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ON RECOGNIZABLE LANGUAGES OF INFINITE PICTURES

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ABSTRACT

In a recent paper, Altenbernd, Thomas and Wöhrle have considered acceptance of languages of infinite two-dimensional words (infinite pictures) by finite tiling systems, with the usual acceptance conditions, such as the Büchi and Muller ones, firstly used for infinite words. The authors asked for comparing the tiling system acceptance with an acceptance of pictures row by row using an automaton model over ordinal words of length ω^2 . We give in this paper a solution to this problem, showing that all languages of infinite pictures which are accepted row by row by Büchi or Choueka automata reading words of length ω^2 are Büchi recognized by a finite tiling system, but the converse is not true. We give also the answer to two other questions which were raised by Altenbernd, Thomas and Wöhrle, showing that it is undecidable whether a Büchi recognizable language of infinite pictures is E-recognizable (respectively, A-recognizable).

Keywords: Languages of infinite pictures; tiling systems; automata reading ordinal words of length ω^2 ; topological complexity; Borel and analytic sets; E-recognizable; A-recognizable; decision problems.

1. Introduction

In a recent paper [1], Altenbernd, Thomas and Wöhrle have considered acceptance of languages of infinite two-dimensional words (infinite pictures) by finite tiling systems, with the usual acceptance conditions, such as the Büchi and Muller ones, firstly used for acceptance of infinite words. This way they extended the classical theory of recognizable languages of finite pictures, [13], to the case of infinite pictures.

On the other hand automata reading ordinal words have been first considered by Büchi in order to study the decidability of the monadic second order theory on countable ordinals. In particular he defined automata reading words of length ω^2 , [5, 6]. Another model of automaton reading words of length ω^2 has been studied by

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Choueka in [7] and it has been shown by Bedon that these two models are equivalent [2, 3]. They accept the so called regular ω^2 -languages which can also be defined by generalized regular expressions, see also the work of Wojciechowski [26, 27].

In [1] the authors asked for comparing the tiling system acceptance with an acceptance of pictures row by row using an automaton model over ordinal words of length ω^2 .

We give in this paper a solution to this problem, showing that the class of languages of infinite pictures which are accepted by Büchi automata reading words of length ω^2 is strictly included in the class of languages of infinite pictures which are Büchi-recognized by some finite tiling system.

Another way to compare these two classes is to compare the topological complexity of languages in each of them, with regard to the Borel and projective hierarchies. We then determine the topological complexity of Büchi-recognized languages of infinite pictures. This way we show that Büchi tiling systems have a much greater accepting power than automata over ordinal words of length ω^2 .

Using topological arguments, we give also the answer to two questions raised in [1], showing that it is undecidable whether a Büchi recognizable language of infinite pictures is E-recognizable (respectively, A-recognizable). For that purpose we use a very similar technique as in a recent paper where we have proved several undecidability results for infinitary rational relations [12].

The paper is organized as follows. In section 2 we recall basic definitions for pictures and tiling systems. Büchi automata reading words of length ω or ω^2 are introduced in section 3. We compare the two modes of acceptance in section 4. Undecidability results are proved in section 5.

2. Tiling Systems

Let Σ be a finite alphabet and $\#$ be a letter not in Σ and let $\hat{\Sigma} = \Sigma \cup \{\#\}$. If m and n are two integers > 0 or if $m = n = 0$, a picture of size (m, n) over Σ is a function p from $\{0, 1, \dots, m+1\} \times \{0, 1, \dots, n+1\}$ into $\hat{\Sigma}$ such that $p(0, i) = p(m+1, i) = \#$ for all integers $i \in \{0, 1, \dots, n+1\}$ and $p(i, 0) = p(i, n+1) = \#$ for all integers $i \in \{0, 1, \dots, m+1\}$ and $p(i, j) \in \Sigma$ if $i \notin \{0, m+1\}$ and $j \notin \{0, n+1\}$. The empty picture is the only picture of size $(0, 0)$ and is denoted by λ . Pictures of size $(n, 0)$ or $(0, n)$, for $n > 0$, are not defined. $\Sigma^{*,*}$ is the set of pictures over Σ . A picture language L is a subset of $\Sigma^{*,*}$.

An ω -picture over Σ is a function p from $\omega \times \omega$ into $\hat{\Sigma}$ such that $p(i, 0) = p(0, i) = \#$ for all $i \geq 0$ and $p(i, j) \in \Sigma$ for $i, j > 0$. $\Sigma^{\omega, \omega}$ is the set of ω -pictures over Σ . An ω -picture language L is a subset of $\Sigma^{\omega, \omega}$.

For Σ a finite alphabet we call Σ^{ω^2} the set of functions from $\omega \times \omega$ into Σ . So the set $\Sigma^{\omega, \omega}$ of ω -pictures over Σ is a strict subset of $\hat{\Sigma}^{\omega^2}$.

We shall say that, for each integer $j \geq 1$, the j^{th} row of an ω -picture $p \in \Sigma^{\omega, \omega}$ is the infinite word $p(1, j).p(2, j).p(3, j) \dots$ over Σ and the j^{th} column of p is the infinite word $p(j, 1).p(j, 2).p(j, 3) \dots$ over Σ .

As usual, one can imagine that, for integers $j > k \geq 1$, the j^{th} column of p is on the right of the k^{th} column of p and that the j^{th} row of p is “above” the k^{th} row of p . This representation will be used in the sequel.

We introduce now tiling systems as in the paper [1].

A tiling system is a tuple $\mathcal{A}=(Q, \Sigma, \Delta)$, where Q is a finite set of states, Σ is a finite alphabet, $\Delta \subseteq (\hat{\Sigma} \times Q)^4$ is a finite set of tiles.

A Büchi tiling system is a pair (\mathcal{A}, F) where $\mathcal{A}=(Q, \Sigma, \Delta)$ is a tiling system and $F \subseteq Q$ is the set of accepting states.

A Muller tiling system is a pair $(\mathcal{A}, \mathcal{F})$ where $\mathcal{A}=(Q, \Sigma, \Delta)$ is a tiling system and $\mathcal{F} \subseteq 2^Q$ is the set of accepting sets of states.

Tiles are denoted by $\begin{pmatrix} (a_3, q_3) & (a_4, q_4) \\ (a_1, q_1) & (a_2, q_2) \end{pmatrix}$ with $a_i \in \hat{\Sigma}$ and $q_i \in Q$,

and in general, over an alphabet Γ , by $\begin{pmatrix} b_3 & b_4 \\ b_1 & b_2 \end{pmatrix}$ with $b_i \in \Gamma$.

We will indicate a combination of tiles by:

$$\begin{pmatrix} b_3 & b_4 \\ b_1 & b_2 \end{pmatrix} \circ \begin{pmatrix} b'_3 & b'_4 \\ b'_1 & b'_2 \end{pmatrix} = \begin{pmatrix} (b_3, b'_3) & (b_4, b'_4) \\ (b_1, b'_1) & (b_2, b'_2) \end{pmatrix}$$

A run of a tiling system $\mathcal{A}=(Q, \Sigma, \Delta)$ over a (finite) picture p of size (m, n) over Σ is a mapping ρ from $\{0, 1, \dots, m+1\} \times \{0, 1, \dots, n+1\}$ into Q such that for all $(i, j) \in \{0, 1, \dots, m\} \times \{0, 1, \dots, n\}$ with $p(i, j) = a_{i,j}$ and $\rho(i, j) = q_{i,j}$ we have

$$\begin{pmatrix} a_{i,j+1} & a_{i+1,j+1} \\ a_{i,j} & a_{i+1,j} \end{pmatrix} \circ \begin{pmatrix} q_{i,j+1} & q_{i+1,j+1} \\ q_{i,j} & q_{i+1,j} \end{pmatrix} \in \Delta.$$

A run of a tiling system $\mathcal{A}=(Q, \Sigma, \Delta)$ over an ω -picture $p \in \Sigma^{\omega, \omega}$ is a mapping ρ from $\omega \times \omega$ into Q such that for all $(i, j) \in \omega \times \omega$ with $p(i, j) = a_{i,j}$ and $\rho(i, j) = q_{i,j}$ we have

$$\begin{pmatrix} a_{i,j+1} & a_{i+1,j+1} \\ a_{i,j} & a_{i+1,j} \end{pmatrix} \circ \begin{pmatrix} q_{i,j+1} & q_{i+1,j+1} \\ q_{i,j} & q_{i+1,j} \end{pmatrix} \in \Delta.$$

We now recall acceptance of finite or infinite pictures by tiling systems:

Definition 2.1 Let $\mathcal{A}=(Q, \Sigma, \Delta)$ be a tiling system, $F \subseteq Q$ and $\mathcal{F} \subseteq 2^Q$.

- The picture language recognized by \mathcal{A} is the set of pictures $p \in \Sigma^{*,*}$ such that there is some run ρ of \mathcal{A} on p .
- The ω -picture language \mathcal{A} -recognized (respectively, E -recognized, Büchi-recognized) by (\mathcal{A}, F) is the set of ω -pictures $p \in \Sigma^{\omega, \omega}$ such that there is some run ρ of \mathcal{A} on p and $\rho(v) \in F$ for all (respectively, for at least one, for infinitely many) $v \in \omega^2$.
- The ω -picture language Muller-recognized by $(\mathcal{A}, \mathcal{F})$ is the set of ω -pictures $p \in \Sigma^{\omega, \omega}$ such that there is some run ρ of \mathcal{A} on p and $\text{Inf}(\rho) \in \mathcal{F}$ where $\text{Inf}(\rho)$ is the set of states occurring infinitely often in ρ .

As stated in [1], an ω -picture language $L \subseteq \Sigma^{\omega, \omega}$ is recognized by a Büchi tiling system if and only if it is recognized by a Muller tiling system.

We shall denote $TS(\Sigma^{\omega, \omega})$ the class of languages $L \subseteq \Sigma^{\omega, \omega}$ which are recognized by some Büchi (or Muller) tiling system.

3. Büchi Automata

We shall assume the reader to be familiar with the elementary theory of countable ordinals, which may be found in [20]. In fact we shall only need in this section to consider ordinals smaller than $\omega^2 + 1$.

Let Σ be a finite alphabet, and α be a countable ordinal. An α -word x (word of length α) over the alphabet Σ is an α -sequence (sequence of length α) of letters in Σ . It will be denoted by $(x(i))_{0 \leq i < \alpha} = x(0).x(1).x(2) \dots x(i) \dots$, where for all i , $0 \leq i < \alpha$, $x(i)$ is a letter in Σ .

For an ordinal $\alpha \geq \omega$, the set of α -words over Σ will be denoted by Σ^α . An α -language over Σ is a subset of Σ^α .

We assume now that the reader has some familiarity with the notion of Büchi and Muller automata reading infinite words, [24, 23, 19].

Definition 3.1 *A Büchi automaton is a 5-tuple $\mathcal{A} = (\Sigma, Q, q_0, \Delta, F)$ where Q is a finite set of states, $q_0 \in Q$ is the initial state, $\Delta \subseteq Q \times \Sigma \times Q$ is the transition relation, and $F \subseteq Q$ is the set of final states.*

A run of \mathcal{A} on the ω -word $\sigma \in \Sigma^\omega$ is an ω -sequence $x \in Q^\omega$ such that $x(0) = q_0$ and $(x(i), \sigma(i), x(i+1)) \in \Delta$ for $i \geq 0$.

The run is called successful if $\text{Inf}(x) \cap F \neq \emptyset$, where $\text{Inf}(x)$ is the set of elements of Q which appear infinitely often in the ω -sequence x .

An ω -word $\sigma \in \Sigma^\omega$ is accepted by \mathcal{A} if there exists a successful run of \mathcal{A} on σ .

$L_\omega(\mathcal{A}) = \{\sigma \in \Sigma^\omega \mid \mathcal{A} \text{ accepts } \sigma\}$ is the ω -language recognized by \mathcal{A} .

A Muller automaton is defined in a similar way except that F is replaced by a set $\mathcal{F} \subseteq 2^Q$ of accepting sets of states and that a run $x \in Q^\omega$ on an ω -word $\sigma \in \Sigma^\omega$ is said to be successful iff $\text{Inf}(x) \in \mathcal{F}$.

Büchi and Muller automata accept the same class of ω -languages: the class of regular ω -languages which is the ω -Kleene closure of the class of regular finitary languages. It follows from Mac Naughton's Theorem that each regular ω -language is also accepted by a deterministic Muller automaton, [24, 23, 19].

In order to define an automaton reading ordinal words of length $\geq \omega$, we must add to the automaton a transition relation for limit steps: after the reading of a word which length is a limit ordinal, the state of the automaton will depend on the set of states which cofinally appeared during the run of the automaton, [6, 14, 2]. We shall give the following definition in the general case of automata reading ordinal words but in fact we shall only need in the sequel the notion of automata reading words of length ω or ω^2 .

Definition 3.2 *An ordinal Büchi automaton is a sextuple $\mathcal{A} = (\Sigma, Q, q_0, \Delta, \gamma, F)$ where: Σ is a finite alphabet, Q is a finite set of states, $q_0 \in Q$ is the initial state,*

$\Delta \subset Q \times \Sigma \times Q$ is the transition relation, and $\gamma \subset P(Q) \times Q$ is the transition relation for limit steps.

Σ, Q, q_0, Δ and F keep the same meaning as before, the meaning of γ is given by the following definition:

Definition 3.3 A run of the ordinal Büchi automaton $\mathcal{A}=(\Sigma, Q, q_0, \Delta, \gamma, F)$, reading the word $\sigma \in \Sigma^\alpha$, is an $(\alpha + 1)$ -sequence of states x defined by: $x(0) = q_0$ and for $i < \alpha$, $(x(i), \sigma(i), x(i + 1)) \in \Delta$ and for a limit ordinal i : $(\text{Inf}(x, i), x(i)) \in \gamma$, where

$$\text{Inf}(x, i) = \{q \in Q \mid \forall \mu < i, \exists \nu < i \text{ such that } \mu < \nu \text{ and } x(\nu) = q\}$$

is the set of states which cofinally appear during the reading of the i first letters of σ .

A run x of the automaton \mathcal{A} over the word σ is called successful if $x(\alpha) \in F$. A word $\sigma \in \Sigma^\alpha$ is accepted by \mathcal{A} if there exists a successful run of \mathcal{A} over σ . We denote $L_\alpha(\mathcal{A})$ the set of words of length α which are accepted by \mathcal{A} .

In particular the above definition provides a notion of automata reading words of length ω^2 . Later Choueka defined another class of automata reading words of length ω^2 (and even ω^n for an integer $n \geq 2$) now called Choueka automata [7]. Bedon proved that these two classes of automata accept the same class of ω^2 -languages, the class of regular ω^2 -languages which can be also defined by ω^2 -regular expressions [2, 3].

Remark 3.4 When we consider only finite words, the language accepted by an ordinal Büchi automaton is a rational language. And an ω -language is accepted by an ordinal Büchi automaton if and only if it is accepted by a Muller automaton hence also by a Büchi automaton.

We shall use in the sequel another way of generating regular ω^2 -languages which is given by the following proposition. We shall reprove this result although it already appeared in [14] and has been also proved in [10].

Proposition 3.5 An ω^2 -language $L \subseteq \Sigma^{\omega^2}$ is regular iff it is obtained from a regular ω -language $R \subseteq \Gamma^\omega$ by substituting in every ω -word $\sigma \in R$ a regular ω -language $L_a \subseteq \Sigma^\omega$ to each letter $a \in \Gamma$.

Proof. Let $\mathcal{A}=(\Sigma, Q, q_0, \Delta, \gamma, F)$ be an ordinal Büchi automaton, and let $L_{\omega^2}(\mathcal{A})$ be the ω^2 -language recognized by \mathcal{A} .

Consider the reading of a word $\sigma \in \Sigma^{\omega^2}$ by \mathcal{A} : After the reading of the first ω letters, \mathcal{A} is in state $x(\omega)$, after the reading of $\omega.2$ letters, \mathcal{A} is in state $x(\omega.2)$ and so on.

For $q_i \in Q, q_j \in Q$ and $E \subseteq Q$, we denote by $L(q_i, q_j, E)$ the ω -language of words $u \in \Sigma^\omega$ such that there exists a reading of u by \mathcal{A} , beginning in state q_i , ending in state q_j after the reading of u , and going through the set of states E (including q_i and q_j).

We easily see that these ω -languages are recognized by Muller automata therefore also by Büchi automata.

Consider now the new alphabet:

$$\Gamma = Q \times Q \times P(Q) = \{(q_i, q_j, E) \mid q_i \in Q, q_j \in Q, E \subseteq Q\}$$

and let $R \subseteq \Gamma^\omega$ containing an ω -word $\sigma \in \Gamma^\omega$ if and only if σ satisfies the two following properties:

- (1). The first letter of σ is in the form (q_0, q, E) and each letter (q_i, q_j, E) is followed by a letter (q_j, q, G) with $q \in Q, G \subseteq Q$.
- (2). The set

$$X = \{q \in Q \mid \text{some letter } (q_i, q_j, G) \text{ appears infinitely often in } \sigma \text{ and } q \in G\}$$

satisfies $(X, q_f) \in \gamma$ for some $q_f \in F$.

R is a regular ω -language and if we substitute in R the ω -language $L(q_i, q_j, E)$ to each letter (q_i, q_j, E) , we obtain the ω^2 -language recognized by \mathcal{A} , i.e. $L_{\omega^2}(\mathcal{A})$.

We have then proved one implication of Proposition 3.5. In fact we shall only need in the sequel this implication.

We just mention that the converse can be easily proved by using regular expressions defining regular ω -languages and regular ω^2 -languages. \square

We have now to define precisely the acceptance of infinite pictures row by row by an automaton model over ordinal words of length ω^2 .

To an infinite picture $p \in \Sigma^{\omega, \omega}$ we associate an ω^2 -word $\bar{p} \in \Sigma^{\omega^2}$ which is defined by $\bar{p}(\omega \cdot n + m) = p(m + 1, n + 1)$ for all integers $n, m \geq 0$.

This can be extended to languages of infinite pictures: for $L \subseteq \Sigma^{\omega, \omega}$ we denote $\bar{L} = \{\bar{p} \mid p \in L\}$ so \bar{L} is an ω^2 -language over Σ .

We can now set the following definition:

Definition 3.6 *A language of infinite pictures $L \subseteq \Sigma^{\omega, \omega}$ is accepted row by row by an ordinal Büchi automaton if and only if the ω^2 -language \bar{L} is regular.*

We shall denote $BA(\Sigma^{\omega, \omega})$ the class of languages $L \subseteq \Sigma^{\omega, \omega}$ such that \bar{L} is regular, i.e. is accepted by an ordinal Büchi automaton.

Remark 3.7 *We have defined the ω^2 -word \bar{p} without the letters $\#$ appearing in the infinite picture p . It is easy to see that this does not change the notion of acceptance of a language of infinite pictures row by row by an ordinal Büchi automaton.*

4. Comparison of The Two Modes of Acceptance

We can now state our main result.

Theorem 4.1 *Every language of infinite pictures which is accepted row by row by an ordinal Büchi automaton is Büchi-recognized by some finite tiling system, but the converse is not true.*

We are going to split the proof of Theorem 4.1 into the two following lemmas.

Lemma 4.2 *Every language of infinite pictures which is accepted row by row by an ordinal Büchi automaton is Büchi-recognized by some finite tiling system.*

Proof. Let $L \subseteq \Sigma^{\omega, \omega}$ be a language of infinite pictures which is accepted row by row by an ordinal Büchi automaton, i.e. such that the ω^2 -language \bar{L} is regular.

By Proposition 3.5, the ω^2 -language \bar{L} is obtained from a regular ω -language $R \subseteq \Gamma^\omega$, where $\Gamma = \{a_1, a_2, \dots, a_n\}$ is a finite alphabet, by substituting in every ω -word $\sigma \in R$ a regular ω -language $R_i \subseteq \Sigma^\omega$ to each letter $a_i \in \Gamma$.

Let $\mathcal{A} = (\Gamma, Q, q_0, \Delta, F)$ be a Büchi automaton accepting the regular ω -language R and, for each integer $i \in [1; n]$, let $\mathcal{A}^i = (\Sigma, Q^i, q_0^i, \Delta^i, F^i)$ be a Büchi automaton accepting the regular ω -language R_i . We assume, without loss of generality, that for all integers $i, j \in [1; n]$, $Q^i \cap Q^j = \emptyset$ and $Q^i \cap Q = \emptyset$.

We shall describe the behaviour of a tiling system $\mathcal{T} = (K, \Sigma, \Delta^{\mathcal{T}})$ which will accept infinite pictures $p \in L$ with a Muller acceptance condition.

A run ρ of \mathcal{T} on an ω -picture $p \in L$ will guess, for each integer $j \geq 1$, an integer $i_j \in \{1, 2, \dots, n\}$ such that the j^{th} row p_j of p is in R_{i_j} . It will then check that for all $j \geq 1$ the ω -word p_j is in R_{i_j} and that the ω -word $a_{i_1}.a_{i_2} \dots a_{i_j} \dots$ is in R .

We are going now to describe informally a run ρ of \mathcal{T} over an infinite picture $p \in \Sigma^{\omega, \omega}$.

Each state of \mathcal{T} , i.e. each element of K , will consist of five components.

The first component of a state of \mathcal{T} is an integer $i_j \in \{1, 2, \dots, n\}$.

It will be used to guess that the ω -word $p_j = p(1, j).p(2, j).p(3, j) \dots$, forming the j^{th} row of the picture p , is in the regular ω -language R_{i_j} .

This first component will be constant on every row of the run ρ and will be propagated horizontally.

The second component is an element of $\cup_{1 \leq i \leq n} Q^i$.

If on the j^{th} row the first component of the state is equal to i_j then the second component on this row will be in Q^{i_j} . It is used to simulate (by horizontal propagation) a run α_j of the Büchi automaton \mathcal{A}^{i_j} on the ω -word p_j forming the j^{th} row of p .

So the projection of $\rho_j = \rho(1, j).\rho(2, j) \dots$ on the second component of states will be equal to α_j .

In order to check that, for all integers $j \geq 1$, the ω -word p_j is in the regular ω -language R_{i_j} , \mathcal{T} has to check that each run α_j is successful, i.e. that $\text{Inf}(\alpha_j) \cap F^{i_j} \neq \emptyset$, or equivalently that some state of F^{i_j} appears infinitely often in the second component (of the state) on the j^{th} row.

This can be done in the following way. One can imagine an ant which moves on the picture p , but only horizontally from the left to the right or vertically. The movement of the ant will be indicated by the third component of the state which will be an element of $\{B, a, a_d\}$.

Letters a, a_d will represent the trajectory of the ant and the blank symbol B will be used elsewhere. The letter a_d will be only used when the ant goes down vertically on the picture.

We shall need also the fourth component of the state of \mathcal{T} which will be an element of $\{B, \star, \star_1\}$.

The walk of the ant begins at the intersection of the first row and the first column of p , i.e. at the place of the letter $p(1, 1)$ of p .

At the beginning of this walk, the ant moves horizontally to the right on the first row (this way is marked by an a on the third component of the state) until it meets an element $q_1 \in F^{i_1}$ on the second component of the state.

There is also a mark \star on the first row which is propagated to the right following the movement of the ant.

If the ant meets an element $q_1 \in F^{i_1}$ on the second component of the state, then the mark \star is transferred on the second row just above it (on the same column) but with an indice 1, so it becomes \star_1 .

This mark \star_1 will be next forwarded horizontally to the right but without the indice 1.

The ant then goes down vertically until it reaches the first row. In that special beginning of its walk, it is already on the first row!

Next the ant moves again to the right on the first row, until it meets an element $q_2 \in F^{i_1}$ on the second component of the state. At that point it goes up on the second row (which is marked by \star on the fourth component) and moves to the right on this row until it meets an element $q_3 \in F^{i_2}$ on the second component of the state.

At that point the mark \star is transferred on the third row just above it (on the same column) but with an indice 1, so it becomes \star_1 . This mark \star_1 will be next forwarded horizontally to the right but without the indice 1.

The mark \star is now on the third row and it indicates that the ant will have to check successively the three first rows at next ascending moves.

The ant then goes down vertically until it reaches the first row. These movements will be indicated by the letter a_d on the third component of the state. Once on the first row its trajectory is again marked by the letter a . It moves to the right, looking for some state of F^{i_1} on the first row, next goes up, moves to the right, looking for some state of F^{i_2} on the second row, next goes up, moves to the right, again looking for some state of F^{i_3} on the third row.

This way it checks successively the first row, then the second row, and the third row

(marked with \star), looking each time for an element of F^{i_j} on the j^{th} row. When it meets an element $q_6 \in F^{i_3}$ on the second component on the third row, it transfers the mark \star (with an indice, so it becomes \star_1) just above it. This mark \star will be next forwarded horizontally to the right, without the indice 1.

The mark \star is now on the fourth row and it indicates that the ant will have to check successively the four first rows at next ascending moves.

The ant then goes down vertically until it reaches the first row and so on ...

We can see that if the mark \star_1 appears infinitely often, it appears one time on each row, and this means that the ant has successively checked the first row, then the two first rows, then the three first rows, ..., then the n first rows, ..., looking each time for an element of F^{i_j} on the j^{th} row.

This implies that, for a given j^{th} row, the ant has checked that some element of Q^{i_j} appears infinitely often on the second component of the state, hence the ω -word $p_j = p(1, j).p(2, j).p(3, j) \dots$ is in the regular ω -language R_{i_j} .

Conversely if for all integers $j \geq 1$ the ω -word p_j is in R_{i_j} , then there are some successful runs α_j of the Büchi automata \mathcal{A}^{i_j} on the ω -words p_j such that the above defined movements of the ant make the mark \star_1 to appear infinitely often.

Notice that the blank symbol B appears on the fourth component of the state whenever neither \star nor \star_1 is used as explained above.

\mathcal{T} has now to check that the integers $i_j, j \geq 1$, are such that the ω -word $a_{i_1}.a_{i_2} \dots a_{i_j} \dots$ is in R . The fifth component of states of K is used for that purpose. On the first column this fifth component is an element of Q and is used to simulate, by vertical propagation, a run α of \mathcal{A} on the ω -word $a_{i_1}.a_{i_2} \dots a_{i_j} \dots$.

This means that the projection of $\rho(1, 1).\rho(1, 2).\rho(1, 3) \dots$ on this fifth component will be equal to α .

On the other columns the fifth component will be simply the blank symbol B .

We have seen that the set of states of the tiling system \mathcal{T} will be:

$$K = \{1, 2, \dots, n\} \times \cup_{1 \leq i \leq n} Q^i \times \{B, a, a_d\} \times \{B, \star, \star_1\} \times (\{B\} \cup Q)$$

and one can define a set of tiles $\Delta^{\mathcal{T}}$ such that corresponding runs of the tiling system $\mathcal{T}=(K, \Sigma, \Delta^{\mathcal{T}})$ are described informally as above.

A run ρ will be successful if and only if the mark \star_1 appears infinitely often on the fourth component of $\rho(v)$ and some state $q \in F$ appears infinitely often on the fifth component of $\rho(v)$, for $v \in \omega^2$.

This acceptance condition may be written as a Muller condition. As stated in [1] any language of ω -pictures which is Muller recognizable by a tiling system is also Büchi recognizable by a tiling system. \square

Lemma 4.3 *There exists a Büchi-recognizable language of infinite pictures which is not accepted row by row by any ordinal Büchi automaton.*

Proof. The class of languages of infinite pictures which are Büchi-recognizable by tiling systems is not closed under complement, [1]. There exists a language $T \subseteq \Sigma^{\omega, \omega}$ of infinite pictures, (where Σ is a finite alphabet), which is Büchi-recognizable by a tiling system but such that its complement is not Büchi-recognizable by any tiling system.

Then \bar{T} cannot be a regular ω^2 -language. Indeed otherwise its complement would be also a regular ω^2 -language because the class of regular ω^2 -languages is closed under complement, [2, 3]. The preceding proof would imply that the complement of T would be also Büchi-recognizable by a tiling system, towards a contradiction. \square

Theorem 4.1 expresses that the class $BA(\Sigma^{\omega, \omega})$ is strictly included in the class $TS(\Sigma^{\omega, \omega})$. We shall see in the next section that one cannot decide whether a language $L \in TS(\Sigma^{\omega, \omega})$ is in $BA(\Sigma^{\omega, \omega})$.

We are going now to compare the topological complexity of languages in the classes $TS(\Sigma^{\omega, \omega})$ and $BA(\Sigma^{\omega, \omega})$.

From now on we shall assume that the reader is familiar with basic notions of topology and with the Borel and projective hierarchies on a space Σ^ω , (where Σ is a finite alphabet having at least two letters), equipped with the Cantor topology, see for example [17, 23, 19, 15].

We recall that a subset of Σ^ω is a Borel set of rank α , for a countable ordinal α , iff it is in $\Sigma_\alpha^0 \cup \Pi_\alpha^0$ but not in $\bigcup_{\gamma < \alpha} (\Sigma_\gamma^0 \cup \Pi_\gamma^0)$.

Recall also the notion of completeness with regard to reduction by continuous functions. For a countable ordinal $\alpha \geq 1$, a set $F \subseteq \Sigma^\omega$ is said to be a Σ_α^0 (respectively, Π_α^0, Σ_1^1)-complete set iff for any set $E \subseteq \Gamma^\omega$ (with Γ a finite alphabet): $E \in \Sigma_\alpha^0$ (respectively, $E \in \Pi_\alpha^0, E \in \Sigma_1^1$) iff there exists a continuous function $f : \Gamma^\omega \rightarrow \Sigma^\omega$ such that $E = f^{-1}(F)$.

For Γ a finite alphabet having at least two letters, the set $\Gamma^{\omega \times \omega}$ of functions from $\omega \times \omega$ into Γ is usually equipped with the product topology of the discrete topology on Γ . This topology may be defined by the following distance d . Let x and y in $\Gamma^{\omega \times \omega}$ such that $x \neq y$, then

$$d(x, y) = \frac{1}{2^n} \quad \text{where}$$

$$n = \min\{p \geq 0 \mid \exists(i, j) \ x(i, j) \neq y(i, j) \text{ and } i + j = p\}.$$

Then the topological space $\Gamma^{\omega \times \omega}$ is homeomorphic to the topological space Γ^ω , equipped with the Cantor topology. Borel subsets of $\Gamma^{\omega \times \omega}$ are defined from open subsets as in the case of the topological space Γ^ω . Analytic subsets of $\Gamma^{\omega \times \omega}$ are obtained as projections on $\Gamma^{\omega \times \omega}$ of Borel subsets of the product space $\Gamma^{\omega \times \omega} \times \Gamma^\omega$. The set $\Sigma^{\omega, \omega}$ of ω -pictures over Σ , viewed as a topological subspace of $\hat{\Sigma}^{\omega \times \omega}$, is easily seen to be homeomorphic to the topological space $\Sigma^{\omega \times \omega}$, via the mapping $\varphi : \Sigma^{\omega, \omega} \rightarrow \Sigma^{\omega \times \omega}$ defined by $\varphi(p)(i, j) = p(i + 1, j + 1)$ for all $p \in \Sigma^{\omega, \omega}$ and $i, j \in \omega$.

The topological complexity of languages of infinite pictures, accepted row by row by ordinal Büchi automata, is given by the following result which is stated in [9].

Proposition 4.4 ([9]) *Let $L \subseteq \Sigma^{\omega, \omega}$ be a language of infinite pictures which is accepted row by row by an ordinal Büchi automaton. Then L is a Borel set of rank smaller than or equal to 5.*

This result can be easily proved, using Proposition 3.5 and the fact that every regular ω -language $R \subseteq \Gamma^\omega$ is a boolean combination of arithmetical Π_2 -sets, hence a Δ_3 -set, so is definable in first order arithmetic by some first order Σ_3 -sentence and also by some first order Π_3 -sentence. One can then show that every regular ω^2 -language is defined in first order arithmetic by some first order Σ_5 -sentence hence is a Borel set of rank smaller than or equal to 5.

On the other side it has been proved in [1] that there exist some Σ_1^1 -complete, hence non Borel, Büchi recognizable language of ω -pictures. The two following lemmas will provide an alternative proof of this result and will be also useful to determine the Borel ranks of languages in $TS(\Sigma^{\omega, \omega})$.

For an ω -language $L \subseteq \Sigma^\omega$ we denote L^B the language of infinite pictures $p \in \Sigma^{\omega, \omega}$ such that the first row of p is in L and the other rows are labelled with the letter B which is assumed to belong to Σ .

Lemma 4.5 *If $L \subseteq \Sigma^\omega$ is accepted by some Turing machine with a Büchi acceptance condition, then L^B is Büchi recognizable by a finite tiling system.*

Proof. Let $L \subseteq \Sigma^\omega$ be an ω -language accepted by some Turing machine T with a Büchi acceptance condition.

We assume that the Turing machine has a single semi-infinite tape, with one reading head which may also write on the tape. Q is the set of states of T , q_0 is the initial state and $F \subseteq Q$ is the set of accepting states. The input alphabet of T is Σ and its working alphabet is $\Gamma \supseteq \Sigma$.

It has been proved by Cohen and Gold that one can consider only such restricted model of Turing machine [8].

An instantaneous configuration of T is given by an infinite word $u.q.v$ where $u \in \Gamma^*$, $q \in Q$, $v \in \Gamma^\omega$, and the first letter of v is the one scanned by the head of T .

The initial configuration of T reading the infinite word $\sigma \in \Sigma^\omega$ is $q_0.\sigma$.

A computation of T reading $\sigma \in \Sigma^\omega$ is an infinite sequence of configurations $\alpha_0, \alpha_1, \alpha_2, \dots, \alpha_i, \dots$, where $\alpha_0 = q_0.\sigma$ is the initial configuration and for all integers $i \geq 0$, $\alpha_i = u_i.q_i.v_i$ is the $(i+1)^{th}$ configuration.

The computation is successful if and only if there exists a final state $q_f \in F$ and infinitely many integers i such that $q_i = q_f$.

We can now use a similar reasoning as in the classical proof of the undecidability of the emptiness problem for recognizable languages of finite pictures, [13, p. 34].

We can define a set of tiles Δ in such a way that for $\sigma \in \Sigma^\omega$, a run ρ of the tiling

system $\mathcal{T}=(\Sigma, \Gamma \cup Q, \Delta, F)$ over the infinite picture σ^B satisfies:

$$\text{for each integer } i \geq 0 \quad \rho(0, i) \cdot \rho(1, i) \cdot \rho(2, i) \dots = \alpha_i = u_i \cdot q_i \cdot v_i$$

i.e. $\rho(0, i) \cdot \rho(1, i) \cdot \rho(2, i) \dots$ is the $(i + 1)^{th}$ configuration of T reading the ω -word $\sigma \in \Sigma^\omega$.

Thus the Büchi tiling system (\mathcal{T}, F) recognizes the language L^B . \square

The following lemma is easy to prove. Details are left to the reader.

Lemma 4.6 *Let α be a countable ordinal ≥ 2 . If $L \subseteq \Sigma^\omega$ is Σ_α^0 -complete (respectively, Π_α^0 -complete, Σ_1^1 -complete), then L^B is Σ_α^0 -complete (respectively, Π_α^0 -complete, Σ_1^1 -complete).*

In particular, for each alphabet Σ having at least two letters, we get some Σ_1^1 -complete language of ω -pictures in the form L^B because it is well known that there exist some Σ_1^1 -complete ω -languages $L \subseteq \Sigma^\omega$ accepted by some Büchi (or Muller) Turing machine.

Notice that the Σ_1^1 -complete Büchi recognizable language $T_2 \subseteq \{0, 1, \$\}^{\omega, \omega}$ of all ω -pictures encoding an ω -tree with an infinite path given in [1] is also in that form.

To determine the ranks of Borel languages of ω -pictures we shall need to consider the first non-recursive ordinal which is called the Church Kleene ordinal and is usually denoted by ω_1^{CK} [18].

Proposition 4.7 *Let Σ be a finite alphabet having at least two letters.*

- (a) *If $L \subseteq \Sigma^{\omega, \omega}$ is Büchi recognizable by a finite tiling system and is a Borel set of rank α , then α is smaller than ω_1^{CK} .*
- (b) *For every non null countable ordinal $\alpha < \omega_1^{CK}$, there exists some language of infinite pictures $L \subseteq \Sigma^{\omega, \omega}$ which is Büchi recognizable by a finite tiling system and is a Borel set of rank α .*

Proof.

(a). It was proved in [1] that every language L of infinite pictures which is Büchi recognizable by a finite tiling system is definable by an existential second order formula of arithmetic. It is well known that this implies that L is a Σ_1^1 -set (lightface) and that if moreover L is a Borel set then its Borel rank is smaller than ω_1^{CK} , see [18].

(b). For $\alpha = 1$ it is well known that a Σ_1^0 -complete set is simply an open but non closed set and that a Π_1^0 -complete set is simply a closed but non open set. For example $O = \{p \in \Sigma^{\omega, \omega} \mid \exists i \geq 1, \exists j \geq 1 \ p(i, j) = B\}$ is a Σ_1^0 -complete subset of $\Sigma^{\omega, \omega}$, and $C = \{p \in \Sigma^{\omega, \omega} \mid \forall i \geq 1, \forall j \geq 1 \ p(i, j) = B\}$ is a Π_1^0 -complete subset of $\Sigma^{\omega, \omega}$. These two languages are in $TS(\Sigma^{\omega, \omega})$.

On the other hand it is well known that, for every non null countable ordinal $\alpha < \omega_1^{CK}$, there exists some Σ_α^0 -complete S_α and some Π_α^0 -complete P_α , subsets of Σ^ω , which are effective, i.e. which are in the class of Σ_1^1 (lightface) subsets of Σ^ω accepted by some Turing machine with a Büchi acceptance condition, [18, 23].

Then by Lemma 4.6 the language $(S_\alpha)^B \subseteq \Sigma^{\omega,\omega}$ (respectively, $(P_\alpha)^B \subseteq \Sigma^{\omega,\omega}$) is Σ_α^0 -complete (respectively, Π_α^0 -complete) and by Lemma 4.5 these languages are in $TS(\Sigma^{\omega,\omega})$. \square

In conclusion of this section, these results show that Büchi tiling systems have a much greater accepting power than automata reading ω^2 -words for acceptance of languages of infinite pictures.

5. Decision Problems

In a recent paper we have proved several undecidability results for infinitary rational relations [12]. These results were deduced from an extreme separation result, proved using the undecidability of the universality problem for finitary rational relations and the existence of a Σ_1^1 -complete infinitary rational relation stated in another paper [11].

We shall use in this section a very similar technique, using this time the undecidability of the emptiness problem for languages of finite pictures and the existence of a Σ_1^1 -complete language of ω -pictures. In a similar way we shall see that this implies several undecidability results.

Proposition 5.1 *Let $\Gamma = \{0, 1, \#\}$, then there exists a family \mathcal{F} of Büchi-recognizable languages of ω -pictures over Γ , such that, for $L \in \mathcal{F}$, either $L = \emptyset$ or L is a Σ_1^1 -complete subset of $\Gamma^{\omega,\omega}$, but one cannot decide which case holds.*

Proof. We have seen that there exists a Büchi-recognizable language $T \subseteq \{0, 1\}^{\omega,\omega}$ which is Σ_1^1 -complete.

On the other side the emptiness problem for recognizable languages of finite pictures is known to be undecidable: if Σ is an alphabet having at least one letter then it is undecidable whether a given recognizable language $L \subseteq \Sigma^{*,*}$ is empty, see [13].

Let us define, for a finite picture $p \in \Sigma^{*,*}$ over a finite alphabet Σ and an infinite picture $p' \in \Sigma_1^{\omega,\omega}$ over the alphabet $\Sigma_1 = \{0, 1\}$, the infinite picture $p \bullet p'$ over the alphabet $\Gamma = \Sigma \cup \Sigma_1 \cup \{\#\}$. We assume that $\hat{\Sigma} = \Sigma \cup \{\#\}$, $\hat{\Sigma}_1 = \Sigma_1 \cup \{\#\}$, and $\hat{\Gamma} = \Gamma \cup \{\#\}_1$, where $\#\}_1$ is a new letter different from the letter $\#$.

If p is a finite picture of size (m, n) , the ω -picture $p \bullet p'$ over Γ is defined by:

$$\begin{aligned} p \bullet p'(0, i) &= p \bullet p'(i, 0) = \#\}_1 \text{ for all integers } i \geq 0, \\ p \bullet p'(i, j) &= p(i-1, j-1) \text{ for all integers } i \in \{1, \dots, m+2\} \text{ and } j \in \{1, \dots, n+2\}, \\ p \bullet p'(i, j) &= \# \text{ for all integers } i \in \{1, \dots, m+2\} \text{ and } j \geq n+2, \\ p \bullet p'(i, j) &= \# \text{ for all integers } i \geq m+2 \text{ and } j \in \{1, \dots, n+2\}, \\ p \bullet p'(i, j) &= p'(i-(m+2), j-(n+2)) \text{ for all integers } i \geq m+2 \text{ and } j \geq n+2. \end{aligned}$$

The intuitive idea is to construct an infinite picture $p \bullet p'$ having a prefix p “followed” by the infinite picture p' , to “complete” elsewhere by some letters $\#$ and then to border with letters $\#\}_1$ to get an ω -picture in $\Gamma^{\omega,\omega}$.

For a language $L \subseteq \Sigma^{*,*}$ we set $L \bullet T = \{p \bullet p' \mid p \in L \text{ and } p' \in T\}$. It is easy to see that if L is a recognizable language of finite pictures then $L \bullet T$ is a Büchi-

recognizable language of ω -pictures because T is also Büchi-recognizable. There are now two cases:

(1) If L is empty then $L \bullet T$ is empty too.

(2) If L is non-empty there is some finite picture $p \in L \subseteq \Sigma^{*,*}$. Let then ψ_p be the mapping from $\Sigma_1^{\omega,\omega}$ into $(\hat{\Sigma} \cup \Sigma_1)^{\omega,\omega}$ defined by $\psi_p(p') = p \bullet p'$.

It is easy to see that the mapping ψ_p is continuous and that $\psi_p^{-1}(L \bullet T) = T$. But T is Σ_1^1 -complete and $L \bullet T$, as well as every Büchi-recognizable language of infinite pictures, is a Σ_1^1 -set because it is definable by an existential second order monadic formula, [1]. This implies that $L \bullet T$ is a Σ_1^1 -complete set.

We can now choose the family \mathcal{F} to be the family of languages $L \bullet T$ obtained with $\Sigma = \{0\}$ and L running over recognizable languages of pictures over Σ . \square

In order to disprove the existence of decision procedures which test Büchi-recognizable ω -picture languages for E-, respectively A-recognizability, we shall need the following lemmas.

Lemma 5.2 *Let Σ be an alphabet having at least two letters and $L \subseteq \Sigma^{\omega,\omega}$ be a E-recognized language of ω -pictures. Then L is a Σ_2^0 -subset of $\Sigma^{\omega,\omega}$.*

Proof. Let Σ be an alphabet having at least two letters and $L \subseteq \Sigma^{\omega,\omega}$ be a language of ω -pictures which is E-recognized by (\mathcal{A}, F) , where $\mathcal{A} = (Q, \Sigma, \Delta)$ is a tiling system and $F \subseteq Q$.

Let $R = \{(p, \rho) \in \Sigma^{\omega,\omega} \times Q^{\omega^2} \mid \rho \text{ is a run of } \mathcal{A} \text{ on } p\}$. It is easy to see that R is a closed subset of $\Sigma^{\omega,\omega} \times Q^{\omega^2}$ where $\Sigma^{\omega,\omega} \times Q^{\omega^2}$ is equipped with the classical product topology.

Let $R_E = \{(p, \rho) \in \Sigma^{\omega,\omega} \times Q^{\omega^2} \mid \exists v \in \omega^2 \rho(v) \in F\}$. It is easy to see that R_E is an open subset of $\Sigma^{\omega,\omega} \times Q^{\omega^2}$.

Then the set $R \cap R_E$ is a boolean combination of open sets. In particular it is a Σ_2^0 -subset of $\Sigma^{\omega,\omega} \times Q^{\omega^2}$, i.e. a countable union of closed subsets of $\Sigma^{\omega,\omega} \times Q^{\omega^2}$. But the topological space $\Sigma^{\omega,\omega} \times Q^{\omega^2}$ is compact because it is the product of two compact spaces. Thus every closed subset of $\Sigma^{\omega,\omega} \times Q^{\omega^2}$ is also compact. Therefore $R \cap R_E$ is a countable union of compact subsets of $\Sigma^{\omega,\omega} \times Q^{\omega^2}$.

But the language L is E-recognized by (\mathcal{A}, F) so it is the projection of the set $R \cap R_E$ onto $\Sigma^{\omega,\omega}$. The projection from $\Sigma^{\omega,\omega} \times Q^{\omega^2}$ onto $\Sigma^{\omega,\omega}$ is continuous and the continuous image of a compact set is a compact set. Thus the language L is a countable union of compact sets hence it is a countable union of closed sets, i.e. a Σ_2^0 -subset of $\Sigma^{\omega,\omega}$. \square

Lemma 5.3 *Let Σ be an alphabet having at least two letters and $L \subseteq \Sigma^{\omega,\omega}$ be a A-recognized language of ω -pictures. Then L is a closed subset of $\Sigma^{\omega,\omega}$.*

Proof. Let Σ be an alphabet having at least two letters and $L \subseteq \Sigma^{\omega, \omega}$ be a language of ω -pictures which is A-recognized by (\mathcal{A}, F) , where $\mathcal{A} = (Q, \Sigma, \Delta)$ is a tiling system and $F \subseteq Q$.

We call $cl(L)$ the topological closure of L and we are going to prove that $L = cl(L)$. For that purpose consider an ω -picture p in $cl(L)$. For all integers $i \geq 1$ there is some ω -picture $p_i \in L$ such that $p \upharpoonright_{\{0,1,\dots,i\} \times \{0,1,\dots,i\}} = p_i \upharpoonright_{\{0,1,\dots,i\} \times \{0,1,\dots,i\}}$

For each integer $i \geq 1$, $p_i \in L$ thus there is some run ρ_i of \mathcal{A} on p_i such that for all $v \in \omega^2$ $\rho_i(v) \in F$.

Consider now the partial runs $\rho'_{i,j} = \rho_i \upharpoonright_{\{0,1,\dots,j\} \times \{0,1,\dots,j\}}$, for $j \leq i$, of \mathcal{A} on the restriction of p_i (hence also of p) to $\{0, 1, \dots, j\} \times \{0, 1, \dots, j\}$.

We can now reason as in the proof of Theorem 4 (a) in [1]. These partial runs $\rho'_{i,j}$ are arranged in a finitely branching tree, via the extension relation. By construction this tree is infinite so by König's Lemma there is an infinite path. This infinite path determines a run of \mathcal{A} on p which is A-accepting thus $p \in L$.

We have then proved that $cl(L) \subseteq L$ so $L = cl(L)$ and L is a closed subset of $\Sigma^{\omega, \omega}$. \square

We can now infer the following result.

Proposition 5.4 *There are no decision procedures which test Büchi-recognizable ω -picture languages for E-, respectively A-recognizability.*

Proof. Consider the family \mathcal{F} of ω -picture Büchi-recognizable languages over Γ , such that, for $L \in \mathcal{F}$, either $L = \emptyset$ or L is a Σ_1^1 -complete subset of $\Gamma^{\omega, \omega}$.

In the first case, L is obviously A-recognizable and E-recognizable. In the second case L is Σ_1^1 -complete so in particular it is not a Borel subset of $\Gamma^{\omega, \omega}$. By Lemmas 5.2 and 5.3 it cannot be E-recognizable (respectively, A-recognizable). But one cannot decide which case holds. \square

As remarked in [1] Staiger-Wagner and co-Büchi recognizability reduces to E-recognizability so the above proof can be applied to Staiger-Wagner and co-Büchi recognizability instead of E-recognizability.

Proposition 5.1 gives an extreme separation result which implies other undecidability results. For example for any Borel class Σ_α^0 or Π_α^0 , α being a countable ordinal ≥ 1 , it is undecidable whether a given Büchi-recognizable language of ω -pictures is in Σ_α^0 (respectively Π_α^0). It is even undecidable whether a given Büchi-recognizable language of ω -pictures is a Borel set or a Σ_1^1 -complete set.

Remark that the same result holds if we replace Borel classes by arithmetical classes Σ_i or Π_i , $i \geq 1$, and the class of Borel sets by the class of arithmetical sets $\cup_{n \geq 1} \Sigma_n = \cup_{n \geq 1} \Pi_n$.

These results show a great contrast with the case of recognizable languages of infinite words where such problems are decidable [16].

Recall now the following definition, see [1]: a tiling system is called deterministic if on any picture it allows at most one tile covering the origin, the state assigned to position $(i + 1, j + 1)$ is uniquely determined by the states at positions (i, j) , $(i + 1, j)$, $(i, j + 1)$ and the states at the border positions $(0, j + 1)$ and $(i + 1, 0)$ are determined by the state $(0, j)$, respectively $(i, 0)$.

As remarked in [1], the hierarchy proofs of the classical Landweber hierarchy defined using deterministic ω -automata “carry over without essential changes to pictures”. In particular it is easy to see that a language of ω -pictures which is Büchi-recognized by a deterministic tiling system is a $\mathbf{\Pi}_2^0$ -set.

Remark that if we use the classical Muller acceptance condition instead of the Büchi condition, we can easily show, as in the case of infinite words, that a language of ω -pictures which is Muller-recognized by a deterministic tiling system is a boolean combination of $\mathbf{\Pi}_2^0$ -sets.

We now state the following results.

Proposition 5.5 *Let $\Gamma = \{0, 1, \#\}$ as in Proposition 5.1. It is undecidable for a given Büchi-recognizable language $L \subseteq \Gamma^{\omega, \omega}$ whether:*

- (1) *L is Büchi-recognized by a deterministic tiling system.*
- (2) *L is Muller-recognized by a deterministic tiling system.*
- (3) *its complement $\Gamma^{\omega, \omega} - L$ is Büchi-recognizable.*
- (4) *\bar{L} is ω^2 -regular.*

Proof. Consider the family \mathcal{F} of Büchi-recognizable ω -picture languages given by Proposition 5.1. Then two cases may happen for $L \in \mathcal{F}$: either L is empty or L is $\mathbf{\Sigma}_1^1$ -complete.

In the first case L is obviously recognized by a deterministic Büchi or Muller tiling system; its complement $\Gamma^{\omega, \omega} - L = \Gamma^{\omega, \omega}$ is Büchi-recognizable and \bar{L} is ω^2 -regular. In the second case L is $\mathbf{\Sigma}_1^1$ -complete. Thus L is not a Borel set hence it is neither Büchi nor Muller-recognized by any deterministic tiling system and \bar{L} is not ω^2 -regular. Moreover in this second case its complement $\Gamma^{\omega, \omega} - L$ is a $\mathbf{\Pi}_1^1$ -complete subset of $\Gamma^{\omega, \omega}$. It is well known that a $\mathbf{\Pi}_1^1$ -complete set is not a $\mathbf{\Sigma}_1^1$ -set thus it cannot be Büchi-recognizable.

But Proposition 5.1 states that one cannot decide which case holds. \square

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**ANNEXE :
ERRATUM TO THE PAPER :**

**ON RECOGNIZABLE LANGUAGES
OF INFINITE PICTURES**

Recall first that the following result was stated as Proposition 4.7 in [Fin04].

Proposition 4.7 *Let Σ be a finite alphabet having at least two letters.*

- (a) *If $L \subseteq \Sigma^{\omega, \omega}$ is Büchi recognizable by a finite tiling system and is a Borel set of rank α , then α is smaller than ω_1^{CK} .*
- (b) *For every non null countable ordinal $\alpha < \omega_1^{\text{CK}}$, there exists some language of infinite pictures $L \subseteq \Sigma^{\omega, \omega}$ which is Büchi recognizable by a finite tiling system and is a Borel set of rank α .*

Item (a) of this result was deduced from the fact that if L is a (lightface) Σ_1^1 -set and that if moreover L is a Borel set then its Borel rank is smaller than ω_1^{CK} . This fact, which is true if we replace the (lightface) class Σ_1^1 by the (lightface) class Δ_1^1 , is actually not true and the given reference [Mos80] does not contain this result.

Kechris, Marker and Sami proved in [KMS89] that the supremum of the set of Borel ranks of (lightface) Π_1^1 , so also of (lightface) Σ_1^1 , sets is the ordinal γ_2^1 . This ordinal is precisely defined in [KMS89]. Kechris, Marker and Sami proved that the ordinal γ_2^1 is strictly greater than the ordinal δ_2^1 which is the first non Δ_2^1 ordinal. Thus in particular it holds that $\omega_1^{\text{CK}} < \gamma_2^1$. Notice that the exact value of the ordinal γ_2^1 may depend on axioms of set theory. For more details, the reader is referred to [KMS89] and to a textbook of set theory like [Jec02].

Notice that it seems still unknown whether *every* non null ordinal $\gamma < \gamma_2^1$ is the Borel rank of a (lightface) Π_1^1 (or Σ_1^1) set. On the other hand, for every non null ordinal $\alpha < \omega_1^{\text{CK}}$, there exist some Σ_α^0 -complete and some Π_α^0 -complete sets in the class $\Delta_1^1 \subset \Sigma_1^1$. This is a well known fact of Effective Descriptive Set Theory which is proved in detail in [FL07].

We can now state the following result which corrects the above false Proposition 4.7.

Theorem

- (a) *The Borel hierarchy of the class \mathcal{C} of Büchi recognizable language of infinite pictures is equal to the Borel hierarchy of the class Σ_1^1 .*
- (b) $\gamma_2^1 = \text{Sup } \{ \alpha \mid \exists L \in \mathcal{C} \text{ such that } L \text{ is a Borel set of rank } \alpha \}$.

- (c) For every non null ordinal $\alpha < \omega_1^{\text{CK}}$, there exists some Σ_α^0 -complete and some Π_α^0 -complete ω -languages in the class \mathcal{C} .

This result follows easily from the proof of Lemmas 4.5 and 4.6 of [Fin04] and from the above cited result of Kechris, Marker and Sami proved in [KMS89].

Notice that a very similar result was obtained in [Fin06] for the class of ω -languages accepted by (real time) one counter Büchi automata, and in [Fin08] for the class of infinitary rational relations accepted by 2-tape Büchi automata.

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