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# COMMON FIXED POINT THEOREMS FOR PAIRS OF SUBCOMPATIBLE MAPS

H. BOUHADJERA † & C. GODET-THOBIE ‡

1-Université européenne de Bretagne, France.

2-Université de Bretagne Occidentale,

Laboratoire de Mathématiques de Brest; Unité CNRS: UMR 6205

6, avenue Victor Le Gorgeu, CS 93837, F-29238 BREST Cedex 3 FRANCE

E-Mail: †hakima.bouhadjera@univ-brest.fr; ‡christiane.godet-thobie@univ-brest.fr

## Abstract

In this paper, we introduce the new concepts of subcompatibility and subsequentially continuity which are respectively weaker than occasionally weakly compatibility and reciprocally continuity. With them, we establish a common fixed point theorem for four maps in a metric space which improves a recent result of Jungck and Rhoades [7]. Also we give another common fixed point theorem for two pairs of subcompatible maps of Greguš type which extends results of the same authors, Djoudi and Nisse [3], Pathak et al. [12] and others and we end our work by giving a third result which generalizes results of Mbarki [8] and others.

**Key words and phrases:** Commuting and weakly commuting maps, compatible and compatible maps of type  $(A)$ ,  $(B)$ ,  $(C)$  and  $(P)$ , weakly compatible maps, occasionally weakly compatible maps, subcompatible maps, reciprocally continuous maps, subsequentially continuous maps, coincidence point, common fixed point, Greguš type, near-contractive common fixed point theorem.

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## 1 Historical introduction and new definitions

Let  $(\mathcal{X}, d)$  be a metric space and let  $f$  and  $g$  be two maps from  $(\mathcal{X}, d)$  into itself.  $f$  and  $g$  are commuting if  $fgx = gfx$  for all  $x$  in  $\mathcal{X}$ .

To generalize the notion of commuting maps, Sessa [15] introduced the concept of weakly commuting maps. He defines  $f$  and  $g$  to be weakly commuting if

$$d(fgx, gfx) \leq d(fx, gx)$$

for all  $x \in \mathcal{X}$ . Obviously, commuting maps are weakly commuting but the converse is not true.

In 1986, Jungck [4] gave more generalized commuting and weakly commuting maps called compatible maps.  $f$  and  $g$  above are called compatible if

$$(1) \quad \lim_{n \rightarrow \infty} d(fgx_n, gfx_n) = 0$$

whenever  $(x_n)$  is a sequence in  $\mathcal{X}$  such that  $\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} gx_n = t$  for some  $t \in \mathcal{X}$ . Clearly, weakly commuting maps are compatible, but the implication is not reversible (see [4]).

Afterwards, the same author with Murthy and Cho [6] made another generalization of weakly commuting maps by introducing the concept of compatible maps of type (A). Previous  $f$  and  $g$  are said to be compatible of type (A) if in place of (1) we have two following conditions:

$$\lim_{n \rightarrow \infty} d(fgx_n, g^2x_n) = 0 \text{ and } \lim_{n \rightarrow \infty} d(gfx_n, f^2x_n) = 0.$$

It is clear to see that weakly commuting maps are compatible of type (A), from [6] it follows that the implication is not reversible.

In their paper [11], Pathak and Khan extended type (A) maps by introducing the concept of compatible maps of type (B) and compared these maps with compatible and compatible maps of type (A) in normed spaces. To be compatible of type (B),  $f$  and  $g$  above have to satisfy, in lieu of condition (1), the inequalities:

$$\begin{aligned} \lim_{n \rightarrow \infty} d(fgx_n, g^2x_n) &\leq \frac{1}{2} \left[ \lim_{n \rightarrow \infty} d(fgx_n, ft) + \lim_{n \rightarrow \infty} d(ft, f^2x_n) \right] \\ &\text{and} \\ \lim_{n \rightarrow \infty} d(gfx_n, f^2x_n) &\leq \frac{1}{2} \left[ \lim_{n \rightarrow \infty} d(gfx_n, gt) + \lim_{n \rightarrow \infty} d(gt, g^2x_n) \right]. \end{aligned}$$

It is clear that compatible maps of type (A) are compatible of type (B), to show that the converse is not true (see [11]).

Further, in 1998, Pathak et al. [12] introduced another generalization of compatibility of type (A) by giving the concept of compatible maps of type (C).  $f$  and  $g$  are said to be compatible of type (C) if they satisfy the two inequalities:

$$\begin{aligned} \lim_{n \rightarrow \infty} d(fgx_n, g^2x_n) &\leq \frac{1}{3} \left[ \lim_{n \rightarrow \infty} d(fgx_n, ft) + \lim_{n \rightarrow \infty} d(ft, f^2x_n) \right. \\ &\quad \left. + \lim_{n \rightarrow \infty} d(ft, g^2x_n) \right] \\ &\text{and} \\ \lim_{n \rightarrow \infty} d(gfx_n, f^2x_n) &\leq \frac{1}{3} \left[ \lim_{n \rightarrow \infty} d(gfx_n, gt) + \lim_{n \rightarrow \infty} d(gt, g^2x_n) \right. \\ &\quad \left. + \lim_{n \rightarrow \infty} d(gt, f^2x_n) \right]. \end{aligned}$$

The same authors gave some examples to show that compatible maps of type (C) need not be neither compatible nor compatible of type (A) (resp. type (B)).

In [10] the concept of compatible maps of type  $(P)$  was introduced and compared with compatible and compatible maps of type  $(A)$ .  $f$  and  $g$  are compatible of type  $(P)$  if instead of (1) we have

$$\lim_{n \rightarrow \infty} d(f^2 x_n, g^2 x_n) = 0.$$

Note that compatibility, compatibility of type  $(A)$  (resp.  $(B)$ ,  $(C)$  and  $(P)$ ) are equivalent if  $f$  and  $g$  are continuous.

In his paper [5], Jungck generalized the compatibility, the compatibility of type  $(A)$  (resp. type  $(B)$ ,  $(C)$  and  $(P)$ ) by introducing the concept of weak compatibility. He defines  $f$  and  $g$  to be weakly compatible if  $ft = gt$  for some  $t \in \mathcal{X}$  implies that  $fgt = gft$ .

It is known that all of the above compatibility notions imply weakly compatible notion, however, there exist weakly compatible maps which are neither compatible nor compatible of type  $(A)$ ,  $(B)$ ,  $(C)$  and  $(P)$  (see [1]).

Recently, Al-Thagafi and Shahzad [2] weakened the concept of weakly compatible maps by giving the new concept of occasionally weakly compatible maps (owc). Two self-maps  $f$  and  $g$  of a set  $\mathcal{X}$  to be owc if and only if there is a point  $x$  in  $\mathcal{X}$  which is a coincidence point of  $f$  and  $g$  at which  $f$  and  $g$  commute; i.e., there exists a point  $x$  in  $\mathcal{X}$  such that  $fx = gx$  and  $fgx = gfx$ .

In this paper, we weaken the above notion by introducing a new concept called **subcompatible maps**.

**1.1 Definition** Let  $(\mathcal{X}, d)$  be a metric space. Maps  $f$  and  $g : \mathcal{X} \rightarrow \mathcal{X}$  are said to be **subcompatible** if and only if there exists a sequence  $(x_n)$  in  $\mathcal{X}$  such that  $\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} gx_n = t$ ,  $t \in \mathcal{X}$  and which satisfy  $\lim_{n \rightarrow \infty} d(fgx_n, gfx_n) = 0$ .

Obviously, two owc maps are subcompatible, however the converse is not true in general. The example below shows that there exist subcompatible maps which are not owc.

**1.2 Example** Let  $\mathcal{X} = [0, \infty[$  with the usual metric  $d$ . Define  $f$  and  $g$  as follows:

$$fx = x^2 \text{ and } gx = \begin{cases} x + 2 & \text{if } x \in [0, 4] \cup ]9, \infty[, \\ x + 12 & \text{if } x \in ]4, 9]. \end{cases}$$

Let  $(x_n)$  be a sequence in  $\mathcal{X}$  defined by  $x_n = 2 + \frac{1}{n}$  for  $n \in \mathbb{N}^* = \{1, 2, \dots\}$ . Then,

$$\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} x_n^2 = 4 = \lim_{n \rightarrow \infty} gx_n = \lim_{n \rightarrow \infty} (x_n + 2),$$

and

$$\begin{aligned} fgx_n &= f(x_n + 2) = (x_n + 2)^2 \rightarrow 16 \text{ when } n \rightarrow \infty \\ gfx_n &= g(x_n^2) = x_n^2 + 12 \rightarrow 16 \text{ when } n \rightarrow \infty \end{aligned}$$

thus,  $\lim_{n \rightarrow \infty} d(fgx_n, gfx_n) = 0$ ; that is,  $f$  and  $g$  are subcompatible.

On the other hand, we have  $fx = gx$  if and only if  $x = 2$  and

$$\begin{aligned} fg(2) &= f(4) = 4^2 = 16 \\ gf(2) &= g(4) = 4 + 2 = 6 \end{aligned}$$

then,  $f(2) = 4 = g(2)$  but  $fg(2) = 16 \neq 6 = gf(2)$ , hence maps  $f$  and  $g$  are not owc. ■

Clearly, we can resume implications between previous notions by the following list:

- Commuting maps  $\Rightarrow$  Weakly commuting maps
- Weakly commuting maps  $\Rightarrow$  Compatible maps
- Weakly commuting maps  $\Rightarrow$  Compatible maps of type  $(A)$
- Compatible maps of type  $(A) \Rightarrow$  Compatible maps of type  $(B)$
- Compatible maps of type  $(A) \Rightarrow$  Compatible maps of type  $(C)$
- Compatible maps (resp. Compatible of type  $(A)$ ,  $(B)$ ,  $(C)$ ,  $(P)$ )  $\Rightarrow$  Weakly compatible maps
- Weakly compatible maps  $\Rightarrow$  Occasionally Weakly compatible maps
- Occasionally weakly compatible maps  $\Rightarrow$  **Subcompatible maps**.

In his paper [9], Pant introduced the concept of reciprocally continuity as follows: Self-maps  $f$  and  $g$  of a metric space  $(\mathcal{X}, d)$  are reciprocally continuous if and only if  $\lim_{n \rightarrow \infty} fgx_n = ft$  and  $\lim_{n \rightarrow \infty} gfx_n = gt$  whenever  $(x_n) \subset \mathcal{X}$  is such that  $\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} gx_n = t \in \mathcal{X}$ . Clearly, any continuous pair is reciprocally continuous but, the converse is not true in general.

Our second objective here is to introduce a new concept called the notion of **subsequentially continuous maps** which weakens the concepts of continuity and reciprocally continuity given above.

**1.3 Definition** *Two self-maps  $f$  and  $g$  of a metric space  $(\mathcal{X}, d)$  are said to be **subsequentially continuous** if and only if there exists a sequence  $(x_n)$  in  $\mathcal{X}$  such that  $\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} gx_n = t$  for some  $t$  in  $\mathcal{X}$  and satisfy  $\lim_{n \rightarrow \infty} fgx_n = ft$  and  $\lim_{n \rightarrow \infty} gfx_n = gt$ .*

If  $f$  and  $g$  are both continuous or reciprocally continuous then they are obviously subsequentially continuous. The next example shows that there exist subsequentially continuous pairs of maps which are neither continuous nor reciprocally continuous.

**1.4 Example** Let  $\mathcal{X}$  be  $[0, \infty[$  endowed with the usual metric  $d$  and define  $f$  and  $g : \mathcal{X} \rightarrow \mathcal{X}$  by

$$fx = \begin{cases} 1+x & \text{if } 0 \leq x \leq 1 \\ 2x-1 & \text{if } 1 < x < \infty, \end{cases} \quad gx = \begin{cases} 1-x & \text{if } 0 \leq x < 1 \\ 3x-2 & \text{if } 1 \leq x < \infty. \end{cases}$$

Obviously,  $f$  and  $g$  are discontinuous at  $x = 1$ .

Let us consider the sequence  $x_n = \frac{1}{n}$  for  $n = 1, 2, \dots$ . We have

$$\begin{aligned} fx_n = 1 + x_n &\rightarrow 1 = t && \text{when } n \rightarrow \infty, \\ gx_n = 1 - x_n &\rightarrow 1 && \text{when } n \rightarrow \infty, \end{aligned}$$

and

$$\begin{aligned} fgx_n = f(1 - x_n) &= 2 - x_n \rightarrow 2 = f(1), \\ gfx_n = g(1 + x_n) &= 1 + 3x_n \rightarrow 1 = g(1), \end{aligned}$$

therefore  $f$  and  $g$  are subsequentially continuous.

Now, let  $(x_n) = \left(1 + \frac{1}{n}\right)$  for  $n = 1, 2, \dots$ . We have

$$\begin{aligned} fx_n = 2x_n - 1 &\rightarrow 1 = t, \\ gx_n = 3x_n - 2 &\rightarrow 1 = t, \end{aligned}$$

and

$$fgx_n = f(3x_n - 2) = 6x_n - 5 \rightarrow 1 \neq 2 = f(1),$$

so  $f$  and  $g$  are not reciprocally continuous. ■

Now, we show the interest of these two definitions by giving three main results.

## 2 A general common fixed point theorem

We begin by a general common fixed point theorem which improves a result of [7].

**2.1 Theorem** *Let  $f, g, h$  and  $k$  be four self-maps of a metric space  $(\mathcal{X}, d)$ . If pairs of maps  $(f, h)$  and  $(g, k)$  are subcompatible and reciprocally continuous, then*

- (a)  $f$  and  $h$  have a coincidence point;
- (b)  $g$  and  $k$  have a coincidence point.

*Further, let  $\varphi : (\mathbb{R}^+)^6 \rightarrow \mathbb{R}$  be a continuous function satisfying the following condition:*

$$(\varphi_1) : \varphi(u, u, 0, 0, u, u) > 0 \quad \forall u > 0.$$

*We suppose that  $(f, h)$  and  $(g, k)$  satisfy, for all  $x$  and  $y$  in  $\mathcal{X}$ ,*

$$(\varphi_2) : \varphi(d(fx, gy), d(hx, ky), d(fx, hx), d(gy, ky), d(hx, gy), d(ky, fx)) \leq 0.$$

*Then,  $f, g, h$  and  $k$  have a unique common fixed point.*

**Proof**

Since pairs of maps  $(f, h)$  and  $(g, k)$  are subcompatible and reciprocally continuous, then, there exist two sequences  $(x_n)$  and  $(y_n)$  in  $\mathcal{X}$  such that

$$\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} hx_n = t \text{ for some } t \in \mathcal{X} \text{ and which satisfy}$$

$$\lim_{n \rightarrow \infty} d(fhx_n, hfx_n) = d(ft, ht) = 0;$$

$$\lim_{n \rightarrow \infty} gy_n = \lim_{n \rightarrow \infty} ky_n = z \text{ for some } z \in \mathcal{X} \text{ and which satisfy}$$

$$\lim_{n \rightarrow \infty} d(gky_n, kgy_n) = d(gz, kz) = 0.$$

Therefore  $ft = ht$  and  $gz = kz$ ; that is,  $t$  is a coincidence point of  $f$  and  $h$  and  $z$  is a coincidence point of  $g$  and  $k$ .

Now, we prove that  $t = z$ . Indeed, by inequality  $(\varphi_2)$ , we have

$$\begin{aligned} & \varphi(d(fx_n, gy_n), d(hx_n, ky_n), d(fx_n, hx_n), \\ & d(gy_n, ky_n), d(hx_n, gy_n), d(ky_n, fx_n)) \\ & \leq 0. \end{aligned}$$

Since  $\varphi$  is continuous, taking the limit as  $n \rightarrow \infty$  yields

$$\varphi(d(t, z), d(t, z), 0, 0, d(t, z), d(z, t)) \leq 0$$

which contradicts  $(\varphi_1)$  if  $t \neq z$ . Hence  $t = z$ .

Also, we claim that  $ft = t$ . If  $ft \neq t$ , using  $(\varphi_2)$ , we get

$$\begin{aligned} & \varphi(d(ft, gy_n), d(ht, ky_n), d(ft, ht), \\ & d(gy_n, ky_n), d(ht, gy_n), d(ky_n, ft)) \\ & \leq 0. \end{aligned}$$

Since  $\varphi$  is continuous, at infinity, we obtain

$$\varphi(d(ft, t), d(ft, t), 0, 0, d(ft, t), d(t, ft)) \leq 0$$

contradicts  $(\varphi_1)$ . Hence  $t = ft = ht$ .

Again, suppose that  $gt \neq t$ , using inequality  $(\varphi_2)$ , we get

$$\begin{aligned} & \varphi(d(ft, gt), d(ht, kt), d(ft, ht), d(gt, kt), d(ht, gt), d(kt, ft)) \\ & = \varphi(d(t, gt), d(t, gt), 0, 0, d(t, gt), d(gt, t)) \leq 0 \end{aligned}$$

contradicts  $(\varphi_1)$ . Thus  $t = gt = kt$ . Therefore  $t = ft = gt = ht = kt$ ; i.e.,  $t = z$  is a common fixed point of maps  $f, g, h$  and  $k$ .

Finally, suppose that there exists another common fixed point  $w$  of maps  $f, g, h$  and  $k$  such that  $w \neq t$ . Then, by inequality  $(\varphi_2)$ , we have

$$\begin{aligned} & \varphi(d(ft, gw), d(ht, kw), d(ft, ht), d(gw, kw), d(ht, gw), d(kw, ft)) \\ & = \varphi(d(t, w), d(t, w), 0, 0, d(t, w), d(w, t)) \leq 0 \end{aligned}$$

which contradicts  $(\varphi_1)$ . Hence  $w = t$ . ■

If we let in Theorem 2.1,  $f = g$  and  $h = k$ , we get the next corollary:

**2.2 Corollary** *Let  $f$  and  $h$  be self-maps of a metric space  $(\mathcal{X}, d)$  such that  $f$  and  $h$  are subcompatible and reciprocally continuous, then, maps  $f$  and  $h$  have a coincidence point.*

*Further let  $\varphi : (\mathbb{R}^+)^6 \rightarrow \mathbb{R}$  be a continuous function satisfying condition  $(\varphi_1)$  and*

$$\varphi(d(fx, fy), d(hx, hy), d(fx, hx), d(fy, hy), d(hx, fy), d(hy, fx)) \leq 0$$

*for every  $x$  and every  $y$  in  $\mathcal{X}$ , then there exists a unique point  $t \in \mathcal{X}$  such that  $ft = ht = t$ .*

If we put  $h = k$ , we get the following result:

**2.3 Corollary** *Let  $f$ ,  $g$  and  $h$  be three self-maps of a metric space  $(\mathcal{X}, d)$ . Suppose that pairs of maps  $(f, h)$  and  $(g, h)$  are subcompatible and reciprocally continuous, then,*

*(a)  $f$  and  $h$  have a coincidence point;*

*(b)  $g$  and  $h$  have a coincidence point.*

*Let  $\varphi : (\mathbb{R}^+)^6 \rightarrow \mathbb{R}$  be a continuous function satisfying condition  $(\varphi_1)$  and*

$$\varphi(d(fx, gy), d(hx, hy), d(fx, hx), d(gy, hy), d(hx, gy), d(hy, fx)) \leq 0$$

*for all  $x, y$  in  $\mathcal{X}$ , then maps  $f$ ,  $g$  and  $h$  have a unique common fixed point  $t \in \mathcal{X}$ .*

Now, with different choices of the real continuous function  $\varphi$ , we obtain the following corollary which contains several already published results.

**2.4 Corollary** *If in the hypotheses of Theorem 2.1, we have instead of  $(\varphi_2)$  one of the following inequalities, for all  $x$  and  $y$  in  $\mathcal{X}$ , then the four maps have a unique common fixed point*

$$(a) \quad d(fx, gy) \leq \alpha \max\{d(hx, ky), d(hx, fx), d(gy, ky), \frac{1}{2}(d(hx, gy) + d(ky, fx))\}$$

*where  $\alpha \in ]0, 1[$ ,*

$$(b) \quad \begin{aligned} & d(fx, gy)(1 + \alpha d(hx, ky)) \\ & \leq \alpha \max\{d(hx, fx)d(gy, ky), d(hx, gy)d(ky, fx)\} \\ & + \beta \max\{d(hx, ky), d(hx, fx), d(gy, ky), \frac{1}{2}(d(hx, gy) + d(ky, fx))\} \end{aligned}$$

*where  $\alpha \geq 0$  and  $0 < \beta < 1$ ,*

$$(c) \quad d^3(fx, gy) \leq \frac{d^2(hx, fx)d^2(gy, ky) + d^2(hx, gy)d^2(ky, fx)}{1 + d(hx, ky) + d(hx, fx) + d(gy, ky)},$$

$$(d) \quad d(fx, gy) \leq F[\max\{d(hx, ky), d(hx, fx), d(gy, ky), \frac{1}{2}(d(hx, gy) + d(ky, fx))\}]$$

*where  $F : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is an upper semi-continuous function such that, for every  $t > 0$ ,  $0 < F(t) < t$ .*



**Proof**

For proof of (a), (b), (c) and (d), we use Theorem 2.1 with the next functions  $\varphi$  which satisfy, for every case, hypothesis  $(\varphi_1)$ .

For (a):

$$\begin{aligned} & \varphi(d(fx, gy), d(hx, ky), d(fx, hx), d(gy, ky), d(hx, gy), d(ky, fx)) \\ = & d(fx, gy) - \alpha \max\{d(hx, ky), d(fx, hx), d(gy, ky), \\ & \frac{1}{2}(d(hx, gy) + d(ky, fx))\} \end{aligned}$$

this function  $\varphi$  is used by many authors, for example Example 1 of Popa [13].

For (b):

$$\begin{aligned} & \varphi(d(fx, gy), d(hx, ky), d(fx, hx), d(gy, ky), d(hx, gy), d(ky, fx)) \\ = & (1 + \alpha d(hx, ky))d(fx, gy) - \alpha \max\{d(fx, hx)d(gy, ky), \\ & d(hx, gy)d(ky, fx)\} - \beta \max\{d(hx, ky), d(fx, hx), d(gy, ky), \\ & \frac{1}{2}(d(hx, gy) + d(ky, fx))\} \end{aligned}$$

for  $\beta = 1$ , we have Example 3 of Popa [14].

For (c):

$$\begin{aligned} & \varphi(d(fx, gy), d(hx, ky), d(fx, hx), d(gy, ky), d(hx, gy), d(ky, fx)) \\ = & d^3(fx, gy) - \frac{d^2(fx, hx)d^2(gy, ky) + d^2(hx, gy)d^2(ky, fx)}{1 + d(hx, ky) + d(fx, hx) + d(gy, ky)} \end{aligned}$$

this function  $\varphi$  is the one of Example 5 of [13] with  $c = 1$ .

For (d):

$$\begin{aligned} & \varphi(d(fx, gy), d(hx, ky), d(fx, hx), d(gy, ky), d(hx, gy), d(ky, fx)) \\ = & d(fx, gy) - F[\max\{d(hx, ky), d(fx, hx), d(gy, ky), \\ & \frac{1}{2}(d(hx, gy) + d(ky, fx))\}]. \end{aligned}$$

■

Now, using the recurrence on  $n$ , we get the following theorem:

**2.5 Theorem** *Let  $h, k$  and  $\{f_n\}_{n \in \mathbb{N}^*}$  be maps from a metric space  $(\mathcal{X}, d)$  into itself such that pairs of maps  $(f_n, h)$  and  $(f_{n+1}, k)$  are subcompatible and reciprocally continuous, then*

(a)  $(f_n, h)$  have a coincidence point;

(b)  $(f_{n+1}, k)$  have a coincidence point.

Suppose that maps  $f_n, f_{n+1}, h$  and  $k$  satisfy the inequality:

$$\begin{aligned} (\varphi_2) \quad & \varphi(d(f_n x, f_{n+1} y), d(hx, ky), d(f_n x, hx), \\ & d(f_{n+1} y, ky), d(hx, f_{n+1} y), d(ky, f_n x)) \\ & \leq 0 \end{aligned}$$

for all  $x$  and  $y$  in  $\mathcal{X}$ , for every  $n \in \mathbb{N}^*$ , where  $\varphi$  is as in Theorem 2.1, then,  $h, k$  and  $\{f_n\}_{n \in \mathbb{N}^*}$  have a unique common fixed point.

**Proof**

By letting  $n = 1$ , we get the assumptions of Theorem 2.1 for maps  $h, k, f_1$  and  $f_2$  with the unique common fixed point  $t$ . Now,  $t$  is a common fixed point of  $h, k, f_1$  and of  $h, k, f_2$ . Otherwise, if  $z$  is another common fixed point of  $h, k$  and  $f_1$ , then by inequality  $(\varphi_2)$ , we have

$$\begin{aligned} & \varphi(d(f_1z, f_2t), d(hz, kt), d(hz, f_1z), \\ & \quad d(kt, f_2t), d(hz, f_2t), d(kt, f_1z)) \\ &= \varphi(d(z, t), d(z, t), 0, 0, d(z, t), d(t, z)) \leq 0 \end{aligned}$$

contradicts  $(\varphi_1)$ , then  $z = t$ .

By the same manner, we prove that  $t$  is the unique common fixed point of maps  $h, k$  and  $f_2$ .

Now, letting  $n = 2$ , we obtain the hypotheses of Theorem 2.1 for maps  $h, k, f_2$  and  $f_3$  and then, they have a unique common fixed point  $z$ . Analogously,  $z$  is the unique common fixed point of  $h, k, f_2$  and of  $h, k, f_3$ . Thus  $z = t$ . Continuing by this method, we clearly see that  $t$  is the required element.  $\blacksquare$

**2.6 Remark** We can also have common fixed point by using only four distances instead of six. The next theorem shows this fact.

**2.7 Theorem** *Let  $f, g, h$  and  $k$  be self-maps of a metric space  $(\mathcal{X}, d)$ . If pairs of maps  $(f, h)$  and  $(g, k)$  are subcompatible and reciprocally continuous, then,*

(a)  *$f$  and  $h$  have a coincidence point;*

(b)  *$g$  and  $k$  have a coincidence point.*

*Let  $\psi : (\mathbb{R}^+)^4 \rightarrow \mathbb{R}$  be a continuous function such that*

$(\psi_1) : \psi(u, u, u, u) > 0 \ \forall u > 0.$

*Suppose that  $(f, h)$  and  $(g, k)$  satisfy the following inequality  $(\psi_2)$ , for all  $x$  and  $y$  in  $\mathcal{X}$ ,*

$(\psi_2) : \psi(d(fx, gy), d(hx, ky), d(hx, gy), d(ky, fx)) \leq 0.$

*Then,  $f, g, h$  and  $k$  have a unique common fixed point.*

**Proof**

First, proof of (a) and (b) is similar to proof of first part of Theorem 2.1.

Now, suppose that  $d(t, z) > 0$ , then, using inequality  $(\psi_2)$ , we get

$$\psi(d(fx_n, gy_n), d(hx_n, ky_n), d(hx_n, gy_n), d(ky_n, fx_n)) \leq 0.$$

Since  $\psi$  is continuous, we obtain at infinity

$$\psi(d(t, z), d(t, z), d(t, z), d(z, t)) \leq 0$$

which contradicts  $(\psi_1)$ , therefore  $z = t$ .

If  $d(ft, t) > 0$ , by inequality  $(\psi_2)$ , we have

$$\psi(d(ft, gy_n), d(ht, ky_n), d(ht, gy_n), d(ky_n, ft)) \leq 0.$$

Since  $\psi$  is continuous, when  $n$  tends to infinity, we get

$$\psi(d(ft, t), d(ft, t), d(ft, t), d(t, ft)) \leq 0$$

which contradicts  $(\psi_1)$ , hence  $t = ft = ht$ .

Similarly, we have  $t = gt = kt$ .

The uniqueness of the common fixed point  $t$  follows easily from inequality  $(\psi_2)$  and condition  $(\psi_1)$ . ■

### 3 A type Greguš common fixed point theorem

In 1998, Pathak et al. [12] introduced an extension of compatibility of type (A) by giving the notion of compatibility of type (C) and they proved a common fixed point theorem of Greguš type for four compatible maps of type (C) in a Banach space. Further, Djoudi and Nisse [3] generalized the result of [12] by weakening compatibility of type (C) to weak compatibility without continuity. In 2006, Jungck and Rhoades [7] extended the result of Djoudi and Nisse by using an idea called occasional weak compatibility of Al-Thagafi and Shahzad [2] which will be published in 2008.

In this part, we establish a common fixed point theorem for four subcompatible maps of Greguš type in a metric space which extends the results of [3], [7] and [12].

Let  $\mathcal{F}$  be the family of maps  $F$  from  $\mathbb{R}^+$  into itself such that  $F$  is upper semi-continuous and  $F(t) < t$  for any  $t > 0$ .

**3.1 Theorem** *Let  $f, g, h$  and  $k$  be maps from a metric space  $(\mathcal{X}, d)$  into itself. If pairs of maps  $(f, h)$  and  $(g, k)$  are compatible and subsequentially continuous, then,*

(a)  $(f, h)$  has a coincidence point;

(b)  $(g, k)$  has a coincidence point.

Moreover, suppose that the four maps satisfy the following inequality:

$$(2) \quad d^p(fx, gy) \leq F(ad^p(hx, ky) + (1-a) \max\{\alpha d^p(fx, hx), \beta d^p(gy, ky), d^{\frac{p}{2}}(fx, hx)d^{\frac{p}{2}}(fx, ky), d^{\frac{p}{2}}(fx, ky)d^{\frac{p}{2}}(hx, gy), \frac{1}{2}(d^p(fx, hx) + d^p(gy, ky))\})$$

for all  $x$  and  $y$  in  $\mathcal{X}$ , where  $0 < a < 1$ ,  $\{\alpha, \beta\} \subset ]0, 1]$ ,  $p \in \mathbb{N}^*$  and  $F \in \mathcal{F}$ . Then  $f, g, h$  and  $k$  have a unique common fixed point.

**Proof**

Since pairs of maps  $(f, h)$  and  $(g, k)$  are compatible and subsequentially continuous, then, there exist two sequences  $(x_n)$  and  $(y_n)$  in  $\mathcal{X}$  such that

$$\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} hx_n = t \text{ for some } t \in \mathcal{X} \text{ and which satisfy}$$

$$\lim_{n \rightarrow \infty} d(fhx_n, hfx_n) = d(ft, ht) = 0;$$

$$\lim_{n \rightarrow \infty} gy_n = \lim_{n \rightarrow \infty} ky_n = z \text{ for some } z \in \mathcal{X} \text{ and which satisfy}$$

$$\lim_{n \rightarrow \infty} d(gky_n, kgy_n) = d(gz, kz) = 0.$$

Therefore  $ft = ht$  and  $gz = kz$ ; that is,  $t$  is a coincidence point of maps  $f$  and  $h$  and  $z$  is a coincidence point of  $g$  and  $k$ .

Furthermore, we prove that  $t = z$ . Suppose that  $d(t, z) > 0$ , indeed by inequality (2) we have

$$\begin{aligned} d^p(fx_n, gy_n) \leq & F(ad^p(hx_n, ky_n) + (1-a) \max\{\alpha d^p(fx_n, hx_n), \\ & \beta d^p(gy_n, ky_n), d^{\frac{p}{2}}(fx_n, hx_n)d^{\frac{p}{2}}(fx_n, ky_n), \\ & d^{\frac{p}{2}}(fx_n, ky_n)d^{\frac{p}{2}}(hx_n, gy_n), \\ & \frac{1}{2}(d^p(fx_n, hx_n) + d^p(gy_n, ky_n))\}). \end{aligned}$$

By properties of  $F$ , we get at infinity

$$\begin{aligned} d^p(t, z) & \leq F(ad^p(t, z) + (1-a)d^p(t, z)) \\ & = F(d^p(t, z)) < d^p(t, z) \end{aligned}$$

this contradiction implies that  $t = z$ .

Now, if  $ft \neq t$ , the use of condition (2) gives

$$\begin{aligned} d^p(ft, gy_n) \leq & F(ad^p(ht, ky_n) + (1-a) \max\{\alpha d^p(ft, ht), \\ & \beta d^p(gy_n, ky_n), d^{\frac{p}{2}}(ft, ht)d^{\frac{p}{2}}(ft, ky_n), \\ & d^{\frac{p}{2}}(ft, ky_n)d^{\frac{p}{2}}(ht, gy_n), \\ & \frac{1}{2}(d^p(ft, ht) + d^p(gy_n, ky_n))\}). \end{aligned}$$

By properties of  $F$ , we obtain at infinity

$$\begin{aligned} d^p(ft, t) & \leq F(ad^p(ft, t) + (1-a)d^p(ft, t)) \\ & = F(d^p(ft, t)) < d^p(ft, t) \end{aligned}$$

this contradiction implies that  $t = ft = ht$ .

Similarly, we have  $gt = kt = t$ . Therefore  $t = z$  is a common fixed point of both  $f, g, h$  and  $k$ .

Suppose that maps  $f, g, h$  and  $k$  have another common fixed point  $w \neq t$ . Then, by (2) we get

$$\begin{aligned} d^p(ft, gw) \leq & F(ad^p(ht, kw) + (1-a) \max\{\alpha d^p(ft, ht), \beta d^p(gw, kw), \\ & d^{\frac{p}{2}}(ft, ht)d^{\frac{p}{2}}(ft, kw), d^{\frac{p}{2}}(ft, kw)d^{\frac{p}{2}}(ht, gw), \\ & \frac{1}{2}(d^p(ft, ht) + d^p(gw, kw))\}); \end{aligned}$$

that is,

$$\begin{aligned} d^p(t, w) & \leq F(ad^p(t, w) + (1-a) \max\{0, d^p(t, w)\}) \\ & = F(d^p(t, w)) < d^p(t, w) \end{aligned}$$

this contradiction implies that  $w = t$ . ■

**3.2 Corollary** *Let  $f$  and  $h$  be two self-maps of a metric space  $(\mathcal{X}, d)$ . If pair of maps  $(f, h)$  is compatible and subsequentially continuous, then  $f$  and  $h$  have a coincidence point.*

Suppose that  $(f, h)$  satisfies the following inequality:

$$d^p(fx, fy) \leq F(ad^p(hx, hy) + (1-a) \max\{\alpha d^p(fx, hx), \beta d^p(fy, hy), d^{\frac{p}{2}}(fx, hx)d^{\frac{p}{2}}(fx, hy), d^{\frac{p}{2}}(fx, hy)d^{\frac{p}{2}}(hx, fy), \frac{1}{2}(d^p(fx, hx) + d^p(fy, hy))\})$$

for all  $x, y$  in  $\mathcal{X}$ , where  $0 < a < 1$ ,  $\{\alpha, \beta\} \subset ]0, 1]$ ,  $p \in \mathbb{N}^*$  and  $F \in \mathcal{F}$ , then  $f$  and  $h$  have a unique common fixed point.

**3.3 Corollary** Let  $f, g$  and  $h$  be three self-maps of a metric space  $(\mathcal{X}, d)$ . Suppose that pairs of maps  $(f, h)$  and  $(g, h)$  are compatible and subsequentially continuous, then,

- (a)  $f$  and  $h$  have a coincidence point;
- (b)  $g$  and  $h$  have a coincidence point.

Further, if the three maps satisfy the next inequality:

$$d^p(fx, gy) \leq F(ad^p(hx, hy) + (1-a) \max\{\alpha d^p(fx, hx), \beta d^p(gy, hy), d^{\frac{p}{2}}(fx, hx)d^{\frac{p}{2}}(fx, hy), d^{\frac{p}{2}}(fx, hy)d^{\frac{p}{2}}(hx, gy), \frac{1}{2}(d^p(fx, hx) + d^p(gy, hy))\})$$

for all  $x$  and  $y$  in  $\mathcal{X}$ , where  $0 < a < 1$ ,  $\{\alpha, \beta\} \subset ]0, 1]$ ,  $p \in \mathbb{N}^*$  and  $F \in \mathcal{F}$ , then  $f, g$  and  $h$  have a unique common fixed point.

Again, using the recurrence on  $n$ , we get the next theorem:

**3.4 Theorem** Let  $h, k$  and  $\{f_n\}_{n \in \mathbb{N}^*}$  be self-maps of a metric space  $(\mathcal{X}, d)$ . Suppose that  $(f_n, h)$  and  $(f_{n+1}, k)$  are compatible and subsequentially continuous, then,

- (a) maps  $f_n$  and  $h$  have a coincidence point;
- (b)  $f_{n+1}$  and  $k$  have a coincidence point.

Furthermore, if the maps satisfying the inequality:

$$d^p(f_n x, f_{n+1} y) \leq F(ad^p(hx, ky) + (1-a) \max\{\alpha d^p(f_n x, hx), \beta d^p(f_{n+1} y, ky), d^{\frac{p}{2}}(f_n x, hx)d^{\frac{p}{2}}(f_n x, ky), d^{\frac{p}{2}}(f_n x, ky)d^{\frac{p}{2}}(hx, f_{n+1} y), \frac{1}{2}(d^p(f_n x, hx) + d^p(f_{n+1} y, ky))\})$$

for all  $x$  and  $y$  in  $\mathcal{X}$ , where  $0 < a < 1$ ,  $\{\alpha, \beta\} \subset ]0, 1]$ ,  $p \in \mathbb{N}^*$  and  $F \in \mathcal{F}$ , then  $h, k$  and  $\{f_n\}_{n \in \mathbb{N}^*}$  have a unique common fixed point.

## 4 A near-contractive common fixed point theorem

We end our work by establishing the next result which especially improves the main result of [8].

**4.1 Theorem** Let  $(\mathcal{X}, d)$  be a metric space,  $f, g, h$  and  $k$  be maps from  $\mathcal{X}$  into itself. If  $(f, h)$  and  $(g, k)$  are compatible and subsequentially continuous or subcompatible and reciprocally continuous, then,

- (a)  $f$  and  $h$  have a coincidence point;
- (b)  $g$  and  $k$  have a coincidence point.

Let  $\Phi$  be a continuous function of  $[0, \infty[$  into itself such that  $\Phi(t) = 0$  if and only if  $t = 0$  and satisfying inequality

$$(3) \quad \Phi(d(fx, gy)) \leq a(d(hx, ky))\Phi(d(hx, ky)) \\ + b(d(hx, ky)) \min\{\Phi(d(hx, gy)), \Phi(d(ky, fx))\}$$

for all  $x$  and  $y$  in  $\mathcal{X}$ , where  $a, b : [0, \infty[ \rightarrow [0, 1[$  are upper semi-continuous and satisfying the condition:

$$a(t) + b(t) < 1 \quad \forall t > 0.$$

Then  $f, g, h$  and  $k$  have a unique common fixed point.

**Proof**

First, proof of parts (a) and (b) is similar to proof of Theorem 2.1.

Now, suppose that  $d(t, z) > 0$ , using inequality (3), we get

$$\Phi(d(fx_n, gy_n)) \\ \leq a(d(hx_n, ky_n))\Phi(d(hx_n, ky_n)) \\ + b(d(hx_n, ky_n)) \min\{\Phi(d(hx_n, gy_n)), \Phi(d(ky_n, fx_n))\}.$$

By properties of  $\Phi$ ,  $a$  and  $b$ , we get at infinity

$$\Phi(d(t, z)) \leq [a(d(t, z)) + b(d(t, z))]\Phi(d(t, z)) \\ < \Phi(d(t, z))$$

which is a contradiction. Hence  $\Phi(d(t, z)) = 0$  which implies that  $d(t, z) = 0$ , thus  $t = z$ .

Next, if  $ft \neq t$ , the use of condition (3) gives

$$\Phi(d(ft, gy_n)) \\ \leq a(d(ht, ky_n))\Phi(d(ht, ky_n)) \\ + b(d(ht, ky_n)) \min\{\Phi(d(ht, gy_n)), \Phi(d(ky_n, ft))\}.$$

By properties of  $\Phi$ ,  $a$  and  $b$ , we get at infinity

$$\Phi(d(ft, t)) \leq [a(d(ft, t)) + b(d(ft, t))]\Phi(d(ft, t)) \\ < \Phi(d(ft, t))$$

this contradiction implies that  $\Phi(d(ft, t)) = 0$  and hence  $t = ft = ht$ .

Similarly, we have  $gt = kt = t$ .

Now, assume that there exists another common fixed point  $w$  of maps  $f, g, h$  and  $k$  such that  $w \neq t$ . By inequality (3) and properties of functions  $\Phi, a$  and  $b$ , we obtain

$$\begin{aligned}\Phi(d(t, w)) &= \Phi(d(ft, gw)) \leq a(d(ht, kw))\Phi(d(ht, kw)) \\ &\quad + b(d(ht, kw)) \min\{\Phi(d(ht, gw)), \Phi(d(kw, ft))\} \\ &= [a(d(t, w)) + b(d(t, w))]\Phi(d(t, w)) \\ &< \Phi(d(t, w))\end{aligned}$$

this contradiction implies that  $\Phi(d(t, w)) = 0 \Leftrightarrow d(t, w) = 0$ , hence  $w = t$ .  $\blacksquare$

**4.2 Remark** A slight improvement is achieved by replacing inequality (3) in Theorem 4.1 by the following one:

$$\begin{aligned}\Phi(d(fx, gy)) &\leq a(d(hx, ky))\Phi(d(hx, ky)) \\ &\quad + b(d(hx, ky)) \left[ \frac{\Phi^{\frac{1}{p}}(d(hx, gy)) + \Phi^{\frac{1}{p}}(d(ky, fx))}{2} \right]^p,\end{aligned}$$

with  $p$  is an integer such that  $p \geq 1$ .

As particular cases, we immediately obtain the two following corollaries:

**4.3 Corollary** *Let  $f$  and  $h$  be self-maps of a metric space  $(\mathcal{X}, d)$ . Assume that pair of maps  $(f, h)$  is compatible and subsequentially continuous or subcompatible and reciprocally continuous, then,  $f$  and  $h$  have a coincidence point. Further, suppose that pair of maps  $(f, h)$  satisfies the inequality:*

$$\begin{aligned}\Phi(d(fx, fy)) &\leq a(d(hx, hy))\Phi(d(hx, hy)) \\ &\quad + b(d(hx, hy)) \min\{\Phi(d(hx, fy)), \Phi(d(hy, fx))\}\end{aligned}$$

for all  $x$  and  $y$  in  $\mathcal{X}$ , where  $\Phi, a$  and  $b$  are as in Theorem 4.1. Then,  $f$  and  $h$  have a unique common fixed point.

**4.4 Corollary** *Let  $f, g, h : \mathcal{X} \rightarrow \mathcal{X}$  be maps. If pairs of maps  $(f, h)$  and  $(g, h)$  are compatible and subsequentially continuous or subcompatible and reciprocally continuous, then,*

- (a)  $f$  and  $h$  have a coincidence point;
- (b)  $g$  and  $h$  have a coincidence point.

Moreover, suppose that maps  $f, g$  and  $h$  satisfy the following inequality:

$$\begin{aligned}\Phi(d(fx, gy)) &\leq a(d(hx, hy))\Phi(d(hx, hy)) \\ &\quad + b(d(hx, hy)) \min\{\Phi(d(hx, gy)), \Phi(d(hy, fx))\}\end{aligned}$$

for all  $x$  and  $y$  in  $\mathcal{X}$ , where  $\Phi, a$  and  $b$  are as in Theorem 4.1, then,  $f, g$  and  $h$  have a unique common fixed point.

We end our work by giving the following result which concern a common fixed point of a sequence of maps. Its proof is easily obtained from Theorem 4.1 by recurrence.

**4.5 Theorem** *Let  $(\mathcal{X}, d)$  be a metric space,  $h, k, \{f_n\}_{n \in \mathbb{N}^*}$  be maps from  $\mathcal{X}$  into itself. If pairs of maps  $(f_n, h)$  and  $(f_{n+1}, k)$  are compatible and subsequentially continuous or subcompatible and reciprocally continuous, then,*

(a)  $f_n$  and  $h$  have a coincidence point;

(b)  $f_{n+1}$  and  $k$  have a coincidence point.

Let  $\Phi$  be a continuous function of  $[0, \infty[$  into itself such that  $\Phi(t) = 0$  if and only if  $t = 0$  and satisfying the following inequality:

$$\begin{aligned} & \Phi(d(f_n x, f_{n+1} y)) \\ \leq & a(d(hx, ky))\Phi(d(hx, ky)) \\ & + b(d(hx, ky)) \min\{\Phi(d(hx, f_{n+1} y)), \Phi(d(ky, f_n x))\} \end{aligned}$$

for all  $x$  and  $y$  in  $\mathcal{X}$ , where  $a$  and  $b : [0, \infty[ \rightarrow [0, 1[$  are upper semi-continuous and satisfying the condition

$$a(t) + b(t) < 1 \quad \forall t > 0.$$

Then,  $h, k$  and all maps  $\{f_n\}_{n \in \mathbb{N}^*}$  have a unique common fixed point.

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