

Bounds on the degree of APN polynomials The Case of $x^{-1} + g(x)$

Gregor Leander* François Rodier†

Abstract

We prove that functions $f : \mathcal{F}_{2^m} \rightarrow \mathcal{F}_{2^m}$ of the form $f(x) = x^{-1} + g(x)$ where g is any non-affine polynomial are APN on at most a finite number of fields \mathcal{F}_{2^m} . Furthermore we prove that when the degree of g is less than 7 such functions are APN only if $m \leq 3$ where these functions are equivalent to x^3 .

1 Introduction

For a given integer m denote by $q = 2^m$ and let \mathcal{F}_q be the finite field with q elements. We study functions $f : \mathcal{F}_q \rightarrow \mathcal{F}_q$ given by their polynomial representation. Such a function, or polynomial, is called *almost perfect nonlinear* (APN) if for every non-zero $a \in \mathcal{F}_q$ and every $b \in \mathcal{F}_q$ the equation

$$f(x) + f(x + a) = b$$

admits at most two solutions $x \in \mathcal{F}_{2^m}$. Amongst others, APN functions have applications in cryptography. Namely, when used as an S-box in a block cipher being APN ensures a good resistance against differential attacks.

Until 2006, there were only very few APN functions known and all of them were power mappings. It was even conjectured that any APN function is equivalent to one of the known APN power functions. Here equivalence is usually defined by saying that two functions $f, g : \mathcal{F}_{2^m} \rightarrow \mathcal{F}_{2^m}$ are equivalent

*Faculty of Mathematics, Technical University of Denmark, e-mail: g.leander@mat.dtu.dk

†Institut of Mathematiques of Luminy – C.N.R.S. – e-mail: rodier@iml.univ-mrs.fr

if there exist an affine permutation on $\mathcal{F}_{2^m} \times \mathcal{F}_{2^m}$ such that the graph of f , i.e. the set $\{(x, f(x)) : x \in \mathcal{F}_{2^m}\}$, is mapped to the graph of g . This equivalence is called CCZ-equivalence (see [6]) and preserves the APN property.

In [8] the first APN function which was not equivalent to any power function was found. Shortly after this, several infinite families of APN functions have been discovered, see for example [1, 2, 3, 4, 5].

The problem of classifying all APN functions seems elusive today. Even the problem of classifying all APN power functions is an open problem and not a lot progress has been made here. However, there are possible steps that can be taken towards a full classification. One approach that already proved to be successful is to show that certain polynomials are not APN for infinitely many extensions of \mathcal{F}_2 . So here one first fixes a finite field \mathcal{F}_q and a function $f : \mathcal{F}_q \rightarrow \mathcal{F}_q$ given as a polynomial in $\mathcal{F}_q[x]$ and poses the question if this function can be APN on infinitely many extensions of \mathcal{F}_q . There is a variety of classes of functions for which it can be shown that each function is APN at most for a finite number of extensions. For example, Jedlicka [10] studied the case of power functions and Voloch [15] focused on binomials. Moreover, Rodier studied general polynomials with the same approach in [12, 13].

2 Our Results

So far the above question has only be treated when the polynomial description of f does not depend on n . In this paper we tackle the question if a given class of functions can be APN infinitely many often for functions of the form

$$\begin{aligned} f : \mathcal{F}_q &\rightarrow \mathcal{F}_q \\ f(x) &= x^{q-2} + g(x) \end{aligned}$$

where g is a fixed polynomial. The description of those functions clearly depends on q , and so these functions do not fall into the classes considered so far. However for every nonzero $x \in \mathcal{F}_q$ we have

$$f(x) = x^{-1} + g(x)$$

and so the equation

$$xf(x) = 1 + xg(x)$$

actually does not depend on q anymore, a fact we will use below. Functions of the form $f(x) = x^{q-2} + g(x)$ are in particular interesting for cryptography

as another important criterion for functions used in symmetric ciphers is a high algebraic degree and functions of the studied form provide the maximal degree possible for a balanced function. Moreover, it turns out that these functions are in particular suitable for the question posed above.

The main idea to prove that a given polynomial is APN only for finitely many extensions involves the estimation of the number of points on the following surface X' of the affine equation

$$\frac{g(x_0) + g(x_1) + g(x_2) + g(x_0 + x_1 + x_2)}{(x_0 + x_1)(x_2 + x_1)(x_0 + x_2)} x_0 x_1 x_2 (x_0 + x_1 + x_2) = 1$$

associated to a function f . There are three major steps to derive these estimations.

The first, and for most of the classes of functions studied so far the most involved step, is to show that the number of points of the associated surface can be bounded by applying the Weil bound (or improvements of this bound). The second step is the observation that the number of \mathcal{F}_q -rational points on a certain surface associated to an APN function can be upper bounded. The third step consists of applying the Weil bound (or improvements of this bound) to get a lower bound on the number of points on the same surface. These bounds will involve the field size q . Moreover, if q is large enough, the derived lower bound exceeds the upper bound for APN functions and this in turn applies that the function can only be APN up to a certain field size. This is the approach taken by all the papers that deal with this kind of questions.

For the first step, i.e. for the Weil bound to be applicable to X' , the surface X' has to fulfill certain properties. One possibility is to show that the surface is absolutely irreducible. For most of the classes studied so far the question of absolute irreducibility was the part where one had to make further restrictions on the studied functions which were not always fully satisfactory. However, for the functions we study the question of absolute irreducibility can be answered completely, see the Theorem below.

Theorem 2.1 *Let $g \in \mathcal{F}_q[x]$ be any non-affine polynomial. The surface X' defined by the affine equation*

$$\frac{g(x_0) + g(x_1) + g(x_2) + g(x_0 + x_1 + x_2)}{(x_0 + x_1)(x_2 + x_1)(x_0 + x_2)} x_0 x_1 x_2 (x_0 + x_1 + x_2) = 1$$

is absolutely irreducible.

The proof of this Theorem can be found in Section 3.

The second step, i.e. the upper bound on the number of \mathcal{F}_q -rational points on the surface X' is stated in the corollary below and proven in Section 4.

Corollary 2.1 *If the polynomial mapping $f(x) = x^{q-2} + g(x)$ (g of degree $d \geq 3$) is APN then the projective surface X' with affine equation*

$$\frac{g(x_0) + g(x_1) + g(x_2) + g(x_0 + x_1 + x_2)}{(x_0 + x_1)(x_2 + x_1)(x_0 + x_2)} x_0 x_1 x_2 (x_0 + x_1 + x_2) = 1$$

has at most $4dq + 4q + 8$ rational points.

The final, and straight forward step, is a lower bound on the number of points on X' . One possible lower bound is given below.

Corollary 2.2 *The projective surface X' with affine equation*

$$\frac{g(x_0) + g(x_1) + g(x_2) + g(x_0 + x_1 + x_2)}{(x_0 + x_1)(x_2 + x_1)(x_0 + x_2)} x_0 x_1 x_2 (x_0 + x_1 + x_2) = 1$$

has at least $q^2 + q + 1 - d(d-1)q^{3/2} - 18(d+4)^4q$ rational points.

PROOF. As it is proven in Theorem 2.1 that X' is absolute irreducible we can apply a result of Lang-Weil [11] improved by Ghorpade-Lachaud ([9, section 11]), and deduce

$$|X'(\mathcal{F}_q) - q^2 - q - 1| \leq d(d-1)q^{3/2} + 18(d+4)^4q.$$

Hence

$$-X'(\mathcal{F}_q) + q^2 + q + 1 \leq d(d-1)q^{3/2} + 18(d+4)^4q.$$

that is

$$X'(\mathcal{F}_q) \geq q^2 + q + 1 - d(d-1)q^{3/2} - 18(d+4)^4q.$$

□

These results are put together in the next theorem.

Theorem 2.2 *Let g be a polynomial from \mathcal{F}_q to \mathcal{F}_q , d its degree. Then, if $d < 0.45q^{1/4} - 3.5$ and $d \geq 5$, f is not APN.*

PROOF. From the above it follows that if $q^2 + q + 1 - d(d-1)q^{3/2} - 18(d+4)^4q > 4dq + 4q + 8$, then $X'(\mathcal{F}_q) > 4dq + 4q + 8$, and thus f is not APN. This condition can be written as

$$q^2 - d(d-1)q^{3/2} - (18(d+4)^4 + 4d + 3)q + 1 > 0$$

or

$$q - d(d-1)q^{1/2} - (18(d+4)^4 + 4d + 3) + \frac{1}{q} > 0$$

This condition is fulfilled for $q^{1/2} > 70 + 33.15d + 4.773d^2$. Or for $d < 0.45q^{1/4} - 3.5$ and $d \geq 5$. \square

Our main result is a corollary of this.

Corollary 2.3 *Let g be any fixed non-affine polynomial function in $\mathcal{F}_q[x]$. Then the functions*

$$\begin{aligned} f : \mathcal{F}_{q^n} &\rightarrow \mathcal{F}_{q^n} \\ f(x) &= x^{q^n-2} + g(x) \end{aligned}$$

are APN on at most a finite number of fields \mathcal{F}_{q^n} .

2.1 Special cases

If we fix the degree of g a closer analysis of the surface X' allows to derive better bounds on the maximal field size such that the given function can be APN. If this maximal size is not too big it can be checked with the help of computers if f of the given form can ever be APN by just checking up to the maximal possible extension. Following this approach to combine theoretical results and computer experiments we prove the following theorem as shown in Section 5.

Theorem 2.3 *Let $m > 3$ and $q = 2^m$ be given. Furthermore, let g be any non-affine polynomial of degree at most 6 in $\mathcal{F}_q[x]$. Then the function*

$$\begin{aligned} f : \mathcal{F}_q &\rightarrow \mathcal{F}_q \\ f(x) &= x^{q-2} + g(x) \end{aligned}$$

is not APN.

Another special case is the case of binomials, i.e. the case where g is a monomial ax^d . This case is particularly suitable for checking the APN property with the help of computers, as there are at most $\gcd(d, q-1)$ non equivalent functions of the form $x^{q-2} + ax^d$. This in turn implies that one can check all functions of the form at least for field sizes smaller or equal to 2^{24} within hours on a standard PC. For this case Theorem 2.2 states that if d is smaller than 30 the function $x^{q-2} + ax^d$ can be APN only on fields up to a size of 2^{24} . The following result therefor is again a combination of computer search on fields up to that size and Theorem 2.2.

Theorem 2.4 *Let $q = 2^m$ and d a nonzero integer not a power of 2 be given. Then the function*

$$\begin{aligned} f : \mathcal{F}_q &\rightarrow \mathcal{F}_q \\ f(x) &= x^{q-2} + ax^d \end{aligned}$$

is not APN for $a \in \mathcal{F}_q^$ and $d \leq 29$.*

3 Proof of Theorem 2.1: Irreducibility of X'

As stated above, one key tool is to apply the Weil bound, or variants, on the number of points on certain surfaces. In order for this bound to be applicable the surface has to fulfill certain properties. The purpose of this section is to show that the surface X' defined by the affine equation

$$\frac{g(x_0) + g(x_1) + g(x_2) + g(x_0 + x_1 + x_2)}{(x_0 + x_1)(x_2 + x_1)(x_0 + x_2)} x_0 x_1 x_2 (x_0 + x_1 + x_2) = 1$$

is absolutely irreducible where g is any fixed polynomial.

Assume that X' is not absolutely irreducible and denote

$$\phi(x_0, x_1, x_2) = \frac{g(x_0) + g(x_1) + g(x_2) + g(x_0 + x_1 + x_2)}{(x_0 + x_1)(x_2 + x_1)(x_0 + x_2)}.$$

Then one may write, with P_i (resp. Q_i) polynomials in 2 (resp. 3) variables:

$$\begin{aligned} &1 + \phi(x_0, x_1, x_2) x_0 x_1 x_2 (x_0 + x_1 + x_2) \\ &= (P_1(x_1, x_2) + x_0 Q_1(x_0, x_1, x_2))(P_2(x_1, x_2) + x_0 Q_2(x_0, x_1, x_2)) \\ &= P_1(x_1, x_2) P_2(x_1, x_2) \\ &\quad + x_0(Q_1(x_0, x_1, x_2) P_2(x_1, x_2) + Q_2(x_0, x_1, x_2) P_1(x_1, x_2)) + \\ &\quad + x_0^2 Q_1(x_0, x_1, x_2) Q_2(x_0, x_1, x_2) \end{aligned}$$

One has $P_1(x_1, x_2)P_2(x_1, x_2) = 1$ so $P_1(x_1, x_2)$ is a nonzero constant, and one can suppose that $P_1(x_1, x_2) = 1$ and $P_2(x_1, x_2) = 1$. Consequently

$$\begin{aligned}
& 1 + \phi(x_0, x_1, x_2)x_0x_1x_2(x_0 + x_1 + x_2) \\
&= 1 + x_0(Q_1(x_0, x_1, x_2) + Q_2(x_0, x_1, x_2)) + x_0^2Q_1(x_0, x_1, x_2)Q_2(x_0, x_1, x_2) \\
&= 1 + x_0(Q_1(0, x_1, x_2) + Q_2(0, x_1, x_2)) + \\
& \quad x_0^2 \left(\frac{Q_1(0, x_1, x_2) + Q_2(0, x_1, x_2) + Q_1(x_0, x_1, x_2) + Q_2(x_0, x_1, x_2)}{x_0} \right) + \\
& \quad x_0^2Q_1(x_0, x_1, x_2)Q_2(x_0, x_1, x_2)
\end{aligned}$$

and

$$\begin{aligned}
& 1 + \phi(x_0, x_1, x_2)x_0x_1x_2(x_0 + x_1 + x_2) \\
&= 1 + \phi(0, x_1, x_2)x_0x_1x_2(x_1 + x_2) + \phi(0, x_1, x_2)x_0^2x_1x_2 + \\
& \quad x_0^2x_1x_2(x_0 + x_1 + x_2) \left(\frac{\phi(x_0, x_1, x_2) + \phi(0, x_1, x_2)}{x_0} \right)
\end{aligned}$$

where the fractions with denominators x_0 are actually polynomials. Hence

$$Q_1(0, x_1, x_2) + Q_2(0, x_1, x_2) = \phi(0, x_1, x_2)x_1x_2(x_1 + x_2)$$

and

$$\begin{aligned}
& \frac{Q_1(0, x_1, x_2) + Q_2(0, x_1, x_2) + Q_1(x_0, x_1, x_2) + Q_2(x_0, x_1, x_2)}{x_0} + \\
& \quad Q_1(x_0, x_1, x_2)Q_2(x_0, x_1, x_2) \\
&= x_1x_2(x_0 + x_1 + x_2) \frac{\phi(x_0, x_1, x_2) + \phi(0, x_1, x_2)}{x_0} + \phi(0, x_1, x_2)x_1x_2
\end{aligned}$$

Let d be the degree of g . Remark that the degree of $\phi(0, x_1, x_2)$ is equal to $d - 3$ (see Lemma 4.2). Hence

$$d = \deg(\phi(0, x_1, x_2)x_1x_2(x_1 + x_2)) \leq \sup(\deg Q_1, Q_2)$$

and $\deg Q_1 + \deg Q_2 \leq 3 + d - 3 - 1$. If $\deg Q_1 \geq \deg Q_2$, one has

$$d \leq \sup(\deg Q_1, Q_2) \leq \deg Q_1 \leq \deg Q_1 + \deg Q_2 \leq d - 1$$

We obtain a contradiction to the assumption that X' is reducible.

4 Proof of Corollary 2.1: The Upper Bound

The purpose of this section is to give a proof of Corollary 2.1, i.e. to show that the number of rational points on the surface X' can be upper bounded if f is APN. The main tool is the following Lemma (see for example [12]).

Let f be a polynomial mapping from \mathcal{F}_q to itself which has no terms of degree a power of 2.

Proposition 4.1 *The function $f : \mathcal{F}_q \rightarrow \mathcal{F}_q$ is APN if and only if the surface $f(x_0) + f(x_1) + f(x_2) + f(x_0 + x_1 + x_2) = 0$ has all of its rational points contained in the surface $(x_0 + x_1)(x_2 + x_1)(x_0 + x_2) = 0$.*

Before we prove Corollary 2.1, remark that the polynomial $f(x_0) + f(x_1) + f(x_2) + f(x_0 + x_1 + x_2)$ is divisible by $(x_0 + x_1)(x_2 + x_1)(x_0 + x_2)$, therefore the quotient

$$\frac{f(x_0) + f(x_1) + f(x_2) + f(x_0 + x_1 + x_2)}{(x_0 + x_1)(x_2 + x_1)(x_0 + x_2)}$$

defines a polynomial which is the affine equation of a surface X if the polynomial is not constant, that is if f is not a q -affine polynomial or a polynomial of degree 3.

We will make use of the following lemmata.

Lemma 4.1 *If d is not a power of 2 and at least 3, and an integer c at least 2 then the polynomial $\sum_{i=1}^c x_i^d + (\sum_{i=1}^c x_i)^d$ is non zero, hence of degree d .*

PROOF. Write $d = 2^a b$ with b odd. The polynomial can be written $(\sum_{i=1}^c x_i^b + (\sum_{i=1}^c x_i)^b)^{2^a}$. The inner polynomial contains a monomial (say $bx_1 x_2^{b-1}$) of degree b . Hence the conclusion. \square

Lemma 4.2 *If $\deg g = d$ is not a power of 2, then*

$$\phi(x_0, x_1, x_2) = \frac{g(x_0) + g(x_1) + g(x_2) + g(x_0 + x_1 + x_2)}{(x_0 + x_1)(x_2 + x_1)(x_0 + x_2)}$$

is a polynomial of degree $d - 3$.

PROOF. Denote by ϕ_d the term of highest degree of ϕ . As it is nonzero and as ϕ_d is a rational homogeneous fraction, ϕ_d is of degree $d - 3$. \square

Now we can prove Corollary 2.1: Let the polynomial mapping $f(x) = x^{q-2} + g(x)$ (g of degree $d \geq 3$) be APN and X' be the surface with affine equation

$$\frac{g(x_0) + g(x_1) + g(x_2) + g(x_0 + x_1 + x_2)}{(x_0 + x_1)(x_2 + x_1)(x_0 + x_2)} x_0 x_1 x_2 (x_0 + x_1 + x_2) = 1$$

Due to Theorem 2.1 X' is absolutely irreducible. We have to show that the corresponding projective surface has at most $4dq + 4q + 8$ rational points.

If the surface X' contained the plane $x_0 + x_1 = 0$, it would contain also the planes $x_2 + x_1 = 0$ and $x_0 + x_2 = 0$ by symmetry, which is impossible as the surface X' is irreducible. So its intersection with the plane $x_0 + x_1 = 0$ is a curve of degree $d + 1$. This curve has at most $(d + 1)q + 1$ rational points from Serre [14]. The same argument works for the plane at l'infinity.

If f is APN, the affine surface X has no other rational points than those of the surface $(x_0 + x_1)(x_2 + x_1)(x_0 + x_2) = 0$, which is union of a plane $x_0 + x_1 = 0$ and of its symmetrical plane.

The set $\overline{X}'(\mathcal{F}_q)$ decomposes as follows:

$$\overline{X}'(\mathcal{F}_q) = X'_{x_0} \cup X'_{x_1} \cup X'_{x_2} \cup X'_{x_0+x_1+x_2} \cup X_{\text{aff}}^* \cup X_{\infty}^*$$

where $X'_a = \overline{X}'(\mathcal{F}_q) \cap (a = 0)$, and X_{aff}^* is the affine complement. The equation of surface X' may be written as follows

$$\frac{x_0^{-1} + g(x_0) + x_1^{-1} + g(x_1) + x_2^{-1} + g(x_2) + (x_0 + x_1 + x_2)^{-1} + g(x_0 + x_1 + x_2)}{(x_0 + x_1)(x_2 + x_1)(x_0 + x_2)} \times x_0 x_1 x_2 (x_0 + x_1 + x_2) = 0.$$

It means that for $x_0 x_1 x_2 (x_0 + x_1 + x_2) \neq 0$, the element of the set X_{aff}^* fulfill

$$\frac{f(x_0) + f(x_1) + f(x_2) + f(x_0 + x_1 + x_2)}{(x_0 + x_1)(x_2 + x_1)(x_0 + x_2)} \times x_0 x_1 x_2 (x_0 + x_1 + x_2) = 0$$

which proves that X_{aff}^* is contained in $X(\mathcal{F}_q)$ hence, as f is APN, in the union of the three planes $(x_0 + x_1)(x_2 + x_1)(x_0 + x_2) = 0$. Therefore the number of points in X_{aff}^* is bounded by $3((d + 1)q + 1)$.

The intersection of the surface \overline{X}' with the plane $x_0 = 0$ is the line $x_0 = 0$ in the plane at infinity, which has $q + 1$ rational points.

The equation of the intersection of the surface \overline{X}' with the plane at infinity is

$$\frac{g(x_0) + g(x_1) + g(x_2) + g(x_0 + x_1 + x_2)}{(x_0 + x_1)(x_2 + x_1)(x_0 + x_2)} x_0 x_1 x_2 (x_0 + x_1 + x_2) = 0$$

It contains the lines $x_0 x_1 x_2 (x_0 + x_1 + x_2) = 0$, for which we have already taken in consideration the points, and the curve

$$\frac{g(x_0) + g(x_1) + g(x_2) + g(x_0 + x_1 + x_2)}{(x_0 + x_1)(x_2 + x_1)(x_0 + x_2)} = 0$$

which has at most $(d - 3)q + 1$ rational points.

So

$$\begin{aligned} \#\overline{X}'(\mathcal{F}_q) &= \#X'_{x_0} + \#X'_{x_1} + \#X'_{x_2} + \#X'_{x_0+x_1+x_2} + X_{\text{aff}}^* + \#X_{\infty}^* \\ &= 4(q + 1) + 3((d + 1)q + 1) + (d - 3)q + 1 \\ &= 4dq + 4q + 8 \end{aligned}$$

which proves the result.

5 Improvements in specific cases

Under certain conditions, we can obtain a better bound on the dimension.

Theorem 5.1 *Let g a polynomial mapping from \mathcal{F}_{2^m} to itself, d its degree. Let us suppose that the surface X' defined by*

$$\frac{g(x_0) + g(x_1) + g(x_2) + g(x_0 + x_1 + x_2)}{(x_0 + x_1)(x_2 + x_1)(x_0 + x_2)} x_0 x_1 x_2 (x_0 + x_1 + x_2) = 1$$

of degree $d' = d + 1$ has only isolated singular points. Then if $d \geq 3$ and $d < q^{1/4}$, f is not APN.

PROOF.

From an improvement of a result of Deligne [7] by Ghorpade-Lachaud ([9], corollaire 7.2), we deduce

$$\begin{aligned} |X'(\mathcal{F}_q) - q^2 - q - 1| &\leq b_1(2, d')q^{3/2} + (b_2(3, d') + 1)q \\ &\leq (d' - 1)(d' - 2)q^{3/2} + (d'^3 - 4d'^2 + 6d' - 1)q \\ &\leq d(d - 1)q^{3/2} + (2 + d - d^2 + d^3)q \end{aligned}$$

where b'_1 and $b_2(3, d')$ are Betti numbers (see [9]). Hence

$$X'(\mathcal{F}_q) \geq q^2 + q + 1 - d(d-1)q^{3/2} - (2 + d - d^2 + d^3)q$$

Therefore if

$$q^2 + q + 1 - d(d-1)q^{3/2} - (2 + d - d^2 + d^3)q > 4dq + 4q + 8,$$

then $X'(\mathcal{F}_q) > 4dq + 4q + 8$, and f is not APN due to Corollary 2.1. This condition can be rewritten as

$$q - d(d-1)q^{1/2} - (5 + 5d - d^2 + d^3) - 7/q > 0.$$

It is fulfilled if $q > d^4$ as soon as $d \geq 3$. □

5.1 Polynomials g of small degree

As stated in the introduction the APN property is invariant under the so called CCZ-equivalence. As adding affine functions is a special case of CCZ-equivalence it is clear that given two functions f and g that differ by an affine function f is APN if and only if g is APN. Moreover, multiplying an APN polynomial function by a constant or replacing x by any non-constant linear polynomial yields again an APN polynomial. These well known observations are summarized in the proposition below.

Proposition 5.1 *A polynomial function f is APN if and only if the polynomial f_0 obtained by removing all monomials of degree a power of 2 and by removing the constant term is APN. Moreover, if f is APN then for any nonzero $a, c \in \mathcal{F}_{2^m}$ and any element $b \in \mathcal{F}_{2^m}$ the polynomial function*

$$cf(ax + b)$$

is APN.

This proposition will be used in the reminder of the paper to simplify the polynomials we have to study.

Polynomials of degree 3 We first focus on polynomials of degree 3. Here, the general form of f is

$$f(x) = x^{q-2} + a_3x^3 + a_2x^2 + a_1x + a_0$$

where $a_3 \neq 0$ which is clearly equivalent to

$$f(x) = x^{q-2} + a_3x^3.$$

Moreover, replacing x by $a_3^{-1/4}x$ and multiplying across by $a_3^{1/4}$ we see that f is in any case equivalent to

$$f(x) = x^{q-2} + x^3.$$

In this case the affine equation for X' becomes

$$\begin{aligned} & 1 + \frac{g(x_0) + g(x_1) + g(x_2) + g(x_0 + x_1 + x_2)}{(x_0 + x_1)(x_2 + x_1)(x_0 + x_2)} x_0x_1x_2(x_0 + x_1 + x_2) \\ = & 1 + \frac{((x_0^3 + x_1^3 + x_2^3 + (x_0 + x_1 + x_2)^3)x_0x_1x_2(x_0 + x_1 + x_2))}{(x_0 + x_1)(x_2 + x_1)(x_0 + x_2)} \\ = & 1 + x_0x_1x_2(x_0 + x_1 + x_2) \end{aligned}$$

The search of singular points on the surface $z^4 + x_0x_1x_2(x_0 + x_1 + x_2) = 0$ gives a finite number of points.

Polynomials of degree 5 Next, we study polynomials of degree 5. The general form of f (up to adding affine equivalence) is given by

$$f(x) = x^{q-2} + a_5x^5 + a_3x^3.$$

Furthermore we can assume without loss of generality that $x_3 \in \mathcal{F}_2$. We have to study the surface

$$\begin{aligned} X' = & 1 + \left(a_3 \frac{x_0^3 + x_1^3 + x_2^3 + (x_0 + x_1 + x_2)^3}{(x_0 + x_1)(x_0 + x_2)(x_1 + x_2)} \right. \\ & \left. + a_5 \frac{x_0^9 + x_1^3 + x_2^9 + (x_0 + x_1 + x_2)^9}{(x_0 + x_1)(x_0 + x_2)(x_1 + x_2)} \right) x_0x_1x_2(x_0 + x_1 + x_2) \end{aligned}$$

and show that there are only a finite number of singular points. The lengthy – but straight forward – computation for showing this can be found in Appendix A.

Polynomials of degree 6 Next, we study polynomials of degree 6. The general form of f (up to adding affine equivalence) is given by

$$f(x) = x^{q-2} + a_6x^6 + a_5x^5 + a_3x^3.$$

Furthermore we can assume without loss of generality that $x_3 \in \mathcal{F}_2$.

In this case we have to study the surface

$$\begin{aligned} X' = 1 + & \left(a_3 \frac{x_0^3 + x_1^3 + x_2^3 + (x_0 + x_1 + x_2)^3}{(x_0 + x_1)(x_0 + x_2)(x_1 + x_2)} \right. \\ & + a_6 \frac{x_0^6 + x_1^3 + x_2^6 + (x_0 + x_1 + x_2)^6}{(x_0 + x_1)(x_0 + x_2)(x_1 + x_2)} \\ & \left. + a_5 \frac{x_0^9 + x_1^3 + x_2^9 + (x_0 + x_1 + x_2)^9}{(x_0 + x_1)(x_0 + x_2)(x_1 + x_2)} \right) x_0x_1x_2(x_0 + x_1 + x_2) \end{aligned}$$

and show that there are only a finite number of singular points. We refer to Appendix B for the proof.

Conclusion As we have seen above for any non-affine polynomial of degree less than 6 the corresponding surface contains only isolated singularities. Therefore, Proposition 5.1 applies. Thus if f is APN it holds that $q = 2^m \leq 6^4$ which implies $m \leq 10$. It can easily be checked that functions form $x^{q-2} + a_6x^6 + a_5x^5 + a_3x^3$ are APN only if $m \leq 3$. Note that, for $m \leq 3$ the APN functions are quadratic and moreover for $m \leq 3$ all APN functions are CCZ equivalent to x^3 . These considerations finally prove Theorem 2.3 stated above.

References

- [1] C. Bracken, E. Byrne, N. Markin, and G. McGuire. A few more quadratic apn functions. *Arxiv preprint arXiv:0804.4799*, 2008.
- [2] C. Bracken, E. Byrne, N. Markin, and G. McGuire. New families of quadratic almost perfect nonlinear trinomials and multinomials. *Finite Fields and Their Applications*, 14(3):703–714, 2008.
- [3] L. Budaghyan and C. Carlet. Classes of quadratic apn trinomials and hexanomials and related structures. *Information Theory, IEEE Transactions on*, 54(5):2354–2357, 2008.

- [4] L. Budaghyan, C. Carlet, and G. Leander. Constructing new apn functions from known ones. *Finite Fields and Applications*, to appear.
- [5] L. Budaghyan, C. Carlet, and G. Leander. Two classes of quadratic apn binomials inequivalent to power functions. *Information Theory, IEEE Transactions on*, 54(9):4218–4229, 2008.
- [6] Claude Carlet, Pascale Charpin, and Victor Zinoviev. Codes, bent functions and permutations suitable for DES-like cryptosystems. *Des. Codes Cryptography*, 15(2):125–156, 1998.
- [7] Pierre Deligne. La conjecture de Weil : I. *Publications Mathematiques of l'IHES*, 43:273–307, 1974.
- [8] Y. Edel, G. Kyureghyan, and A. Pott. A new apn function which is not equivalent to a power mapping. *IEEE Transactions on Information Theory*, 52(2):744–747, 2006.
- [9] Sudhir R. Ghorpade and Gilles Lachaud. Etale cohomology, Lefschetz theorems and the number of points of singular varieties over finite fields. *Moscow Mathematical Journal*, 2:589–631, 2002.
- [10] David Jedlicka. Apn monomials over $\text{GF}(2^n)$ for infinitely many n . *Finite Fields and Their Applications*, 13(4):1006–1028, 2007.
- [11] Serge Lang and Andre Weil. Number of points of varieties in finite fields. *American Journal of Mathematics*, 76(4):819–827, 1954.
- [12] F. Rodier. Borne sur le degré des polynômes presque parfaitement non-linéaires. *Arxiv preprint math.AG/0605232, to be published with the proceedings of the conference AGCT-11*, 2006.
- [13] F. Rodier. Bounds on the degrees of apn polynomials. *to be published with the proceedings of the workshop BFCA08, Copenhagen, 2008*, 2006.
- [14] J. P. Serre. Lettre à M. Tsfasman. *Asterisque*, 198-199-200:351–353, 1991.
- [15] Felipe Voloch. Symmetric cryptography and algebraic curves. *Proceedings of the First SAGA Conference, Papeete, France, 2007*.

A Singular points for g of degree 5

We have to study the surface

$$X' = 1 + \left(a_3 \frac{x_0^3 + x_1^3 + x_2^3 + (x_0 + x_1 + x_2)^3}{(x_0 + x_1)(x_0 + x_2)(x_1 + x_2)} + a_5 \frac{x_0^9 + x_1^9 + x_2^9 + (x_0 + x_1 + x_2)^9}{(x_0 + x_1)(x_0 + x_2)(x_1 + x_2)} \right) x_0 x_1 x_2 (x_0 + x_1 + x_2)$$

and show that there are only a finite number of singular points. For this we compute the derivatives of the projective version of X'

$$\begin{aligned} X' &= a_5 x_2 x_1 x_0 (x_0 + x_1 + x_2) (x_0^2 + x_0 x_1 + x_0 x_2 + x_1^2 + x_1 x_2 + x_2^2) \\ &\quad + a_3 z^2 x_2 x_1 x_0 (x_0 + x_1 + x_2) \\ &\quad + z^6 \end{aligned}$$

The derivatives of the projective version of X' are as follows

$$\begin{aligned} \frac{\partial X'}{\partial x_0} &= x_2 x_1 (x_1 + x_2) P(x_1, x_2, z) \\ \frac{\partial X'}{\partial x_1} &= x_0 x_2 (x_0 + x_2) P(x_0, x_2, z) \\ \frac{\partial X'}{\partial x_2} &= x_0 x_1 (x_0 + x_1) P(x_0, x_1, z) \\ \frac{\partial X'}{\partial z} &= 0 \end{aligned}$$

where

$$P(x, y, z) = a_5 (x^2 + xy + y^2) + a_3 z^2$$

To study the singular points of these equations, we make some case distinction.

Case $x_0 = 0$: In this case X' simplifies to $z = 0$ and we have

$$\frac{\partial X'}{\partial x_0}(0, x_1, x_2, 0) = a_5 x_2 x_1 (x_1 + x_2) (x_1^2 + x_1 x_2 + x_2^2)$$

which, up to equivalence, implies a finite number of singularities .

Case $x_1 = 0$ or $x_2 = 0$: Due to symmetries the cases can be handled exactly like the first case.

Case $x_1 = x_2$: Here we are left with the following system of equations

$$\begin{aligned} a_5 x_2^2 x_0^2 (x_0 + x_2)^2 + a_3 z^2 x_2^2 x_0^2 + z^6 &= 0 \\ x_0 x_2 (x_0 + x_2) (a_5 (x_0^2 + x_0 x_2 + x_2^2) + a_3 z^2) &= 0 \end{aligned}$$

Now if $x_0 = x_2$ the first of this equation becomes

$$(a_3 x_0^4 + z^4) z^2 = 0 \quad (1)$$

If $a_3 = 0$ then $z = 0$ and there are, up to equivalence, at most two points $(1, 1, 1, 0)$ and $(0, 0, 0, 0)$. For $a_3 \neq 0$ we can assume $a_3 = 1$, see above. Now, if $z \neq 0$ then $x_0 = z$ and, up to equivalence, there is only one point $(1, 1, 1, 1)$.

The case $z = 0$ and $x_0 = x_1 = x_2$ correspond, up to equivalence to at most two points $(1, 1, 1, 0)$ and $(0, 0, 0, 0)$.

For $x_0 = 0$ or $x_2 = 0$ see the first case above. Thus we are left with the case

$$\begin{aligned} a_5 (x_0^2 + x_0 x_2 + x_2^2) + a_3 z^2 &= 0 \\ a_5 x_2^2 x_0^2 (x_0 + x_2)^2 + a_3 x_0^2 x_2^2 z^2 + z^6 &= 0 \end{aligned}$$

Computing the resultants of both equations with respect to x_0 and x_2 we get

$$a_5^3 Q(x_0) = 0 \text{ and } a_5^3 Q(x_2) = 0$$

where

$$Q(x) = a_5^3 x^{12} + a_3 a_5^2 x^{10} z^2 + a_3^2 a_5 x^8 z^4 + a_3^3 x^6 z^6 + a_3 a_5 x^4 z^8 + a_5 z^{12}.$$

The fact that Q is a non-zero polynomial (as $a_5 \neq 0$) implies only a finite number of solutions (up to equivalence).

Case $x_0 = x_2$ or $x_0 = x_1$: Due to symmetries the cases can be handled exactly like the case above.

None of the above cases: Here we get the following equations.

$$\begin{aligned} X' &= 0 \\ P(x_1, x_2, z) &= 0 \\ P(x_0, x_2, z) &= 0 \\ P(x_0, x_1, z) &= 0 \end{aligned}$$

In particular the singular points have to fulfil

$$\begin{aligned} P(x_1, x_2, z) + P(x_0, x_2, z) &= a_5(x_0 + x_1)(x_0 + x_1 + x_2) = 0 \\ P(x_1, x_2, z) + P(x_0, x_2, z) &= a_5(x_0 + x_2)(x_0 + x_1 + x_2) = 0 \\ P(x_1, x_2, z) + P(x_0, x_2, z) &= a_5(x_1 + x_2)(x_0 + x_1 + x_2) = 0. \end{aligned}$$

The cases $x_0 = x_1$ or $x_0 = x_2$ or $x_1 = x_2$ have been studied above. The case $x_0 + x_1 + x_2 = 0$ implies $z = 0$ (as $X' = 0$) and then

$$P(x_0, x_2, 0) = a_5(x_0^2 + x_0x_1 + x_1^2) = 0$$

which leaves only a finite number of singularities (up to equivalence).

B Singular points for g of degree 6

We have to study the surface

$$\begin{aligned} X' = 1 + & \left(a_3 \frac{x_0^3 + x_1^3 + x_2^3 + (x_0 + x_1 + x_2)^3}{(x_0 + x_1)(x_0 + x_2)(x_1 + x_2)} \right. \\ & + a_6 \frac{x_0^6 + x_1^3 + x_2^6 + (x_0 + x_1 + x_2)^6}{(x_0 + x_1)(x_0 + x_2)(x_1 + x_2)} \\ & \left. + a_5 \frac{x_0^9 + x_1^3 + x_2^9 + (x_0 + x_1 + x_2)^9}{(x_0 + x_1)(x_0 + x_2)(x_1 + x_2)} \right) x_0x_1x_2(x_0 + x_1 + x_2) \end{aligned}$$

and show that there are only a finite number of singular points. The computations and case distinctions are very similar to the case where g is of degree 5.

We compute the derivatives of the projective version of X'

$$\begin{aligned} X' &= a_6x_2x_1x_0(x_1 + x_2)(x_0 + x_2)(x_0 + x_1)(x_0 + x_1 + x_2) \\ &+ a_5zx_2x_1x_0(x_0 + x_1 + x_2)(x_0^2 + x_0x_1 + x_0x_2 + x_1^2 + x_1x_2 + x_2^2) \\ &+ a_3z^3x_2x_1x_0(x_0 + x_1 + x_2) \\ &+ z^7 \end{aligned}$$

with respect to x_0, x_1, x_2 and z . We get

$$\begin{aligned}\frac{\partial X'}{\partial x_0} &= x_2 x_1 (x_1 + x_2) P(x_1, x_2, z) \\ \frac{\partial X'}{\partial x_1} &= x_0 x_2 (x_0 + x_2) P(x_0, x_2, z) \\ \frac{\partial X'}{\partial x_2} &= x_0 x_1 (x_0 + x_1) P(x_0, x_1, z) \\ \frac{\partial X'}{\partial z} &= a_5 x_2 x_1 x_0 (x_0 + x_1 + x_2) (x_0^2 + x_0 x_1 + x_0 x_2 + x_1^2 + x_1 x_2 + x_2^2) \\ &\quad + a_3 z^2 x_2 x_1 x_0 (x_0 + x_1 + x_2) + z^6\end{aligned}$$

where

$$P(x, y, z) = a_6 xy(x + y) + a_5 z(x^2 + xy + y^2) + a_3 z^3$$

To study the singular points of these equations, we make some case distinction.

Case $x_0 = 0$: Then $\frac{\partial X'}{\partial z}(0, x_1, x_2) = 0$ implies $z = 0$ and this simplifies to

$$\begin{aligned}\frac{\partial X'}{\partial x_0}(0, x_1, x_2, 0) &= a_6 x_2^2 x_1^2 (x_1 + x_2)^2 \\ &= 0\end{aligned}$$

which for $a_6 \neq 0$ implies, up to equivalence, a finite number of singularities.

Case $x_1 = 0$ or $x_2 = 0$: Due to symmetries the cases can be handled exactly like the first case.

Case $x_1 = x_2$: In this case we are left with only two non-zero equations, namely

$$\begin{aligned}\frac{\partial X'}{\partial x_1}(x_0, x_2, x_2, z) &= x_0 x_2 (x_0 + x_2) P(x_0, x_2, z) \\ \frac{\partial X'}{\partial z}(x_0, x_2, x_2, z) &= a_5 x_2^2 x_0^2 (x_0 + x_2)^2 + a_3 x_0^2 x_2^2 z^2 + z^6\end{aligned}$$

Now, if $x_0 = x_2$ then $x_0 = x_1 = x_2$ and we get

$$\frac{\partial X'}{\partial z}(x_0, x_0, x_0, z) = (a_3 x_0^4 + z^4) z^2.$$

which has been studied already in the case of a degree 5 polynomial (see Equation (1))

For $x_0 = 0$ or $x_2 = 0$ see the first case above. Thus we are left with the case

$$\begin{aligned} P(x_0, x_2, z) &= 0 \\ a_5 x_2^2 x_0^2 (x_0 + x_2)^2 + a_3 x_0^2 x_2^2 z^2 + z^6 &= 0 \end{aligned}$$

For this we again distinguish two cases: $z = 0$ and $z \neq 0$. For $z = 0$ we get

$$P(x_0, x_2, 0) = a_6 x_2 x_0 (x_0 + x_2).$$

Thus we have either $x_0 = 0$ or $x_2 = 0$ which have been handled above, or $x_0 = x_2$ which gives again at most two points up to equivalence, see above.

For $z \neq 0$ we can restrict, up to equivalence, to the case $z = 1$. In this case we get

$$\begin{aligned} a_6 x_0 x_2 (x_0 + x_2) + a_5 (x_0^2 + x_0 x_2 + x_2^2) + a_3 &= 0 \\ a_5 x_2^2 x_0^2 (x_0 + x_2)^2 + a_3 x_0^2 x_2^2 + 1 &= 0 \end{aligned}$$

Computing the resultant of these two equations with respect to x_0 and x_2 we get

$$Q(x_0) = 0 \text{ and } Q(x_2) = 0$$

where

$$\begin{aligned} Q(x) &= (a_3 a_5^3 a_6^2 + a_5^6) x^{12} + (a_3^2 a_5^2 a_6^2 + a_3 a_5^5) x^{10} + (a_3^3 a_5 a_6^2 + a_3^2 a_5^4 + a_3 a_6^4) x^8 \\ &\quad + (a_3^4 a_6^2 + a_3^3 a_5^3) x^6 + (a_3 a_5^4 + a_6^4) x^4 + a_5^4 \end{aligned}$$

For $a_5 \neq 0$ Q is a non-zero polynomial as its constant term is non-zero. For $a_5 = 0$ Q is non-zero as the degree 4 term is non-zero. Therefore, in any case we get at most finitely many points.

Case $x_0 = x_2$ or $x_0 = x_1$: Due to symmetries the cases can be handled exactly like the case above.

None of the above cases: In this case we have to study

$$\begin{aligned} P(x_1, x_2, z) &= 0 \\ P(x_0, x_2, z) &= 0 \\ P(x_0, x_1, z) &= 0 \\ \frac{\partial X'}{\partial z} &= 0 \end{aligned}$$

This implies that x_0, x_1, x_2, z are solutions to the following set of equations:

$$\begin{aligned} P(x_1, x_2, z) + P(x_0, x_2, z) &= (a_6x_2 + a_5z)(x_0 + x_1)(x_0 + x_1 + x_2) = 0 \\ P(x_1, x_2, z) + P(x_0, x_2, z) &= (a_6x_1 + a_5z)(x_0 + x_2)(x_0 + x_1 + x_2) = 0 \\ P(x_1, x_2, z) + P(x_0, x_2, z) &= (a_6x_0 + a_5z)(x_1 + x_2)(x_0 + x_1 + x_2) = 0 \end{aligned}$$

The cases $x_0 = x_1$ or $x_0 = x_2$ or $x_1 = x_2$ have been studied above. The case $x_0 + x_1 + x_2 = 0$ implies $z = 0$ (as $X' = 0$) and as seen above

$$P(x_0, x_2, 0) = a_6x_2x_0(x_0 + x_2),$$

and we are back to cases studied before. Thus the only case left is

$$\begin{aligned} (a_6x_2 + a_5z) &= 0 \\ (a_6x_1 + a_5z) &= 0 \\ (a_6x_0 + a_5z) &= 0 \end{aligned}$$

which implies $x_0 = x_1 = x_2$ and, up to equivalence, at most two points.