xi}_{k_{n-1}}(i_{n-1}) \cdot \zeta_0^{k_0} \dots \zeta_{n-1}^{k_{n-1}} \end{aligned} \quad (6.15)$$

where $\rho(k_0, \dots, k_{n-1}) = \frac{1}{|\Sigma_{\mathbf{k}}|}$ and $\Sigma_{\mathbf{k}}$ is the 'index-fixing' subgroup of permutations $\varepsilon' : \{0, \dots, n-1\} \rightarrow \{0, \dots, n-1\}$ such that $(k_0, \dots, k_{n-1}) = (k_{\varepsilon'(0)}, \dots, k_{\varepsilon'(n-1)})$, and

$$\tilde{\mathbf{B}}_{\tau\sigma}^{n,r,\varepsilon}(i_0, \dots, i_{n-1}) := I_{\tau\sigma}^n(f_{\mathbf{i}}^{n,r,\varepsilon}). \quad (6.16)$$

2. *Let, for $V = \{v_0, \dots, v_{j-1}\} \subset \{0, \dots, n-1\}$,*

$$\begin{aligned} f_{\mathbf{i}}^{n,r,\varepsilon}|_V(\zeta_{v_0}, \dots, \zeta_{v_{j-1}}) &:= \sum_{(k_{v_0}, \dots, k_{v_{j-1}}) \in S} \rho_V(k_{v_0}, \dots, k_{v_{j-1}}) \\ & r^{|k_{v_0}| + \dots + |k_{v_{j-1}}|} a_{k_{v_0}} \dots a_{k_{v_{j-1}}} \tilde{\xi}_{k_{v_0}}(i_{v_0}) \dots \tilde{\xi}_{k_{v_{j-1}}}(i_{v_{j-1}}) \cdot \zeta_{v_0}^{k_{v_0}} \dots \zeta_{v_{j-1}}^{k_{v_{j-1}}} \end{aligned} \quad (6.17)$$

where $S = \{(k_{v_0}, \dots, k_{v_{j-1}}) \in \mathbb{Z}^j \mid \varepsilon^{-1}(v_i) \leq \varepsilon^{-1}(v_j) \Rightarrow |k_{v_i}| \leq |k_{v_j}|\}$ and $\rho_V(k_{v_0}, \dots, k_{v_{j-1}}) = \frac{1}{|\Sigma_{(k_{v_0}, \dots, k_{v_{j-1}})}|}$.

If $\mathbf{i} = \{0, \dots, n-1\}$ or if no confusion is possible, we shall drop the lower index \mathbf{i} and write simply $f^{n,r,\varepsilon}$ or $f^{n,r,\varepsilon}|_V$.

By definition,

$$\tilde{\mathbf{B}}_{\tau\sigma}^{n,r} = \sum_{\varepsilon \in \Sigma_n} \tilde{\mathbf{B}}_{\tau\sigma}^{n,r,\varepsilon}. \quad (6.18)$$

Note that the coefficient ρ is generically 1, so that we may essentially forget about it. Seeing $\tilde{\mathbf{B}}^{n,r,\varepsilon}$ as $I_{\mathbb{T}^\varepsilon}(f^{n,r,\varepsilon} \circ \varepsilon)$ will allow us to define its regularization in the next subsection.

Definition 6.9 (well-labeled tree) *Let \mathbb{T} be a decorated rooted tree with distinct labels. We shall say that \mathbb{T} is well-labeled if $\ell(v) < \ell(w)$ whenever $v, w \in \mathbb{T}$ and $w \rightarrow v$.*

Then ℓ induces a total ordering on $V(\mathbb{T})$, compatible with the tree partial ordering given by \rightarrow . By a slight abuse of notation, we shall index the vertices by their label and write $v < w$ if $\ell(v) < \ell(w)$.

The above definition extends naturally to forests. Those appearing in the decomposition of \mathbb{T}^ε (see Definition 6.5 and Lemma 6.7) are naturally well-labeled. As we shall see, Hölder estimates hold when one integrates an appropriate regularization of $f^{n,r,\varepsilon} \circ \varepsilon$ along *well-labeled* trees.

6.2 Tree multiplicative property

Definition 6.10 (admissible cuts) *1. Let \mathbb{T} be a tree, with set of vertices $V(\mathbb{T})$ and root denoted by 0. If $\mathbf{v} = (v_1, \dots, v_J)$ is any totally disconnected subset of $V(\mathbb{T}) \setminus \{0\}$, i.e. $v_i \not\rightarrow v_j$ for all $i, j = 1, \dots, J$, then we shall say that \mathbf{v} is an admissible cut of \mathbb{T} , and write $\mathbf{v} \models V(\mathbb{T})$. We let $R_{\mathbf{v}}\mathbb{T}$ be the sub-forest (or sub-tree if $J = 1$) obtained by keeping only the vertices above \mathbf{v} , i.e. $V(R_{\mathbf{v}}\mathbb{T}) = \mathbf{v} \cup \{w \in V(\mathbb{T}) : \exists j = 1, \dots, J, w \rightarrow v_j\}$, and $L_{\mathbf{v}}\mathbb{T}$ be the sub-tree obtained by keeping all other vertices.*

2. Let $\mathbb{T} = \mathbb{T}_1 \dots \mathbb{T}_l$ be a forest, together with its decomposition into trees. Then an admissible cut of \mathbb{T} is a disjoint union $\mathbf{v}_1 \cup \dots \cup \mathbf{v}_l$, $\mathbf{v}_i \subset \mathbb{T}_i$, where \mathbf{v}_i is either \emptyset , $\{0_i\}$ (root of \mathbb{T}_i) or an admissible cut of \mathbb{T}_i . By definition, we let $L_{\mathbf{v}}\mathbb{T} = L_{\mathbf{v}_1}\mathbb{T}_1 \dots L_{\mathbf{v}_l}\mathbb{T}_l$, $R_{\mathbf{v}}\mathbb{T} = R_{\mathbf{v}_1}\mathbb{T}_1 \dots R_{\mathbf{v}_l}\mathbb{T}_l$ (if $\mathbf{v}_i = \emptyset$, resp. $\{0_i\}$, then $(L_{\mathbf{v}_i}\mathbb{T}_i, R_{\mathbf{v}_i}\mathbb{T}_i) := (\mathbb{T}_i, \emptyset)$, resp. $(\emptyset, \mathbb{T}_i)$).

We exclude by convention the two trivial cuts $\emptyset \cup \dots \cup \emptyset$ and $\{0_1\} \cup \dots \cup \{0_l\}$.

See Fig. 1 and 2.

Let us give two explicit examples (the notation Sk (skeleton) is defined in Definition 6.12 below):

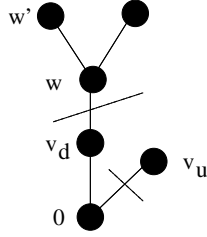


Figure 1: Admissible cut (notations refer to the proof of Lemma 6.11).

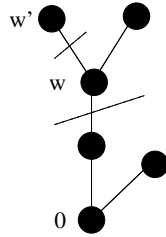


Figure 2: Non-admissible cut.

Example 1. $\mathbb{T} = \mathbb{T}^{\text{Id}_n} = \begin{pmatrix} \tau & 0 & 1 & \dots & n-2 \\ \sigma & \sigma & \sigma & \dots & \sigma \end{pmatrix}$. Then the tree \mathbb{T} is simply a trunk (see Fig. 3), so admissible cuts are made of just one cut at height $n_1 = 1, \dots, n-1$. Then $L_{n_1}\mathbb{T} = \mathbb{T}^{\text{Id}_{n_1}}$, $\mathbb{R}_{n_1}\mathbb{T} = \mathbb{T}^{\text{Id}_{n-n_1}}$, while $[\text{Sk}\mathbb{T}]_\sigma = - \begin{pmatrix} \emptyset & 0 & 1 & \dots & n-2 \\ \sigma & \emptyset & \emptyset & \dots & \emptyset \end{pmatrix}$.

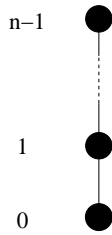


Figure 3: Trunk tree. Vertices are always indexed by their labels.

Example 2 (see Fig. 4).

Let $\varepsilon : (0, 1, 2) \rightarrow (1, 2, 0)$. Then $\mathbb{T}_\varepsilon = \begin{pmatrix} \tau & 0 & \tau \\ \sigma & \sigma & 0 \end{pmatrix} = \begin{pmatrix} \tau & 0 \\ \sigma & \sigma \end{pmatrix} \cdot \begin{pmatrix} \tau \\ \sigma \end{pmatrix}$ —

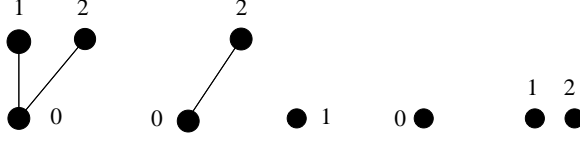


Figure 4: From left to right: \mathbb{T}' ; $L_{\{1\}}\mathbb{T}' \otimes R_{\{1\}}\mathbb{T}'$; $L_{\{1,2\}}\mathbb{T}' \otimes R_{\{1,2\}}\mathbb{T}'$.

\mathbb{T}' where

$$\mathbb{T}' = \begin{pmatrix} \tau & 0 & 0 \\ \sigma & \sigma & \sigma \end{pmatrix}.$$

Consider \mathbb{T}' . Then $\mathbf{v} = \{1\}$, $\{2\}$ or $\{1, 2\}$ and

$$L_{\{1\}}\mathbb{T}' = \begin{pmatrix} \tau & \cdot & 0 \\ \sigma & \cdot & \sigma \end{pmatrix}, \quad R_{\{1\}}\mathbb{T}' = \begin{pmatrix} \tau \\ \sigma \end{pmatrix}, \quad \text{Sk}R_{\{1\}}\mathbb{T}' = - \begin{pmatrix} \emptyset \\ \sigma \end{pmatrix}; \quad (6.19)$$

$$L_{\{2\}}\mathbb{T}' = \begin{pmatrix} \tau & 0 & \cdot \\ \sigma & \sigma & \cdot \end{pmatrix}, \quad R_{\{2\}}\mathbb{T}' = \begin{pmatrix} \tau \\ \sigma \end{pmatrix}; \quad (6.20)$$

$$L_{\{1,2\}}\mathbb{T}' = \begin{pmatrix} \tau \\ \sigma \end{pmatrix}, \quad R_{\{1,2\}}\mathbb{T}' = \begin{pmatrix} \cdot & \tau & \tau \\ \cdot & \sigma & \sigma \end{pmatrix} = \begin{pmatrix} \tau \\ \sigma \end{pmatrix} \begin{pmatrix} \tau \\ \sigma \end{pmatrix},$$

$$\text{Sk}(R_{\{1,2\}}\mathbb{T}') = \begin{pmatrix} \emptyset \\ \sigma \end{pmatrix} \begin{pmatrix} \emptyset \\ \sigma \end{pmatrix}. \quad (6.21)$$

Lemma 6.11 (tree multiplicative property) (see [17]) *Let \mathbb{T} be a forest. Then*

$$([\delta\mathbb{T}])_{\tau u \sigma} := [\mathbb{T}]_{\tau \sigma} - [\mathbb{T}]_{\tau u} - [\mathbb{T}]_{u \sigma} = \sum_{\mathbf{v} \models V(\mathbb{T})} [L_{\mathbf{v}}\mathbb{T}]_{\tau u} [R_{\mathbf{v}}\mathbb{T}]_{u \sigma}. \quad (6.22)$$

Proof.

Assume first that \mathbb{T} is a tree. We shall prove this relation by induction on the number of vertices. Let \mathbb{T}' be the tree obtained by adding to \mathbb{T} a vertex w' such that $w' \rightarrow w \in \mathbb{T}$.

Let $\mathbf{v} \models V(\mathbb{T})$. There are two cases (see Fig. 1):

- (i) ('down' case) either there is some $v_d \in \mathbf{v}$ such that $v_d = w$ or $w \rightarrow v_d$. Then v_d is necessarily unique and $\mathbf{v} = \{v_d\} \cup \mathbf{v}_u$, $\mathbf{v}_u = \{v_1, \dots, v_J\}$ totally disconnected, such that $v_j \not\leftrightarrow v_d$ and $v_d \not\leftrightarrow v_j$ for $j = 1, \dots, J$;

- (ii) ('up' case) or $\mathbf{v} = \mathbf{v}_u$ with $\mathbf{v}_u = \{v_1, \dots, v_J\}$ totally disconnected such that $v_j \neq w$ and $w \not\rightsquigarrow v_j$ for $j = 1, \dots, J$.

Note that an admissible cut of \mathbb{T} is *ipso facto* also an admissible cut of \mathbb{T}' . If $\mathbf{v} = \{v_d\} \cup \mathbf{v}_u$ (down case), then $\mathbf{v} \cup \{w'\}$ is not an admissible cut of \mathbb{T}' . On the contrary (up case), $\mathbf{v}' := \mathbf{v} \cup \{w'\}$ is admissible. Finally, the single cut $\{w'\}$ is also an admissible cut of \mathbb{T}' .

Assume property (6.22) is true for \mathbb{T} . Then

$$\begin{aligned}
[\mathbb{T}']_{\tau\sigma} - [\mathbb{T}']_{u\sigma} - [\mathbb{T}']_{\tau u} &= \left([\mathbb{T}]_{\tau\sigma} \mid \begin{array}{c} w \\ \sigma \end{array} \right) - \left([\mathbb{T}]_{u\sigma} \mid \begin{array}{c} w \\ \sigma \end{array} \right) - \left([\mathbb{T}]_{\tau u} \mid \begin{array}{c} w \\ u \end{array} \right) \\
&= \left([\mathbb{T}]_{\tau\sigma} - [\mathbb{T}]_{u\sigma} - [\mathbb{T}]_{\tau u} \mid \begin{array}{c} w \\ \sigma \end{array} \right) + \left([\mathbb{T}]_{\tau u} \mid \begin{array}{c} u \\ \sigma \end{array} \right) \\
&= \left([\mathbb{T}]_{\tau u} \mid \begin{array}{c} u \\ \sigma \end{array} \right) + \sum_{\mathbf{v} \models V(\mathbb{T})} A_{\mathbf{v}} \tag{6.23}
\end{aligned}$$

where

$$A_{\mathbf{v}} = \left([L_{\mathbf{v}}\mathbb{T}]_{\tau u} [R_{\mathbf{v}}\mathbb{T}]_{u\sigma} \mid \begin{array}{c} w \\ \sigma \end{array} \right). \tag{6.24}$$

- (i) (down case) Assume $\mathbf{v} = \{v_d\} \cup \mathbf{v}_u$. Then $L_{\mathbf{v}}\mathbb{T}' = L_{\mathbf{v}}\mathbb{T}$, $[R_{\mathbf{v}}\mathbb{T}']_{u\sigma} = \left(R_{\mathbf{v}}\mathbb{T} \mid \begin{array}{c} w \\ \sigma \end{array} \right)$
hence $[L_{\mathbf{v}}\mathbb{T}']_{\tau u} [R_{\mathbf{v}}\mathbb{T}']_{u\sigma} = A_{\mathbf{v}}$.

- (ii) (up case) Assume $\mathbf{v} = \mathbf{v}_u$ and let as before $\mathbf{v}' = \mathbf{v}_u \cup \{w'\}$. Then

$$[L_{\mathbf{v}}\mathbb{T}']_{\tau u} = \left([L_{\mathbf{v}}\mathbb{T}]_{\tau u} \mid \begin{array}{c} w \\ u \end{array} \right), \quad R_{\mathbf{v}}\mathbb{T}' = R_{\mathbf{v}}\mathbb{T}; \tag{6.25}$$

$$[L_{\mathbf{v}'}\mathbb{T}'] = L_{\mathbf{v}}\mathbb{T}, \quad [R_{\mathbf{v}'}\mathbb{T}']_{u\sigma} = \left([R_{\mathbf{v}}\mathbb{T}]_{u\sigma} \mid \begin{array}{c} u \\ \sigma \end{array} \right) \tag{6.26}$$

hence $[L_{\mathbf{v}}\mathbb{T}']_{\tau u} [R_{\mathbf{v}}\mathbb{T}']_{u\sigma} + [L_{\mathbf{v}'}\mathbb{T}']_{\tau u} [R_{\mathbf{v}'}\mathbb{T}']_{u\sigma} = A_{\mathbf{v}}$.

- (iii) Assume $\mathbf{v} = \{w'\}$. Then $[L_{\{w'\}}\mathbb{T}']_{\tau u} [R_{\{w'\}}\mathbb{T}']_{u\sigma} = \left([\mathbb{T}]_{\tau u} \mid \begin{array}{c} u \\ \sigma \end{array} \right)$.

If now $\mathbb{T} = \mathbb{T}_1 \dots \mathbb{T}_l$, $l \geq 1$ is a forest, then (as a straightforward computation proves)

$$\begin{aligned}
[\delta(\mathbb{T} \cdot \mathbb{T}_{l+1})]_{\tau u\sigma} &= ([\mathbb{T}]_{\tau u} + [\mathbb{T}]_{u\sigma} + [\delta\mathbb{T}]_{\tau u\sigma})([\mathbb{T}_{l+1}]_{\tau u} + [\mathbb{T}_{l+1}]_{u\sigma} + [\delta\mathbb{T}_{l+1}]_{\tau u\sigma}) \\
&\quad - [\mathbb{T}]_{\tau u} [\mathbb{T}_{l+1}]_{\tau u} - [\mathbb{T}]_{u\sigma} [\mathbb{T}_{l+1}]_{u\sigma}
\end{aligned} \tag{6.27}$$

hence (by induction on l)

$$[\delta\mathbb{T}]_{\tau u \sigma} = \prod_{j=1}^l ([\mathbb{T}_j]_{\tau u} + [\mathbb{T}_j]_{u \sigma} + [\delta\mathbb{T}_j]_{\tau u \sigma}) - \prod_{j=1}^l [\mathbb{T}_j]_{\tau u} - \prod_{j=1}^l [\mathbb{T}_j]_{u \sigma}, \quad (6.28)$$

hence the result (the last two terms corresponding to the two forbidden trivial cuts, see Definition 6.10). \square

Definition 6.12 (skeleton of a graph) 1. Let \mathbb{T} be a tree. Then the skeleton of \mathbb{T} is the truncated graph $[\text{Sk}(\mathbb{T})]_{\sigma}$ obtained by replacing all occurrences of the vertex σ in \mathbb{T} by \emptyset , except for the first row which we set equal to $\text{Sk}(\mathbb{T}).0 = - \begin{pmatrix} \emptyset \\ \sigma \end{pmatrix}$.

2. If $\mathbb{T} = \mathbb{T}_1 \dots \mathbb{T}_d$ is a forest together with its tree decomposition, then define $\text{Sk}\mathbb{T} = \text{Sk}(\mathbb{T}_1) \dots \text{Sk}(\mathbb{T}_d)$.

See Examples 1 and 2 above.

Lemma 6.13 (skeleton decomposition) Let \mathbb{T} be a tree. Then

$$[\mathbb{T}]_{\tau \sigma} = [\delta\text{Sk}(\mathbb{T})]_{\tau \sigma} - \sum_{v \models V(\mathbb{T})} [L_v \mathbb{T}]_{\tau \sigma} [\text{Sk}(R_v \mathbb{T})]_{\sigma} \quad (6.29)$$

where $[\delta(\text{Sk}\mathbb{T})]_{\tau \sigma} := [\text{Sk}\mathbb{T}]_{\tau} - [\text{Sk}\mathbb{T}]_{\sigma}$.

Proof.

The proof is similar to that of the preceding lemma and we shall use the same notations. One has by induction hypothesis:

$$[\mathbb{T}']_{\tau \sigma} = \left([\mathbb{T}]_{\tau \sigma} \mid \begin{matrix} w \\ \sigma \end{matrix} \right) = \left([\delta\text{Sk}(\mathbb{T})]_{\tau \sigma} \mid \begin{matrix} w \\ \sigma \end{matrix} \right) - \sum_{v \models V(\mathbb{T})} A_v, \quad (6.30)$$

where

$$A_v = \left([L_v \mathbb{T}]_{\tau \sigma} [\text{Sk}(R_v \mathbb{T})]_{\sigma} \mid \begin{matrix} w \\ \sigma \end{matrix} \right). \quad (6.31)$$

Note that

$$[\delta\text{Sk}(\mathbb{T}')]_{\tau \sigma} = \left([\delta\text{Sk}(\mathbb{T})]_{\tau \sigma} \mid \begin{matrix} w \\ \emptyset \end{matrix} \right). \quad (6.32)$$

(i) (down case) Then $L_{\mathbf{v}}\mathbb{T}' = L_{\mathbf{v}}\mathbb{T}$, $[\text{Sk}(R_{\mathbf{v}}\mathbb{T}')]_{\sigma} = \left([\text{Sk}(R_{\mathbf{v}}\mathbb{T})]_{\sigma} \mid \begin{matrix} w \\ \emptyset \end{matrix} \right)$.

(ii) (up case) Then

$$\begin{aligned} [L_{\mathbf{v}}\mathbb{T}']_{\tau\sigma} &= \left([L_{\mathbf{v}}\mathbb{T}]_{\tau\sigma} \mid \begin{matrix} w \\ \sigma \end{matrix} \right), \quad \text{Sk}(R_{\mathbf{v}}\mathbb{T}') = \text{Sk}(R_{\mathbf{v}}\mathbb{T}) \\ L_{\mathbf{v}'}\mathbb{T}' &= L_{\mathbf{v}}\mathbb{T}, \quad [\text{Sk}(R_{\mathbf{v}'}\mathbb{T}')]_{\sigma} = - \left([\text{Sk}(R_{\mathbf{v}}\mathbb{T})]_{\sigma} \mid \begin{matrix} \emptyset \\ \sigma \end{matrix} \right) \end{aligned} \quad (6.33)$$

(iii) Let $\mathbf{v} = \{w'\}$,

$$[L_{\{w'\}}\mathbb{T}']_{\tau\sigma} = [\mathbb{T}]_{\tau\sigma} = [\delta\text{Sk}\mathbb{T}]_{\tau\sigma} - \sum_{\mathbf{v} \models V(\mathbb{T})} [L_{\mathbf{v}}\mathbb{T}]_{\tau\sigma} [\text{Sk}(R_{\mathbf{v}}\mathbb{T})]_{\sigma}, \quad [\text{Sk}(R_{\{w'\}}\mathbb{T}')]_{\sigma} = - \left(\begin{matrix} \emptyset \\ \sigma \end{matrix} \right). \quad (6.34)$$

Recombining all these identities gives

$$[\mathbb{T}']_{\tau\sigma} = [\delta\text{Sk}(\mathbb{T}')]_{\tau\sigma} - \sum_{\mathbf{v} \models V(\mathbb{T}')} [L_{\mathbf{v}}\mathbb{T}']_{\tau\sigma} [\text{Sk}(R_{\mathbf{v}}\mathbb{T}')]_{\sigma}. \quad (6.35)$$

□

Consider now $f = f^{n,r,\varepsilon}$ as defined in Definition 6.8, and assume (as in subsection 5.1) some regularization procedure $I_{\text{Sk}\mathbb{T}}(f^{n,r,\varepsilon} \circ \varepsilon|_{V(\mathbb{T})}) \rightarrow \mathcal{R}I_{\text{Sk}\mathbb{T}}(f^{n,r,\varepsilon} \circ \varepsilon|_{V(\mathbb{T})})$, commuting with the projection onto the Fourier components, has been defined for any well-labeled tree \mathbb{T} (see Definition 6.9), such that

$$[\delta\mathcal{R}I_{\text{Sk}\mathbb{T}}(f^{n,r,\varepsilon} \circ \varepsilon|_{V(\mathbb{T})})]_{\tau\sigma} = [\delta I_{\text{Sk}\mathbb{T}}(f^{n,r,\varepsilon} \circ \varepsilon|_{V(\mathbb{T})})]_{\tau\sigma} = \delta B_{\tau\sigma}^r(\varepsilon(v)) \quad (6.36)$$

if $\mathbb{T} = \{v\}$ is the trivial tree with one vertex, v .

Lemma 6.14 (regularization) *Let $\mathbb{T} = \mathbb{T}_1 \dots \mathbb{T}_l$ be a well-labeled forest, together with its tree decomposition. Define by induction the regularized integral $[\mathcal{R}I_{\mathbb{T}}]_{\tau\sigma}$ by*

$$\prod_{j=1}^l \left\{ [\delta\mathcal{R}I_{\text{Sk}\mathbb{T}_j}]_{\tau\sigma} - \sum_{\mathbf{v} \models V(\mathbb{T}_j)} [\mathcal{R}I_{L_{\mathbf{v}}\mathbb{T}_j}]_{\tau\sigma} [\mathcal{R}I_{\text{Sk}R_{\mathbf{v}}\mathbb{T}_j}]_{\sigma} \right\} \quad (6.37)$$

where the argument $f = f^{n,r,\varepsilon} \circ \varepsilon$ has been omitted (see Lemma 6.4 (2)), and where $I_{\text{Sk}\mathbb{T}'} \rightarrow \mathcal{R}I_{\text{Sk}\mathbb{T}'}$ is any regularization procedure commuting with

the projections onto the Fourier components such that eq. (6.36) is satisfied. Then $[\mathcal{R}I_{\mathbb{T}}]_{\tau\sigma}$ satisfies the following tree multiplicative property:

$$[\delta\mathcal{R}I_{\mathbb{T}}]_{\tau u\sigma} = \sum_{\mathbf{v} \models V(\mathbb{T})} [\mathcal{R}I_{L_{\mathbf{v}}\mathbb{T}}]_{\tau u} \cdot [\mathcal{R}I_{R_{\mathbf{v}}\mathbb{T}}]_{u\sigma}. \quad (6.38)$$

By analogy with section 5, $[\mathcal{R}I_{\mathbb{T}_j}(\delta)]_{\tau\sigma} := [\delta\mathcal{R}I_{\text{Sk}\mathbb{T}_j}]_{\tau\sigma}$, resp. $[\mathcal{R}I_{\mathbb{T}_j}(\partial)]_{\tau\sigma} := -\sum_{\mathbf{v} \models V(\mathbb{T}_j)} [\mathcal{R}I_{L_{\mathbf{v}}\mathbb{T}_j}]_{\tau\sigma} [\mathcal{R}I_{\text{Sk}R_{\mathbf{v}}\mathbb{T}_j}]_{\sigma}$ will be called the increment, resp. boundary term associated to the tree \mathbb{T}_j .

Remarks.

- If $\varepsilon = \text{Id}_n$, and $\mathbb{T} = \mathbb{T}^{\text{Id}_n}$ is the trunk tree, then eq. (6.37) coincides with eq. (5.10), and the tree multiplicative property (6.38) is identical to the multiplicative property (5.14).
- Mind that $[\mathcal{R}I_{L_{\mathbf{v}}\mathbb{T}}]_{\tau u} [\mathcal{R}I_{R_{\mathbf{v}}\mathbb{T}}]_{u\sigma}(f)$ is a *bilinear expression*, not a product. However, projecting $f = f^{n,r,\varepsilon} \circ \varepsilon$ onto any of its Fourier components transforms the expressions (6.37) and (6.38) into products and the 'projected' multiplicative property (5.14) into the usual multiplicative property (ii) of the Introduction. Identities (5.14) and (6.38) may be seen as 'tensorized' (or also 'projected') versions of the ordinary/tree multiplicative property which hold for functions f which are of the product form $f_0 \otimes \dots \otimes f_{n-1}$ (see Lemma 6.4, (1) and (2)).

Proof. If the multiplicative property (6.38) holds for trees, then it holds automatically for forests since $[\mathcal{R}I_{\mathbb{T}_1 \dots \mathbb{T}_l}]_{\tau\sigma}$ is the product $\prod_{j=1}^l [\mathcal{R}I_{\mathbb{T}_j}]_{\tau\sigma}$. Hence we may assume that \mathbb{T} is a tree, say, with n vertices. Suppose (by induction) that the above multiplicative property (6.38) holds for all trees with $\leq n-1$ vertices. Then

$$\begin{aligned} [\delta\mathcal{R}I_{\mathbb{T}}]_{\tau u\sigma} &= \sum_{\mathbf{v} \models V(\mathbb{T})} \left(-[\delta\mathcal{R}I_{L_{\mathbf{v}}\mathbb{T}}]_{\tau u\sigma} [\mathcal{R}I_{\text{Sk}(R_{\mathbf{v}}\mathbb{T})}]_{\sigma} + [\mathcal{R}I_{L_{\mathbf{v}}\mathbb{T}}]_{\tau u} [\delta\mathcal{R}I_{\text{Sk}(R_{\mathbf{v}}\mathbb{T})}]_{u\sigma} \right) \\ &= \sum_{\mathbf{v} \models V(\mathbb{T})} \sum_{\mathbf{w} \models V(L_{\mathbf{v}}\mathbb{T})} \left(-[\mathcal{R}I_{L_{\mathbf{w}} \circ L_{\mathbf{v}}(\mathbb{T})}]_{\tau u} [\mathcal{R}I_{R_{\mathbf{w}} \circ L_{\mathbf{v}}(\mathbb{T})}]_{u\sigma} [\mathcal{R}I_{\text{Sk}R_{\mathbf{v}}\mathbb{T}}]_{\sigma} \right. \\ &\quad \left. + [\mathcal{R}I_{L_{\mathbf{v}}\mathbb{T}}]_{\tau u} [\delta\mathcal{R}I_{\text{Sk}(R_{\mathbf{v}}\mathbb{T})}]_{u\sigma} \right). \end{aligned} \quad (6.39)$$

Let $\mathbf{x} = \mathbf{v} \amalg \mathbf{w} := \mathbf{v} \cup \mathbf{w} \setminus \{i \in \mathbf{v} \cup \mathbf{w} \mid \exists j \in \mathbf{v} \cup \mathbf{w} \mid i \rightarrow j\}$. Then one easily proves that $L_{\mathbf{w}} \circ L_{\mathbf{v}}(\mathbb{T}) = L_{\mathbf{x}}(\mathbb{T})$, $R_{\mathbf{v}}(\mathbb{T}) = R_{\mathbf{v}} \circ R_{\mathbf{x}}(\mathbb{T})$ and

$R_{\mathbf{w}} \circ L_{\mathbf{v}}(\mathbb{T}) = L_{\mathbf{v}} \circ R_{\mathbf{x}}(\mathbb{T})$. Hence

$$\begin{aligned} [\delta \mathcal{R}I_{\mathbb{T}}]_{\tau u \sigma} &= \sum_{\mathbf{x} \models V(\mathbb{T})} [\mathcal{R}I_{L_{\mathbf{x}}\mathbb{T}}]_{\tau u} \left(- \sum_{\mathbf{v} \models V(R_{\mathbf{x}}\mathbb{T})} [\mathcal{R}I_{L_{\mathbf{v}}(R_{\mathbf{x}}\mathbb{T})}]_{u \sigma} [\mathcal{R}I_{\text{Sk}R_{\mathbf{v}}(R_{\mathbf{x}}\mathbb{T})}]_{\sigma} + [\delta \mathcal{R}I_{\text{Sk}(R_{\mathbf{x}}\mathbb{T})}]_{u \sigma} \right) \\ &= \sum_{\mathbf{x} \models V(\mathbb{T})} [\mathcal{R}I_{L_{\mathbf{x}}\mathbb{T}}]_{\tau u} [\mathcal{R}I_{R_{\mathbf{x}}\mathbb{T}}]_{u \sigma}. \end{aligned} \quad (6.40)$$

□

We shall now prove that property (6.38) implies (after summing over all permutations ε) the usual multiplicative property (ii) of the Introduction for $\mathcal{R}\tilde{\mathbf{B}}^{n,r}$. Since \mathcal{R} commutes with the projections onto the Fourier components, it is enough to prove this for any Fourier component of $\mathcal{R}\tilde{\mathbf{B}}^{n,r}$. The following Lemma proves that the non-tensorized versions of the usual/tree multiplicative properties are equivalent.

Lemma 6.15 (tree versus usual multiplicative property) *Let $\varepsilon \in \Sigma_n$ and \mathcal{T}_N be the set of well-labeled forests with $\leq N$ vertices. Assume the tree functional $(\mathbf{X}_{\tau\sigma}^{\varepsilon}(\mathbb{T}))_{\mathbb{T} \in \mathcal{T}_N}$ satisfies the (non-tensorized) tree multiplicative property, namely,*

$$[\delta \mathbf{X}_{\tau\sigma}^{\varepsilon}(\mathbb{T})]_{\tau u \sigma} = \sum_{\mathbf{v} \models V(\mathbb{T})} \mathbf{X}_{\tau u}^{\varepsilon}(L_{\mathbf{v}}\mathbb{T}) \cdot \mathbf{X}_{u \sigma}^{\varepsilon}(R_{\mathbf{v}}\mathbb{T}), \quad (6.41)$$

and that $\mathbf{X}_{\tau\sigma}^{\varepsilon}(\mathbb{T}) = \mathbf{X}_{\tau\sigma}^{\varepsilon}(\mathbb{T}_1) \dots \mathbf{X}_{\tau\sigma}^{\varepsilon}(\mathbb{T}_L)$ if $\mathbb{T} = \mathbb{T}_1 \dots \mathbb{T}_L$ is a forest.

Let $\mathbf{i} = (i_1, \dots, i_l) \in \{0, \dots, n-1\}$, $i_1 \neq \dots \neq i_l$. Denote by $\varepsilon_{\mathbf{i}}$ the restriction of ε to (i_1, \dots, i_l) ; one may see $\varepsilon_{\mathbf{i}}$ as an element of Σ_l by renumbering (i_1, \dots, i_l) and $(\varepsilon(i_1), \dots, \varepsilon(i_l))$ as $(0, \dots, l-1)$ without changing their ordering. Decompose the graph $\mathbb{T}^{\varepsilon_{\mathbf{i}}}$ as a sum $\sum_l \pm \mathbb{T}_{l,1} \dots \mathbb{T}_{l,L}$ of well-labeled forests as in Lemma 6.7, and define

$$\mathbf{X}_{\tau\sigma}^l(i_1, \dots, i_l) := \sum_l \pm \mathbf{X}_{\tau\sigma}^{\varepsilon}(\mathbb{T}_{l,1}) \dots \mathbf{X}_{\tau\sigma}^{\varepsilon}(\mathbb{T}_{l,L}). \quad (6.42)$$

Then $(\mathbf{X}_{\tau\sigma}^1, \dots, \mathbf{X}_{\tau\sigma}^N)$ satisfies the usual multiplicative property (ii) of the Introduction.

Proof. Let $\varepsilon \in \Sigma_n$ and $\varepsilon' := \tau_{\varepsilon(i), \varepsilon(i+1)} \circ \varepsilon$ where $\tau_{\varepsilon(i), \varepsilon(i+1)}$ is the transposition of the neighbours $\varepsilon(i)$ and $\varepsilon(i+1)$. Let $\mathbb{T}^{\varepsilon} = \sum_l \pm \mathbb{T}_l$, $\mathbb{T}_l = \mathbb{T}_{l,1} \dots \mathbb{T}_{l,L}$ be the tree decomposition of \mathbb{T}^{ε} as in Lemma 6.7. Fix some index l . Then the vertices $\varepsilon(i)$, $\varepsilon(i+1)$ may:

- (i) either belong to two unconnected branches of $\mathbb{T}_{l,j}$ for some j ;
- (ii) or be successive vertices on the same branch of $\mathbb{T}_{l,j}$ for some j ;
- (iii) or belong to two different tree components $\mathbb{T}_{l,j}, \mathbb{T}_{l,j'}$ of \mathbb{T}_l .

If $\varepsilon(i) \in \mathbb{T}_{l,j}$, we let $\varepsilon(i)^\uparrow = \{\varepsilon(i)\} \cup \{v \in \mathbb{T}_{l,j} \mid v \rightarrow \varepsilon(i), v \neq \varepsilon(i+1), v \not\rightarrow \varepsilon(i+1)\}$ be the set of vertices lying 'above' $\varepsilon(i)$ but not on or above the branch containing both $\varepsilon(i)$ and $\varepsilon(i+1)$, and $\varepsilon(i)^\downarrow$ be the vertex below $\varepsilon(i)$, i.e. $\varepsilon(i) \rightarrow \varepsilon(i)^\downarrow$ (unless $\varepsilon(i)$ is the root of $\mathbb{T}_{l,j}$). Let also $\check{\mathbb{T}}_{l,j} = \mathbb{T}_{l,j} \setminus \varepsilon(i)^\uparrow$ be the tree obtained from $\mathbb{T}_{l,j}$ by 'skipping' $\varepsilon(i)^\uparrow$ while going down $\mathbb{T}_{l,j}$.

Now $\mathbb{T}^{\varepsilon'}$ may be gotten from \mathbb{T}^ε by transforming each \mathbb{T}_l in the following way:

- if $\varepsilon(i), \varepsilon(i+1)$ belong to different trees or to unconnected branches of the same tree (cases (i), (iii)), do nothing;
- if $\varepsilon(i), \varepsilon(i+1) \in \mathbb{T}_{l,j}$ for some j and, say, $\varepsilon(i+1) \rightarrow \varepsilon(i)$ (case (ii)) then split $\mathbb{T}_{l,j}$ into $\mathbb{T}_{l,j}^+ - \mathbb{T}_{l,j}^-$, where $\mathbb{T}_{l,j}^+$, resp. $\mathbb{T}_{l,j}^-$ is obtained from $\check{\mathbb{T}}_{l,j}$ by grafting the branch $\varepsilon(i)^\uparrow$ at $\varepsilon(i)^\downarrow$, resp. $\varepsilon(i+1)$. See Fig. 5 and 6.

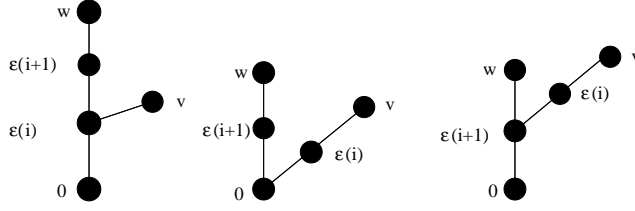


Figure 5: $\mathbb{T} := \mathbb{T}_{l,j}, \mathbb{T}^+$ and \mathbb{T}^- . In this case $\varepsilon(i)^\uparrow = \{\varepsilon(i), v\}$ and $\varepsilon(i)^\downarrow = 0$.

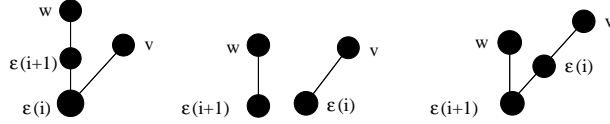


Figure 6: $R_v \mathbb{T}, (R_v \mathbb{T})^+ = (R_v \mathbb{T} \setminus \varepsilon(i)^\uparrow) \cdot \varepsilon(i)^\uparrow$, and $(R_v \mathbb{T})^- = Gr_{\varepsilon(i)^\uparrow}^{\varepsilon(i+1)}(R_v \mathbb{T} \setminus \varepsilon(i)^\uparrow)$ with $v = \{\varepsilon(i)\}$.

In the special case when $\varepsilon(i)$ is the root of $\mathbb{T}_{l,j}$, then we set $\mathbb{T}_{l,j}^+ = \check{\mathbb{T}}_{l,j} \cdot \varepsilon(i)^\uparrow$ ($\mathbb{T}_{l,j}^+$ is then a forest).

These transformations are simply the tree counterpart of the following identities obtained by exchanging the integrations with respect to $\zeta_{\varepsilon(i)}$ and $\zeta_{\varepsilon(i+1)}$:

$$\begin{aligned}
& \dots \int_{\sigma}^{\zeta_{\varepsilon(i)}^-} d\zeta_{\varepsilon(i)} \int_{\sigma}^{\zeta_{\varepsilon(i)}} d\zeta_{\varepsilon(i+1)} \dots = \dots \int_{\sigma}^{\zeta_{\varepsilon(i)}^-} d\zeta_{\varepsilon(i+1)} \int_{\zeta_{\varepsilon(i+1)}}^{\zeta_{\varepsilon(i)}^-} d\zeta_{\varepsilon(i)} \dots \\
& = \dots \int_{\sigma}^{\zeta_{\varepsilon(i)}^-} d\zeta_{\varepsilon(i+1)} \left(\int_{\sigma}^{\zeta_{\varepsilon(i)}^-} d\zeta_{\varepsilon(i)} - \int_{\sigma}^{\zeta_{\varepsilon(i+1)}} d\zeta_{\varepsilon(i)} \right) \dots \quad (6.43)
\end{aligned}$$

and similarly (if $\varepsilon(i)$ is a root)

$$\begin{aligned}
& \dots \int_{\sigma}^{\tau} d\zeta_{\varepsilon(i)} \int_{\sigma}^{\zeta_{\varepsilon(i)}} d\zeta_{\varepsilon(i+1)} \dots = \dots \int_{\sigma}^{\tau} d\zeta_{\varepsilon(i+1)} \left(\int_{\sigma}^{\tau} d\zeta_{\varepsilon(i)} - \int_{\sigma}^{\zeta_{\varepsilon(i+1)}} d\zeta_{\varepsilon(i)} \right) \dots \quad (6.44)
\end{aligned}$$

Since every permutation ε may be written as a product of transpositions between neighbours, and the lemma is trivially true for $\varepsilon = 1$, all we need to prove is that (assuming we are in case (ii) and, say, $\varepsilon(i+1) \rightarrow \varepsilon(i)$)

$$[\delta \mathbf{X}^{\varepsilon}(\mathbb{T}_{l,j})]_{\tau u \sigma} = [\delta \mathbf{X}^{\varepsilon'}(\mathbb{T}_{l,j}^+)]_{\tau u \sigma} - [\delta \mathbf{X}^{\varepsilon'}(\mathbb{T}_{l,j}^-)]_{\tau u \sigma}. \quad (6.45)$$

Consider $\mathbf{v} \models V(\mathbb{T}_{l,j})$ as in the proof of Lemma 6.11. There are three cases:

- (i) $\varepsilon(i), \varepsilon(i+1) \notin \mathbf{v}$: then both $\varepsilon(i)$ and $\varepsilon(i+1)$ are on the same side of the cut, i.e. $\varepsilon(i), \varepsilon(i+1) \in L_{\mathbf{v}}\mathbb{T}_{l,j}$ or $\varepsilon(i), \varepsilon(i+1) \in R_{\mathbf{v}}\mathbb{T}_{l,j}$, hence (applying the same rules as for the transformation $\mathbb{T}^{\varepsilon} \rightarrow \mathbb{T}^{\varepsilon'}$ either to $L_{\mathbf{v}}\mathbb{T}_{l,j}$ or to $R_{\mathbf{v}}\mathbb{T}_{l,j}$) $\mathbf{X}_{\tau u}^{\varepsilon}(L_{\mathbf{v}}\mathbb{T}_{l,j})\mathbf{X}_{u\sigma}^{\varepsilon}(R_{\mathbf{v}}\mathbb{T}_{l,j})$ is equal to its right hand-side counterpart $\left\{ \mathbf{X}_{\tau u}^{\varepsilon'}(L_{\mathbf{v}}(\mathbb{T}_{l,j}^+)) - \mathbf{X}_{\tau u}^{\varepsilon'}(L_{\mathbf{v}}(\mathbb{T}_{l,j}^-)) \right\} \mathbf{X}_{u\sigma}^{\varepsilon}(R_{\mathbf{v}}(\mathbb{T}_{l,j}))$ or $\mathbf{X}_{\tau u}^{\varepsilon}(L_{\mathbf{v}}(\mathbb{T}_{l,j})) \left\{ \mathbf{X}_{u\sigma}^{\varepsilon'}(R_{\mathbf{v}}(\mathbb{T}_{l,j}^+)) - \mathbf{X}_{u\sigma}^{\varepsilon'}(R_{\mathbf{v}}(\mathbb{T}_{l,j}^-)) \right\}$.
- (ii) $\varepsilon(i) \in \mathbf{v}$ (hence $\varepsilon(i+1) \notin \mathbf{v}$): then the corresponding cut on the right hand-side gives no contribution, since $(L_{\mathbf{v}}(\mathbb{T}_{l,j}^+), R_{\mathbf{v}}(\mathbb{T}_{l,j}^+)) = (L_{\mathbf{v}}(\mathbb{T}_{l,j}^-), R_{\mathbf{v}}(\mathbb{T}_{l,j}^-))$ (namely, grafting $\varepsilon(i)^{\uparrow}$ at $\varepsilon(i)^-$ and then stripping it is equivalent to grafting it at $\varepsilon(i+1)$ and stripping it). As for the left hand-side, it writes

$$\begin{aligned}
& [\mathbf{X}^{\varepsilon}(L_{\mathbf{v}}\mathbb{T}_{l,j})]_{\tau u} [\mathbf{X}^{\varepsilon}(R_{\mathbf{v}}\mathbb{T}_{l,j})]_{u\sigma} = [\mathbf{X}^{\varepsilon'}(L_{\mathbf{v}}\mathbb{T}_{l,j})]_{\tau u} \\
& \left\{ [\mathbf{X}^{\varepsilon'}(R_{\mathbf{v}}\mathbb{T}_{l,j} \setminus \varepsilon(i)^{\uparrow})]_{u\sigma} [\mathbf{X}^{\varepsilon'}(\varepsilon(i)^{\uparrow})]_{u\sigma} - [\mathbf{X}^{\varepsilon'}(Gr_{\varepsilon(i)^{\uparrow}}^{\varepsilon(i+1)}(R_{\mathbf{v}}\mathbb{T}_{l,j} \setminus \varepsilon(i)^{\uparrow}))]_{u\sigma} \right\} \\
& =: A_1 + A_2 \quad (6.46)
\end{aligned}$$

where $R_{\mathbf{v}}\mathbb{T}_{l,j} \setminus \varepsilon(i)^{\uparrow}$, resp. $Gr_{\varepsilon(i)^{\uparrow}}^{\varepsilon(i+1)}(R_{\mathbf{v}}\mathbb{T}_{l,j} \setminus \varepsilon(i)^{\uparrow})$ is obtained from $R_{\mathbf{v}}\mathbb{T}_{l,j}$ by skipping the branch $\varepsilon(i)^{\uparrow}$, resp. by grafting it at $\varepsilon(i+1)$.

- (iii) $\varepsilon(i+1) \in \mathbf{v}$ (hence $\varepsilon(i) \notin \mathbf{v}$): then the contribution to $\delta \mathbf{X}^\varepsilon(\mathbb{T}_{l,j})$ and $\delta \mathbf{X}^{\varepsilon'}(\mathbb{T}_{l,j}^+)$ are the same, while the contribution to $\delta \mathbf{X}^{\varepsilon'}(\mathbb{T}_{l,j}^-)$ is equal to A_2 .

Finally, adding $\varepsilon(i+1)$ to the cut \mathbf{v} of case (ii) yields an admissible cut of $\mathbb{T}_{l,j}^+$ and leads to a contribution in the right hand-side which is equal to A_1 .

□

6.3 Definition and estimates of the regularized increment term

Recall (see Lemma 6.7) that \mathbb{T}^ε is a finite sum $\sum_l \pm \mathbb{T}_{l,1} \dots \mathbb{T}_{l,L(l)}$. Let $\mathbb{T} = \mathbb{T}_{l,j}$ be any of the trees appearing in this decomposition (for the sake of brevity, we shall simply write $\mathbb{T} \subset \mathbb{T}^\varepsilon$). The estimates of $\mathcal{R}I_{\mathbb{T}}$ we obtain in the present and the next subsection imply in a straightforward way Theorem 0.1 in the Introduction. We shall now prove estimates for the regularized increment term $\mathcal{R}I_{\mathbb{T}}(\delta) = \delta \mathcal{R}I_{\text{Sk}\mathbb{T}}$. Recall (see Definition 6.9) that the vertices of \mathbb{T} are indexed by $-$ and totally ordered according to $-$ their labels; by definition also, if (k_0, \dots, k_{n-1}) is a multi-index in the defining function $f^{n,r,\varepsilon} \circ \varepsilon$, then $v < w \Rightarrow |k_v| \leq |k_w|$.

Definition 6.16 Fix $C_{reg} \in (0, 1)$. Let, for $\mathbb{T} \subset \mathbb{T}^\varepsilon$ with set of vertices $V(\mathbb{T}) = \{v_1 < \dots < v_j\}$,

$$\begin{aligned} \mathbb{Z}_{reg}^{\mathbb{T}} &:= \{(k_{v_1}, \dots, k_{v_j}) \in \mathbb{Z}^j \mid |k_{v_1}| \leq \dots \leq |k_{v_j}| \text{ and} \\ &\forall v \in V(\mathbb{T}), |k_v + \sum_{w \rightarrow v} k_w| > C_{reg} \max\{|k_w|; w \rightarrow v\}\}. \end{aligned} \quad (6.47)$$

Let $\mathcal{R}I_{\text{Sk}\mathbb{T}}(f^{n,r,\varepsilon} \circ \varepsilon|_{V(\text{Sk}\mathbb{T})}) := I_{\text{Sk}\mathbb{T}}(f_{reg}^{n,r,\varepsilon} \circ \varepsilon|_{V(\text{Sk}\mathbb{T})})$ where the regularized defining function $f_{reg}^{n,r,\varepsilon}$ is defined as follows:

$$\begin{aligned} f_{reg}^{n,r,\varepsilon} \circ \varepsilon|_{V(\text{Sk}\mathbb{T})}(\zeta_{v_1}, \dots, \zeta_{v_j}) &= \\ &\sum_{(k_{v_1}, \dots, k_{v_j}) \in \mathbb{Z}_{reg}^{\mathbb{T}}} r^{|k_{v_1}| + \dots + |k_{v_j}|} a_{k_{v_1}} \dots a_{k_{v_j}} \tilde{\xi}_{k_{v_1}}(\varepsilon(v_1)) \dots \tilde{\xi}_{k_{v_j}}(\varepsilon(v_j)) \zeta_{v_1}^{k_{v_1}} \dots \zeta_{v_j}^{k_{v_j}}. \end{aligned} \quad (6.48)$$

Remark 6.17 Since \mathbb{T} is well-labeled, by definition, $|k_v| \leq |k_w|$ whenever $v, w \in \mathbb{T}$ and $w \rightarrow v$; in other words, the multi-index $(|k_v|, v \in V(\mathbb{T}))$ is increasing with respect to the partial ordering \rightarrow . It is actually totally

ordered with respect to the label ordering (and we shall use this fact in the course of the proofs), but any total ordering compatible with the tree partial ordering would give the same results. Hence the only essential property is the fact that \mathbb{T} is well-labeled.

Lemma 6.18 (Hölder estimate and rate of convergence) *Let $\mathbb{T} \subset \mathbb{T}^\varepsilon$ and $\alpha < 1/n$.*

1. The skeleton term $\mathcal{R}I_{\text{Sk}\mathbb{T}} := \mathcal{R}I_{\text{Sk}\mathbb{T}}(f^{n,r,\varepsilon} \circ \varepsilon|_{V(\mathbb{T})})$ writes

$$[\mathcal{R}I_{\text{Sk}\mathbb{T}}]_z = \sum_{(k_{v_1}, \dots, k_{v_j}) \in \mathbb{Z}_{\text{reg}}^{\mathbb{T}}} r^{|k_{v_1}| + \dots + |k_{v_j}|} \frac{a_{k_{v_1}} \dots a_{k_{v_j}} \tilde{\xi}_{k_{v_1}}(\varepsilon(v_1)) \dots \tilde{\xi}_{k_{v_j}}(\varepsilon(v_j)) z^{k_{v_1} + \dots + k_{v_j}}}{\prod_{v \in V(\mathbb{T})} [k_v + \sum_{w \rightarrow v} k_w]}.$$

(6.49)

2. It satisfies the required Hölder estimate:

$$\mathbb{E} |[\delta \mathcal{R}I_{\text{Sk}\mathbb{T}}]_{\tau\sigma}|^2 \leq Cd(\sigma, \tau)^{2\alpha|V(\mathbb{T})|}.$$

(6.50)

3. (rate of convergence) : there exists a constant $C > 0$ such that, for every $\frac{1}{2} < r_1, r_2 < 1$ and $\sigma, \tau \in \mathcal{C}$,

$$\mathbb{E} \left| [\delta \mathcal{R}I_{\text{Sk}\mathbb{T}}(f^{n,r_1,\varepsilon} \circ \varepsilon|_{V(\mathbb{T})})]_{\tau\sigma} - [\delta \mathcal{R}I_{\text{Sk}\mathbb{T}}(f^{n,r_2,\varepsilon} \circ \varepsilon|_{V(\mathbb{T})})]_{\tau\sigma} \right|^2 \leq C|r_1 - r_2|^{2\alpha}.$$

(6.51)

Proof.

1. Easy computation (generalization of eq. (5.9)).
2. (Hölder estimate)

Let $V(\mathbb{T}) = \{v_1 < \dots < v_j\}$, so that $|k_{v_1}| \leq \dots \leq |k_{v_j}|$ (see preceding Remark). Since every vertex $v \in V(\mathbb{T}) \setminus \{v_1\}$ connects to the root v_1 , one has $j|k_{v_j}| \geq |k_{v_1} + \dots + k_{v_j}| > C_{\text{reg}}|k_{v_j}|$, so the exponent of z in eq. (6.49) is comparable to k_{v_j} , i.e. belongs to $[C^{-1}k_{v_j}, Ck_{v_j}]$ if C is some large enough positive constant. Write $\mathcal{R}I_{\text{Sk}\mathbb{T}} = \sum_{K \in \mathbb{Z}} C_K z^K$.

Vertices at which 2 or more branches join are called *nodes*, and vertices to which no vertex is connected are called *leaves* (see Fig. 7).

The set $Br(v_1 \rightarrow v_2)$ of vertices from a leaf or a node v_1 to a node v_2 (or to the root) is called a *branch* if it does not contain any other node. By convention, $Br(v_1 \rightarrow v_2)$ includes v_1 and excludes v_2 .

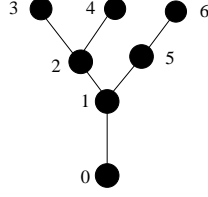


Figure 7: 3,4,6 are leaves; 1, 2 and 5 are nodes, 2 and 5 are uppermost; branches are e.g. $Br(2 \rightarrow 1)$ or $Br(6 \rightarrow 1)$.

Consider an uppermost node n , i.e. a node to which no other node is connected, together with the set of leaves $\{w_1 < \dots < w_J\}$ above n . Let $p_j = |V(Br(w_j \rightarrow n))|$. Note that $\left| \frac{a_{k_n}}{k_n + \sum_{w \rightarrow n} k_w} \right|^2 \lesssim k_{w_j}^{-1-2\alpha}$. Now we proceed to estimate $\text{Var}C_K$. On the branch number j ,

$$\begin{aligned} & \sum_{|k_v| \leq |k_{w_j}|, v \in Br(w_j \rightarrow n) \setminus \{w_j\}} \left[\prod_{v \in Br(w_j \rightarrow n)} \frac{r^{|k_v|} a_{k_v}}{k_v + \sum_{w \rightarrow v} k_w} \right]^2 \\ & \lesssim |k_{w_j}|^{p_j-1} |k_{w_j}|^{-p_j(1+2\alpha)} = |k_{w_j}|^{-1-2\alpha p_j} \end{aligned} \quad (6.52)$$

and (summing over $k_{w_1}, \dots, k_{w_{J-1}}$ and over k_n)

$$\begin{aligned} & |k_{w_J}|^{-1-2\alpha} \sum_{|k_{w_{J-1}}| \leq |k_{w_J}|} |k_{w_J}|^{-1-2\alpha p_J} \\ & \left(\dots \left(\sum_{|k_{w_1}| \leq |k_{w_2}|} |k_{w_2}|^{-1-2\alpha p_2} \left(\sum_{|k_n| \leq |k_{w_1}|} |k_{w_1}|^{-1-2\alpha p_1} \right) \right) \dots \right) \\ & \lesssim |k_{w_J}|^{-1-2\alpha} |k_{w_J}|^{-2\alpha(p_1 + \dots + p_J)} = |k_{w_J}|^{-1-2\alpha W(n)}, \end{aligned} \quad (6.53)$$

where $W(n) = p_1 + \dots + p_J + 1 = |\{v : v \rightarrow n\}| + 1$ is the *weight* of n .

One may then consider the reduced tree \mathbb{T}_n obtained by shrinking all vertices above n (including n) to *one* vertex with weight $W(n)$ and perform the same operations on \mathbb{T}_n . Repeat this inductively until \mathbb{T} is shrunk to one point. In the end, one gets $\text{Var}C_K \lesssim |k_{v_j}|^{-1-2\alpha|V(\mathbb{T})|} \lesssim K^{-1-2\alpha|V(\mathbb{T})|}$. Now apply Lemma 2.1.

3. (rate of convergence)

Let $X_z^{n,r_1,r_2,+} := \mathcal{R}I_{\text{SkT}}(f^{n,r_1,\varepsilon} \circ \varepsilon|_{V(\text{SkT})}) - \mathcal{R}I_{\text{SkT}}(f^{n,r_2,\varepsilon} \circ \varepsilon|_{V(\text{SkT})})$.

Proceeding as in the proof of Lemma 5.5 gives $X_z^{n,r_1,r_2,+}$ as a sum,

$X_z^{n,r_1,r_2,+} = \sum_{v \in V(\mathbb{T})} X_z^{n,r_1,r_2,+}(v)$, where $X_z^{n,r_1,r_2,+}(v) = \sum_{k_v \in \mathbb{Z}} C'_{k_v}(z)(r_1^{|k_v|} z^{k_v} - r_2^{|k_v|} z^{k_v})$ is obtained from $\mathcal{R}I_{\text{SkT}}(f^{n,r,\varepsilon} \circ \varepsilon|_{V(\text{SkT})})$ by replacing $r^{|k_v|}$ with $r_1^{|k_v|} - r_2^{|k_v|}$, and $r^{|k_w|}$, $w \neq v$ either by $r_1^{|k_w|}$ or by $r_2^{|k_w|}$. We want to estimate $\text{Var}C'_{k_v}(z)$ uniformly in z .

Fix the value of k_v in the computations in the above proof for the Hölder estimate. Let w_J be the maximal leaf above v , and $n \rightarrow v$ be the node just above v . Summing over all nodes above v and taking the variance leads to an expression bounded by $|k_{w_J}|^{-1-2\alpha W(n)}$, where $W(n)$ is the weight of n . Consider now the corresponding shrunk tree \mathbb{T}_n . Sum over all vertices $w \in \mathbb{T}_n$ such that $w \rightarrow v$ or $v \rightarrow w$. These make up a trunk tree, which allows us to apply the same arguments as in the proof of Lemma 5.5 and get a variance bounded by $|k_v|^{-1-2\alpha|\bar{W}(v)|}$, where $\bar{W}(v) = \{w \in \mathbb{T} : w \rightarrow v \text{ or } v \rightarrow w\} \cup \{v\}$. Removing the vertices belonging to $\bar{W}(v)$ from \mathbb{T} leads to a forest which gives a finite contribution to the variance. Hence (by Lemma 2.1) $\mathbb{E}|X_z^{n,r_1,r_2,+}(v)|^2 \lesssim |r_1 - r_2|^{2\alpha|\bar{W}(v)|}$.

□

The notion of weight $W(v)$ of a vertex v introduced in this proof will be used again in subsection 7.1.

6.4 Estimates for boundary terms

Let $\mathbb{T} \subset \mathbb{T}^\varepsilon$ as in subsection 6.3. We shall now prove estimates for the boundary term $\mathcal{R}I_{\mathbb{T}}(\partial)$ associated to \mathbb{T} (see Lemma 6.14).

Lemma 6.19 *Let $\mathbb{T} \subset \mathbb{T}^\varepsilon$.*

1. (Hölder estimate) *The regularized boundary term $\mathcal{R}I_{\mathbb{T}}(\partial) := \mathcal{R}I_{\mathbb{T}}(f^{n,r,\varepsilon} \circ \varepsilon|_{V(\mathbb{T})})(\partial)$ satisfies:*

$$\mathbb{E} |[\mathcal{R}I_{\mathbb{T}}(\partial)]_{\tau\sigma}|^2 \leq Cd(\sigma, \tau)^{2\alpha|V(\mathbb{T})|} \quad (6.54)$$

for a certain constant C .

2. (rate of convergence) *There exists a positive constant C such that, for every $\frac{1}{2} < r_1, r_2 < 1$,*

$$\mathbb{E} |[\mathcal{R}I_{\mathbb{T}}(f^{n,r_1,\varepsilon} \circ \varepsilon|_{V(\mathbb{T})})(\partial)]_{\tau\sigma} - [\mathcal{R}I_{\mathbb{T}}(f^{n,r_2,\varepsilon} \circ \varepsilon|_{V(\mathbb{T})})(\partial)]_{\tau\sigma}|^2 \leq C|r_1 - r_2|^{2\alpha}. \quad (6.55)$$

Proof.

1. Apply repeatedly Lemma 6.14 to \mathbb{T} : in the end, $[\mathcal{R}I_{\mathbb{T}}(\partial)]_{\tau\sigma}$ appears as a sum of 'skeleton-type' terms of the form (see Figure 8)

$$A_{\tau\sigma} := [\delta\mathcal{R}I_{\text{Sk}L\mathbb{T}}]_{\tau\sigma} \cdot [\mathcal{R}I_{\text{Sk}R_{v_1}\mathbb{T}}]_{\sigma} [\mathcal{R}I_{\text{Sk}R_{v_2}\circ L_{v_1}(\mathbb{T})}]_{\sigma} \cdots [\mathcal{R}I_{\text{Sk}R_{v_l}\circ L_{v_{l-1}}\circ \dots \circ L_{v_1}(\mathbb{T})}]_{\sigma} (f^{n,r,\varepsilon} \circ \varepsilon|_{V(\mathbb{T})}), \quad (6.56)$$

where $\mathbf{v}_1 = (v_{1,1} < \dots < v_{1,J_1}) \models \mathbb{T}$, $\mathbf{v}_2 \models L_{v_1}\mathbb{T}$, \dots , $\mathbf{v}_l = (v_{l,1}, \dots, v_{l,J_l}) \models L_{v_{l-1}} \circ \dots \circ L_{v_1}(\mathbb{T})$ and $L\mathbb{T} := L_{v_l} \circ \dots \circ L_{v_1}(\mathbb{T})$. In eq. (6.56) the tree \mathbb{T} has been split into a number of subtrees, $L\mathbb{T} \cup \cup_{j=1}^J \mathbb{T}_j$; we call this splitting the splitting associated to $A_{\tau\sigma}$ for further reference. Note $A_{\tau\sigma}$ may also be written as $[\delta\mathcal{R}I_{\text{Sk}L\mathbb{T}}]_{\tau\sigma} \prod_{j=1}^J [\mathcal{R}I_{\text{Sk}\mathbb{T}_j}]_{\sigma}$.

First step.

Fix $\mathbf{k} = (k_{v_{1,1}}, \dots, k_{v_{1,J_1}})$ such that $|k_{v_{1,1}}| \leq \dots \leq |k_{v_{1,J_1}}|$, and let $[\mathcal{R}I_{\text{Sk}R_{v_1}\mathbb{T}}[\mathbf{k}]]_z$ be the expression obtained by keeping all terms in the series $\mathcal{R}I_{\text{Sk}R_{v_1}\mathbb{T}}$ such that $\mathbf{k} = (k_{v_{1,1}}, \dots, k_{v_{1,J_1}})$ (with $|k_{v_{1,1}}| \leq \dots \leq |k_{v_{1,j}}|$) is fixed. (Note that the regularized integral $\mathcal{R}I_{\text{Sk}R_{v_1}\mathbb{T}}[\mathbf{k}]$ plays the same role as $\mathcal{R}\tilde{\mathbf{H}}[k]$ in the proof of Lemma 5.6.) Then

$$\text{Var}[\mathcal{R}I_{\text{Sk}R_{v_1}\mathbb{T}}[\mathbf{k}]]_{\sigma} \lesssim \prod_{v \in \mathbf{v}_1} \left[|k_v|^{-1-2\alpha} \sum_{|k_w| \geq |k_v|, w \in R_v\mathbb{T} \setminus \{v\}} |k_w|^{-1-2\alpha} \right], \quad (6.57)$$

hence

$$\text{Var}[\mathcal{R}I_{\text{Sk}R_{v_1}\mathbb{T}}[\mathbf{k}]]_{\sigma} \lesssim \prod_{v \in \mathbf{v}_1} |k_v|^{-2|V(R_v\mathbb{T})|\alpha-1}. \quad (6.58)$$

Second step.

More generally, let $B_{\sigma}[\mathbf{k}]$ be the expression obtained by keeping all terms in the series $[\mathcal{R}I_{\text{Sk}R_{v_1}\mathbb{T}}]_{\sigma} [\mathcal{R}I_{\text{Sk}R_{v_2}\circ L_{v_1}(\mathbb{T})}]_{\sigma} \cdots [\mathcal{R}I_{\text{Sk}R_{v_l}\circ L_{v_{l-1}}\circ \dots \circ L_{v_1}(\mathbb{T})}]_{\sigma}$ with fixed value of the indices $\mathbf{k} = (k_{v_{1,1}}, \dots, k_{v_{l,J_l}})$. Then

$$\text{Var}(B_{\sigma}[\mathbf{k}]) \lesssim \prod_{v \in \mathbf{v}_l} |k_v|^{-2|V(R_v\mathbb{T})|\alpha-1} \quad (6.59)$$

(proof by induction on l).

Third step.

Let $V(L\mathbb{T}) = \{w_1 < \dots < w_{max}\}$. By definition, $A_{\tau\sigma} = \sum_{K \in \mathbb{Z}} C_K(\sigma)(\tau^K - \sigma^K)$, with

$$C_K(\sigma) = \sum_{\mathbf{k}=(k_{v_{l,1}}, \dots, k_{v_{l, J_l}})} \sum_{((k_w)_{w \in V(L\mathbb{T})}) \in S_{\mathbf{k}}} \frac{\prod_{w \in V(L\mathbb{T})} r^{|k_w|} a_{k_w} \tilde{\xi}_{k_w}(\varepsilon(k_w))}{\prod_{w \in V(L\mathbb{T})} (k_w + \sum_{w' \rightarrow w, w' \in V(L\mathbb{T})} k_{w'})} B_{\sigma}[\mathbf{k}] \quad (6.60)$$

where indices in $S_{\mathbf{k}}$ satisfy in particular the following conditions:

- $|k_w + \sum_{w' \rightarrow w, w' \in V(L\mathbb{T})} k_{w'}| > C_{reg} \max\{|k_{w'}| : w' \rightarrow w, w' \in V(L\mathbb{T})\}$; in particular, $\left| \frac{a_{k_w}}{k_w + \sum_{w' \rightarrow w} k_{w'}} \right|^2 \lesssim |k_w|^{-1-2\alpha}$;
- $\sum_{w \in V(L\mathbb{T})} k_w = K$;
- for every $w \in V(L\mathbb{T})$, $|k_w| \leq |k_{w_{max}}|$ and $|k_w| \leq |k_v|$ for every $v \in R(w) := \{v = v_{l,1}, \dots, v_{l, J_l} \mid v \rightarrow w\}$ (note that $R(w)$ may be empty). See Fig. 8.

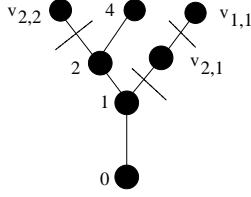


Figure 8: Here $V(L\mathbb{T}) = \{0, 1, 2, 4\}$, $R(0) = R(4) = \emptyset$, $R(1) = \{v_{2,1}\}$, $R(2) = \{v_{2,2}\}$.

Note that $|K| \lesssim |k_{w_{max}}| \lesssim |K|$ since every vertex in $V(L\mathbb{T})$ connects to the root (see first lines of the proof of Lemma 6.18), hence the condition $\sum_{w \in V(L\mathbb{T})} k_w = K$ is more or less equivalent to fixing $k_{w_{max}} \simeq K$ and letting k_w , $w \in V(L\mathbb{T}) \setminus \{w_{max}\}$ range over $[-|K|, |K|]$.

If $w \in L\mathbb{T}$, split $R(w)$ into $R(w)_{>} \cup R(w)_{<}$, where $R(w)_{\geq} := \{v \in R(w) \mid v \geq w_{max}\}$. Summing over indices corresponding to vertices in $R\mathbb{T}_{>} := \{v = v_{l,1}, \dots, v_{l, J_l} \mid v > w_{max}\} = \cup_{w \in L\mathbb{T}} R(w)_{>}$, one gets

$$\prod_{v \in R\mathbb{T}_{>}} \sum_{|k_v| \geq |K|} |k_v|^{-2|V(R_v\mathbb{T})|^{\alpha-1}} \lesssim |K|^{-2\alpha \sum_{v \in R\mathbb{T}_{>}} |V(R_v\mathbb{T})|}. \quad (6.61)$$

Let $w \in L\mathbb{T} \setminus \{w_{max}\}$ such that $R(w)_{<} \neq \emptyset$. Let $R(w)_{<} = \{v_{i_1} < \dots <$

v_{i_j} } (note that $R(w_{max})_{<} = \emptyset$). Then

$$\begin{aligned} & |k_w|^{-1-2\alpha} \sum_{|k_{v_{i_1}}|=|k_w|}^{\infty} \sum_{|k_{v_{i_2}}|=|k_{v_{i_1}}|}^{\infty} \cdots \sum_{|k_{v_{i_j}}|=|k_{v_{i_{j-1}}}|}^{\infty} \\ & |k_{v_{i_1}}|^{-2|V(R_{v_{i_1}}\mathbb{T})|\alpha-1} \cdots |k_{v_{i_j}}|^{-2|V(R_{v_{i_j}}\mathbb{T})|\alpha-1} \lesssim |k_w|^{-1-2\alpha(1+\sum_{v \in R(w)_{<}} |V(R_v\mathbb{T})|)}. \end{aligned} \quad (6.62)$$

In other words, each vertex $w \in L\mathbb{T}$ 'behaves' as if it had a weight $1 + \sum_{v \in R(w)_{<}} |V(R_v\mathbb{T})|$. Hence (by the same method as in the proof of Lemma 6.18) $\text{Var}(C_K(\sigma)) \lesssim |K|^{-1-2\alpha(|V(L\mathbb{T})| + \sum_{v \in R\mathbb{T}_{<}} |V(R_v\mathbb{T})|)}$. $|K|^{-2\alpha \sum_{v \in R\mathbb{T}_{>}} |V(R_v\mathbb{T})|} = |K|^{-1-2\alpha|V(\mathbb{T})|}$. Now apply Lemma 2.1.

2. Use the definition of $\mathcal{R}I_{\mathbb{T}}(\partial)$ given in Lemma 6.14 and mimick the proof of Lemma 5.7. The result follows by induction on $|V(\mathbb{T})|$ using the rate of convergence estimate for increments given in Lemma 6.18.

□

Remark. Note that $[\mathcal{R}I_{\mathbb{T}_{l,1}}]_{\tau\sigma} \cdots [\mathcal{R}I_{\mathbb{T}_{l,L(l)}}]_{\tau\sigma}(f^{n,r,\varepsilon} \circ \varepsilon)$ is the projection of the product $\prod_{j=1}^{L(l)} [\mathcal{R}I_{\mathbb{T}_{l,j}}(f^{n,r,\varepsilon} \circ \varepsilon)]_{V(\mathbb{T}_{l,j})\tau\sigma}$ onto the subspace of the n -th chaos of Γ generated by $\{\xi_{k_0}(\varepsilon(0)) \otimes \cdots \otimes \xi_{k_{n-1}}(\varepsilon(n-1)) \mid |k_0| \leq \cdots \leq |k_{n-1}|\}$. The projection decreases the L^2 -norm, which yields the estimates of Theorem 0.1 for $\mathcal{R}\tilde{\mathbf{B}}^{n,r}(i_0, \dots, i_{n-1})$ with $i_0 \neq \dots \neq i_{n-1}$.

7 End of proof and final remarks

7.1 Case of coinciding indices

Our previous estimates for $\mathbb{E}|\mathcal{R}\tilde{\mathbf{B}}_{\tau\sigma}^{n,r}(i_0, \dots, i_{n-1})|^2$ (Hölder estimate) and $\mathbb{E}|\mathcal{R}\tilde{\mathbf{B}}_{\tau\sigma}^{n,r_1}(i_0, \dots, i_{n-1}) - \mathcal{R}\tilde{\mathbf{B}}_{\tau\sigma}^{n,r_2}(i_0, \dots, i_{n-1})|^2$ (rate of convergence) with $i_j = j$ rest on the essential independence assumption for the variables $\xi_i(0), \dots, \xi_i(n-1)$. We claim that the same estimates also hold true for $\mathbb{E}|\mathcal{R}\tilde{\mathbf{B}}^{n,r}(i_0, \dots, i_{n-1})|^2$ and $\mathbb{E}|\mathcal{R}\tilde{\mathbf{B}}_{\tau\sigma}^{n,r_1}(i_0, \dots, i_{n-1}) - \mathcal{R}\tilde{\mathbf{B}}_{\tau\sigma}^{n,r_2}(i_0, \dots, i_{n-1})|^2$ if some of the indices (i_0, \dots, i_{n-1}) coincide, with the *same definition* of the regularization procedure \mathcal{R} . The key Lemma for the proof is

Lemma 7.1 (Wick's lemma) (see [21], §5.1.2 and 9.3.4)

Let (X_1, \dots, X_n) be a centered Gaussian vector. Denote by $X_{i_1} \diamond \dots \diamond X_{i_k}$ ($1 \leq i_1, \dots, i_k \leq n$) or $:X_{i_1} \dots X_{i_k}$: the Wick product of X_{i_1}, \dots, X_{i_k} (also

called: normal ordering of the product $X_{i_1} \dots X_{i_k}$), i.e. the projection of the product $X_{i_1} \dots X_{i_k}$ onto the k -th chaos of the Gaussian space generated by X_1, \dots, X_n . Then:

1.

$$\begin{aligned}
X_1 \dots X_n &= X_1 \diamond \dots \diamond X_n + \sum_{(i_1, i_2)} \mathbb{E}[X_{i_1} X_{i_2}] X_1 \diamond \dots \diamond \check{X}_{i_1} \diamond \dots \diamond \check{X}_{i_2} \diamond \dots \diamond X_n \\
&+ \dots + \sum_{(i_1, i_2), \dots, (i_{2k+1}, i_{2k+2})} \mathbb{E}[X_{i_1} X_{i_2}] \dots \mathbb{E}[X_{i_{2k+1}} X_{i_{2k+2}}] \\
&X_1 \diamond \dots \diamond \check{X}_{i_1} \diamond \dots \diamond \check{X}_{i_2} \diamond \dots \diamond \check{X}_{i_{2k+1}} \diamond \dots \diamond \check{X}_{i_{2k+2}} \diamond \dots \diamond X_n \\
&+ \dots, \tag{7.1}
\end{aligned}$$

where the sum ranges over all partial pairings of indices $(i_1, i_2), \dots, (i_{2k+1}, i_{2k+2})$ ($1 \leq k \leq \lfloor \frac{n}{2} \rfloor - 1$).

2. For every set of indices $i_0, \dots, i_{j-1}, i'_0, \dots, i'_{j-1}$,

$$\mathbb{E} \left[(X_{i_0} \diamond \dots \diamond X_{i_{j-1}}) (X_{i'_0} \diamond \dots \diamond X_{i'_{j-1}}) \right] = \sum_{\varepsilon \in \Sigma_j} \prod_{m=0}^{j-1} \mathbb{E}[X_{i_m} X_{i'_{\varepsilon(m)}}]. \tag{7.2}$$

In our case (considering $\mathcal{R}\tilde{\mathbf{B}}_{\tau\sigma}^{n,r}(i_0, \dots, i_{n-1})$) we get a decomposition of the product $\tilde{\xi}_{k_0}(i_0) \dots \tilde{\xi}_{k_{n-1}}(i_{n-1})$ into $\tilde{\xi}_{k_0}(i_0) \diamond \dots \diamond \tilde{\xi}_{k_{n-1}}(i_{n-1})$, plus the sum over all possible non-trivial pair contractions $\langle \tilde{\xi}_{k_j}(i_j) \tilde{\xi}_{k_{j'}}(i_{j'}) \rangle = \mathbb{E}[\tilde{\xi}_{k_j}(i_j) \tilde{\xi}_{k_{j'}}(i_{j'})] = 1$ with $i_j = i_{j'}$, $k_j = -k_{j'}$.

Consider first the normal ordering of $\mathcal{R}\tilde{\mathbf{B}}_{\tau\sigma}^{n,r}(i_0, \dots, i_{n-1})$. As in the proof of Lemma 5.10 in [31], let $\Sigma_{\mathbf{i}}$ be the 'index-fixing' subgroup of Σ_n such that $\varepsilon' \in \Sigma_{\mathbf{i}} \iff \forall j = 0, \dots, n-1, i_{\varepsilon'(j)} = i_j$. Then (by Wick's lemma and the Cauchy-Schwarz inequality) :

$$\begin{aligned}
\text{Var} : \mathcal{R}\tilde{\mathbf{B}}_{\tau\sigma}^{n,r}(i_0, \dots, i_{n-1}) &:= \mathbb{E} \left[: \mathcal{R}\tilde{\mathbf{B}}_{\tau\sigma}^{n,r}(i_0, \dots, i_{n-1}) : \right]^2 \\
&= \sum_{\varepsilon' \in \Sigma_{\mathbf{i}}} \mathbb{E} \left[: \mathcal{R}\tilde{\mathbf{B}}_{\tau\sigma}^{n,r}(0, \dots, n-1) : : \overline{\mathcal{R}\tilde{\mathbf{B}}_{\tau\sigma}^{n,r}(\varepsilon'(0), \dots, \varepsilon'(n-1))} : \right] \\
&\leq |\Sigma_{\mathbf{i}}| \cdot \mathbb{E} |\mathcal{R}\tilde{\mathbf{B}}_{\tau\sigma}^{n,r}(0, \dots, n-1)|^2, \tag{7.3}
\end{aligned}$$

hence the Hölder and rate estimates of sections 4, 5, 6 also hold for $\mathcal{R}\tilde{\mathbf{B}}_{\tau\sigma}^{n,r}(i_0, \dots, i_{n-1})$:

One must now prove that the estimates of sections 4, 5, 6 hold true for all possible contractions of $\mathcal{R}\tilde{\mathbf{B}}^{n,r}(i_0, \dots, i_{n-1})$. Fixing some non-trivial contraction $(j_1, j_2), \dots, (j_{2l-1}, j_{2l})$, $l \geq 1$, results in some expression $X_{\tau\sigma} =: I_{\tau\sigma}^n(\check{f}_{reg}^{n-2l,r})$ belonging to the chaos of order $n - 2l$. By necessity, $i_{j_1} = i_{j_2}, \dots, i_{j_{2l-1}} = i_{j_{2l}}$, but it may well be that there are other index coincidences. The same reasoning as in the case of $:\mathcal{R}\tilde{\mathbf{B}}_{\tau\sigma}^{n,r}(i_0, \dots, i_{n-1})$: (see eq. (7.3)) shows that one may actually assume $i_m \neq i_{m'}$ if $m \neq m'$ and $\{m, m'\} \neq \{j_1, j_2\}, \dots, \{j_{2l-1}, j_{2l}\}$. Now (as we shall presently prove) the expression obtained after summing over $k_{j_1}, \dots, k_{j_{2l}}$ is more or less equivalent to considering $\check{X}_{\tau\sigma} := \mathcal{R}\tilde{\mathbf{B}}_{\tau\sigma}^{n-2l,r}(i_0, \dots, i_{\check{j}_1}, \dots, i_{\check{j}_{2l}}, \dots, i_{n-1})$ (which has same law as $\mathcal{R}\tilde{\mathbf{B}}_{\tau\sigma}^{n-2l,r}(0, \dots, n - 2l - 1)$) and (following the idea introduced in Lemma 6.18) increasing by one the weight W of some other (possibly coinciding) indices $j'_1, \dots, j'_{2l} \neq j_1, \dots, j_{2l}$ – or, in other words, ‘inserting’ a factor $|k_{j'_1}|^{-2\alpha} \dots |k_{j'_{2l}}|^{-2\alpha}$ in the variance series –. This amounts in the end to increasing the Hölder regularity $(n - 2l)\alpha^-$ of $\check{X}_{\tau\sigma}$ by $2l\alpha$, which gives the expected regularity.

Fix some permutation $\varepsilon : \{0, \dots, n - 1\} \rightarrow \{0, \dots, n - 1\}$, and consider the sum over the restricted index set $|k_{\varepsilon(0)}| \leq \dots \leq |k_{\varepsilon(n-1)}|$ as in section 6. Change as before the order of integration and the names of the indices so that $\xi_{k_{\varepsilon(j)}}(i_j) \rightarrow \xi_{k_j}(i_{\varepsilon(j)})$; for convenience, we shall still index the pairings as $(j_1, j_2), \dots, (j_{2l-1}, j_{2l})$. If contractions operate between non-neighbouring indices $(i_{\varepsilon(j)}, i_{\varepsilon(j')})$, i.e. such that $|\varepsilon(j') - \varepsilon(j)| \geq 2$, this means that the sequence of indices (which is increasing in absolute value) is actually *constant* in absolute value in-between. By using some other equivalent choice of the permutation ε , one may then assume that these indices become neighbours, i.e. $|\varepsilon(j') - \varepsilon(j)| = 1$.

Let $\mathbb{T}_l = \mathbb{T}_{l,1} \dots \mathbb{T}_{l,L}$ be a forest appearing in the decomposition of \mathbb{T}^ε as in subsection 6.3. Then (see also proof of Lemma 6.15) $\varepsilon(j)$ and $\varepsilon(j')$ may

- (i) either belong to two unconnected branches of $\mathbb{T}_{l,j}$ for some j ;
- (ii) or be successive vertices on the same branch of $\mathbb{T}_{l,j}$ for some j ;
- (iii) or belong to two different tree components $\mathbb{T}_{l,j}, \mathbb{T}_{l,j'}$ of \mathbb{T}_l .

Applying repeatedly Lemma 6.14 to $\mathbb{T}_{l,1} \dots \mathbb{T}_{l,L}$ leads to a sum of product of skeleton-type terms of the same type as $A_{\tau\sigma}$ (see eq. (6.56)). Let us denote by $A_{\tau\sigma} = A_{\tau\sigma}(1) \dots A_{\tau\sigma}(L)$ one of these terms, and by $L\mathbb{T}_{l,1}, \dots, L\mathbb{T}_{l,L}, \mathbb{T}'_1, \dots, \mathbb{T}'_J$ all subtrees appearing in the splittings associated to $A_{\tau\sigma}(1), \dots, A_{\tau\sigma}(L)$, so

that

$$A_{\tau\sigma} = \prod_{j=1}^L [\delta \mathcal{R} I_{\text{Sk}L\mathbb{T}_{l,j}}]_{\tau\sigma} \prod_{j=1}^J [\mathcal{R} I_{\text{Sk}\mathbb{T}'_j}]_{\sigma} (\check{f}^{n-2l,r,\varepsilon} \circ \varepsilon). \quad (7.4)$$

Let \mathbb{T} be one of the above trees, either $L\mathbb{T}_{l,j}$ or \mathbb{T}'_j . Reconsider the proof of the Hölder estimate in Lemma 6.18 or Lemma 6.19. The contribution to $X_{\tau\sigma}$ of the $(\varepsilon, \mathbb{T})$ -skeleton term is (some projection of) $I_{\text{Sk}\mathbb{T}}(\check{f}_{reg}^{n-2l,r,\varepsilon} \circ \varepsilon|_{V(\mathbb{T})})$ for some function $\check{f}_{reg}^{n-2l,r,\varepsilon}$ (that we shall not need to write down explicitly), abbreviated as $\mathcal{R} I_{\text{Sk}\mathbb{T}}^{\vee}$. Assume for a moment that all contractions are of type (i) or (ii). Since $k_{j_1} + k_{j_2} = \dots = k_{j_{2l-1}} + k_{j_{2l}} = 0$, the exponent of z in $[\mathcal{R} I_{\text{Sk}\mathbb{T}}^{\vee}]_z$ is the same as in $\mathcal{R} I_{\text{Sk}\check{\mathbb{T}}}$, where $\check{\mathbb{T}}$ is the contracted tree obtained by 'skipping' $\{j_1, \dots, j_{2l}\} \cap V(\mathbb{T})$ while going down the tree \mathbb{T} (see Fig. 9, 10, 11).

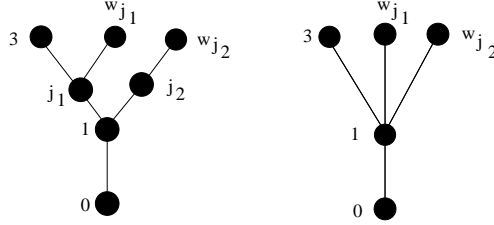


Figure 9: Case (i-a). \mathbb{T} and $\check{\mathbb{T}}$.

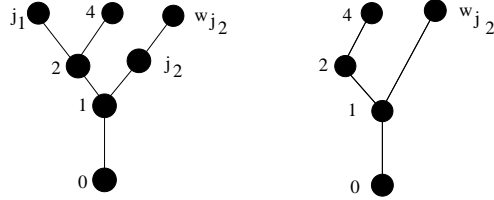


Figure 10: Case (i-b). \mathbb{T} and $\check{\mathbb{T}}$.

In any case (going back to arbitrary contractions) $\mathcal{R} I_{\text{Sk}\check{\mathbb{T}}}$ makes part of $\check{X}_{\tau\sigma}$. Now

$$[\mathcal{R} I_{\text{Sk}\mathbb{T}}^{\vee}]_z = \sum_{(k_v), v \in V(\mathbb{T}), v \neq j_1, \dots, j_{2l}} \prod_{v \in V(\mathbb{T}), v \neq j_1, \dots, j_{2l}} \frac{r^{|k_v|} a_{k_v} \tilde{\xi}_{k_v}(\varepsilon(v)) z^{k_v}}{k_v + \sum_{w \in \mathbb{T}, w \rightarrow v} k_w} \cdot \sum_{(k_{j_m}), j_m \in V(\mathbb{T})} \prod_{j_m \in V(\mathbb{T})} r^{|k_{j_m}|} \frac{a_{k_{j_m}} z^{k_{j_m}}}{k_{j_m} + \sum_{w \in \mathbb{T}, w \rightarrow j_m} k_w} \quad (7.5)$$

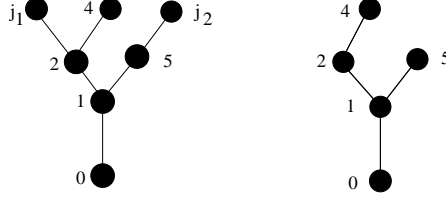


Figure 11: Case (i-c). \mathbb{T} and $\check{\mathbb{T}}$.

The sum is over the subset of indices $\mathbb{Z}_{reg}^{\mathbb{T}}$ with the supplementary restriction $k_{j_{2m-1}} = -k_{j_{2m}}$ if $j_{2m-1}, j_{2m} \in V(\mathbb{T})$.

The denominator $|k_v + \sum_{w \in \mathbb{T}, w \rightarrow v} k_w|$ for $m \neq j_1, \dots, j_{2l}$ is larger (up to a constant) than the denominator $|k_v + \sum_{w \in \check{\mathbb{T}}, w \rightarrow v} k_w|$ obtained by considering the same term in the contracted tree integral $\check{X}_{\tau\sigma}$ (namely, $|k_v + \sum_{w \in \mathbb{T}, w \rightarrow v} k_w|$ is of the same order as $\max\{|k_w|; w \in \mathbb{T}, w \rightarrow v\} \geq \max\{|k_w|; w \in \check{\mathbb{T}}, w \rightarrow v\}$). Hence $\mathbb{E}|\delta\mathcal{R}I_{\text{Sk}\mathbb{T}}^\vee|^2$ may be bounded in the same way as $\mathbb{E}|\delta\mathcal{R}I_{\text{Sk}\check{\mathbb{T}}}|^2$ in the proof of Lemma 6.18, except that each term in the sum over $(k_v, v \in V(\mathbb{T}), v \neq j_1, \dots, j_{2l})$ comes with an extra multiplicative pre-factor $S = S((k_v), v \in V(\mathbb{T}), v \neq k_{j_1}, \dots, k_{j_{2l}})$ – due to the sum over $(k_{j_m})_m$ – which may be seen as an ‘insertion’. The same remark also holds for the variances considered in the proof of Lemma 6.19.

Let us estimate this prefactor. We shall assume for the sake of clarity that there is a single contraction $(j_1, j_2) = (j, j')$ (otherwise the prefactor should be evaluated by contracting each tree in several stages, ‘skipping’ successively $(j_1, j_2), \dots, (j_{2l-1}, j_{2l})$ by pairs).

Case (i): (j, j') belong to unconnected branches of \mathbb{T} . This case splits into three different subcases:

- (i-a) neither j nor j' is a leaf. Let w_1 , resp. w_2 be the leaf above j , resp. j' of maximal index and assume (without loss of generality) that $|k_{w_1}| \leq |k_{w_2}|$. Then

$$S \lesssim \left(\sum_{|k_j| \leq |k_{w_1}|} \frac{a_{k_j}^2}{|k_{w_1} k_{w_2}|} \right)^2 \lesssim \left(\sum_{|k_j| \leq |k_{w_1}|} |k_{w_1}|^{-1-2\alpha} \right)^2 \lesssim |k_{w_1}|^{-4\alpha} \quad (7.6)$$

which has the effect of increasing the weight $W(w_1)$ by 2.

(i-b) j is a leaf, j' is not. Let w_2 be the leaf above j' . Then

$$S \leq \left(\sum_{|k_j| \leq |k_{w_2}|} \frac{a_{k_j}^2}{|k_j k_{w_2}|} \right)^2 \lesssim \left(\frac{1}{|k_{w_2}|} \sum_{|k_j| \leq |k_{w_2}|} |k_j|^{-2\alpha} \right)^2 \lesssim |k_{w_2}|^{-4\alpha}. \quad (7.7)$$

(i-c) both j and j' are leaves. Let v_1 , resp. v_2 be the vertex below j , resp. j' , i.e. $j \rightarrow v_1$, $j' \rightarrow v_2$. Then

$$S \lesssim \left(\sum_{|k_j| \geq \max(|k_{v_1}|, |k_{v_2}|)} \left(\frac{a_{k_j}}{k_j} \right)^2 \right)^2 \lesssim |k_{v_1}|^{-4\alpha} \quad (7.8)$$

which has the effect of increasing $W(v_1)$ by 2.

Case (ii): (j, j') are successive vertices on the same branch. Assume (without loss of generality) that $j \rightarrow j'$. Then $S = 0$ if j is a leaf (since $k_{j'} + \sum_{w \rightarrow j'} k_w = k_j + k_{j'} = 0$ and such indices fail to meet the condition defining $\mathbb{Z}_{reg}^{\mathbb{T}}$), otherwise $S \lesssim |k_w|^{-4\alpha}$ if w is the leaf of maximal index above j (by the same argument as in case (i-a)).

Case (iii).

This case is a variant of case (i) and splits into three subcases (iii-a), (iii-b), (iii-c). Nothing changes compared to case (i) unless (as in the proof of Lemma 6.18 or in the 3rd step of Lemma 6.19) one needs to compute the variance of the coefficient C_K or $C_K(\sigma)$ of z^K for K fixed. Assume j , resp. j' belongs to the tree \mathbb{T} , resp. \mathbb{T}' and let w , resp. w' be the leaf above j , resp. j' if j , resp. j' is not a leaf. The presence of the extra vertices j, j' modifies the exponent K , resp. K' of z , resp. z' in $[I_{\text{SkT}}(\check{f}_{reg}^{n-2l, r, \varepsilon} \circ \varepsilon|_{V(\mathbb{T})})]_z$, resp. $[I_{\text{SkT}'}(\check{f}_{reg}^{n-2l, r, \varepsilon} \circ \varepsilon|_{V(\mathbb{T}')})]_{z'}$ by a factor which is bounded and bounded away from 0 in case (iii-a), hence $S \lesssim |k_w|^{-4\alpha}$ as in case (i-a). In case (iii-b), considering for instance the proof of Lemma 6.18 for simplicity, and letting $j \rightarrow v$, $v \in \mathbb{T}$, one has: $|k_j| \lesssim |K| \lesssim |k_j|$. Hence the sum over $k_j, k_{j'}$ contributes an extra multiplicative pre-factor S to the variance of the coefficient of $z^K(z')^{K'}$ of order

$$S \lesssim \left(\sum_{|k_j|=|K|/2} \left| \frac{a_{k_j}^2}{k_j k_{w'}} \right| \right)^2 \lesssim \left(\sum_{|k_j|=|K|/2} \left(\frac{a_{k_j}}{k_j} \right)^2 \right)^2 \lesssim |K|^{-4\alpha}. \quad (7.9)$$

Case (iii-c) is similar and left to the reader.

This concludes the proof of Theorem 5.1 and Theorem 5.2.

7.2 Geometric property

We shall prove in this subsection the following Lemma:

Lemma 7.2 *The rough path $\mathcal{R}\tilde{\mathbf{B}}^r$ satisfies the geometric property (iii) in the Introduction.*

Proof. Let $\mathbf{i} = (i_0, \dots, i_{n_1+n_2-1})$ be some multi-index. As we have seen in subsection 7.1 (see Lemma 6.14), $\mathcal{R}\tilde{\mathbf{B}}^{n_1+n_2,r}(i_0, \dots, i_{n_1+n_2-1})$ may be written as $\sum_{\varepsilon \in \Sigma_{n_1+n_2}} \mathcal{R}\tilde{\mathbf{B}}^{n_1+n_2,r,\varepsilon}(i_0, \dots, i_{n_1+n_2-1})$, and $\mathcal{R}\tilde{\mathbf{B}}^{n_1+n_2,r,\varepsilon}(i_0, \dots, i_{n_1+n_2-1})$ as a sum of split terms of the form

$$A_{\tau\sigma}^\varepsilon := \prod_{l=1}^L [\delta \mathcal{R}I_{\text{Sk}L\mathbb{T}_l}]_{\tau\sigma} \prod_{j=1}^J [\mathcal{R}I_{\text{Sk}\mathbb{T}'_j}]_{\sigma} (f_{\mathbf{i}}^{n_1+n_2,r,\varepsilon} \circ \varepsilon) \quad (7.10)$$

(see Definition 6.8) where $\cup_{l=1}^L V(L\mathbb{T}_l) \cup \cup_{j=1}^J V(\mathbb{T}'_j) = \{0, \dots, n_1 + n_2 - 1\}$. There are two cases:

- (pure case) each tree $\mathbb{T} = L\mathbb{T}_l$ or \mathbb{T}'_j is such that either $V(\mathbb{T}) \subset \{0, \dots, n_1 - 1\}$ or $V(\mathbb{T}) \subset \{n_1, \dots, n_1 + n_2 - 1\}$;
- (mixed case) some tree \mathbb{T} contains some vertex in $\{0, \dots, n_1 - 1\}$ and some vertex in $\{n_1, \dots, n_1 + n_2 - 1\}$.

Consider first the *non-regularized* iterated integrals $\tilde{\mathbf{B}}_{\tau\sigma}^{n_1,r}(i_0, \dots, i_{n_1-1})$ and $\tilde{\mathbf{B}}_{\tau\sigma}^{n_2,r}(i_{n_1}, \dots, i_{n_1+n_2-1})$. Since \tilde{B}^r is a smooth path, the geometric property

$$\begin{aligned} & \tilde{\mathbf{B}}_{\tau\sigma}^{n_1,r}(i_0, \dots, i_{n_1-1}) \tilde{\mathbf{B}}_{\tau\sigma}^{n_2,r}(i_{n_1}, \dots, i_{n_1+n_2-1}) - \\ & \sum_{\mathbf{k} \in \text{Sh}((i_0, \dots, i_{n_1-1}), (i_{n_1}, \dots, i_{n_1+n_2-1}))} \tilde{\mathbf{B}}_{\tau\sigma}^{n_1+n_2,r}(\mathbf{k}_0, \dots, \mathbf{k}_{n_1+n_2-1}) = 0 \end{aligned} \quad (7.11)$$

holds. Replacing $\tilde{\mathbf{B}}^{n_1,r}$, $\tilde{\mathbf{B}}^{n_2,r}$ and $\tilde{\mathbf{B}}^{n_1+n_2,r}$ by the \mathcal{R} -regularized quantities in eq. (7.11) yields some extra counterterms whose sum (as we shall presently show) vanishes.

To begin with, one easily sees that the counterterms due to the substitution $\tilde{\mathbf{B}}^{n_1,r} \rightarrow \mathcal{R}\tilde{\mathbf{B}}^{n_1,r}$ and $\tilde{\mathbf{B}}^{n_2,r} \rightarrow \mathcal{R}\tilde{\mathbf{B}}^{n_2,r}$ are equal to the pure case counterterms obtained by replacing $\tilde{\mathbf{B}}^{n_1+n_2,r}$ with $\mathcal{R}\tilde{\mathbf{B}}^{n_1+n_2,r}$. So we must prove that the sum of the mixed case counterterms vanishes.

Let $\varepsilon \in \Sigma_{n_1+n_2}$ and $A_{\tau\sigma}^\varepsilon$ be a mixed term. Keeping the same notations as above, we may choose $v_1, v_2 \in \mathbb{T}$ such that $v_2 \rightarrow v_1$ and (without loss of

generality) $v_1 \in \{0, \dots, n_1 - 1\}$, $v_2 \in \{n_1, \dots, n_1 + n_2 - 1\}$. Let (as in the proof of Lemma 6.15) v_1^\uparrow be the set of vertices lying above v_1 , but not on or above the branch containing both v_1 and v_2 , and similarly, $v_2^\uparrow := \{v_2\} \cup \{w \in \mathbb{T} \mid w \rightarrow v_2\}$. Let $\tilde{\mathbb{T}}$ be the tree obtained from \mathbb{T} by stripping the branches $v_1^\uparrow \setminus \{v_1\}$ and $v_2^\uparrow \setminus \{v_2\}$, exchanging v_1 and v_2 and putting back the two branches at their previous places, i.e. grafting $v_1^\uparrow \setminus \{v_1\}$ at v_1 and $v_2^\uparrow \setminus \{v_2\}$ at v_2 (see Fig. 12).

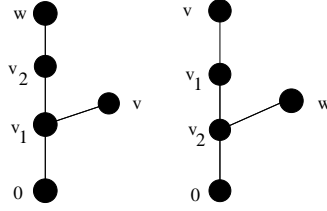


Figure 12: \mathbb{T} and $\tilde{\mathbb{T}}$.

The tree $\tilde{\mathbb{T}}$ shows up when splitting some forest appearing in the decomposition of $\mathbb{T}^{\tau_{v_1, v_2} \circ \varepsilon}$, but with the opposite sign. Hence the counterterm coming from the regularization of $I_{\text{Sk}\mathbb{T}}$ with defining function $f_i^{n_1+n_2, r, \varepsilon} \circ \varepsilon$ is the opposite of the counterterm coming from the regularization of $I_{\text{Sk}\tilde{\mathbb{T}}}$ with defining function $f_{\tau_{v_1, v_2}(i)}^{n_1+n_2, r, \tau_{v_1, v_2} \circ \varepsilon} \circ (\tau_{v_1, v_2} \circ \varepsilon)$. \square

7.3 Real-line construction

It is tempting to try and work directly with B (or Γ) instead of \tilde{B} , and use Fourier integrals on the real line instead of Fourier series on the unit circle. This approach is equivalent to the former one, but presents (as we shall presently show) some (minor) drawbacks.

The Fourier transform of the covariance kernel

$$K_\eta'^-(s-t) := \text{Cov}(\Gamma'_{s+i\eta}, \Gamma'_{t+i\eta}) = \frac{\alpha(1-2\alpha)}{2 \cos \pi\alpha} (-i(s-t) + 2\eta)^{2\alpha-2} \quad (7.12)$$

writes [11]

$$\mathcal{F}K_\eta'^-(k) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} K_\eta'^-(x) e^{-ixk} dx = -\frac{\pi\alpha}{2 \cos \pi\alpha \Gamma(-2\alpha)} e^{-2\eta k} k^{1-2\alpha} H(k), \quad (7.13)$$

where $H(k) = 1_{k>0}$ is the Heaviside function. Hence one may realize Γ' on Π^+ as

$$\Gamma'_{t+i\eta} = c_\alpha \int_0^\infty e^{(it-\eta)k} k^{1/2-\alpha} W(dk), \quad (7.14)$$

where $(W_k = W_k^1 + iW_k^2)_{k \geq 0}$ is the standard complex Brownian motion and $c_\alpha = \frac{1}{2} \sqrt{-\frac{\alpha}{\cos \pi \alpha \Gamma(-2\alpha)}}$. Integrating and taking the real part leads to

$$B_{t+i\eta} = c_\alpha \int_{\mathbb{R}} e^{-\eta|k|} |k|^{\frac{1}{2}-\alpha} \frac{e^{itk} - 1}{ik} W(dk) \quad (7.15)$$

if W is extended to \mathbb{R} by setting $W_{-k} = -\overline{W_k}$ ($k \geq 0$). In the limit $\eta \rightarrow 0$, one retrieves the well-known harmonizable representation.

The problem is now to define an analogue of the formal integral $\int^\tau \zeta^k d\zeta$ (see subsection 5.1). The best choice is: $\int^x e^{ikx} dx = \frac{e^{ikx}}{ik}$ (with now an imaginary 'origin' at $x = \pm i\infty$, one might say, see discussion at the very end of section 3, or proof of Lemma 4.2), but then $\int^t \Gamma'_{x+i\eta} dx = -ic_\alpha \int_0^\infty e^{(it-\eta)k} k^{-1/2-\alpha} W(dk)$ is ill-defined because of the infra-red singularity when $k \rightarrow 0$. This problem may be evacuated simply by constructing a rough path for the infra-red cut regularized fBm process

$$B_{t+i\eta}^{cut} = c_\alpha \int_{|k| \geq 1} e^{-\eta|k|} |k|^{\frac{1}{2}-\alpha} \frac{e^{itk} - 1}{ik} W(dk). \quad (7.16)$$

The neglected part $B - B^{cut}$ is C^∞ , which allows then to deduce very easily a rough path for the original fBm process (see Proposition 1.6). Also, Definition 6.16 should be modified as follows:

$$\begin{aligned} \mathbb{Z}_{reg}^{\mathbb{T}} := & \{ (k_{v_1}, \dots, k_{v_j}) \in \mathbb{R}^j \mid |k_{v_1}| \leq \dots \leq |k_{v_j}| \text{ and} \\ & \forall v \in V(\mathbb{T}), |k_v + \sum_{w \rightarrow v} k_w| > \max(1, C_{reg} \max\{|k_w|; w \rightarrow v\}) \} \end{aligned} \quad (7.17)$$

so that the denominator $|k_v + \sum_{w \rightarrow v} k_w|$ appearing by successive integrations is always larger than 1. Then the proof proceeds exactly in the same way.

7.4 About the two-dimensional antisymmetric fBm

Consider a one-dimensional unit-disk analytic fractional Brownian motion $\tilde{\Gamma}$.

Definition 7.3 *Let $\tilde{Z}_z = (\tilde{Z}_z(1), \tilde{Z}_z(2)) = (2\text{Re } \tilde{\Gamma}_z, 2\text{Im } \tilde{\Gamma}_z)$, $z \in \mathcal{D}$. We call this new process living on the unit disk the two-dimensional unit-disk antisymmetric fBm.*

Its restriction to the unit circle is an a.s. α^- -Hölder Gaussian process, whose marginal laws $\tilde{Z}(1), \tilde{Z}(2)$ coincide with that of the unit-disk fBm \tilde{B} (the same observation is true for $Z_z = (2\operatorname{Re} \Gamma_z, 2\operatorname{Im} \Gamma_z)$ constructed out of the original afBm process Γ , as proved in [31]). The covariance between $\tilde{Z}(1)$ and $\tilde{Z}(2)$ writes

$$\operatorname{Cov}(\tilde{Z}_z(1), \tilde{Z}_w(2)) = -i\mathbb{E}\left[\left(\tilde{\Gamma}_z + \overline{\left(\tilde{\Gamma}_z\right)}\right)\left(\tilde{\Gamma}_w - \overline{\left(\tilde{\Gamma}_w\right)}\right)\right] = 2\operatorname{Im} \mathbb{E}\left[\overline{\left(\tilde{\Gamma}_z\right)}\tilde{\Gamma}_w\right] \quad (7.18)$$

hence it is antisymmetric in (z, w) .

As shown in [31], the covariance of the corresponding process $Z = (Z(1), Z(2))$ on the real line writes

$$\operatorname{Cov}(Z_s(1), Z_t(2)) = -\frac{\tan \pi\alpha}{2}[-\operatorname{sgn}(s)|s|^{2\alpha} + \operatorname{sgn}(t)|t|^{2\alpha} - \operatorname{sgn}(t-s)|t-s|^{2\alpha}]. \quad (7.19)$$

Note that we never used any particular linear combination of the analytic/anti-analytic components of \tilde{B} in the estimates of sections 4, 5, 6. Hence these also hold for \tilde{Z} , which gives for free a rough path over \tilde{Z} (or also Z) satisfying Theorem 0.1 of the Introduction.

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