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# $C^k$ -SMOOTH APPROXIMATIONS OF LUR NORMS

PETR HÁJEK AND ANTONÍN PROCHÁZKA

ABSTRACT. Let  $X$  be a WCG Banach space admitting a  $C^k$ -Fréchet smooth norm. Then  $X$  admits an equivalent norm which is simultaneously  $C^1$ -Fréchet smooth, LUR, and a uniform limit of  $C^k$ -Fréchet smooth norms. If  $X = C([0, \alpha])$ , where  $\alpha$  is an ordinal, then the same conclusion holds true with  $k = \infty$ .

## 1. INTRODUCTION

The theory of  $C^k$ -Fréchet smooth approximations of continuous functions on Banach spaces is well-developed, thanks to the work of many mathematicians, whose classical results and references can be found in the authoritative monograph [2]. The known techniques rely on the use of  $C^k$ -Fréchet smooth partitions of unity, resp. certain coordinatewise smooth embeddings into the space  $c_0(\Gamma)$  (due to Toruńczyk [23]). They are highly nonlinear, and even non-Lipschitz in nature. For example, if the given function is Lipschitz or has some uniform continuity, trying to preserve the Lipschitzness of the approximating smooth functions leads to considerable additional technical difficulties (e.g. [10], [11], [12], [13], [18], [19]).

It is well-known that the (apparently harder, and less developed) parallel theory of approximations of norms on a Banach space by  $C^k$ -Fréchet smooth renormings requires different techniques.

Several open problems proposed in [2] are addressing these issues. In particular, if a Banach space admits an equivalent  $C^k$ -Fréchet smooth renorming, is it possible to approximate (uniformly on bounded sets) all norms by  $C^k$ -Fréchet smooth norms? Even in the separable case, the answer is not known in full generality, although the positive results in [3] and [4] are quite strong, and apply to most classical Banach spaces. In the nonseparable setting, no general results are known, with a small exception of [7]. In particular, one of the open problems in [2] is whether on a given WCG Banach space with an equivalent  $C^k$ -Fréchet smooth norm, there exists an equivalent locally uniformly rotund (LUR) norm which is a uniform limit on bounded sets of  $C^k$ -Fréchet smooth norms. The notion of LUR is of fundamental importance for renorming theory, and we refer to [2] and the more recent [20] for an extensive list of authors and results.

Such a result is of interest for several reasons. It can be used to obtain rather directly the uniform approximations of general continuous operators, by  $C^k$ -Fréchet smooth ones. Moreover, since LUR norms form a residual set in the metric space of all equivalent norms on a Banach space, a positive answer is to be expected. There is a closely related problem of obtaining a norm which shares simultaneously good rotundity and smoothness properties. By a famous result of Asplund [1], on every separable Asplund space there exists an equivalent norm which is simultaneously  $C^1$ -Fréchet smooth and LUR. A clever proof using Baire category, and disposing of the separability condition on the underlying Banach space, was devised in [6] ([2], II.4.3). The theorem holds in particular in all WCG Asplund spaces (in particular all reflexive spaces). Its proof works under the assumption that the space admits an LUR norm, as well as a norm whose dual is LUR. It is well-known that dual LUR implies that the original norm is  $C^1$ -Fréchet smooth. However, using this approach one cannot in general handle norms with higher degree

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of differentiability, even in the separable case. Indeed, by [9] ([2], Proposition V.1.3), a space admitting a LUR and simultaneously  $C^2$ -Fréchet smooth norm is superreflexive. There is not even a rotund and  $C^2$ -Fréchet smooth norm on  $c_0(\Gamma)$  ([14], [15]). In fact, one cannot even handle the proper case of LUR and  $C^1$ -Fréchet smooth norms. Indeed, Talagrand [22] proved that  $C([0, \omega_1])$  admits an equivalent  $C^\infty$ -Fréchet smooth norm, although it admits no dual LUR renorming. The existence of LUR renorming of this space follows from Troyanski's theorem [24]. In light of the previous results it is natural to ask whether this space has a  $C^1$ -Fréchet smooth and simultaneously LUR renorming. This question was posed on various occasions, e.g. in [8].

Our main result addresses both of the above mentioned open problems, namely higher smoothness approximation and simultaneous LUR and  $C^1$ -smoothness. Under reasonable assumptions (e.g. for WLD,  $C(K)$  where  $K$  is Valdivia compact, or  $C([0, \alpha])$ , i.e. the space of continuous functions on the ordinal interval  $[0, \alpha]$ ), it gives a renorming which is simultaneously  $C^1$ -Fréchet smooth and LUR, and admits a uniform approximation on bounded sets by  $C^k$ -Fréchet norms. As a corollary we obtain a positive solution to both of the mentioned problems. We should emphasize that it is unknown whether  $C^1$ -Fréchet smooth norms are residual, or even dense, in the space of all equivalent norms on  $C([0, \alpha])$ .

The paper is organized as follows. In Section 2, we introduce our notation and we present some auxiliary lemmata. We include the easy proofs for reader's convenience. The main result, its corollaries and the frame of the proof of the main result are gathered in Section 3. Sections 4 and 5 then contain the details of the construction.

## 2. PRELIMINARIES

The closed unit ball of a Banach space  $(X, \|\cdot\|)$  is denoted by  $B_{(X, \|\cdot\|)}$ , or  $B_X$  for short. Similarly, the open unit ball of  $X$  is  $B_{(X, \|\cdot\|)}^O = B_X^O$ . By  $\Gamma$  we denote an index set. Smoothness and higher smoothness is meant in the Fréchet sense.

**Definition 2.1.** Let  $A \subset \ell^\infty(\Gamma)$ . We say that a function  $f : \ell^\infty(\Gamma) \rightarrow \mathbb{R}$  in  $A$  *locally depends on finitely many coordinates* (LFC) if for each  $x \in A$  there exists a neighborhood  $U$  of  $x$ , a finite  $M = \{\gamma_1, \dots, \gamma_n\} \subset \Gamma$  and a function  $g : \mathbb{R}^{|M|} \rightarrow \mathbb{R}$  such that  $f(y) = g(y(\gamma_1), \dots, y(\gamma_n))$  for each  $y \in U$ .

**Definition 2.2.** Let  $X$  be a vector space. A function  $g : X \rightarrow \ell^\infty(\Gamma)$  is said to be *coordinatewise convex* if, for each  $\gamma \in \Gamma$ , the function  $x \mapsto g_\gamma(x)$  is convex. We use the terms as *coordinatewise non-negative* or *coordinatewise  $C^k$ -smooth* in a similar way.

**Lemma 2.3.** Let  $X$  be a Banach space and let  $h : X \rightarrow \ell^\infty(\Gamma)$  be a continuous function which is coordinatewise  $C^k$ -smooth,  $k \in \mathbb{N} \cup \{\infty\}$ . Let  $f : \ell^\infty(\Gamma) \rightarrow \mathbb{R}$  be a  $C^k$ -smooth function which locally depends on finitely many coordinates. Then  $f \circ h$  is  $C^k$ -smooth.

*Proof.* Let  $x \in X$  be fixed. Since  $f$  is LFC, there is a neighborhood  $U$  of  $h(x)$ ,  $M = \{\gamma_1, \dots, \gamma_n\} \subset \Gamma$  and  $g : \mathbb{R}^{|M|} \rightarrow \mathbb{R}$  as in Definition 2.1. The function  $g$  is  $C^k$ -smooth, because  $f$  is  $C^k$ -smooth. As  $h$  is continuous, there exists a neighborhood  $V$  of  $x$  such that  $h(V) \subset U$ . Since  $h$  is coordinatewise  $C^k$ -smooth, it follows that  $h(\cdot) \upharpoonright_M := (h(\cdot)(\gamma_1), \dots, h(\cdot)(\gamma_n))$  is  $C^k$ -smooth from  $X$  to  $\mathbb{R}^{|M|}$ . Finally, we have for each  $y \in V$  that  $f(h(y)) = g(h(y) \upharpoonright_M)$  and the claim follows.  $\square$

**Lemma 2.4.** Let  $\Phi : \ell^\infty(\Gamma) \rightarrow \mathbb{R}$  and let  $x \in \ell^\infty(\Gamma)$  be such that

- a)  $\Phi$  is LFC at  $x$ ,
- b)  $\Phi'(x)x \neq 0$ ,
- c)  $\Phi(\cdot)$  and  $\Phi'(\cdot)$  are continuous at  $x$ .

Then there is a neighborhood  $U$  of  $x$  and a unique function  $F : U \rightarrow \mathbb{R}$  which is continuous at  $x$  and satisfies  $F(x) = 1$  and  $\Phi(\frac{y}{F(y)}) = 1$  for all  $y \in U$ . Moreover  $F$  is LFC at  $x$ .

*Proof.* The first part of the assertion follows immediately from the Implicit Function Theorem. We will show that  $F$  is LFC at  $x$ . From the assumption a) we know that there is a neighborhood  $V$  of

$x$ ,  $M = \{\gamma_1, \dots, \gamma_n\} \subset \Gamma$ , and  $g : \mathbb{R}^n \rightarrow \mathbb{R}$  such that  $\Phi(y) = g(y \upharpoonright_M)$  for all  $y \in V$ . It is obvious that  $g'(x \upharpoonright_M)x \upharpoonright_M = \Phi'(x)x$  so it is possible to apply the Implicit Function Theorem to the equation  $g\left(\frac{y}{h(y)}\right) = 1$  to get  $h : V' \rightarrow \mathbb{R}$ , where  $V'$  is a neighborhood of  $x \upharpoonright_M$ , such that  $h(x \upharpoonright_M) = 1$  and  $h$  is continuous at  $x \upharpoonright_M$ . There is a neighborhood  $U' \subset U \cap V$  of  $x$  such that we may define  $H : U' \rightarrow \mathbb{R}$  by  $H(y) := h(y \upharpoonright_M)$  for  $y \in U'$ . Then  $H(x) = 1$  and  $H$  is continuous at  $x$ . Also,  $\Phi\left(\frac{y}{H(y)}\right) = g\left(\frac{y \upharpoonright_M}{h(y \upharpoonright_M)}\right) = 1$ . The uniqueness of  $F$  implies that  $F = H$  on  $U'$  so  $F$  is LFC at  $x$ .  $\square$

The following lemma is a variant of Fact II.2.3(i) in [2].

**Lemma 2.5.** *Let  $\varphi : X \rightarrow \mathbb{R}$  be a convex non-negative function,  $x_r, x \in X$  for  $r \in \mathbb{N}$ . Then the following conditions are equivalent:*

- (i)  $\frac{\varphi^2(x_r) + \varphi^2(x)}{2} - \varphi^2\left(\frac{x+x_r}{2}\right) \rightarrow 0$ ,
- (ii)  $\lim \varphi(x_r) = \lim \varphi\left(\frac{x+x_r}{2}\right) = \varphi(x)$ .

If  $\varphi$  is homogeneous, the above conditions are also equivalent to

- (iii)  $2\varphi^2(x_r) + 2\varphi^2(x) - \varphi^2(x+x_r) \rightarrow 0$ .

*Proof.* Since  $\varphi$  is convex and non-negative, and  $y \mapsto y^2$  is increasing for  $y \in [0, +\infty)$ , it holds

$$\frac{\varphi^2(x_r) + \varphi^2(x)}{2} - \varphi^2\left(\frac{x+x_r}{2}\right) \geq \frac{\varphi^2(x_r) + \varphi^2(x)}{2} - \left(\frac{\varphi(x) + \varphi(x_r)}{2}\right)^2 = \left(\frac{\varphi(x) - \varphi(x_r)}{2}\right)^2$$

which proves (i)  $\Rightarrow$  (ii). The implication (ii)  $\Rightarrow$  (i) is trivial and so is the equivalence (i)  $\Leftrightarrow$  (iii).  $\square$

**Lemma 2.6.** *Let  $f, g$  be twice differentiable, convex, non-negative, real functions of one real variable. Let  $F : \mathbb{R}^2 \rightarrow \mathbb{R}$  be given as  $F(x, y) := f(x)g(y)$ . For  $F$  to be convex in  $\mathbb{R}^2$ , it is sufficient that  $g$  is convex and*

$$(f'(x))^2(g'(y))^2 \leq f''(x)f(x)g''(y)g(y). \quad (1)$$

for all  $(x, y) \in \mathbb{R}^2$ .

*Proof.* Let  $(x, y) \in \mathbb{R}^2$  be fixed. Since  $g$  is convex, the function  $F$  is convex when restricted to the vertical line going through  $(x, y)$ . Let  $s = at + b$  ( $a, b \in \mathbb{R}$ ) be a line going through  $(x, y)$ , i.e.  $y = ax + b$ . The second derivative at a point  $(x, y)$  of  $F$  restricted to this line is given as:

$$f(x)g''(y)a^2 + 2f'(x)g'(y)a + f''(x)g(y).$$

In order for the second derivative to be non-negative for all  $a \in \mathbb{R}$ , it is sufficient that the discriminant  $(2f'(x)g'(y))^2 - 4f(x)g''(y)f''(x)g(y)$  of the above quadratic term be non-positive, which occurs exactly when our condition (1) holds for  $(x, y)$ .  $\square$

**Definition 2.7.** We say that a function  $f : \ell^\infty(\Gamma) \rightarrow \mathbb{R}$  is *strongly lattice* if  $f(x) \leq f(y)$  whenever  $|x(\gamma)| \leq |y(\gamma)|$  for all  $\gamma \in \Gamma$ .

**Lemma 2.8.** *Let  $f : \ell^\infty(\Gamma) \rightarrow \mathbb{R}$  be convex and strongly lattice. Let  $g : X \rightarrow \ell^\infty(\Gamma)$  be coordinatewise convex and coordinatewise non-negative. Then  $f \circ g : X \rightarrow \mathbb{R}$  is convex.*

*Proof.* Let  $a, b \geq 0$  and  $a + b = 1$ . Since  $g$  is coordinatewise convex and non-negative, we have

$$0 \leq g_\gamma(ax + by) \leq ag_\gamma(x) + bg_\gamma(y)$$

for each  $\gamma \in \Gamma$ . The strongly lattice property and the convexity of  $f$  yield

$$f(g(ax + by)) \leq f(ag(x) + bg(y)) \leq af(g(x)) + bf(g(y))$$

so  $f \circ g$  is convex.  $\square$

**Definition 2.9.** Let us define  $[\cdot] : \ell^\infty(\Gamma) \rightarrow \mathbb{R}$  by  $[x] = \inf \{t; \{\gamma; |x(\gamma)| > t\} \text{ is finite}\}$ . Then  $[\cdot]$  is 1-Lipschitz, strongly lattice seminorm on  $(\ell^\infty(\Gamma), \|\cdot\|_\infty)$ .

*Proof.* In fact  $\lceil x \rceil = \|q(x)\|_{\ell^\infty/c_0}$ , where  $q : \ell^\infty(\Gamma) \rightarrow \ell^\infty(\Gamma)/c_0(\Gamma)$  is the quotient map and  $\|\cdot\|_{\ell^\infty/c_0}$  the canonical norm on the quotient  $\ell^\infty(\Gamma)/c_0(\Gamma)$ . Clearly,  $\lceil x \rceil = 0$  if and only if  $x \in c_0(\Gamma)$ . Let us assume that  $\lceil x \rceil = t > 0$ . Then, for every  $0 < s < t$ , there are infinitely many  $\gamma \in \Gamma$  such that  $|x(\gamma)| > s$ . It follows that  $\|x - y\|_\infty > s$  for every  $y \in c_0(\Gamma)$  and consequently  $\|q(x)\|_{\ell^\infty/c_0} \geq t$ . On the other hand, we may define  $y \in c_0(\Gamma)$  as

$$y(\gamma) := \begin{cases} x(\gamma) - t & \text{if } x(\gamma) > t, \\ x(\gamma) + t & \text{if } x(\gamma) < -t, \\ 0 & \text{otherwise.} \end{cases}$$

Obviously  $\|x - y\|_\infty \leq t$ , so  $\|q(x)\|_{\ell^\infty/c_0} \leq t$ . The strongly lattice property of  $\lceil \cdot \rceil$  follows directly from the definition.  $\square$

**Definition 2.10.** Let  $(X, \|\cdot\|)$  be a Banach space and let  $\mu$  be the smallest ordinal such that  $|\mu| = \text{dens}(X)$ . A system  $\{P_\alpha\}_{\omega \leq \alpha \leq \mu}$  of projections from  $X$  into  $X$  is called a *projectional resolution of identity* (PRI) provided that, for every  $\alpha \in [\omega, \mu]$ , the following conditions hold true

- (a)  $\|P_\alpha\| = 1$ ,
- (b)  $P_\alpha P_\beta = P_\beta P_\alpha = P_\alpha$  for  $\omega \leq \alpha \leq \beta \leq \mu$ ,
- (c)  $\text{dens}(P_\alpha X) \leq |\alpha|$ ,
- (d)  $\bigcup \{P_{\beta+1}X : \beta < \alpha\}$  is norm-dense in  $P_\alpha X$ ,
- (e)  $P_\mu = \text{id}_X$ .

If  $\{P_\alpha\}_{\omega \leq \alpha \leq \mu}$  is a PRI on a Banach space  $X$ , we use the following notation:  $\Lambda := \{0\} \cup [\omega, \mu]$ ,  $Q_\gamma := P_{\gamma+1} - P_\gamma$  for all  $\gamma \in [\omega, \mu)$  while  $Q_0 := P_\omega$ , and  $P_A := \sum_{\gamma \in A} Q_\gamma$  for any finite subset  $A$  of  $\Lambda$ .

**Lemma 2.11.** *Let  $X$  be a Banach space with a PRI  $\{P_\alpha\}_{\omega \leq \alpha \leq \mu}$ . Then for each  $x \in X$ ,  $\varepsilon > 0$ ,  $\alpha \in [\omega, \mu]$  there is a finite set  $A_\varepsilon^\alpha(x) \subset \Lambda$  such that*

$$\|P_{A_\varepsilon^\alpha(x)}x - P_\alpha x\| < \varepsilon.$$

*We may choose  $A = A_\varepsilon^\alpha(x)$  in such a way that  $Q_\beta x \neq 0$  for  $\beta \in A$  since  $P_A = \sum_{\gamma \in A} Q_\gamma$ .*

*Proof.* We will proceed by a transfinite induction on  $\alpha$ . If  $\alpha = \omega$ , then  $A_\varepsilon^\omega(x) := \{0\}$  for any  $\varepsilon > 0$ . If  $\alpha = \beta + 1$  for some ordinal  $\beta$ , then  $A_\varepsilon^\alpha(x) := A_\varepsilon^\beta(x) \cup \{\beta\}$  for all  $\varepsilon > 0$ . Finally, if  $\alpha$  is a limit ordinal, we will use the continuity of the mapping  $\gamma \mapsto P_\gamma x$  at  $\alpha$  [2, Lemma VI.1.2] to find  $\beta < \alpha$  such that  $\|P_\beta x - P_\alpha x\| < \varepsilon/2$ . Thus it is possible to set  $A_\varepsilon^\alpha(x) := A_{\varepsilon/2}^\beta(x)$ .  $\square$

### 3. MAIN RESULT

Let us recall that a norm  $\|\cdot\|$  in a Banach space  $X$  is *locally uniformly rotund* (LUR) if  $\lim_r \|x_r - x\| = 0$  whenever  $\lim_r (2\|x_r\|^2 + 2\|x\|^2 - \|x_r + x\|^2) = 0$ .

**Theorem 3.1.** *Let  $k \in \mathbb{N} \cup \{\infty\}$ . Let  $(X, |\cdot|)$  be a Banach space with a PRI  $\{P_\alpha\}_{\omega \leq \alpha \leq \mu}$  such that each  $Q_\gamma X$  admits a  $C^1$ -smooth, LUR equivalent norm which is a limit (uniform on bounded sets) of  $C^k$ -smooth norms. Let  $X$  admit an equivalent  $C^k$ -smooth norm  $\|\cdot\|$ .*

*Then  $X$  admits an equivalent  $C^1$ -smooth, LUR norm  $\|\|\cdot\|\|$  which is a limit (uniform on bounded sets) of  $C^k$ -smooth norms.*

Our first corollary provides a positive solution of Problem 8.2 (c) in [8].

**Corollary 3.2.** *Let  $\alpha$  be an ordinal. Then the space  $C([0, \alpha])$  admits an equivalent norm which is  $C^1$ -smooth, LUR and a limit of  $C^\infty$ -smooth norms.*

*Proof of Corollary 3.2.* By a result of Talagrand [22] and Haydon [17],  $C([0, \alpha])$  admits an equivalent  $C^\infty$ -smooth norm. On the other hand, the natural PRI on  $C([0, \alpha])$  defined as

$$(P_\gamma x)(\beta) = \begin{cases} x(\beta) & \text{if } \beta \leq \gamma, \\ x(\gamma) & \text{if } \beta \geq \gamma \end{cases}$$

has  $(P_{\gamma+1} - P_\gamma)X$  one-dimensional for each  $\gamma \in [\omega_0, \alpha)$ .  $\square$

**Theorem 3.3.** *Let  $k \in \mathbb{N} \cup \{\infty\}$ . Let  $\mathcal{P}$  be a class of Banach spaces such that every  $X$  in  $\mathcal{P}$*

- *admits a PRI  $\{P_\alpha\}_{\omega \leq \alpha \leq \mu}$  such that  $(P_{\alpha+1} - P_\alpha)X \in \mathcal{P}$ ,*
- *admits a  $C^k$ -smooth equivalent norm.*

*Then each  $X$  in  $\mathcal{P}$  admits an equivalent, LUR,  $C^1$ -smooth norm which is a limit (uniform on bounded sets) of  $C^k$ -smooth norms.*

*Proof.* We will carry out induction on the density of  $X$ . Let  $X \in \mathcal{P}$  be separable, i.e.  $\text{dens}(X) = \omega$ . Then we get the result from the theorem of McLaughlin, Poliquin, Vanderwerff and Zizler [21] or [2, Theorem V.1.7].

Next, we assume for  $X \in \mathcal{P}$  that  $\text{dens}(X) = \mu$  and that every Banach space  $Y \in \mathcal{P}$  with  $\text{dens}(Y) < \mu$  admits a  $C^1$ -smooth, LUR norm which is a limit of  $C^k$ -smooth norms. Let  $\{P_\alpha\}_{\omega \leq \alpha \leq \mu}$  be a PRI on  $X$  such that  $Q_\alpha X \in \mathcal{P}$  for each  $\alpha \in \Lambda$ . Then  $\text{dens}(Q_\alpha X) \leq |\alpha + 1| = |\alpha| < \mu$ . Thus the inductive hypothesis enables us to use Theorem 3.1.  $\square$

The above theorem has immediate corollaries for each  $\mathcal{P}$ -class (see [16] for this notion). The following Corollary 3.4 solves in the affirmative Problem 8.8 (s) in [8] (see also Problem VIII.4 in [2]).

**Corollary 3.4.** *Let  $X$  admit a  $C^k$ -smooth norm for some  $k \in \mathbb{N} \cup \infty$ . If  $X$  is Vařák (i.e. WCD) or WLD or  $C(K)$  where  $K$  is a Valdivia compact, then  $X$  admits a  $C^1$ -smooth, LUR equivalent norm which is a limit (uniform on bounded sets) of  $C^k$ -smooth norms.*

*Proof of Theorem 3.1.* Let  $0 < c < 1$ . It follows from the hypothesis that, for each  $\gamma \in \Lambda$ , there are a  $C^1$ -smooth, LUR norm  $\|\cdot\|_\gamma$  on  $Q_\gamma X$  and  $C^k$ -smooth norms  $(\|\cdot\|_{\gamma,i})_{i \in \mathbb{N}}$  on  $Q_\gamma X$  such that

$$c \|x\| \leq \|x\|_\gamma \leq \|x\| \tag{2}$$

for all  $x \in Q_\gamma X$  and such that  $(1 - \frac{1}{i^2}) \|x\|_\gamma \leq \|x\|_{\gamma,i} \leq \|x\|_\gamma$  for all  $x \in Q_\gamma X$ .

We seek the new norm on  $X$  in the form

$$\|x\|^2 := N(x)^2 + J(x)^2 + \|x\|^2.$$

We will insure during the construction that both  $N$  and  $J$  are  $C^1$ -smooth and approximated by  $C^k$ -smooth norms. In order to see that  $\|\cdot\|$  is LUR, we are going to show that  $\|x - x_r\| \rightarrow 0$  provided that

$$2\|x_r\|^2 + 2\|x\|^2 - \|x + x_r\|^2 \rightarrow 0 \text{ as } r \rightarrow \infty. \tag{3}$$

Consider the following two statements:

- a)  $\|P_A x_r - P_A x\| \rightarrow 0$  for each finite  $A \subset \Lambda$  with  $0 \notin \{Q_\gamma x : \gamma \in A\}$ ,
- b) for every  $\varepsilon > 0$  there exists a finite  $A \subset \Lambda$  with  $0 \notin \{Q_\gamma x : \gamma \in A\}$  and such that  $\|P_A x - x\| < \varepsilon$  and  $\|P_A x_r - x_r\| < \varepsilon$  for all but finitely many  $r \in \mathbb{N}$ .

Clearly, the simultaneous validity of a) and b) implies that  $\|x - x_r\| \rightarrow 0$  as

$$\|x - x_r\| \leq \|P_A x_r - P_A x\| + \|P_A x - x\| + \|P_A x_r - x_r\|.$$

We construct  $N$  in such a way that we can prove in Lemma 4.1 that (3) implies a). Consequently, we construct  $J$  in such a way that we can prove in Lemma 5.8 that (3) implies b).  $\square$

#### 4. ABOUT $N$

We may and do assume that the equivalent norms  $|\cdot|$  and  $\|\cdot\|$  satisfy

$$|\cdot| \leq \|\cdot\| \leq C|\cdot|$$

for some  $C \geq 1$ .

The basic properties of PRI [2, Lemma VI.1.2] and the above equivalence of norms yield  $(\|Q_\gamma x\|)_{\gamma \in \Lambda} \in c_0(\Lambda)$ , and using  $\|Q_\gamma\| \leq 2C$  with the second inequality of (2), it follows that  $T : x \in (X, \|\cdot\|) \mapsto (\|Q_\gamma x\|)_{\gamma \in \Lambda} \in (c_0(\Lambda), \|\cdot\|_\infty)$  is a  $2C$ -Lipschitz mapping. Similarly for  $T_i : x \in X \mapsto (\|Q_\gamma x\|)_{\gamma \in \Lambda} \in c_0(\Lambda)$ .

For each  $n \in \mathbb{N}$ , we will consider an equivalent norm on  $c_0(\Lambda)$  given as

$$\zeta_n(x) := \sup_{M \in \Lambda_n} \sqrt{\sum_{\gamma \in M} x(\gamma)^2}$$

where  $\Lambda_n := \{M \in 2^\Lambda : |M| = n\}$ . It is easily seen that  $\zeta_n$  is  $n$ -Lipschitz with respect to the usual norm on  $c_0(\Lambda)$ . Also,  $\zeta_n$  is obviously strongly lattice, so by Theorem 1 in [7], for each  $\varepsilon > 0$  there is a  $C^\infty$ -smooth equivalent norm  $N_{n,\varepsilon}$  on  $c_0(\Lambda)$  such that  $(1 - \varepsilon)\zeta_n(x) \leq N_{n,\varepsilon}(x) \leq \zeta_n(x)$  for all  $x \in c_0(\Lambda)$  with  $\|x\|_\infty \leq 1$ . Finally, we define

$$N(x)^2 := \sum_{m,n \in \mathbb{N}} \frac{1}{2^{n+m}} N_{n,\frac{1}{m}}^2(T(x))$$

Now the norm  $N(\cdot)$  is  $C^1$ -smooth since each  $N_{n,\frac{1}{m}} \circ T$  is  $2nC$ -Lipschitz and  $C^1$ -smooth. The latter property follows since each  $N_{n,\frac{1}{m}}$  is not only LFC but it depends on nonzero coordinates only (cf. Remark on page 461 in [17]). This fact is not explicitly mentioned in [7] but follows from the proof there (see [7, p. 270]). We may define the approximating norms as

$$N_i(x)^2 := \sum_{m,n=1}^i \frac{1}{2^{n+m}} N_{n,\frac{1}{m}}^2(T_i(x)).$$

As a finite sum of  $C^k$ -smooth norms,  $N_i$  is  $C^k$ -smooth. Using  $\|T_i(x) - T(x)\|_\infty \leq \frac{2C}{i^2}$  for  $\|x\| \leq 1$ , it is standard to check that  $N_i(x) \rightarrow N(x)$  uniformly for  $\|x\| \leq 1$ . We carry out some similar considerations in more detail on page 14 when we demonstrate that  $J$  is approximated by  $C^k$ -smooth norms.

**Lemma 4.1.** *Let us assume that (3) holds for  $x, x_r \in X$ ,  $r \in \mathbb{N}$ , and let  $\tilde{A} \subset \Lambda$  be a finite set such that  $Q_\gamma x \neq 0$  for  $\gamma \in \tilde{A}$ . Then  $\|P_{\tilde{A}}x - P_{\tilde{A}}x_r\| \rightarrow 0$  as  $k \rightarrow \infty$ .*

*Proof.* Let  $A := \{\gamma \in \Lambda : \|Q_\gamma x\|_\gamma \geq \min_{\alpha \in \tilde{A}} \|Q_\alpha x\|_\alpha\}$ . Let  $n := |A|$ . We may assume that  $\|x\| \leq 1$  which implies  $\|Tx\|_\infty \leq 2C$ . Using (3) and Lemma 2.5 we may assume that  $\|x_r\| \leq 2$  thus  $\|Tx_r\|_\infty \leq 4C$ . The convergence (3) and convexity (see Fact II.2.3 in [2]) imply that

$$2N_{n,\frac{1}{m}}^2(T(x_r)) + 2N_{n,\frac{1}{m}}^2(T(x)) - N_{n,\frac{1}{m}}^2(T(x + x_r)) \xrightarrow{r} 0$$

for all  $m \in \mathbb{N}$ . This further yields that

$$2\zeta_n^2(T(x_r)) + 2\zeta_n^2(T(x)) - \zeta_n^2(T(x + x_r)) \xrightarrow{r} 0$$

as well. Indeed, let  $\varepsilon > 0$  be given. We use that  $N_{n,\frac{1}{m}} \rightarrow \zeta_n$  uniformly on bounded sets of  $c_0(\Lambda)$  to find  $m_0 \in \mathbb{N}$  such that  $|N_{n,\frac{1}{m}}^2(y) - \zeta_n^2(y)| < \varepsilon/6$  for all  $y \in 6CB_{c_0(\Lambda)}$  and all  $m \geq m_0$ . Now let  $r_0 \in \mathbb{N}$  satisfy that for all  $r \geq r_0$  it holds  $2N_{n,\frac{1}{m_0}}^2(T(x_r)) + 2N_{n,\frac{1}{m_0}}^2(T(x)) - N_{n,\frac{1}{m_0}}^2(T(x + x_r)) < \varepsilon/6$ . For each  $r \geq r_0$  we obtain  $2\zeta_n^2(T(x_r)) + 2\zeta_n^2(T(x)) - \zeta_n^2(T(x + x_r)) < \varepsilon$ .

Let  $B \in \Lambda_n$  be arbitrary and let  $A_r \in \Lambda_n$  such that

$$\sqrt{\sum_{\gamma \in A_r} \|Q_\gamma(x + x_r)\|_\gamma^2} = \zeta_n(x + x_r).$$

Then

$$\begin{aligned} 2\zeta_n^2(T(x_r)) + 2\zeta_n^2(T(x)) - \zeta_n^2(T(x + x_r)) &\geq 2 \sum_{\gamma \in B} \|Q_\gamma x\|_\gamma^2 + 2 \sum_{\gamma \in A_r} \|Q_\gamma x_r\|_\gamma^2 - \sum_{\gamma \in A_r} \|Q_\gamma(x + x_r)\|_\gamma^2 \\ &= 2 \sum_{\gamma \in A_r} \|Q_\gamma x\|_\gamma^2 + 2 \sum_{\gamma \in A_r} \|Q_\gamma x_r\|_\gamma^2 - \sum_{\gamma \in A_r} \|Q_\gamma(x + x_r)\|_\gamma^2 \\ &\quad + 2 \left( \sum_{\gamma \in B} \|Q_\gamma x\|_\gamma^2 - \sum_{\gamma \in A_r} \|Q_\gamma x\|_\gamma^2 \right) \end{aligned} \quad (4)$$

Since

$$2 \sum_{\gamma \in A_r} \|Q_\gamma x\|_\gamma^2 + 2 \sum_{\gamma \in A_r} \|Q_\gamma x_r\|_\gamma^2 - \sum_{\gamma \in A_r} \|Q_\gamma(x + x_r)\|_\gamma^2 \geq 0$$

we get from (4) that

$$\liminf_r \sum_{\gamma \in A_r} \|Q_\gamma x\|_\gamma^2 \geq \sup \left\{ \sum_{\gamma \in B} \|Q_\gamma x\|_\gamma^2 : B \in \Lambda_n \right\} = \zeta_n(T(x)) = \sum_{\gamma \in A} \|Q_\gamma x\|_\gamma^2 \quad (5)$$

where the last equality follows from the definition of  $A$ . Equation (5) together with the definition of  $A$  show that  $A = A_r$  for all  $r$  sufficiently large. We continue with such  $r$  and we choose  $B := A$  in (4) to get that

$$2 \sum_{\gamma \in A} \|Q_\gamma x\|_\gamma^2 + 2 \sum_{\gamma \in A} \|Q_\gamma x_r\|_\gamma^2 - \sum_{\gamma \in A} \|Q_\gamma(x + x_r)\|_\gamma^2 \xrightarrow{r} 0.$$

Since  $x \mapsto \sqrt{\sum_{\gamma \in A} \|Q_\gamma x\|_\gamma^2}$  is an equivalent LUR norm on  $P_A X$ , it follows that  $\|P_A(x - x_r)\| \rightarrow 0$  and, by continuity of  $P_{\tilde{A}}$ , we obtain the claim of the lemma.  $\square$

## 5. ABOUT $J$

Let  $\{\phi_\eta\}_{0 < \eta < 1}$  be a system of functions satisfying

- (i)  $\phi_\eta : [0, +\infty) \rightarrow [0, +\infty)$ , for  $0 < \eta < 1$ , is a convex  $C^\infty$ -smooth function such that  $\phi_\eta$  is strictly convex on  $[1 - \eta, +\infty)$ ,  $\phi_\eta([0, 1 - \eta]) = \{0\}$  and  $\phi_\eta(1) = 1$ .
- (ii) If  $0 < \eta_1 \leq \eta_2 < 1$  then  $\phi_{\eta_1}(x) \leq \phi_{\eta_2}(x)$  for any  $x \in [0, 1]$ .

One example of such a system can be constructed as follows: let  $\phi : \mathbb{R} \rightarrow \mathbb{R}$  be  $C^\infty$ -smooth such that  $\phi(x) = 0$  if  $x \leq 0$ ,  $\phi(1) = 1$  and  $\phi$  is increasing and strictly convex on  $[0, +\infty)$ . We define  $\phi_\eta(x) := \phi\left(\frac{x - (1 - \eta)}{\eta}\right)$  for all  $x \in [0, 1]$ . Now the system  $\{\phi_\eta\}$  satisfies (ii) since  $\eta \mapsto \frac{x - (1 - \eta)}{\eta}$  is increasing for every  $x \in [0, 1)$  while the validity of (i) follows from properties of  $\phi$ .

We define a function  $\Phi_\eta : \ell^\infty(\Gamma) \rightarrow (-\infty, +\infty]$  by

$$\Phi_\eta(x) = \sum_{\gamma \in \Gamma} \phi_\eta(|x(\gamma)|).$$

Let us define  $Z_\eta : \ell^\infty(\Gamma) \rightarrow \mathbb{R}$  as the Minkowski functional of the set  $C = \{x \in \ell^\infty(\Gamma); \Phi_\eta(x) < 1/2\}$ .

**Lemma 5.1.** *Let  $0 < \eta < 1$  be fixed. Then  $Z_\eta$  is a strongly lattice seminorm such that  $(1 - \eta)Z_\eta(x) \leq \|x\|_\infty$  and  $Z_\eta$  is LFC,  $C^\infty$ -smooth and strictly positive in the set*

$$A_\eta(\Gamma) := \{x \in \ell^\infty(\Gamma) : [x] < (1 - \eta) \|x\|_\infty\}.$$

Moreover  $(1 - \eta)Z_\eta(x) < \|x\|_\infty$  for all  $x \in A_\eta(\Gamma)$ .



*Proof.* The set  $C$  is symmetric convex with zero as interior point (indeed,  $(1 - \eta)B_{\ell^\infty(\Gamma)} \subset C$ ) so  $Z_\eta$  is  $\frac{1}{1-\eta}$ -Lipschitz and convex.

Let  $A'_\eta(\Gamma) := \{x \in \ell^\infty(\Gamma) : \lceil x \rceil < 1 - \eta\}$ . This set is convex and open since  $\lceil \cdot \rceil$  is a continuous and convex (seminorm). The function  $\Phi_\eta$  is in  $A'_\eta(\Gamma)$  a locally finite sum of convex  $C^\infty$ -smooth functions, thus it is a convex function which is LCF and  $C^\infty$ -smooth in  $A'_\eta(\Gamma)$ .

Let us fix  $x_0 \in A'_\eta(\Gamma)$  such that  $\Phi_\eta(x_0) = 1/2$ . Then, since  $\phi_\eta$  is increasing at the points where it is not zero, we get  $Z_\eta(x_0) = 1$  and  $\Phi'_\eta(x_0)x_0 > 0$ . As is usual, we consider the equation  $\Phi_\eta\left(\frac{x}{Z_\eta(x)}\right) = \frac{1}{2}$ . By the Implicit Function Theorem, this equation locally redefines  $Z_\eta$  and proves that  $Z_\eta$  is  $C^\infty$ -smooth on some neighborhood  $U$  of  $x_0$  since  $\Phi_\eta$  is. Moreover by application of Lemma 2.4 we get that  $Z_\eta$  is LFC at  $x_0$ .

To prove that  $Z_\eta$  is LFC, strictly positive and  $C^\infty$ -smooth in  $A_\eta(\Gamma)$  it is enough to show that for each  $x \in A_\eta(\Gamma)$  there is  $\lambda > 0$  such that  $\lambda x \in A'_\eta(\Gamma)$  and  $\Phi_\eta(\lambda \cdot x) = 1/2$  and then use the homogeneity of  $Z_\eta$ .

Let  $x \in A_\eta(\Gamma)$ . Then  $\left\lceil \frac{x}{\|x\|_\infty} \right\rceil < 1 - \eta$  and since  $A'_\eta(\Gamma)$  is convex, it follows that  $[0, \frac{x}{\|x\|_\infty}] \subset A'_\eta(\Gamma)$ . We have for such  $x$  that  $\Phi_\eta(\frac{x}{\|x\|_\infty}) \geq 1$ ,  $\Phi_\eta(0 \cdot x) = 0$  and the mapping  $\lambda \mapsto \Phi_\eta(\lambda x)$  is continuous for  $\lambda \in [0, \frac{1}{\|x\|_\infty}]$ . Hence there must exist  $\lambda \in (0, \frac{1}{\|x\|_\infty})$  such that  $\lambda x \in A'_\eta(\Gamma)$  and  $\Phi_\eta(\lambda \cdot x) = 1/2$ .

We continue showing that  $Z_\eta$  is strongly lattice. First observe that  $\Phi_\eta$  is strongly lattice as  $\phi_\eta$  is nondecreasing. Let  $|x| \leq |y|$  and  $Z_\eta(x) = 1$ . Then  $x \in \partial C$  which implies that  $\lceil x \rceil = 1 - \eta$  or  $\Phi_\eta(x) = 1/2$ . Since both functions  $\lceil \cdot \rceil$  and  $\Phi_\eta$  are strongly lattice, we conclude that  $\lceil y \rceil \geq 1 - \eta$  or  $\Phi_\eta(y) \geq 1/2$  which in turn implies that  $Z_\eta(y) \geq 1$ . For a general  $x$  we employ the homogeneity of  $Z_\eta$ , so  $Z_\eta$  is strongly lattice.

Finally, if  $x \in A_\eta(\Gamma)$ , then the above considerations imply that  $\Phi_\eta\left(\frac{x}{Z_\eta(x)}\right) = 1/2$ . This is possible only if there is some  $\gamma \in \Gamma$  such that  $\frac{x(\gamma)}{Z_\eta(x)} > 1 - \eta$ , and the moreover claim follows.  $\square$

**Lemma 5.2.** *Let  $0 < \eta_1 \leq \eta_2 < 1$ . Then  $Z_{\eta_1}(x) \leq Z_{\eta_2}(x)$  for every  $x \in A_{\eta_2}(\Gamma)$ .*

*Proof.* First of all, if  $x \in A_{\eta_2}(\Gamma)$ , then  $x \in A_{\eta_1}(\Gamma)$ . So the equivalence  $Z_{\eta_i}(\lambda x) = 1 \Leftrightarrow \Phi_{\eta_i}(\lambda x) = 1/2$  holds for both  $i = 1, 2$ . Let us assume that  $Z_{\eta_1}(\lambda x) = 1$  for some  $\lambda > 0$ . Then the ordering of functions  $\phi_\eta$  yields  $1/2 = \Phi_{\eta_1}(\lambda x) \leq \Phi_{\eta_2}(\lambda x)$  which results in  $Z_{\eta_2}(\lambda x) \geq 1$ .  $\square$

**Lemma 5.3.** *Let  $0 < \eta < 1$  be given and let  $x_r, x \in A_\eta(\Gamma)$  ( $r \in \mathbb{N}$ ) be non-negative (in the lattice  $\ell^\infty(\Gamma)$ ) such that*

$$2Z_\eta^2(x) + 2Z_\eta^2(x_r) - Z_\eta^2(x + x_r) \rightarrow 0 \text{ as } r \rightarrow \infty.$$

*Then  $x_r(\gamma) \rightarrow x(\gamma)$  for any  $\gamma \in \Gamma$  such that  $x(\gamma) > Z_\eta(x)(1 - \eta)$ .*

*Proof.* The assumption and Lemma 2.5 yield

$$Z_\eta(x_r) \rightarrow Z_\eta(x) \quad \text{and} \quad Z_\eta\left(\frac{x + x_r}{2}\right) \rightarrow Z_\eta(x). \quad (6)$$

Let us put  $\tilde{x} := \frac{x}{Z_\eta(x)}$  and  $\tilde{x}_r := \frac{x_r}{Z_\eta(x_r)}$ . We get from (6) that

$$2Z_\eta^2(\tilde{x}) + 2Z_\eta^2(\tilde{x}_r) - Z_\eta^2(\tilde{x} + \tilde{x}_r) \rightarrow 0.$$

Since  $Z_\eta(\tilde{x}) = Z_\eta(\tilde{x}_r) = 1$ , the above implies that

$$\lambda_r := Z_\eta(\tilde{x} + \tilde{x}_r) \rightarrow 2.$$

We may deduce from  $x, x_r \in A_\eta(\Gamma)$  that  $\Phi_\eta(\tilde{x}) = 1/2 = \Phi_\eta(\tilde{x}_r)$  for all  $k \in \mathbb{N}$ . Also,  $\Phi_\eta(\lambda_r^{-1}(\tilde{x} + \tilde{x}_r)) = 1/2$  for all but finitely many  $k \in \mathbb{N}$ . Indeed, if  $\Phi_\eta(\lambda_r^{-1}(\tilde{x} + \tilde{x}_r)) \neq 1/2$ , then  $\lambda_r^{-1}(\tilde{x} + \tilde{x}_r) \in \partial A'_\eta(\Gamma)$ . Then in fact  $\lceil \lambda_r^{-1}(\tilde{x} + \tilde{x}_r) \rceil = 1 - \eta$ . As  $\tilde{x} \in A'_\eta(\Gamma)$ , there is  $\xi > 0$  such that  $\lceil \tilde{x} \rceil + \xi < 1 - \eta$ . By the same reasoning  $\lceil \tilde{x}_r \rceil < 1 - \eta$ . By the convexity (subadditivity) of  $\lceil \cdot \rceil$  and these estimates one has

$$\lceil \tilde{x} + \tilde{x}_r \rceil \leq \lceil \tilde{x} \rceil + \lceil \tilde{x}_r \rceil < 2(1 - \eta) - \xi.$$

Finally,  $\lambda_r < \frac{2(1-\eta)-\xi}{1-\eta}$  which can happen only for finitely many  $r$  as  $\lambda_r \rightarrow 2$ .

As  $\Phi_\eta$  is continuous at  $\tilde{x}$  and  $\lambda_r \rightarrow 2$ , it follows

$$\Phi_\eta((\lambda_r - 1)^{-1}\tilde{x}) \rightarrow 1/2.$$

Consequently

$$(1 - \lambda_r^{-1})\Phi_\eta((\lambda_r - 1)^{-1}\tilde{x}) + \lambda_r^{-1}\Phi_\eta(\tilde{x}_r) - \Phi_\eta(\lambda_r^{-1}(\tilde{x} + \tilde{x}_r)) \rightarrow 0. \quad (7)$$

Let  $a > 1 - \eta$ . The definition of  $\phi_\eta$  and a compactness argument imply that for each  $\varepsilon > 0$  there exists  $\Delta > 0$  such that if for reals  $r, s, \alpha$  it holds

- $0 \leq r \leq 4 \max\{\|\tilde{x}\|_\infty, \sup_r \|\tilde{x}_r\|_\infty\}$ ,
- $a \leq s \leq 4 \max\{\|\tilde{x}\|_\infty, \sup_r \|\tilde{x}_r\|_\infty\}$ ,
- $\frac{1}{4} \leq \alpha \leq \frac{3}{4}$ , and
- $\alpha\phi_\eta(r) + (1 - \alpha)\phi_\eta(s) - \phi_\eta(\alpha r + (1 - \alpha)s) < \Delta$ ,

then  $|r - s| < \varepsilon$ .

In particular, let  $a > 1 - \eta$  be such that  $\{\gamma \in \Gamma; \tilde{x}(\gamma) > 1 - \eta\} = \{\gamma \in \Gamma; \tilde{x}(\gamma) > a\}$  and let  $\gamma \in \Gamma$  be such that  $\tilde{x}(\gamma) > a$ . Then for  $r$  large enough we have  $(\lambda_r - 1)^{-1}\tilde{x}(\gamma) > a$  so we may substitute  $r := \tilde{x}_r(\gamma)$ ,  $s := (\lambda_r - 1)^{-1}\tilde{x}(\gamma)$  and  $\alpha := \lambda_r^{-1}$ . It follows from (7) that one has  $|(\lambda_r - 1)^{-1}\tilde{x}(\gamma) - \tilde{x}_r(\gamma)| \rightarrow 0$  as  $k \rightarrow \infty$ . Since  $\lambda_r \rightarrow 2$  and using (7), we finally get that  $x_r(\gamma) \rightarrow x(\gamma)$  as  $k \rightarrow \infty$ .  $\square$

The following system of convex functions is at the heart of our construction. We recall that  $C \geq 1$  is the constant of equivalence between the norms  $|\cdot|$  and  $\|\cdot\|$ , which was introduced in Section 4.

**Lemma 5.4.** *There exist*

- a decreasing sequence of positive numbers  $\delta_n \searrow 0$ ;  $\delta_1 < 2C$ ;
- a decreasing sequence of positive numbers  $\rho_n \searrow 0$ ;
- positive numbers  $\kappa_{n,m} > 0$  such that for each  $n \in \mathbb{N}$  the sequence  $(\kappa_{n,m})_m$  is decreasing and  $\kappa_{n,m} \xrightarrow{m} 0$ ; for each  $n, m \in \mathbb{N}$  one has  $\rho_n > 2\kappa_{n,m}$ ;
- an equi-Lipschitz system of non-negative,  $C^\infty$ -smooth, 1-bounded, convex functions

$$\{g_{n,m,l} : \mathcal{D}_{n,l} \rightarrow \mathbb{R} : n, m \in \mathbb{N}, l = 1, \dots, n\},$$

where  $\mathcal{D}_{n,l} := [0, 2nC - \delta_n(n-l)] \times [0, 1 + 2nC]$ , satisfying (with  $n, m, l \in \mathbb{N}$ ,  $l \leq n$ , resp.  $l < n$  in (A2), (A5))

(A1)  $g_{n,m,l}(t, s) = 0$  iff  $(t, s) \in [0, l\delta_n] \times [0, 1 + 2nC] =: \mathcal{N}_{n,l}$ ;

(A2)  $g_{n,m,l}(t, s) \geq g_{n,m,l+1}(t, s) + \rho_n$  whenever  $(t, s) \in \mathcal{D}_{n,l} \setminus \mathcal{N}_{n,l+1}$ ;

(A3) if  $(t, 0) \in \mathcal{D}_{n,l} \setminus \mathcal{N}_{n,l}$ , then  $s \mapsto g_{n,m,l}(t, s)$  is increasing on  $[0, 1 + 2nC]$  and

$$g_{n,m,l}(t, 1 + 2nC) - g_{n,m,l}(t, 0) \leq \kappa_{n,m};$$

(A4) if  $(t, 0) \in \mathcal{D}_{n,l}$ , then  $g_{n,m,l}(t, 0) = g_{n,m+1,l}(t, 0)$ ;

(A5) for all  $(t, s) \in \mathcal{D}_{n,l} \setminus \mathcal{N}_{n,l}$  it holds  $g_{n,m,l}(t, s) < g_{n,m,l+1}(t + r, s)$  provided  $r > \delta_n$ .

(A6) Let  $(t, s) \in \mathcal{D}_{n,l} \setminus \mathcal{N}_{n,l}$ . If  $(t_r, s_r) \in \mathcal{D}_{n,l}$  and  $t_r \rightarrow t$  and  $g_{n,m,l}(t_r, s_r) \rightarrow g_{n,m,l}(t, s)$  as  $r \rightarrow \infty$ , then  $s_r \rightarrow s$ .

(A7) The mapping  $(t, s) \mapsto g_{n,m,l}(|t|, |s|)$  is strongly lattice in  $\mathcal{D}_{n,l}$ .

*Proof.* Let  $f : \mathbb{R} \rightarrow [0, +\infty)$  be defined as

$$f(t) := \begin{cases} 0, & \text{for } t \leq 0, \\ \exp(-\frac{1}{t^2}) & \text{for } t > 0. \end{cases}$$

It is elementary (one may use Lemma 2.6) to check that  $f(t) \cdot (s^2 + s + 1)$  is convex in the strip  $(-\infty, 10^{-1}] \times [0, 10^{-1}]$  so the function

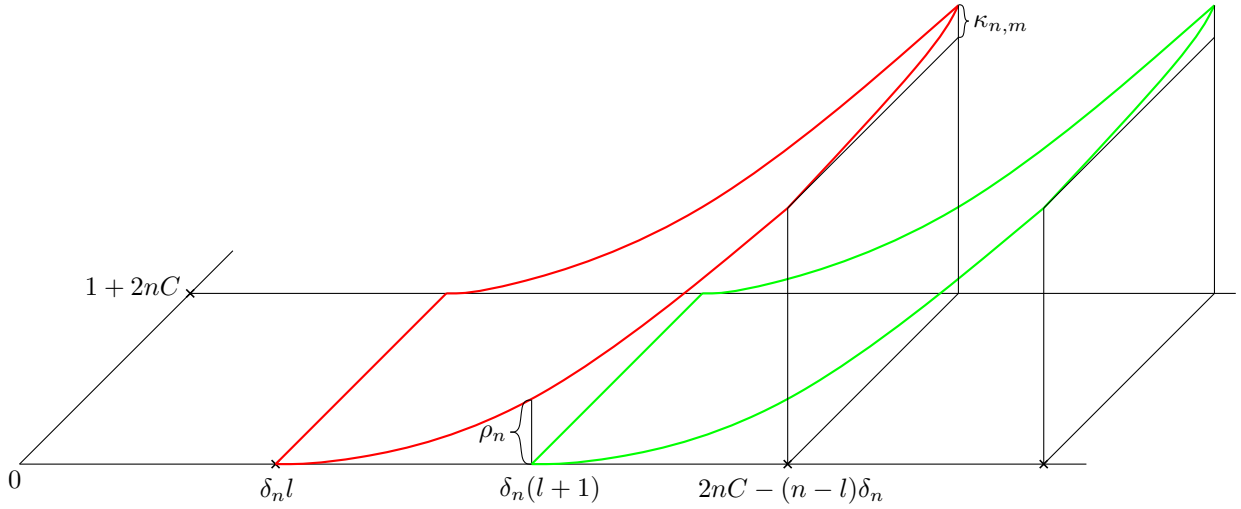
$$g(t, s) := f(10^{-1}t) \cdot ((10^{-1}s)^2 + 10^{-1}s + 1)$$

is convex in the strip  $(-\infty, 1] \times [0, 1]$ . We take for  $(\delta_n)_n$  just any decreasing null sequence of positive numbers such that  $\delta_1 < 2C$ , and we define

$$g_{n,m,l}(t, s) := g\left(\frac{t - \delta_n l}{(2C - \delta_n)n}, \theta_{n,m} \frac{s}{1 + 2nC}\right).$$

where  $\theta_{n,m} \in (0, 1)$  will be chosen later. Now since our functions  $g_{n,m,l}$  are just shifts and stretches of one non-negative,  $C^\infty$ -smooth, 1-bounded, Lipschitz, convex function, it follows that all  $g_{n,m,l}$  share these properties (with the same Lipschitz constant).

Properties (A1), (A4) and (A5) are straightforward, see also Figure 5. Notice that, when  $t > 0$ , the



function  $s \mapsto g(t, s)$  is increasing on  $[0, 1]$ . This implies the first part of (A3). In order to satisfy (A2), we may define  $\rho_n$  as

$$\rho_n := \inf \{g_{n,m,l}(t, s) - g_{n,m,l+1}(t, s) : l, m \in \mathbb{N}, l < n, (t, s) \in \mathcal{D}_{n,l} \setminus \mathcal{N}_{n,l+1}\}$$

which evaluates as  $\rho_n = g_{n,1,1}(2\delta_n, 0) = f\left(\frac{\delta_n}{(2-\delta_n)n}\right) \searrow 0$  as  $n \rightarrow \infty$ . Notice that this  $\rho_n$  does not depend on the choice of  $\theta_{n,m}$ . On the other hand, in order to fulfill (A3),  $\kappa_{n,m}$  may be defined as

$$\kappa_{n,m} := \sup \{g_{n,m,l}(t, 1 + 2nC) - g_{n,m,l}(t, 0) : l \leq n, (t, 0) \in \mathcal{D}_{n,l}\}$$

which evaluates as  $\kappa_{n,m} = g_{n,m,n}(2nC, 1 + nC) - g_{n,m,n}(2nC, 0)$ . We see that, by an appropriate choice of  $\theta_{n,m}$  (in particular, for each  $n \in \mathbb{N}$ , the sequence  $(\theta_{n,m})_m$  should be decreasing to zero), one may satisfy the requirements  $\rho_n > 2\kappa_{n,m}$  and  $\kappa_{n,m} \searrow 0$  as  $m \rightarrow \infty$ .

For the proof of (A6) let us assume that  $s_r \not\rightarrow s$ . The fact that  $g_{n,m,l}(t_r, \cdot) \rightarrow g_{n,m,l}(t, \cdot)$  uniformly on  $[0, 1 + 2nC]$  leads quickly to a contradiction.

Finally (A7) follows since  $g$  is non-decreasing in  $\mathcal{D}_{n,l}$  in each variable.  $\square$

Let us fix, for each  $\delta > 0$ , some  $C^\infty$ -smooth, convex mapping  $\xi_\delta$  from  $[0, +\infty)$  to  $[0, +\infty)$  which satisfies  $\xi_\delta([0, \delta]) = \{0\}$ ,  $\xi_\delta(t) > 0$  for  $t > \delta$  and  $\xi_\delta(t) = t - 2\delta$  for  $t \geq 3\delta$ . Such a mapping can be constructed e.g. by integrating twice a  $C^\infty$ -smooth, non-negative bump.

**Lemma 5.5.** *Let  $n, m \in \mathbb{N}$  be fixed and let us define a mapping  $H_{n,m} : B_{(X, \|\cdot\|)}^O \rightarrow \ell^\infty(F_n)$  where  $F_n = \{(A, B) \in 2^\Lambda \times 2^\Lambda : |A| \leq n, B \subset A, A \neq \emptyset \neq B\}$  by*

$$H_{n,m}x(A, B) := g_{n,m,|A|}\left(\sum_{\gamma \in A} \xi_{\delta_n}(\|Q_\gamma x\|_\gamma), \xi_{\delta_n}(\|P_B x - x\|)\right).$$

Then  $H_{n,m}$  is a continuous, coordinatewise convex and coordinatewise  $C^1$ -smooth mapping, and for each  $x \in X$  such that  $\|x\| < 1$  it holds  $H_{n,m}x \in A_{\rho_n/2 - \kappa_{n,m}}(F_n) \cup \{0\}$  (see the definition of the set  $A_{\rho_n/2 - \kappa_{n,m}}(F_n)$  in Lemma 5.1).

Notice that, by the definition of  $\kappa_{n,m}$  in Lemma 5.4, we have always  $\rho_n/2 - \kappa_{n,m} > 0$ . We will use the notation  $\eta_{n,m} := \rho_n/2 - \kappa_{n,m}$ .

*Proof.* When  $\|x\| < 1$ , then (2) yields  $(\sum_{\gamma \in A} \xi_{\delta_n}(\|Q_\gamma x\|_\gamma), \xi_{\delta_n}(\|P_B x - x\|)) \in [0, 1+2|A|C] \times [0, 2|A|C] \subset \mathcal{D}_{n,|A|}$ . So for each  $(A, B) \in F_n$  the mapping  $x \mapsto H_{n,m}x(A, B)$  is  $C^1$ -smooth as a composition of such mappings. Also,  $\{x \mapsto H_{n,m}x(A, B) : (A, B) \in F_n\}$  is equi-Lipschitz thus  $H_{n,m}$  is continuous. Each  $x \mapsto H_{n,m}x(A, B)$  is convex by application of Lemma 2.8 since  $g_{n,m,l}$  is convex and strongly lattice. Because  $\sup g_{n,m,l}(\mathcal{D}_{n,|A|}) < 1$  for each  $l \leq n$ , we get that  $\|H_{n,m}x\|_\infty < 1$ .

We are going to prove that  $\lceil H_{n,m}x \rceil < \|H_{n,m}x\|_\infty (1 - \rho_n/2 + \kappa_{n,m})$  or  $\|H_{n,m}x\|_\infty = 0$ . For any  $x \in X$  and  $\delta > 0$ , let  $\Lambda(x, \delta) := \{\gamma \in \Lambda : \|Q_\gamma x\|_\gamma > \delta\}$ . Let  $x \in B_{(X, \|\cdot\|)}^O$  be fixed and let us define a set  $E \subset F_n$  as  $E := \{(A, B) \in F_n : A \subset \Lambda(x, \delta_n)\}$ . Since  $E$  is finite, it holds

$$\lceil H_{n,m}x \rceil = \lceil H_{n,m}x \upharpoonright_{F_n \setminus E} \rceil \leq \sup \{H_{n,m}x(A, B) : (A, B) \in F_n \setminus E\}. \quad (8)$$

If there is no  $(A, B) \in F_n \setminus E$  such that  $H_{n,m}(A, B) > 0$ , then  $\lceil H_{n,m}x \rceil = 0$  and our claim is trivially true. We proceed assuming that  $H_{n,m}x(A, B) > 0$  for some  $(A, B) \in F_n \setminus E$ . Then

$$\left( \sum_{\gamma \in A} \xi_{\delta_n}(\|Q_\gamma x\|_\gamma), \xi_{\delta_n}(\|P_B x - x\|) \right) \notin \mathcal{N}_{n,|A|}$$

which, by (A1) in Lemma 5.4, can happen only if  $C := A \cap \Lambda(x, \delta_n) \neq \emptyset$ . Since  $(A, B) \notin E$ , we have  $|C| < |A|$ . It follows from Lemma 5.4 (A2) and (A3) that

$$\begin{aligned} g_{n,m,|A|} \left( \sum_{\gamma \in A} \xi_{\delta_n}(\|Q_\gamma x\|_\gamma), \xi_{\delta_n}(\|P_B x - x\|) \right) &\leq g_{n,m,|C|} \left( \sum_{\gamma \in C} \xi_{\delta_n}(\|Q_\gamma x\|_\gamma), \xi_{\delta_n}(\|P_B x - x\|) \right) - \rho_n \\ &\leq g_{n,m,|C|} \left( \sum_{\gamma \in C} \xi_{\delta_n}(\|Q_\gamma x\|_\gamma), \xi_{\delta_n}(\|P_D x - x\|) \right) - \rho_n + \kappa_{n,m} \end{aligned}$$

for any  $D \subset C$ . Of course, since  $\Lambda(x, \delta_n)$  is finite, there are only finitely many couples  $(C, D)$  such that  $D \subset C \subset \Lambda(x, \delta_n)$ . We may therefore write

$$H_{n,m}x(A, B) \leq \max_{D \subset C \subset \Lambda(x, \delta_n)} H_{n,m}x(C, D) - \rho_n + \kappa_{n,m} \leq \|H_{n,m}\|_\infty (1 - \rho_n + \kappa_{n,m})$$

for any  $(A, B) \in F_n \setminus E$ . This together with (8) gives  $\lceil H_{n,m}x \rceil < \|H_{n,m}\|_\infty (1 - (\rho_n/2 - \kappa_{n,m}))$ .  $\square$

**Lemma 5.6.** *Let  $0 \neq x \in B_{(X, \|\cdot\|)}^O$  and let  $A$  be a finite subset of  $\Lambda$  such that  $Q_\gamma x \neq 0$  when  $\gamma \in A$ . We claim that, for all  $n, m \in \mathbb{N}$  sufficiently large, there exists a finite  $C_{n,m} \subset \Lambda$  such that*

- $A \subset C_{n,m}$ , and
- $H_{n,m}x(C_{n,m}, A) > (1 - \eta_{n,m})Z_{\eta_{n,m}}(H_{n,m}x)$ .

*Proof.* We start by defining  $A^* := \{\gamma \in \Lambda : \|Q_\gamma x\|_\gamma \geq \min_{\alpha \in A} \|Q_\alpha x\|_\alpha\}$  and we set out for finding  $C_{n,m}$  so that in fact  $A^* \subset C_{n,m}$ .

Let us investigate the mapping  $L_n : B_{(X, \|\cdot\|)}^O \rightarrow \ell^\infty(F_n)$  defined as

$$L_n y(D, E) := g_{n,1,|D|} \left( \sum_{\gamma \in D} \xi_{\delta_n}(\|Q_\gamma y\|_\gamma), 0 \right).$$

By the same argument as in the proof of Lemma 5.5, we get that  $[L_n x] \leq (1 - \rho_n) \|L_n x\|_\infty$  or  $L_n x = 0$ . Hence  $L_n x \in A_{\rho_n/2} \cup \{0\}$ . If  $n$  is large enough, necessarily  $L_n x \neq 0$ . It follows that  $L_n x$  attains a nonzero maximum. For  $n \in \mathbb{N}$ , let  $C_n$  be such that  $L_n x(C_n, D) = \|L_n x\|_\infty$  for some (and all) non-empty  $D \subset C_n$ . We claim that, for  $n$  sufficiently large,  $A^* \subset C_n$ .

Let us denote  $b := \min \left\{ \|Q_\gamma x\|_\gamma : \gamma \in A^* \right\} - \max \left\{ \|Q_\gamma x\|_\gamma : \gamma \in \Lambda \setminus A^* \right\}$ . Since  $Q_\gamma x \neq 0$  for all  $\gamma \in A$ , and for the  $c_0$ -nature of  $(\|Q_\gamma x\|_\gamma)_{\gamma \in \Lambda}$ , it follows that  $b > 0$ . Notice that

$$b_n := \xi_{\delta_n} \left( \min \left\{ \|Q_\gamma x\|_\gamma : \gamma \in A^* \right\} \right) - \xi_{\delta_n} \left( \max \left\{ \|Q_\gamma x\|_\gamma : \gamma \in \Lambda \setminus A^* \right\} \right) \rightarrow b \text{ as } n \rightarrow \infty.$$

Let  $n \geq |A^*|$  be so large that  $\delta_n < \xi_{\delta_n} \left( \min \left\{ \|Q_\gamma x\|_\gamma : \gamma \in A^* \right\} \right)$  and  $\delta_n < b_n$ .

If  $A^* \not\subset C_n$ , there exists  $\gamma_1 \in A^* \setminus C_n$ . If  $|C_n| < n$ , then we define  $\tilde{C}_n := \{\gamma_1\} \cup C_n$ . By our choice of  $n$ , we have that  $\xi_{\delta_n}(\|Q_{\gamma_1} x\|_{\gamma_1}) > \delta_n$  and so by the property (A5) in Lemma 5.4 we get that

$$g_{n,1,|C_n|} \left( \sum_{\gamma \in C_n} \xi_{\delta_n}(\|Q_\gamma x\|_\gamma), 0 \right) < g_{n,1,|\tilde{C}_n|} \left( \sum_{\gamma \in \tilde{C}_n} \xi_{\delta_n}(\|Q_\gamma x\|_\gamma), 0 \right)$$

contradicting that any couple  $(C_n, D) \in F_n$  maximizes  $L_n x$ .

If  $|C_n| = n$ , then there exists  $\gamma_2 \in C_n \setminus A^*$  and we define  $\tilde{C}_n := \{\gamma_1\} \cup C_n \setminus \gamma_2$ . Our choice of  $n$  yields that  $\xi_{\delta_n}(\|Q_{\gamma_1} x\|_{\gamma_1}) - \xi_{\delta_n}(\|Q_{\gamma_2} x\|_{\gamma_2}) > \delta_n$  so (A5) in Lemma 5.4 implies

$$g_{n,1,n} \left( \sum_{\gamma \in C_n} \xi_{\delta_n}(\|Q_\gamma x\|_\gamma), 0 \right) < g_{n,1,n} \left( \sum_{\gamma \in \tilde{C}_n} \xi_{\delta_n}(\|Q_\gamma x\|_\gamma), 0 \right)$$

once again contradicting that any couple  $(C_n, D) \in F_n$  maximizes  $L_n x$ . So  $A^* \subset C_n$ .

At this moment, we leave  $n$  fixed according to the choices above and we start tuning  $m$ . First of all, let us observe that  $L_n x(C_n, A) > Z_{\rho_n/2}(L_n x)(1 - \rho_n/2)$  by the moreover part of Lemma 5.1. Since  $\eta_{n,m} \nearrow \rho_n/2$  as  $m \rightarrow \infty$ , we deduce that there is some  $p \in \mathbb{N}$  such that  $L_n x(C_n, A) > Z_{\rho_n/2}(L_n x)(1 - \eta_{n,p})$ . We will work, for  $\gamma \in F_n$ , with the set  $M_\gamma = \{u \in \ell^\infty(F_n) : |u(\gamma)| > Z_{\rho_n/2}(u)(1 - \eta_{n,p})\}$ . The set  $M_\gamma$  is open and, in particular,  $L_n x \in M_{(C_n, A)}$ .

Using (A3) and (A4) in Lemma 5.4 we may see that  $H_{n,m} x \rightarrow L_n x$  in  $(\ell^\infty(F_n), \|\cdot\|_\infty)$  as  $m \rightarrow \infty$ . Since  $L_n x$  is a member of the open set  $A_{\rho_n/2}(F_n)$ , so will be  $H_{n,m} x$  for  $m$  large enough. Similarly, the openness of  $M_{(C_n, A)}$  insures that  $H_{n,m} x \in M_{(C_n, A)}$  for  $m \geq p$  and large enough. This means that

$$H_{n,m} x(C_n, A) > Z_{\rho_n/2}(H_{n,m} x)(1 - \eta_{n,p}) \geq Z_{\eta_{n,m}}(H_{n,m} x)(1 - \eta_{n,m})$$

where the second inequality follows from Lemma 5.2 as  $\rho_n/2 \geq \eta_{n,m}$  and  $\eta_{n,m} \geq \eta_{n,p}$  for all  $m \geq p$ . So we may define  $C_{n,m} := C_n$  for  $m$  sufficiently large.  $\square$

We came close to the definition of the norm  $J$ . First, we choose some decreasing sequence of positive numbers  $\sigma_j \searrow 0$  and we define  $J_{n,m} : B_{(X, \|\cdot\|)}^O \rightarrow \mathbb{R}$  as

$$J_{j,n,m}(x) := \xi_{\sigma_j}(Z_{\eta_{n,m}}(H_{n,m} x)).$$

Next, let  $\tilde{J} : B_{(X, \|\cdot\|)}^O \rightarrow \mathbb{R}$  be defined as

$$\tilde{J}^2(x) := \|x\|^2 + \sum_{j,n,m \in \mathbb{N}} \frac{1}{2^{j+n+m}} J_{j,n,m}^2(x)$$

and finally let  $J : X \rightarrow \mathbb{R}$  be defined as the Minkowski functional of  $\{x \in X : \tilde{J}(x) \leq 1/2\}$ .

**Lemma 5.7.** *The function  $J$  is an equivalent norm on  $X$  which is  $C^1$ -smooth away from the origin.*

*Proof.* By the Implicit Function Theorem, to see the differentiability it is sufficient to show that for each  $x \in X$  such that  $J(x) = 1$ , the function  $\tilde{J}$  is Fréchet differentiable on some neighborhood of  $x$  with  $\tilde{J}'(x)x \neq 0$ .

First of all let us observe that, for each  $\sigma > 0$  and  $\eta > 0$ , the composed function  $\xi_\sigma \circ Z_\eta : \ell^\infty(F_n) \rightarrow \mathbb{R}$  is  $C^\infty$ -smooth and LFC in  $A_\eta(F_n) \cup \{0\}$ . Of course it is – we know it already for points in  $A_\eta(F_n)$  and clearly, there is a neighborhood  $U$  of  $0 \in F_n$  such that  $\xi_\sigma \circ Z_\eta$  is constant in  $U$ .

Let  $x \in X$  such that  $J(x) = 1$ . Then  $\|x\| \leq 1/2$ . It follows from Lemma 5.5 and from Lemma 2.3 that each  $J_{j,n,m}$  is  $C^1$ -smooth at  $x$ . Further we claim that there is a constant  $K > 0$  such that each  $J_{j,n,m}$  is  $nK$ -Lipschitz. Indeed, there is a constant  $K' > 0$  such that  $H_{n,m}$  is  $(1 + 2nC)K'$ -Lipschitz for all  $n, m \in \mathbb{N}$ ;  $Z_\eta$  is 2-Lipschitz for each  $0 < \eta < 1/2$  and  $\xi_\sigma$  is 1-Lipschitz for each  $\sigma > 0$ . It follows that  $\tilde{J}$  is  $K''$ -Lipschitz for some  $K'' > 0$ . The calculus rules lead to the conclusion that  $\tilde{J}$  is Fréchet differentiable on a neighborhood of any  $x \in X$  such that  $\|x\| < 1$ ; then the convexity of all terms implies that  $\tilde{J}'(x)x > 0$ .

Finally,  $2\|x\| \leq J(x) \leq 2K''\|x\|$  where the second inequality follows from the  $K''$ -Lipschitzness of  $\tilde{J}$ .  $\square$

**Lemma 5.8.** *Whenever  $x_r, x \in X$ ,  $r \in \mathbb{N}$ , are such that (3) holds, then for each  $\varepsilon > 0$  there is a finite subset  $A$  of  $\Lambda$  such that  $Q_\gamma x \neq 0$  for  $\gamma \in A$ ,  $\|P_A x - x\| < \varepsilon$  and  $\|P_A x_r - x_r\| < \varepsilon$  for all  $r$  sufficiently large.*

*Proof.* We may assume, that  $J(x) = 1$ . We start by finding a finite  $A \subset \Lambda$  such that  $\|P_A x - x\| < \varepsilon/2$  and such that  $Q_\gamma x \neq 0$  for  $\gamma \in A$ . This is possible by Lemma 2.11. Now we will just show that  $\|P_A x_r - x_r\| \rightarrow \|P_A x - x\|$ .

It follows from (3) and from the uniform continuity of  $\tilde{J}$  on bounded sets that

$$\frac{\tilde{J}^2(x_r) + \tilde{J}^2(x)}{2} - \tilde{J}^2\left(\frac{x + x_r}{2}\right) \rightarrow 0 \text{ as } k \rightarrow \infty. \quad (9)$$

By the convexity of the terms in the definition of  $\tilde{J}$ , we get that

$$\frac{J_{j,n,m}^2(x_r) + J_{j,n,m}^2(x)}{2} - J_{j,n,m}^2\left(\frac{x + x_r}{2}\right) \rightarrow 0 \text{ as } k \rightarrow \infty. \quad (10)$$

for each  $j, n, m \in \mathbb{N}$ .

Let us borrow the notation  $L_n x$  from the proof of Lemma 5.6. Let us recall that  $H_{n,m} x \geq L_n x \geq 0$  (in the lattice  $\ell^\infty(F_n)$ ) for all  $m \in \mathbb{N}$ . There is some  $n_0 \in \mathbb{N}$  such that for all  $n \geq n_0$  we have that  $L_n x \neq 0$ . Hence  $Z_{\eta_{n,m}}(H_{n,m} x) \geq Z_{\eta_{n,m}}(L_n x) \geq Z_{\eta_{n,1}}(L_n x) > 0$  for  $n \geq n_n$  and  $m \in \mathbb{N}$ . Therefore for each  $n \geq n_0$  there exists  $j_n \in \mathbb{N}$  such that for all  $j \geq j_n$  and all  $m \in \mathbb{N}$  one has  $J_{j,n,m} > 0$ . Since, for  $n \geq n_0, m \in \mathbb{N}$  and  $j \geq j_n$ , (10) is equivalent to

$$\lim_r J_{j,n,m}(x_r) = J_{j,n,m}(x) = \lim_r J_{j,n,m}\left(\frac{x_r + x}{2}\right)$$

and  $\xi_\sigma \upharpoonright_{(\sigma, +\infty)}$  has a continuous inverse, it follows that

$$\frac{Z_{\eta_{n,m}}^2(H_{n,m} x_r) + Z_{\eta_{n,m}}^2(H_{n,m} x)}{2} - Z_{\eta_{n,m}}^2\left(H_{n,m}\left(\frac{x_r + x}{2}\right)\right) \xrightarrow{r} 0$$

for all  $n \geq n_0$  and  $m \in \mathbb{N}$ . Since  $x \mapsto H_{n,m} x(A, B)$  is convex and non-negative for each  $(A, B) \in F_n$  and since  $Z_{\eta_{n,m}}$  is strongly lattice and convex it follows

$$\begin{aligned} 0 &\leftarrow \frac{Z_{\eta_{n,m}}^2(H_{n,m} x_r) + Z_{\eta_{n,m}}^2(H_{n,m} x)}{2} - Z_{\eta_{n,m}}^2\left(H_{n,m}\left(\frac{x_r + x}{2}\right)\right) \geq \\ &\geq \frac{Z_{\eta_{n,m}}^2(H_{n,m} x_r) + Z_{\eta_{n,m}}^2(H_{n,m} x)}{2} - Z_{\eta_{n,m}}^2\left(\frac{H_{n,m} x_r + H_{n,m} x}{2}\right) \geq 0 \end{aligned}$$

for every  $n \geq n_0$  and  $m \in \mathbb{N}$ . Let us fix  $n \geq n_0$  and  $m \in \mathbb{N}$  both large enough in the sense of Lemma 5.6. We also require that  $\delta_n < \|P_A x - x\|$ . By application of Lemma 5.6, we obtain a set  $C_{n,m}$  such that  $\gamma := (C_{n,m}, A) \in F_n$  satisfies the assumptions of Lemma 5.3. Thus, using this last mentioned lemma, we may conclude that  $H_{n,m} x_r(C_{n,m}, A) \rightarrow H_{n,m} x(C_{n,m}, A)$  as  $k \rightarrow \infty$ .

To finish the argument, we employ Lemma 4.1 to see that

$$\sum_{\gamma \in C_{n,m}} \xi_{\delta_n}(\|Q_\gamma x_r\|_\gamma) \rightarrow \sum_{\gamma \in C_{n,m}} \xi_{\delta_n}(\|Q_\gamma x\|_\gamma) \text{ as } r \rightarrow \infty$$

and we apply Lemma 5.4 (A6) on the function  $g_{n,m}|_{C_{n,m}}$ . This leads to  $\xi_{\delta_n}(\|P_A x_r - x_r\|) \rightarrow \xi_{\delta_n}(\|P_A x - x\|)$  which means that  $\|P_A x_r - x_r\| \rightarrow \|P_A x - x\|$  by our choice of  $n$ .  $\square$

In the end of all we are going to show that  $J$  is a limit of  $C^k$ -smooth norms. A self-evident choice for the approximating norms  $J_i$  is as follows. Let us define

$$\begin{aligned} H_{n,m}^i x(A, B) &:= g_{n,m,l} \left( \sum_{\gamma \in A} \xi_{\delta_n}(\|Q_\gamma x\|_{\gamma,i}), \xi_{\delta_n}(\|P_B x - x\|) \right), \\ J_{j,n,m,i}(x) &:= \xi_{\sigma_j}(Z_{\eta_{n,m}}(H_{n,m}^i x)), \\ \tilde{J}_i^2(x) &:= \|x\|^2 + \sum_{1 \leq j,n,m \leq i} \frac{1}{2^{j+n+m}} J_{j,n,m,i}^2(x) \end{aligned}$$

and  $J_i$  as the Minkowski functional of  $\{x \in X : \tilde{J}_i(x) \leq 1/2\}$ . As a finite sum of  $C^k$ -smooth functions,  $\tilde{J}_i$  is  $C^k$ -smooth. The Implicit Function Theorem implies the same about  $J_i$ . Moreover  $2\|x\| \leq J_i(x) \leq 2K''\|x\|$  as in the proof of Lemma 5.7. Let  $\varepsilon > 0$  be given. We will show that there is an index  $i_0 \in \mathbb{N}$  such that  $|\tilde{J}_i^2(x) - \tilde{J}^2(x)| < \varepsilon$  whenever  $\|x\| < 1$  and  $i \geq i_0$ . For this it is sufficient that  $\left(\frac{2C}{i_0}\right)^2 < \varepsilon/2$  and

$$\sum_{\max\{j,n,m\} \geq i_0} \frac{2}{2^{j+n+m}} < \varepsilon/2$$

because then, for each  $i \geq i_0$ ,

$$\begin{aligned} |\tilde{J}_i^2(x) - \tilde{J}^2(x)| &\leq \sum_{1 \leq j,n,m \leq i} \frac{1}{2^{j+n+m}} (J_{j,n,m,i}^2(x) - J_{j,n,m}^2(x)) + \sum_{\max\{j,n,m\} \geq i_0} \frac{1}{2^{j+n+m}} J_{j,n,m}^2(x) \\ &< \sum_{1 \leq j,n,m \leq i} \frac{1}{2^{j+n+m}} \left(\frac{2Ci}{i^2}\right)^2 + \varepsilon/2 < \varepsilon \end{aligned}$$

where in the second inequality we are using (2) and  $(1 - \frac{1}{i^2})\|x\|_\gamma \leq \|x\|_{\gamma,i} \leq \|x\|_\gamma$  to estimate the first term and  $J_{j,n,m}(x) \leq 2$  for  $\|x\| < 1$ . This proves that  $\tilde{J}_i \rightarrow \tilde{J}$  uniformly on  $B_{(X,\|\cdot\|)}^O$ .

Now let us observe that, since  $\tilde{J}(0) = 0$ , we have the estimate

$$\frac{1}{2} |\lambda - 1| \leq \left| \frac{1}{2} - \tilde{J}(\lambda x) \right| \tag{11}$$

for all  $x \in X$  such that  $\tilde{J}(x) = \frac{1}{2}$ , or equivalently such that  $J(x) = 1$ .

We assume that there is a sequence  $(x_i) \subset B_{(X,\|\cdot\|)}^O$  such that  $J_i(x_i) - J(x_i) \not\rightarrow 0$ . Let  $c_i > 0$ , resp.  $d_i > 0$ , be such that  $J(c_i x_i) = 1$ , resp.  $J_i(d_i x_i) = 1$ . It follows that  $\lambda_i := \frac{d_i}{c_i} \not\rightarrow 1$  so we may and do assume that there is some  $\varepsilon > 0$  such that  $|\lambda_i - 1| > 2\varepsilon$  for all  $i \in \mathbb{N}$ . On the other hand, since  $\|d_i x_i\| \leq \frac{1}{2}$  and since  $\tilde{J}_i \rightarrow \tilde{J}$  uniformly on  $B_{(X,\|\cdot\|)}^O$ , we get that  $\left| \tilde{J}(\lambda_i c_i x_i) - \frac{1}{2} \right| = \left| \tilde{J}(\lambda_i c_i x_i) - \tilde{J}_i(d_i x_i) \right| \leq \varepsilon$  for  $i$  large enough. Thus, having in mind (11), we obtain  $|\lambda_i - 1| \leq 2\varepsilon$ . As a result of this contradiction we see immediately that  $J_i \rightarrow J$  uniformly on bounded sets.

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