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CERTIFICATES OF CONVEXITY FOR BASIC SEMI-ALGEBRAIC SETS

JB. LASSERRE

ABSTRACT. We provide two certificates of convexity for arbitrary basic semi-algebraic sets of \mathbb{R}^n . The first one is based on a necessary and sufficient condition whereas the second one is based on a sufficient (but simpler) condition only. Both certificates are obtained from any feasible solution of a related semidefinite program and so can be obtained numerically (however, up to machine precision).

1. INTRODUCTION

With $\mathbb{R}[x]$ being the ring of real polynomials in the variables x_1, \dots, x_n , consider the basic semi-algebraic set $\mathbf{K} \subset \mathbb{R}^n$ defined by:

$$(1.1) \quad \mathbf{K} := \{x \in \mathbb{R}^n : g_j(x) \geq 0, \quad j = 1, \dots, m\}$$

for some given polynomials $g_j \in \mathbb{R}[x]$, $j = 1, \dots, m$.

The classical necessary and sufficient condition for \mathbf{K} to be convex reads

$$(1.2) \quad x, y \in \mathbf{K} \quad \Rightarrow \quad \lambda x + (1 - \lambda)y \in \mathbf{K} \quad \forall \lambda \in [0, 1].$$

This geometric condition (in fact a definition of convexity) does not depend on the representation of \mathbf{K} but requires uncountably many tests and cannot be checked in general.

Of course concavity of g_j for every $j = 1, \dots, m$, provides a certificate of convexity for \mathbf{K} but not every convex set \mathbf{K} in (1.1) is defined by concave polynomials. Hence an important issue is to analyze whether there exists a necessary and sufficient condition of convexity in terms of the representation (1.1) of \mathbf{K} because after all, very often (1.1) is the only information available about \mathbf{K} . Moreover, a highly desirable feature would be that such a condition can be checked, at least numerically.

In a recent work [4], the author has provided an algorithm to obtain a numerical *certificate* of convexity for \mathbf{K} in (1.1) by using the condition:

$$(1.3) \quad \langle \nabla g_j(y), x - y \rangle \geq 0, \quad \forall x, y \in \mathbf{K} \text{ with } g_j(y) = 0,$$

which is equivalent to (1.2) provided that Slater¹ condition holds and the nondegeneracy condition $\nabla g_j(y) \neq 0$ holds whenever $y \in \mathbf{K}$ and $g_j(y) = 0$. This certificate consists of two polynomials $\theta_1, \theta_2 \in \mathbb{R}[x, y]$ whose characterization obviously implies that (1.3) holds true and so \mathbf{K} is convex (whence the name certificate). When

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¹Slater condition holds if there exists $x_0 \in \mathbf{K}$ such that $g_k(x_0) > 0$ for every $k = 1, \dots, m$.

\mathbf{K} is convex, the coefficients of $\theta_1, \theta_2 \in \mathbb{R}[x, y]$ are any feasible solution of what is called a *semidefinite program*, that is, a convex optimization problem which (up to given arbitrary precision) can be solved in time polynomial in the input size of the problem. (For more details on semidefinite programming and its applications, the interested reader is referred to e.g. [8].) Therefore a *numerical certificate* (θ_1, θ_2) obtained by solving such a semidefinite program is only approximate because the coefficients of θ_1, θ_2 , output of a numerical algorithm, are subject to unavoidable numerical inaccuracies.

The present contribution is to provide a certificate of convexity for *arbitrary* basic semi-algebraic sets (1.1), i.e., with no assumption on \mathbf{K} . This time, by certificate we mean an obvious guarantee that the geometric condition (1.2) holds true (instead of (1.3) in [4]). To the best of our knowledge, and despite the result is almost straightforward, it the first of this type for arbitrary basic semi-algebraic sets. As in [4] our certificate also consists of two polynomials (but now in $\mathbb{R}[x, y, \lambda]$ instead of $\mathbb{R}[x, y]$ because one uses (1.2) rather than (1.3)) whose characterization is based on the powerful Stengle's Positivstellensatz in real algebraic geometry. In addition, a numerical certificate can also be obtained as the output of a semidefinite program, but then valid only up to machine precision. We also provide another certificate based on a simpler characterization which now uses only a sufficient condition for a polynomial to be nonnegative on \mathbf{K} ; so in this case, even if \mathbf{K} is convex there is no guarantee to obtain the required certificate. Finally, we also provide a sufficient condition that permits to obtain a numerical certificate of non convexity of \mathbf{K} in the form of points $x, y \in \mathbf{K}$ and $\lambda \in (0, 1)$ which violate (1.2).

2. MAIN RESULT

Given the basic semi-algebraic set \mathbf{K} defined in (1.1), let $\widehat{\mathbf{K}} \subset \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ be the associated basic semi-algebraic set:

$$(2.1) \quad \widehat{\mathbf{K}} := \{(x, y, \lambda) : \hat{g}_j(x, y, \lambda) \geq 0, j = 1, \dots, 2m + 1\}$$

where:

$$(2.2) \quad (x, y, \lambda) \mapsto \hat{g}_j(x, y, \lambda) := g_j(x), \quad j = 1, \dots, m$$

$$(2.3) \quad (x, y, \lambda) \mapsto \hat{g}_j(x, y, \lambda) := g_{m-j}(y), \quad j = m + 1, \dots, 2m$$

$$(2.4) \quad (x, y, \lambda) \mapsto \hat{g}_j(x, y, \lambda) := \lambda(1 - \lambda), \quad j = 2m + 1,$$

and let $P(\hat{g}) \subset \mathbb{R}[x, y, \lambda]$ be the preordering associated with the polynomials \hat{g}_j that define $\widehat{\mathbf{K}}$ in (2.1), i.e.,

$$(2.5) \quad P(g) := \left\{ \sum_{J \subseteq \{1, \dots, 2m+1\}} \sigma_J \left(\prod_{k \in J} \hat{g}_k \right) : \sigma_J \in \Sigma[x, y, \lambda] \right\}.$$

where $\Sigma[x, y, \lambda] \subset \mathbb{R}[x, y, \lambda]$ is the set polynomials that are sums of squares (in short s.o.s.). Our necessary and sufficient condition of convexity is as follows.

Theorem 2.1. *Let $\mathbf{K} \subset \mathbb{R}^n$ be the basic semi-algebraic set defined in (1.1). Then \mathbf{K} is convex if and only if for every $j = 1, \dots, m$ and all $(x, y, \lambda) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$:*

$$(2.6) \quad \sigma_j(x, y, \lambda) g_j(\lambda x + (1 - \lambda) y) = g_j(\lambda x + (1 - \lambda) y)^{2p_j} + h_j(x, y, \lambda)$$

for some polynomials $\sigma_j, h_j \in P(\hat{g})$ and some integer $p_j \in \mathbb{N}$.

Proof. The set \mathbf{K} is convex if and only if (1.2) holds, that is, if and only if for every $j = 1, \dots, m$,

$$g_j(\lambda x + (1 - \lambda)y) \geq 0, \quad \forall x, y \in \mathbf{K}, \lambda \in [0, 1],$$

or, equivalently, if and only if for every $j = 1, \dots, m$,

$$(2.7) \quad g_j(\lambda x + (1 - \lambda)y) \geq 0, \quad \forall (x, y, \lambda) \in \widehat{\mathbf{K}}.$$

But then (2.6) is just application of Stengle's Positivstellensatz [1, Theor. 4.4.2, p. 92] to (2.7) (in fact, a Nichtnegativstellensatz version). \square

The polynomials $\sigma_j, h_j \in P(\hat{g})$, $j = 1, \dots, m$, obtained in (2.6) indeed provide an obvious certificate of convexity for \mathbf{K} . This is because if (2.6) holds then for every $x, y \in \mathbf{K}$ and every $\lambda \in [0, 1]$ one has $\sigma_j(x, y, \lambda) \geq 0$ and $h_j(x, y, \lambda) \geq 0$ because $\sigma_j, h_j \in P(\hat{g})$; and so $\sigma_j(x, y, \lambda)g_j(\lambda x + (1 - \lambda)y) \geq 0$. Therefore if $\sigma_j(x, y, \lambda) > 0$ then $g_j(\lambda x + (1 - \lambda)y) \geq 0$ whereas if $\sigma_j(x, y, \lambda) = 0$ then $g_j(\lambda x + (1 - \lambda)y)^{2p_j} = 0$ which in turn implies $g_j(\lambda x + (1 - \lambda)y) = 0$. Hence for every $j = 1, \dots, m$, $g_j(\lambda x + (1 - \lambda)y) \geq 0$ for every $x, y \in \mathbf{K}$ and every $\lambda \in [0, 1]$, that is, (1.2) holds and so \mathbf{K} is convex.

2.1. A numerical certificate of convexity. In this section we describe how to obtain numerically the polynomial certificate $(\sigma_j, h_j) \in P(\hat{g}) \times P(\hat{g})$, $j = 1, \dots, m$, of Theorem 2.1.

Let $v_s := (x^\alpha y^\beta \lambda^k)$, $(\alpha, \beta \in \mathbb{N}^n, k \in \mathbb{N})$ be the vector of monomials of the canonical basis of $\mathbb{R}[x, y, \lambda]_s$ (the vector space of polynomials of degree at most s), i.e., the vector of all monomials

$$x^\alpha y^\beta \lambda^k = x_1^{\alpha_1} \dots x_n^{\alpha_n} y_1^{\beta_1} \dots y_n^{\beta_n} \lambda^k, \quad (\alpha, \beta, k) \in \mathbb{N}_s^{2n+1},$$

where

$$\mathbb{N}_s^{2n+1} := \{(\alpha, \beta, k) \in \mathbb{N}^{2n+1} : |\alpha| + |\beta| + k = k + \sum_i \alpha_i + \sum_j \beta_j \leq s\}.$$

Let $r := \binom{2n+1+s}{2n+1}$ and write the matrix $v_s v_s^T \in \mathbb{R}[x, y, \lambda]^{r \times r}$ as $\sum_{\alpha, \beta, k} B_{\alpha\beta k} x^\alpha y^\beta \lambda^k$ for some appropriate real symmetric matrices $(B_{\alpha\beta k}) \in \mathbb{R}^{r \times r}$, $(\alpha, \beta, k) \in \mathbb{N}_{2s}^{2n+1}$.

Recall that in (2.6) $\sigma_j, h_j \in P(\hat{g})$ can be written

$$(2.8) \quad \sigma_j = \sum_{J \subseteq \{1, \dots, 2m+1\}} \theta_J^j \left(\prod_{k \in J} \hat{g}_k \right), \quad \theta_J^j \in \Sigma[x, y, \lambda]$$

$$(2.9) \quad h_j = \sum_{J \subseteq \{1, \dots, 2m+1\}} \varphi_J^j \left(\prod_{k \in J} \hat{g}_k \right), \quad \varphi_J^j \in \Sigma[x, y, \lambda],$$

and for every $J \subseteq \{1, \dots, 2m+1\}$, write

$$\begin{aligned} \theta_J^j(x, y, \lambda) &= \sum_{\alpha, \beta \in \mathbb{N}^n, k \in \mathbb{N}} \theta_{J, \alpha\beta k}^j x^\alpha y^\beta \lambda^k \\ \varphi_J^j(x, y, \lambda) &= \sum_{\alpha, \beta \in \mathbb{N}^n, k \in \mathbb{N}} \varphi_{J, \alpha\beta k}^j x^\alpha y^\beta \lambda^k \end{aligned}$$

for some real coefficients $(\theta_{J, \alpha\beta k}^j)$ and $(\varphi_{J, \alpha\beta k}^j)$.

Then for every $j = 1, \dots, m$, checking whether (2.6) holds with an a priori bound $2s$ on $2p_j$ and on the degrees of the polynomials θ_J^j and φ_J^j for all $J \subseteq \{1, \dots, 2m+1\}$, reduces to find a feasible solution to a system \mathbf{Q}_{j_s} where :

- The unknowns are the coefficients $(\theta_{J,\alpha\beta k}^j)$ and $(\varphi_{J,\alpha\beta k}^j)$ as well as real symmetric matrices $X_J, Z_J \in \mathbb{R}^{r \times r}$, $J \subseteq \{1, \dots, 2m+1\}$.
- (2.6) with σ_J and h_J as in (2.8)-(2.9), defines linear equality constraints between the unknown coefficients $(\theta_{J,\alpha\beta k}^j)$ and $(\varphi_{J,\alpha\beta k}^j)$, for all $J \subseteq \{1, \dots, 2m+1\}$ and all $(\alpha, \beta, k) \in \mathbb{N}_{2s}^{2n+1}$,
- For every $J \subseteq \{1, \dots, 2m+1\}$, the s.o.s. constraint on θ_J^j and φ_J^j reads:

$$\begin{aligned} \theta_{J,\alpha,\beta,k}^j &= \text{trace}(X_J^j, B_{\alpha\beta k}), & \forall (\alpha, \beta, k) \in \mathbb{N}_{2s}^{2n+1}, \\ \varphi_{J,\alpha,\beta,k}^j &= \text{trace}(Z_J^j, B_{\alpha\beta k}), & \forall (\alpha, \beta, k) \in \mathbb{N}_{2s}^{2n+1}, \\ X_J^j, Z_J^j &\succeq 0 \quad (\text{linear matrix inequalities}) \end{aligned}$$

(where for a real symmetric matrix A the notation $A \succeq 0$ stands for A is positive semidefinite). Hence, finding a feasible solution $(\theta_J^j, \varphi_J^j, X_J, Z_J)$ of the above system \mathbf{Q}_{j_s} reduces to solving what is called a *semidefinite program*, that is, a convex optimization problem which (up to given arbitrary precision) can be solved in time polynomial in the input size of the problem. For more details on semidefinite programming see e.g. [8].

If (2.6) holds then for every $j = 1, \dots, m$, there is some $s_j \in \mathbb{N}$ such that the semidefinite programs $\mathbf{Q}_{j_{s_j}}$ has a feasible solution. Therefore convexity of \mathbf{K} can be checked by increasing the degree bound s and solving the finitely many semidefinite programs \mathbf{Q}_{j_s} until a feasible solution is found at some $s := s_j$. Then any feasible solution $\sigma_j, h_j \in P(\hat{g})$ of (2.6), $j = 1, \dots, m$, provides a certificate of convexity for \mathbf{K} . However the certificate is only "numerical" as the coefficients of the polynomials σ_j, h_j are obtained numerically and are subject to unavoidable numerical inaccuracies.

2.2. An easier sufficient condition for convexity. While Theorem 2.1 provides a necessary and sufficient condition for convexity, it is very expensive to check because for each $j = 1, \dots, m$, the certificate of convexity $\sigma_j, h_j \in P(\hat{g})$ in (2.6) involves computing $2 \times 2^{2m+1}$ polynomials θ_J^j, φ_J^j . However one also has the following sufficient condition:

Theorem 2.2. *Let $\mathbf{K} \subset \mathbb{R}^n$ be the basic semi-algebraic set defined in (1.1). Then \mathbf{K} is convex if for every $j = 1, \dots, m$ and all $x, y, \lambda \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$:*

$$(2.10) \quad \begin{aligned} g_j(\lambda x + (1-\lambda)y) &= \sum_{k=1}^m \left(\sigma_k^j(x, y, \lambda) g_k(x) + \psi_k^j(x, y, \lambda) g_k(y) \right) \\ &+ \sigma_j(x, y, \lambda) + \omega_j(x, y, \lambda) \lambda(1-\lambda) \end{aligned}$$

for some polynomials $\sigma_j, \omega_j, \sigma_k^j, \psi_k^j \in \Sigma[x, y, \lambda]$.

Proof. Observe that if (2.10) holds then $g_j(\lambda x + (1-\lambda)y) \geq 0$ on $\widehat{\mathbf{K}}$ for every $j = 1, \dots, m$ and so \mathbf{K} is convex. \square

Again, checking whether (2.10) holds with an a priori bound $2s$ on the degrees of the s.o.s. polynomials $\sigma_j, \omega_j, \sigma_k^j, \psi_k^j$, reduces to solving a semidefinite program. But in contrast to the semidefinite program \mathbf{Q}_{j_s} previously defined, it now only involves

$2m + 2$ unknown s.o.s. polynomials (to be compared with $2 \times 2^{2m+1}$ previously). On the other hand, Theorem 2.2 only provides a sufficient condition, that is, even if \mathbf{K} is convex it may happen that (2.10) does not hold.

However, when \mathbf{K} is compact, convex, and if for some $M > 0$ the quadratic polynomial $x \mapsto M - \|x\|^2$ can be written

$$(2.11) \quad M - \|x\|^2 = \sigma_0(x) + \sum_{k=1}^m \sigma_k(x) g_k(x),$$

for some s.o.s. polynomials $(\sigma_k) \subset \Sigma[x]$, then (2.10) is *almost* necessary because for every $\epsilon > 0$:

$$(2.12) \quad \begin{aligned} g_j(\lambda x + (1 - \lambda)y) + \epsilon &= \sum_{k=1}^m \left(\sigma_{k\epsilon}^j(x, y, \lambda) g_k(x) + \psi_{k\epsilon}^j(x, y, \lambda) g_k(y) \right) \\ &+ \sigma_{j\epsilon}(x, y, \lambda) + \omega_{j\epsilon}(x, y, \lambda) \lambda(1 - \lambda) \end{aligned}$$

for some polynomials $\sigma_{j\epsilon}, \omega_{j\epsilon}, \sigma_{k\epsilon}^j, \psi_{k\epsilon}^j \in \Sigma[x, y, \lambda]$.

Indeed, consider the quadratic polynomial

$$(x, y, \lambda) \mapsto \Delta(x, y, \lambda) := 2M + 1 - \|x\|^2 - \|y\|^2 - \lambda^2.$$

From (2.11) and noting that $1 - \lambda^2 = (1 - \lambda)^2 + 2\lambda(1 - \lambda)$, the polynomial Δ can be written as

$$\begin{aligned} \Delta(x, y, \lambda) &= \sigma_0(x) + \sum_{k=1}^m \sigma_k(x) g_k(x) + \sigma_0(y) + \sum_{k=1}^m \sigma_k(y) g_k(y) \\ &+ (1 - \lambda)^2 + 2\lambda(1 - \lambda), \end{aligned}$$

and so Δ belongs to the quadratic module $Q(\hat{g}) \subset \mathbb{R}[x, y, \lambda]$ generated by the polynomials \hat{g}_k that define $\hat{\mathbf{K}}$, that is, the set

$$\begin{aligned} Q(\hat{g}) &:= \left\{ \sum_{k=1}^m \sigma_k(x, y, \lambda) g_k(x) + \psi_k(x, y, \lambda) g_k(y) \right. \\ &\left. + \sigma_0(x, y, \lambda) + \varphi(x, y, \lambda) \lambda(1 - \lambda) : \sigma_k, \psi_k, \varphi \in \Sigma[x, y, \lambda] \right\}. \end{aligned}$$

This implies that $Q(\hat{g})$ is archimedean (see e.g. [7]). Therefore, as $g_j(\lambda x + (1 - \lambda)y) + \epsilon > 0$ on $\hat{\mathbf{K}}$, (2.12) is a consequence of Putinar's Positivstellensatz [6].

2.3. A certificate on non-convexity. In this final section we provide a numerical certificate of *non convexity* of \mathbf{K} when the optimal value of a certain semidefinite program is strictly negative and some moment matrix associated with an optimal solution satisfies a certain rank condition.

Given a sequence $\mathbf{z} = (z_{\alpha\beta k})$ indexed in the canonical basis $(x^\alpha y^\beta \lambda^k)$ of $\mathbb{R}[x, y, \lambda]$, let $L_{\mathbf{z}} : \mathbb{R}[x, y, \lambda] \rightarrow \mathbb{R}$ be the linear functional

$$f \quad (= \sum_{\alpha, \beta, k} f_{\alpha\beta k} x^\alpha y^\beta \lambda^k) \mapsto L_{\mathbf{z}}(f) = \sum_{\alpha, \beta, k} f_{\alpha\beta k} z_{\alpha\beta k}$$

and as in [3], the *moment* matrix $M_s(\mathbf{z}) \in \mathbb{R}^{r \times r}$ associated with \mathbf{z} is the matrix with rows and columns indexed in the canonical basis $(x^\alpha y^\beta \lambda^k)$ and with entries

$$M_s(\mathbf{z})((\alpha, \beta, k), (\alpha', \beta', k')) = z_{(\alpha+\alpha')(\beta+\beta')(k+k')}$$

for every $(\alpha, \beta, k), (\alpha', \beta', k') \in \mathbb{N}_s^{2n+1}$.

Similarly, with a polynomial $(x, y, \lambda) \mapsto \theta(x, y, \lambda) = \sum_{\alpha, \beta, k} \theta_{\alpha\beta k} x^\alpha y^\beta \lambda^k$, the *localizing* matrix $M_s(\theta \mathbf{z}) \in \mathbb{R}^{r \times r}$ associated with θ and \mathbf{z} , is the matrix with rows and columns indexed in the canonical basis $(x^\alpha y^\beta \lambda^k)$ and with entries

$$M_s(\theta \mathbf{z})((\alpha, \beta, k), (\alpha', \beta', k')) = \sum_{\alpha'', \beta'', \lambda''} \theta_{\alpha'' \beta'' k''} z_{(\alpha + \alpha' + \alpha'')(\beta + \beta' + \beta'')(k + k' + k'')},$$

for every $(\alpha, \beta, k), (\alpha', \beta', k') \in \mathbb{N}_s^{2n+1}$.

Let $v_k := \lceil (\deg \hat{g}_k)/2 \rceil$, $k = 1, \dots, 2m+1$, and for every $j = 1, \dots, m$, and $s \geq v := \max_k v_k$, consider the semidefinite program

$$(2.13) \quad \begin{cases} \rho_{js} = \min_{\mathbf{z}} L_{\mathbf{z}}(g_j(\lambda x + (1-\lambda)y)) \\ \text{s.t.} & M_s(\mathbf{z}) \succeq 0 \\ & M_{s-v_k}(\hat{g}_k \mathbf{z}) \succeq 0, \quad k = 1, \dots, 2m+1 \\ & \mathbf{z}_0 = 1. \end{cases}$$

The semidefinite program (2.13) is a relaxation of the global optimization problem

$$g_j^* := \min_{x, y, \lambda} \{g_j(\lambda x + (1-\lambda)y) : (x, y, \lambda) \in \widehat{\mathbf{K}}\}$$

and so $\rho_{js} \leq g_j^*$ for every $s \geq v$. Moreover, $\rho_{js} \uparrow g_j^*$ as $s \rightarrow \infty$; for more details see e.g. [3].

Theorem 2.3. *Let $\mathbf{K} \subset \mathbb{R}^n$ be as in (1.1) and let \mathbf{z} be an optimal solution of the semidefinite program (2.13) with optimal value ρ_{js} . If $\rho_{js} < 0$ and*

$$(2.14) \quad \text{rank } M_s(\mathbf{z}) = \text{rank } M_{s-v}(\mathbf{z}) \quad (=: t)$$

then the set \mathbf{K} is not convex and one may extract t points $(x(i), y(i), \lambda(i)) \in \widehat{\mathbf{K}}$, $i = 1, \dots, t$, such that

$$g_j(\lambda(i)x(i) + (1-\lambda(i))y(i)) < 0, \quad \forall i = 1, \dots, t$$

hence each being a certificate that \mathbf{K} is not convex.

Proof. By the flat extension theorem of Curto and Fialkow [2] (see also Laurent [5]), the rank condition (2.14) ensures that \mathbf{z} is the moment sequence of a t -atomic probability measure μ supported on $\widehat{\mathbf{K}}$. That is:

$$z_{\alpha\beta k} = \int_{\widehat{\mathbf{K}}} x^\alpha y^\beta \lambda^k d\mu, \quad \forall (\alpha, \beta, k) \in \mathbb{N}_{2s}^{2n+1}$$

Let $(x(i), y(i), \lambda(i))_{i=1}^t \subset \widehat{\mathbf{K}}$ be the support of μ which is a positive linear combination of Dirac measures $\delta_{(x(i), y(i), \lambda(i))}$ with positive weights (γ_i) such that $\sum_i \gamma_i = 1$. Then

$$\begin{aligned} g_j^* \geq \rho_{js} = L_{\mathbf{z}}(g_j(\lambda x + (1-\lambda)y)) &= \int_{\widehat{\mathbf{K}}} g_j(\lambda x + (1-\lambda)y) d\mu \\ &= \sum_{i=1}^t \gamma_i g_j(\lambda(i)x(i) + (1-\lambda(i))y(i)) \\ &\geq \sum_{i=1}^t \gamma_i g_j^* = g_j^*, \end{aligned}$$

which shows that $\rho_{js} = g_j^*$ and so, $g_j(\lambda(i)x(i) + (1-\lambda(i))y(i)) = g_j^*$ for every $i = 1, \dots, t$. But then the result follows from $\rho_{js} < 0$. \square

REFERENCES

- [1] J. Bochnak, M. Coste and M-F. Roy. *Real Algebraic Geometry*, Springer Verlag, Berlin (1998).
- [2] R.E. Curto and L. A. Fialkow. The truncated complex K -moment problem, *Trans. Amer. Math. Soc.* **352** (2000), pp. 2825–2855.
- [3] J.B. Lasserre. Global optimization with polynomials and the problem of moments, *SIAM J. Optim.* **11** (2001), pp 796–817.
- [4] J.B. Lasserre. Convexity in semi-algebraic geometry and polynomial optimization, *SIAM J. Optim.*, to appear. [arXiv:0806.3784v3](https://arxiv.org/abs/0806.3784v3)
- [5] M. Laurent. Revisiting two theorems of Curto and Fialkow on moment matrices, *Proc. Amer. Math. Soc.* **133** (2005), pp. 2965-2976.
- [6] M. Putinar. Positive polynomials on compact semi-algebraic sets, *Ind. Univ. Math. J.* **42** (1993), pp. 969–984.
- [7] M. Schweighofer. Optimization of polynomials on compact semialgebraic sets *SIAM J. Optim.* **15** (2005), 805–825.
- [8] L. Vandenberghe, S. Boyd. Semidefinite programming, *SIAM Rev.* **38** (1996), 49–95.

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