Uniform minimality, unconditionality and interpolation in backward shift invariant spaces

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UNIFORM MINIMALITY, UNCONDITIONALITY AND INTERPOLATION IN BACKWARD SHIFT INVARIANT SPACES

ERIC AMAR & ANDREAS HARTMANN

ABSTRACT. We discuss relations between uniform minimality, unconditionality and interpolation for families of reproducing kernels in backward shift invariant subspaces. This class of spaces contains as prominent examples the Paley-Wiener spaces for which it is known that uniform minimality does in general neither imply interpolation nor unconditionality. Hence, contrarily to the situation of standard Hardy spaces (and other scales of spaces), changing the size of the space seems in this context necessary to deduce unconditionality or interpolation from uniform minimality. Such a change can take two directions: lowering the power of integration, or "increasing" the defining inner function (e.g. increasing the type in the case of Paley-Wiener space).

1. INTRODUCTION

A famous result by Carleson states that a sequence of points $S = \{a_k\}$ in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is an interpolating sequence for the space H^∞ of bounded analytic functions on $\mathbb{D}$, meaning that every bounded sequence on S can be interpolated by a function f in H^∞ on S, i.e. $H^\infty|S \supset l^\infty$, if and only if the sequence S satisfies the Carleson condition:

$$\inf_{a\in S} |B_a(a)| = \delta > 0, \tag{1.1}$$

where $B_a = \prod_{u\neq a} b_u$ is the Blaschke product vanishing exactly on $S \setminus \{a\}$, and $b_a(z) = \frac{|a|}{a}\frac{a-z}{1-\overline{a}z}$ (see [Ca58]). We will write $S \in (C)$ for short when S satisfies (1.1). Obviously in this situation we also have the embedding $H^\infty|\Lambda \subset l^\infty$, so that $S \in (C)$ is equivalent to $H^\infty|\Lambda = l^\infty$. Subsequently it was shown by Shapiro and Shields [SS61] that for $p \in (1,\infty)$ a similar result holds:

$$H^p|S \supset l^p(1-|a|^2) = \{(v_a)_{a\in S} : \sum_{a\in S}(1-|a|^2)|v_a|^p < \infty\}$$

if and only if $S \in (C)$. Again, it turns out that we also have $H^p|S \subset l^p(1-|a|^2)$ (the measure $\sum_{a\in S}(1-|a|^2)\delta_a$ is a so-called Carleson measure), so that $S \in (C)$ is equivalent to $H^p|S = l^p(1-|a|^2)$. Considering reproducing kernels $k_a(z) = (1-\overline{a}z)^{-1}$ the interpolation condition and the Carleson condition can be restated in terms of geometric properties of the sequence $(k_a)_{a\in S}$. More precisely the Carleson condition is equivalent to $(k_a/\|k_a\|_{p'})_a$ being uniformly minimal in $H^{p'}$, and the interpolating condition $H^p|S = l^p(1-|a|)$ to $(k_a/\|k_a\|_{p'})_{a\in S}$ being an unconditional sequence in $H^{p'}$ (precise definitions will be given below). Hence, another way of stating the interpolation result in Hardy spaces is to say that a sequence of normalized

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reproducing kernels in $H^{p'}$ is uniformly minimal if and only if it is an unconditional basis in its span (since interpolation in the scale of Hardy spaces does not depend on p, the distinction between p and p' may appear artificial here). This special situation is not isolated. It turns out to be true in the Bergman space (see [SchS98]), and in Fock spaces and Paley-Wiener spaces for certain indices of p (see [SchS00]).

More recently, in [Am08] the first named author has given a method allowing to deduce interpolation from uniform minimality when the size of the space is increased by lowering the power of integration. This result requires that the underlying space is the closure of a uniform algebra, and applies in particular to Hardy spaces on the ball.

We would like to use some of the methods discussed in [Am08] to show that uniform minimality implies unconditionality in a bigger space for certain backward shift invariant subspaces K_I^p for which the Paley-Wiener spaces are a particular instance. Recall that for an inner function I, $K_I^p = H^p \cap I\overline{H_0^p}$ (when considered as a space of functions on $\mathbb{T}$), which is equal to the orthogonal complement of IH^2 when $p = 2$. Note also that these spaces are projected subspaces of H^p ($1 < p < \infty$), and the projection — orthogonal when $p = 2$ — is given by $P_I = IP_-\overline{I}$, where $P_- = Id - P_+$ and P_+ is the Riesz projections of $f(e^{it}) = \sum_{n\in\mathbb{Z}} a_n e^{int} \in L^p(\mathbb{T})$ onto the analytic part $\sum_{n\geq 0} a_n e^{int}$. We would like to draw the attention of the reader to the special situation when $I(z) = I_\tau(z) := \exp(2\tau(z+1)/(z-1))$. Then, the space K_I^p is isomorphic to the Paley-Wiener space PW_τ^p of entire functions of exponential type τ and p-th power integrable on the real line (see Section 3). By the Paley-Wiener theorem, PW_τ^2 is isometrically isomorphic to $L^2(-\tau, \tau)$. Already in this "simple" case the description of interpolating sequences is not known (see more comments below). There exist sufficient density conditions for interpolation (or unconditionality) when $p = 2$. They allow to check that a certain uniform minimal sequence, which is not unconditional, becomes unconditional when we "increase" the inner function meaning that we replace I by $I^{1+\varepsilon}$, $\varepsilon > 0$. (It is well known that $K_I^2 \subset K_{I^{1+\varepsilon}}^2$ and even $K_{I^{1+\varepsilon}}^2 = \overline{K_I^2 + IK_{I^\varepsilon}^2}$.) The density conditions for $p = 2$ do not seem to generalize to $p \neq 2$ (see Proposition 3.2 and comments at the end of Section 3), so that there is no easy argument that could show that lowering the integration power without changing I is sufficient to deduce unconditionality from uniform minimality. This makes the problem extremely delicate. So, in the general situation that we consider and where density or other usable conditions are not known, it seems extremly difficult to deduce interpolation from uniform minimality only be increasing the space in one direction (either adding factors to I or lowering the integration power p). Let us mention however that under the assumption $I(\lambda_n) \to 0$ the equivalence between uniform minimality and unconditionality in K_I^2 has been established in [HNP81] (see also [Fr99] for a vector valued version of this result).

Our results will require some conditions on the inner function such as being one-component. This means that the level set $L(I, \varepsilon) = \{z \in \mathbb{D} : |I(z)| < \varepsilon\}$ of the inner function I is connected for some $\varepsilon \in (0, 1)$ (which is for instance the case for I_τ). One-component inner functions appear in work by Aleksandrov, Treil-Volberg etc. in the connection with embedding theorems and Carleson measures.

As a consequence of our discussions we state here a sample result:

Theorem 1.1. *Let I be a one-component singular inner function, $S \subset \mathbb{D}$, $1 < p \leq 2$. Suppose that* $\sup_{a\in S} |I(a)| < 1$. *If $(k_a^I/\|k_a^I\|_{p'})_{a\in S}$ is uniformly minimal in $K_I^{p'}$, where* $1/p + 1/p' =$

1*, then for every* $\varepsilon > 0$ *and for every* $s < p$*,* S *is an interpolating sequence for* $K^s_{I^{1+\varepsilon}}$ *and* $(k^I_a/\|k^I_a\|_{s'})_{a\in S}$ *is an unconditional sequence in* $K^{s'}_{I^{1+\varepsilon}}$*,* $1/s + 1/s' = 1$.

As already pointed out, a characterization of interpolating sequences already for Paley-Wiener spaces is unknown for general p (when $p = \infty$ Beurling gives a characterization, and for $0 < p \leq 1$, see [Fl95]; a crucial difference between these cases and $1 < p < \infty$ is the boundedness of the Hilbert transform on L^p). For the case of complete interpolating sequences in PW^p_τ, i.e. interpolating sequences for which the interpolating functions are unique, these are characterized in [LS97] appealing to the Carleson condition and the Muckenhoupt A^p-condition for some function associated with the generating function of S. Sufficient conditions are pointed out in [SchS00] using a kind of uniform zero-set condition in the spirit of Beurling. Such a condition cannot be necessary since there are complete interpolating sequences in the Paley-Wiener spaces. Another approach is based on invertibility properties of $P_I|K^p_B$, where $B = \prod_{a\in S} b_a$ and discussed in the seminal paper [HNP81] (see also [Ni02]). Once having observed that the Carleson condition for S is necessary (under the condition $\sup_{a\in S}|I(a)| < 1$), and so $(k_a/\|k_a\|_p)_{a\in S}$ is an unconditional basis for K^p_B, the left invertibility of $P_I|K^p_B$ guarantees that $(k^I_a/\|k^I_a\|_p)_{a\in S}$ is still an unconditional sequence. The invertibility properties of $P_I|K^p_B$ can be reduced to the invertibility properties of a certain Toeplitz operator ($T_{I\overline{B}}$). Again, and also in this approach, one can feel an essential difference between complete interpolating sequences and not necessary complete interpolating sequences. The case of complete interpolating sequences corresponds to invertibility of $T_{I\overline{B}}$, and a criterion of invertibility of Toeplitz operators is known. This is the theorem of Devinatz and Widom (see e.g. [Ni02, Theorem B4.3.1]) for $p = 2$ and Rochberg (see [Ro77]) for $1 < p < \infty$, and again it is based on the Muckenhoupt (A_p) condition (or the Helson-Szegő condition in case $p = 2$), this time for some function $h \in H^p$ such that $I\overline{B} = \overline{h}/h$. A useful description of left-invertibility of Toeplitz operators, the situation corresponding to general not necessarily complete interpolating sequences, is not available. For the case $p = 2$ an implicit condition is given in [HNP81], and a condition based on extremal functions in the kernel of the adjoint $T_{\overline{I}B}$ can be found in [HSS04].

The paper is organized as follows. In the next section we introduce the necessary material on uniform minimality, dual boundedness and unconditionality. A characterization of unconditional bases of point evaluations (or reproducing kernels) will be given in terms of interpolation and embedding. We will also discuss some Carleson-type conditions which are naturally connected with embedding problems. Section 3 is devoted to a longer discussion of the situation in the Paley-Wiener spaces. We essentially put the known material in the perspective of our work. This should convince the reader that it is difficult to get better result. In the last section we give our main result Theorem 4.5 which as a special case contains Theorem 1.1.

2. Preliminaries

2.1. **Geometric properties of families of vectors of Banach spaces.** We begin with some observations in the classical H^p concerning the relation between uniform minimality and unconditionality. Recall that the reproducing kernel of H^p in $a \in \mathbb{D}$ is given by $k_a(z) = (1 - \overline{a}z)^{-1}$. The Carleson condition $\inf_{a\in S}|B_a(a)| \geq \delta > 0$ can then be restated as $(k_a/\|k_a\|_{p'})_{a\in S}$ being a uniformly minimal sequence in $H^{p'}$ (which is equivalent here to $(k_a/\|k_a\|_p)_{a\in S}$ being uniformly minimal in H^p). Let us explain this a little bit more. By definition a sequence of normalized vectors $(x_n)_n$ in a Banach space X is uniformly minimal if

$$\inf_n \operatorname{dist}(x_n, \bigvee_{k\neq n} x_k) = \delta > 0. \tag{2.1}$$

(Here $\bigvee_i x_i$ denotes the closed linear span of the vectors x_i.) By the Hahn-Banach theorem this is equivalent to the existence of a sequence of functionals $(\varphi_n)_n$ in X^* such that $\varphi_n(x_k) = \delta_{nk}$, where δ_{nk} is the usual Kronecker symbol, and $\sup_n \|\varphi\|_{X^*} < \infty$. In our situation, setting

$$\varphi_a = \frac{B_a}{B_a(a)} \frac{k_a}{k_a(a)} \|k_a\|_p,$$

we get

$$\langle \varphi_a, \frac{k_b}{\|k_b\|_p} \rangle = \delta_{ab}.$$

Since $\|k_a\|_s \simeq (1-|a|^2)^{1-1/s}$ we moreover have $\sup_{a\in S} \|\varphi_a\|_q < \infty$. Another way of viewing the uniform minimality condition when $p = 2$ is given in terms of angles: a sequence $(x_n)_n$ of vectors in a Hilbert space is uniformly minimal if the angles between x_n and $\bigvee_{k\neq n} x_k$ are uniformly bounded away from zero.

A notion closely related with uniform minimality is that of dual boundedness (see [Am08]). Let us give a general definition

Definition 2.1. Let $X \subset \operatorname{Hol}(\Omega)$ be a reflexive Banach space of holomorphic functions on a domain Ω. Suppose that the point evaluations E_z are continuous for every $z \in \Omega$. A sequence $S \subset \Omega$ is called dual-bounded if the sequence $(E_a/\|E_a\|_{X^*})_{a\in S}$ of reproducing kernels is uniformly minimal.

Again, by the Hahn-Banach theorem this means that there exists a sequence $(\rho_a)_{a\in S}$ of elements in X $(= X^{**})$ with uniformly bounded norm $\sup_{a\in S} \|\rho_a\|_X < \infty$ and $\langle \rho_a, E_b/\|E_b\|_{X^*}\rangle = \delta_{ab}$, i.e. $\rho_a(b) = \delta_{ab}\|E_b\|_{X^*}$.

This condition is termed weak interpolation in [SchS00].

Let us discuss the unconditionality. Recall that a basis $(x_n)_n$ of vectors in a Banach space X is an unconditional basis if for every $x \in X$, there exists a numerical sequence (α_n) such that the sum $\sum_n \alpha_n x_n$ converges to x, and for every sequence of signs $\varepsilon = (\varepsilon_n)$, the sum $\sum_n \varepsilon_n \alpha_n x_n$ converges in X to a vector x_ε with norm comparable to $\|x\|$. We will discuss the interpolation condition $H^p|\Lambda \supset l^p(1-|a|^2)$ in the light of this definition using reproducing kernels. First recall from [SS61] that we have $H^p|\Lambda = l^p(1-|a|^2)$. Let $B = B_S$ be the Blaschke product vanishing on S. Set $K_B^p = H^p \cap B\overline{H_0^p}$, where $H_0^p = zH^p$. The space K_B^p is a backward shift invariant subspace. Also, $K_B^p = \bigvee_{a\in S} k_a$, and $H^p = K_B^p + BH^p$ ($K_B^p = P_B H^p$ is a projected space). So the interpolation condition is equivalent to $K_B^p|\Lambda = l^p(1-|a|^2)$, and since the interpolation problem has unique solution in K_B^p, we have for every $f \in K_B^p$, $\|f\|_p^p \simeq \sum_n (1-|a|^2)|f(a)|^p$. Clearly under this condition the functions φ_a introduced above exist and are in K_B^p. Then for every finite sequence (v_a) and every sequence of signs (ε_n) we have

$$\|\sum_{a\in S} \varepsilon_a v_a \varphi_a\|_p^p \simeq \sum_{a\in S} (1-|a|^2)|\varepsilon_a|^p |v_n|^p \simeq \sum_{a\in S} (1-|a|^2)|v_n|^p$$

which shows that $(\varphi_a)_a$ is an unconditional basis in K_B^p. Then $(k_a/\|k_a\|_p)$ is also an unconditional basis in K_B^p.

Again, the unconditionality can be expressed in terms of angles when $p = 2$: a sequence $(x_n)_n$ of vectors in a Hilbert space is unconditional if the angles between $\bigvee_{k\in\sigma} x_k$ and $\bigvee_{k\in\mathbb{N}\setminus\sigma} x_k$ is uniformly bounded away from zero for every $\sigma \in \mathbb{N}$.

So the interpolation results tell us that in H^p a sequence of reproducing kernels is uniformly minimal if and only if it is an unconditional sequence. Such results also hold in other spaces like e.g. Bergman spaces (see [SchS98]) and in Fock and Paley-Wiener spaces for certain values of p (see [SchS00]).

We will be interested in the situation in backward shift invariant subspaces K_I^p.

2.2. **Unconditional bases and interpolation.** In this section we will establish a general link between unconditional basis on the one hand and interpolation with an additional embedding property on the other hand. It turns out that this link can be reformulated, in the spirit of [Ni78, Theorem1.2], in abstract terms without appealing to the notion of interpolation. We will start with this general result before coming back to the special context of interpolation.

Suppose that X is a reflexive Banach space, and let $(y_n)_n$ be a sequence of normalized elements in X^* that we suppose at least minimal: $\operatorname{dist}(y_n, \bigvee_{k\neq n} y_k) > 0$ for every $n \in \mathbb{N}$. We set $Y = \bigvee y_n$ and $N := Y^\perp \subset (X^*)^* = X$. By the minimality condition there exists a sequence $(x_n)_n \in X^{**} = X$ such that $\langle x_n, y_k\rangle_{X=X^{**},X^*} = \delta_{n,k}$, $n, k \in \mathbb{N}$.

For a sequence space l, we consider the canonical system $\{e_n\}_n$ where $e_n = (\delta_{nk})_k$. The space l will be called ideal if whenever $(a_n)_n \in l$ and $|b_n| \le |a_n|$, $n \in \mathbb{N}$, then $(b_n) \in l$. Recall also that a family of vectors in a Banach space is called fundamental if it generates a dense set in the Banach space. Observe that the canonical system is an unconditional basis in l if and only if l is ideal and the canonical system is fundamental in l.

We obtain the following result.

Proposition 2.2. *Let X be a reflexive Banach space. With the above notation, the following assertions are equivalent.*

(1) *The sequence $(y_n)_n$ is an unconditional basis in $Y = \bigvee_n y_n$.*
(2) *The sequence $(x_n + N)_n$ is an unconditional basis in X/N (in general not normalized).*
(3) *There exists two reflexive Banach sequence spaces l_1, l_2, in which the respective canonical systems are unconditional bases and such that*
  (i) *The set of generalized Fourier coefficients of X contains l_1:*

$$\{\langle x, y_n\rangle_{X,X^*})_n : x \in X\} \supset l_1,$$

  (ii) *for every $\mu = (\mu_n)_n \in l_2$,*

$$\|\sum_n \mu_n y_n\|_{X^*} \lesssim \|\mu\|_{l_2};$$

moreover $l_2 \simeq l_1^*$ and the duality of l_1 and $l_1^* \simeq l^2$ is given by $\langle(\alpha_n)_n, (\mu_n)_n\rangle_{l_1,l_2} = \sum_n \alpha_n\mu_n$.

This theorem is in the spirit of [Ni78, Theorem 1.2]. However, in Nikolski's theorem there does not really appear the condition (i) together with an embedding of type (ii). The condition (i) will later on play the rôle of the interpolation part.

Proof. . Observe first that $Y^* = (X^*)^*/Y^\perp = X/N$. Moreover, for every $u \in N = Y^\perp$, $\langle x_n + u, y_k\rangle = \langle x_n, y_n\rangle = \delta_{nk}$, and hence $((y_n)_n, (x_n+N)_n)$ is a biorthogonal system in (Y, Y^*). By the general theory (see for instance [Sing70, Corollary I.12.2, Theorem II.17.7]) we obtain the equivalence of (1) and (2).

Let us now prove that (1) and (2) imply (3). By [Ni78, Theorem 1.1] the sequence $(y_n)_n$ is an unconditional sequence in Y if and only if the multiplier space $\mathrm{mult}(y_n) := \{\mu = (\mu_n)_n : T_\mu : \mathrm{Lin}(y_n) \longrightarrow \mathrm{Lin}(y_n), \sum_{finite} \alpha_n y_n \longmapsto \sum_{finite} \mu_n \alpha_n y_n$ extends to a bounded operator on $Y\}$ is equal to l^∞. And this, by [Ni78, Lemma 1.2] is equivalent to the existence of a sequence space l_2 in which the canonical system is an unconditional basis such that $(y_n)_n$ is a l_2-basis, which means that

$$\begin{array}{rcl} T : l_2 & \longrightarrow & Y \\ (\mu_n)_n & \longmapsto & \displaystyle\sum_n \mu_n y_n \end{array}$$

is an isomorphism. Note that Y is reflexive as a closed subspace of the reflexive Banach space X^*, and so is l_2.

For exactly the same reason, by (2) there exists a sequence space l_1 with the required properties such that

$$\begin{array}{rcl} S : l_1 & \longrightarrow & X/N \\ (\alpha_n)_n & \longmapsto & \displaystyle\sum_n \alpha_n x_n =: x_\alpha + N \end{array}$$

is an isomorphism. Note that X/N is reflexive as a quotient space of the reflexive Banach space X, and so is l_1. Take $(\alpha_n)_n \in l_1$, then $S((\alpha_n)_n) = x_\alpha + N \in X/N$ for a suitable $x_\alpha \in X$. Now $(\langle x_\alpha, y_n\rangle)_n = (\langle \sum_k \alpha_k x_k, y_n\rangle)_n = (\alpha_n)_n$ (note that $\sum_k \alpha_k x_k + N$ converges in X/N). So $(\alpha_n)_n \in \{(\langle x, y_n\rangle) : x \in X\}$.

Finally, since $l_1 \simeq X/N$, $l_2 \simeq Y$ and $Y^* = X/N$ we have $l_2^* \simeq l_1$ and by reflexivity $l_2 \simeq l_1^*$. Moreover, by the idenfication maps we can write for $(\alpha_n)_n \in l_1$ and $(\mu_n)_n \in l_2 \simeq l_1^*$:

$$\langle (\alpha_n)_n, (\mu_n)_n\rangle_{l_1,l_2} = \langle \sum_n \alpha_n x_n + N, \mu_k y_n\rangle_{X/N,Y} = \sum_{n,k} \alpha_n \mu_k \langle x_n, y_n\rangle_{X,Y} = \sum_n \alpha_n \mu_n.$$

We finish by showing that (3) implies (1). By (ii), the operator T is bounded and by construction onto, so that we are done if we can show that T is left invertible: $\|\mu\|_{l_2} \lesssim \|T\mu\|_Y$. Now by (i) for $(\alpha_n)_n \in l_1$, there exists $x_\alpha \in X$ such that $\alpha_n = \langle x_\alpha, y_n\rangle$. Let us introduce the operator

$$\begin{array}{rcl} A : l_1 & \longrightarrow & X/N \\ (\alpha_n)_n & \longmapsto & x_\alpha + N. \end{array}$$

This operator is well defined (if we choose x'_α with $\langle x'_\alpha, y_n\rangle = \alpha_n$, then $\langle x'_\alpha - x_\alpha, y_n\rangle = 0$ for every n and $x'_\alpha - x_\alpha \in N$). It is also linear. Let us check that its graph is closed. For this consider a sequence $(\alpha_n^N)_n$ converging to $(\alpha_n)_n$ in l_1. Since the canonical basis is an unconditional basis in l_1, we obtain coordinate-wise convergence: $\alpha_n^N \to \alpha_n$ when $N \to \infty$. We assume that $A((\alpha_n^N)_n) = x_{\alpha^N} + N \to x + N$. Note that $A((\alpha_n)_n) = x_\alpha + N$. Then for every n we have $\langle x, y_n\rangle = \lim_{N\to\infty} \langle x_{\alpha^N}, y_n\rangle = \lim_{N\to\infty} \alpha_n^N = \alpha_n = \langle x_\alpha, y_n\rangle$. So $x - x_\alpha \in N$ and $x + N = A((\alpha_n)_n)$. By the closed graph theorem A is bounded.

Let us show that $A^* : (X/N)^* = Y \to l_1^*$ is the left inverse to T (modulo the isomorphism from l_1^* to l_2). Equivalently it is sufficient to show that $T^*A : l_1 \to l_2^*$ is an isomorphism. Note that for $(\alpha_n)_n \in l_1$ and $(\mu_n)_n \in l_2$, we have

$$\begin{aligned}\langle T^*A(\alpha_n)_n, (\mu_n)_n\rangle_{l_2^*,l_2} &= \langle A(\alpha_n)_n, T(\mu_n)_n\rangle_{X/N,Y} = \langle x_\alpha + N, \sum \mu_n y_n\rangle_{X/N,Y} \\ &= \sum_n \mu_n \langle x_\alpha, y_n\rangle_{X,X^*} \\ &= \sum_n \mu_n \alpha_n\end{aligned}$$

By assumption this is equal to $\langle(\alpha_n)_n, (\mu_n)_n\rangle_{l_1,l_2}$ so that for every $(\alpha_n)_n \in l_1$ and $(\mu_n)_n \in l_2$, we have

$$\langle T^*A(\alpha_n)_n, (\mu_n)_n\rangle_{l_2^*,l_2} = \langle(\alpha_n)_n, (\mu_n)_n\rangle_{l_1,l_2}.$$

Hence T^*A is the identity (modulo the identification between l_1 and l_2^*). ■

It is interesting to note that when X is a Hilbert space more can be said about the structure of l: it is clear that then $l = l^2$. However, by a result of Lindenstrauss and Zippin (see [LZ69]), if in a Banach space X every two normalized unconditional bases are isomorphic to each other, then X is isomorphic to one of the following spaces c_0, l^1 or l^2. In other words the general theory does not yield $l = l^p$ when $(x_n)_n$ is an unconditional basis in (a subspace of) $X = L^p$ (Pelczynski constructed actually unconditional bases in l^p which are not equivalent to the canonical basis, [Pe60]).

Let X be Banach space of holomorphic functions on $\mathbb{D}$, such that the point evaluations E_a in $a \in \mathbb{D}$ are continuous in X. A sequence $S \subset \mathbb{D}$ is called l-interpolating for a sequence space l (defined on S) if for every sequence $v = (v_a)_{a\in S}$ with $(v_a/\|E_a\|_{X^*})_{a\in S} \in l$ there is a function $f \in X$ with $f(a) = v_a$, i.e.

$$X|S \supset l(1/\|E_a\|_{X^*}) := \{v = (v_a)_{a\in S} : (v_a/\|E_a\|_{X^*})_{a\in S} \in l\}.$$

Since, $\|E_a\|_{(H^p)^*} \simeq \|k_a\|_{p'} \simeq (1-|a|^2)^{-1/p}$ ($1 < p < \infty$), this definition is consistent with the definitions we gave before for H^p, in which case we had chosen $l = l^p$.

The reader should also note that in the previous subsection we have repeatedly used the fact that interpolation in H^p, i.e. $H^p|S \supset l^p(1-|a|^2)$ (we will not consider the case $p = \infty$ here) implies in fact the equality $H^p|S = l^p(1-|a|^2)$ (this is Shapiro and Shields' result, [SS61]).

In the general case, without any further information, we have to impose an additional embedding. For the convenience of the proof in the following result we will suppose that X is reflexive (and so l will be). We will also need the notion of ideal space. A sequence space l is called ideal if whenever $v = (v_n)_n \in l$ and $w = (w_n)_n$ is any numerical sequence with $|w_n| \leq |v_n|$ for every n then also $w \in l$. This notion appears naturally in the context of free interpolation and unconditional bases.

Proposition 2.3. *Suppose $X \subset \mathrm{Hol}(\mathbb{D})$ is reflexive and S is a sequence in $\mathbb{D}$. The following assertions are equivalent.*

(1) *There exists a reflexive and ideal sequence space l such that*
   (i) *S is l-interpolating*

(ii) *There is a constant C such that for every finitely supported sequence $\mu = (\mu)_{a\in S}$, we have* $\| \sum_{a\in S} \mu_a \frac{E_a}{\|E_a\|_{X^*}}\|_{X^*} \leq C\|\mu\|_{l^*}$,

(2) $(E_a)_{a\in S}$ *is an unconditional sequence in* X^*.

A sequence satisfying condition (ii) will be called l^*-Carleson or q-Carleson when $l^* = l^q$ (a Carleson embedding for X^* with respect to the sequence space l^*). See Subsection 2.3 for more on Carleson conditions.

Note that another way of writing (ii) is

$$\forall f \in X, \forall \mu \in l^*, \quad |\sum_{a\in S} \mu_a \frac{f(a)}{\|E_a\|_{X^*}}| \leq C\|f\|_X \|\mu\|_{l^*},$$

which means that for every $f \in X$, the sequence $(f(a)/\|E_a\|_{X^*})_{a\in S}$ is in $(l^*)^* = l$, and hence (ii) is equivalent to

$$\|(f(a)/\|E_a\|_{X^*})_{a\in S}\|_l \leq C\|f\|_X, \tag{2.2}$$

which means $X|\Lambda \subset l(1/\|E_a\|_{X^*})$ (there will be more discussions on Carleson measures in Subsection 2.3). We thus have

Corollary 2.4. *Suppose $X \subset \mathrm{Hol}(\mathbb{D})$ is reflexive and S is a sequence in $\mathbb{D}$. The following assertions are equivalent.*

(1) *There exists a reflexive and ideal sequence space l such that* $X|\Lambda = l(1/\|E_a\|_{X^*})$
(2) $(E_a)_{a\in S}$ *is an unconditional sequence in X^* (an l^*-basis in its span).*

Proof of Proposition 2.3. By [Ni78, Theorem 1.1] the sequence $(E_a)_{a\in S}$ is an unconditional sequence in its span if and only if the multiplier space $\mathrm{mult}(E_a) := \{\mu = (\mu_a)_{a\in S} : T_\mu : \mathrm{Lin}(E_a) \longrightarrow \mathrm{Lin}(E_a), \sum_{finite} \alpha_a E_a \longmapsto \sum_{finite} \mu_a \alpha_a E_a$ extends to a bounded operator on $X_0^* := \bigvee_{a\in S} E_a\}$ is equal to l^∞. And this, by [Ni78, Lemma 1.2] is equivalent to the existence of an ideal space l_0 such that $(E_a)_{a\in S}$ is a l_0-basis, which means that $X_0 \simeq l_0(E_a) := \{(\alpha_a)_{a\in S} : (\alpha_a \|E_a\|_{X^*}) \in l_0\}$, in other words the mapping $(\alpha_a)_{a\in S} \longmapsto \sum_{a\in S} \alpha_a E_a$ is an isomorphism from $l_0(E_a)$ onto X_0, or equivalently

$$\begin{aligned} T : l_0 &\longrightarrow X^* \\ (\beta_a)_{a\in S} &\longmapsto \sum_{a\in S} \beta_a \frac{E_a}{\|E_a\|_{X^*}} \end{aligned}$$

is an isomorphism. Note that X_0 is reflexive as a closed subspace of a reflexive Banach space, and so is l_0. Set $l := l_0^*$ (so that $l^* = l_0$). By the preceding argument, $(E_a)_{a\in S}$ is an unconditional sequence in its span if and only if

$$c\|\mu\|_{l^*} \leq \| \sum_{a\in S} \mu_a \frac{E_a}{\|E_a\|_{X^*}} \|_{X^*} \leq C\|\mu\|_{l^*}, \tag{2.3}$$

for some fixed constants c, C. This yields in particular (ii).

We will compute the adjoint operator $T^* : X \longrightarrow l$. Let $\mu \in l^*$,

$$\langle T^* f, \mu\rangle = \langle f, T\mu\rangle = \langle f, \sum_{a\in S} \mu_a \frac{E_a}{\|E_a\|_{X^*}}\rangle = \sum_{a\in S} \mu_a \frac{f(a)}{\|E_a\|_{X^*}}.$$

Hence, the functional T^*f on l^* is represented by a sequence the entries of which are given by $f(a)/\|E_a\|_{X^*}$, $a \in S$. In other words $T^*f = (f(a)/\|E_a\|_{X^*})_{a\in S} \in (l^*)^* = l$.

Now the left hand inequality in (2.3) is equivalent to the left invertibility of T which is equivalent to the surjectivity of T^* i.e. to the fact S is l-interpolating. This show that (2) implies (1).

For the converse implication, note that (ii) implies the right inequality in (2.3). Moreover this inequality shows also that T is well defined and bounded. By the above arguments the surjectivity of T^* is equivalent to the fact that S is interpolating. On the other hand the surjectivity of T^* is equivalent to the left invertibility of T and so to the left inequality in (2.3). ∎

Still the following is true

Corollary 2.5. *If S is interpolating for K_I^p and if there is a constant C such that for every finitely supported sequence $\mu = (\mu)_{a\in S}$, we have $\|\sum_{a\in S}\mu_a k_a^I/\|k_a^I\|_{p'}\|_{p'} \le C\|\mu\|_{l^{p'}}$, then $(k_a^I/\|k_a^I\|_{p'})_{a\in S}$ is an unconditional sequence in $K_I^{p'}$.*

More precisely the conclusion would be that $(k_a^I)_{a\in S}$ is an $l^{p'}$-basis in its span. This conclusion can in general not be deduced only from the condition of unconditionality as explained above. However, in the special situation $\sup_{a\in S}|I(a)| < 1$, [HNP81, Theorem 6.3, Partie II] shows that if the reproducing kernels form an unconditional sequence in $K_I^{p'}$ then automatically they form an $l^{p'}$-basis in their span.

2.3. **Carleson measures.** Let us fix the framework of this subsection. S is a sequence in $\mathbb{D}$, I an inner function and $1 \le q < \infty$. For $a \in S$ we denote by $k_{q,a}^I = k_a^I/\|k_a^I\|_q$ the normalized reproducing kernel.

Let $1 \le q < \infty$. Recall that a sequence S is called q-Carleson if

$$\exists D_q > 0, \forall \mu \in l^q,\ \left\|\sum_{a\in S}\mu_a k_{q,a}^I\right\|_q \le D_q\|\mu\|_q.$$

We will also use the notion of weak q-Carleson sequences:

Definition 2.6. Let $2 \le q < \infty$. The sequence S is called weakly q-Carleson if

$$\exists D_q > 0, \forall \mu \in l^q,\ \left\|\sum_{a\in S}|\mu_a|^2|k_{q,a}^I|^2\right\|_{q/2} \le D_q\|\mu\|_q^2.$$

Note that by [Am08, Lemma 3.2], the q-Carleson property implies the weak q-Carleson property.

Observe also that $(l^q)^* = l^p$, that the dual of K_I^q can be identified with K_I^p, and that the functional of point evaluation E_a can then be identified with k_a^I. Now, using the notation from the preceding subsection, by (2.2), S is q-Carleson if and only if for every $f \in K_I^p$,

$$\sum_{a\in S}\frac{|f(a)|^p}{\|k_a^I\|_p^p} \le c\|f\|_p,$$

which means that $\nu := \sum_{a\in S}\delta_a/\|k_a^I\|_p$ is a K_I^p-Carleson measure: $K_I^p \subset L^p(\nu)$.

In the special situation when I is one-component, then by a result by Aleksandrov (see (4.1)), we have

$$\|k_a^I\|_p \simeq \left(\frac{1-|I(a)|^2}{1-|a|^2}\right)^{1/p'},$$

p' being the conjugated index to p, and so, if S is q-Carleson and I is one-component, then the measure

$$d\nu = \sum_{a\in S} \frac{1-|a|^2}{1-|I(a)|^2}\delta_a$$

is K_I^p-Carleson.

Geometric Carleson conditions

In [TV96], the following geometric notion of Carleson measure appears. For an inner function I and an $\varepsilon > 0$, let $L(I,\varepsilon) = \{z \in \mathbb{D} : |I(z)| < \varepsilon\}$ be the associated level set. In the notation of [Al02], let $\mathcal{C}(I)$ be the set of measures for which there exists $C > 0$ such that

$$|\mu|(S(\zeta, r)) \leq Cr \tag{2.4}$$

for every Carleson window $S(\zeta = e^{it}, h) := \{z = re^{i\theta} \in \mathbb{D} : 1-h < r < 1, |t-\theta| < h\}$ meeting $L(I, 1/2)$ (this is of course a weaker notion than the usual one requiring (2.4) on all Carleson windows; the value $\varepsilon = 1/2$ is of no particular relevance). Let also $\mathcal{C}_p(I)$ be the set of measures for which $K_I^p \subset L^p(\mu)$. Strengthening the results of [TV96], Aleksandrov proved in [Al02, Theorem 1.4] that for one component inner functions $\mathcal{C}(I) = \mathcal{C}_p(I)$. In other words, the geometric Carleson condition (2.4) on Carleson windows meeting the level set $L(I, 1/2)$ characterizes the K_I^p-Carleson measures for one component inner functions.

Combining these observations, we get the following characterization.

Fact 2.7. Let I be a one-component inner function. Then the following assertions are equivalent.

(i) S is p'-Carleson

(ii) $\nu = \sum_{a\in S} \frac{1-|a|^2}{1-|I(a)|^2}\delta_a$ is K_I^p-Carleson

(iii) ν (as defined in point (ii)) satisfies the geometric Carleson condition (2.4) on Carleson windows meeting the level set $L(I, 1/2)$.

Question. Do there exist in backward shift invariant subspaces interpolating sequences S that are not p'-Carleson?

3. Paley-Wiener spaces

We will discuss a special class of backward shift invariant subspaces. Let $I(z) = e^{i2\pi z}$ be the singular inner function in the upper half plane with sole singularity at ∞ (to fix the ideas, we have chosen the mass of the associated singular measure to be 2π). Recall (see [Ni02, B.1]) that the transformation

$$\begin{aligned} U_p : H^p(\mathbb{D}) &\longrightarrow H^p(\mathbb{C}^+) \\ f &\longmapsto \left\{ x \to (U_p f)(x) = \left(\frac{1}{\pi(x+i)^2}\right)^{1/p} f\left(\frac{x-i}{x+i}\right) \right\} \end{aligned}$$

is an isomorphism of the Hardy space on the disk $H^p(\mathbb{D})$ onto the Hardy space $H^p(\mathbb{C}^+)$ of the upper half plane $\mathbb{C}^+ = \{z \in \mathbb{C} : \mathrm{Im} z > 0\}$. This transformation sends the inner function $I_0(z) = \exp(2\pi(z+1)/(z-1))$ on $\mathbb{D}$ to I on $\mathbb{C}^+$.

Let PW_π^p be the Paley-Wiener space of entire functions of type π which are p-th power integrable on the real line. Pick $f \in PW_\pi^p$. By a theorem by Plancherel and Pólya (see [Lev96, Lecture 7, Theorem 4]) we get

$$\int_{\mathbb{R}} |f(x+ia)|^p dx \leq e^{p\pi|a|}\|f\|_p^p \tag{3.1}$$

for every $a \in \mathbb{R}$. Setting $F(z) = e^{i\pi z} f(z)$ (which means that in a sense we compensate the type in the positive imaginary direction) yields

$$\int_{\mathbb{R}} |F(z+iy)|^p dx = \int_{\mathbb{R}} |f(x+iy)|^p e^{-p\pi y} dx \leq \|f\|_p^p$$

in particular for every $y > 0$ which means that $F \in H^p(\mathbb{C}^+)$. Dividing F by I we obtain an analytic function in the lower halfplane $\mathbb{C}_-$ and for every $y < 0$,

$$\int_{\mathbb{R}} |F(x+iy)e^{-i2\pi(x+iy)}|^p dx = \int_{\mathbb{R}} |f(x+iy)|^p e^{p\pi y} dx \leq \|f\|_p^p$$

so that F/I is in the Hardy space of the lower halfplane $H^p(\mathbb{C}_-)$. Hence $F \in H^p(\mathbb{C}^+) \cap \overline{I}H_0^p(\mathbb{C}_-) =: K_{\mathbb{R},I}^p$ (now considered as a space of functions on $\mathbb{R}$, the elements of which can of course be continued analytically to the whole plane). It is clear that $K_{\mathbb{R},I}^p$ can be identified via U_p with K_I^p on $\mathbb{D}$ (or $\mathbb{T}$). Hence there is a natural identification between Paley-Wiener spaces and backward invariant subspaces (on $\mathbb{T}$ or $\mathbb{R}$): $PW_\pi^p = e^{-i\pi z} U_p K_I^p$.

It is well known that in the particular case $p = 2$, PW_π^p is nothing but $\mathcal{F}L^2(-\pi, \pi)$ (this comes from the Paley-Wiener theorem).

Let us make another observation concerning imaginary translations. For $a \in \mathbb{R}$, let

$$\begin{array}{rcl} \Phi_a : PW_\pi^p & \longrightarrow & PW_\pi^p \\ f & \longmapsto & \{\Phi_a f : z \longmapsto f(z - ia)\}. \end{array}$$

Using again the Plancherel-Pólya theorem (see (3.1)), we see that Φ_a is well-defined and bounded (it is clearly linear). It is also invertible with inverse $\Phi_a^{-1} = \Phi_{-a}$. So Φ_a is an isomorphism of PW_π^p onto itself (the type that we fixed to π here does not really matter).

So the Paley-Wiener spaces are special candidates of our spaces K_I^p, which motivates the following important observations. In general it is not true that uniform minimality implies interpolation or unconditionality which we will explain now following [SchS00].

By definition a sequence $\Gamma = \{x_k + iy_k\}_k$ is interpolating for PW_τ^p if for every numerical sequence $(v_k)_k$ with

$$\sum_k |v_k|^p e^{-p\tau|\eta_k|}(1 + |\eta_k|) < \infty \tag{3.2}$$

there exists $f \in PW_\tau^p$ with $f(\gamma_k) = a_k$.

Theorem 3.1 (Schuster-Seip, 2000). *Let $2 \leq p < \infty$. There exists a dual bounded sequence Γ which is not interpolating in PW_π^p.*

We would like to recall here the construction of Schuster and Seip since it will serve later on.

Proof. Define a squence $\Gamma = \{\gamma_k\}_{k\in\mathbb{Z}}$ by $\gamma_0 = 0$ and $\gamma_k(p) = k + \delta_k(p)$, $k \in \mathbb{Z} \setminus \{0\}$, where $\delta_k(p) = \operatorname{sign}(k)/(2p_0)$ and $p_0 = \max(p, p')$, $1/p + 1/p' = 1$. Since this sequence is real, the weight appearing in (3.2) is equal to 1.

Now let $G(z) = z\prod_{k\neq 0}(1 - \frac{z}{\gamma_k})$ which defines an entire function of exponential type π with $|G(x)| \simeq d(x,\Gamma)(1+|x|)^{-1/p_0}$. Note that the p-th power integrability of $|G|$ on $\mathbb{R}$ is determined by $(1+|x|)^{-1/p_0}$, and the latter function is never p-th power integrable on $\mathbb{R}$ (one could distinguish the case $p > 2$ and $p < 2$). Hence, Γ is a uniqueness set and thus interpolating if and only if it is completely interpolating.

We will use the same type of computations as in the proof of [LS97, Theorem 2] to check that Γ is not (completely) interpolating when $p \geq 2$. According to [LS97, Theorem 1], it suffices to check that F^p, where $F(x) = |G(x)/d(x,\Gamma)| \simeq (1+|x|)^{-1/p_0}$, is not (A_p), i.e.

$$\frac{1}{|I|}\int_I F^p dt \left(\frac{1}{|I|}\int_I F^{-p'} dt\right)^{p-1}$$

is not uniformly bounded in the intervals I. For $p \geq 2$, we have $p_0 = p$ and hence we have to consider

$$\frac{1}{|I|}\int_I (1+|t|)^{-1} dt \left(\frac{1}{|I|}\int_I (1+|t|)^{p'/p} dt\right)^{p-1}.$$

This expression behaves like $\log(1+|x|)$ when $I = [0,x]$, which is incompatible with the (A_p)-condition. So the sequence Γ is not interpolating.

On the other hand, $g_k(z) = G(z)/(z-\gamma_k)$ vanishes on $\Gamma \setminus \{\gamma_k\}$ and satisfies

$$|g_k(\gamma_k)| \simeq \|g_k\|_{L^p(\mathbb{R})}. \tag{3.3}$$

Note that $G \in L^p$ (if and) only if $(1+|x|)^{-1/p'} \in L^p$, i.e. $p/p' = p-1 > 1$ or $p > 2$. This implies that the sequence is dual bounded. In fact, note that the reproducing kernel of the Paley-Wiener space PW_π^p in $x \in \mathbb{R}$ is given by $k_x(z) = \operatorname{sinc}(\pi(z-x)) = \sin(\pi(z-x))/(\pi(z-x))$, the norm of which in $L^{p'}(\mathbb{R})$ can be easily estimated to be comparable to a constant independantly of x. Hence (3.3) implies that $\tilde{g}_k := g_k/\|g_k\|_p$ is of uniformly bounded norm and $|\tilde{g}_k(\gamma_k)| \simeq 1 \simeq \|k_{\gamma_k}\|_{L^{p'}(\mathbb{R})}$. Suitably renormed, $(\tilde{g}_k)_k$ thus furnishes the family $(\rho_{\gamma_k})_k$ mentioned after Definition 2.1. ∎

As a consequence, in PW_π^p there exists a sequence Γ such that $\{k_{\gamma_l}/\|k_{\gamma_l}\|_{PW_\pi^{p'}}\}_l$ is uniformly minimal in $PW_\pi^{p'}$ but not unconditional.

Still, it can be observed that Γ is uniformly separated in the euclidean distance and hence by the classical Plancherel-Pólya inequality we have for every $f \in PW_\pi^p$

$$\sum_k |f(\gamma_k)|^p \leq C\|f\|_p^p, \tag{3.4}$$

so that the restriction operator $f \longmapsto f|\Gamma$ is continuous from PW_π^p to l^p (onto when Γ is interpolating), in other words the measure $\sum_{\gamma\in\Gamma}\delta_\gamma$ is PW_π^p-Carleson.

More can be said. The following result is nothing but a re-interpretation of [LS97].

Proposition 3.2. *Let $1 < p \leq 2$. Then for every $1 < s < p$ there exists a sequence Γ that is interpolating for PW_π^p without being interpolating for PW_π^s.*

So, in the scale of Paley-Wiener spaces — which represents a subclass of backward shift invariant subspaces — an interpolating sequence is not necessarily interpolating in an arbitrary bigger space, and so *a fortiori* a dual bounded sequence for a given p is not necessarily interpolating for a bigger space K_I^s, $s < p$. This should motivate why in our main result discussed in the next section we increase the space in two directions to get interpolation from dual boundedness: we increase the space by adding factors to the defining inner function *and* by decreasing p.

Again we translate the result to the language of unconditionality. The sequence constructed in this proposition is again a real sequence which is uniformly separated in the euclidean metric so that (3.4) holds for p and s and hence the measure $\sum_{k\in\mathbb{Z}} \delta_{\gamma_k}$ is a Carleson measure. This implies that if Γ is interpolation for PW_π^p then we do not only have $PW_\pi^p|\Gamma \supset l^p$ (recall that the reproducing kernel is given by the sinc-function in $\gamma_k \in \mathbb{R}$ the norm of which is comparable to a constant) but $PW_\pi^p|\Gamma = l^p$. By Corollary 2.4 this means that $(k_\gamma/\|k_\gamma\|_{p'})_{\gamma\in\Gamma})$ is unconditional in $PW_\pi^{p'}$. Clearly, since Γ is not interpolating for PW_π^s, the sequence $(k_\gamma/\|k_\gamma\|_{s'})_{\gamma\in\Gamma}$ cannot be unconditional in $PW_\pi^{s'}$. We recapitulate these observations in the following result.

Corollary 3.3. *Let $1 < p \leq 2$. Then for every $1 < s < p$ there exists a sequence Γ such that $(k_\gamma/\|k_\gamma\|_{p'})_{\gamma\in\Gamma})$ is unconditional in $PW_\pi^{p'}$ and $(k_\gamma/\|k_\gamma\|_{s'})_{\gamma\in\Gamma}$ is not unconditional for $PW_\pi^{s'}$.*

Recall that k_x, the reproding kernel in PW_τ^2 is given by a sinc-function the norm of which is comparable to a constant when $x \in \mathbb{R}$.

It can be noted that $s' > p'$ so that $PW_\pi^{s'}$ is a smaller space than $PW_\pi^{p'}$.

Proof of Proposition 3.2. Since $1 < p \leq 2$ we have $p_0 := \max(p, p') = p'$ (recall $1/p + 1/p' = 1$). In contrast to the above example where we have 'spread out' slightly the integers (by adding a constant to the positive integers and subtracting the same constant from the negative integers) to obtain a dual bounded sequence which is not interpolating ($p \geq 2$) we will now narrow the integers: let $\delta_k = -\operatorname{sign}(k)/2s'$. We have in particular $s_0 = \max(s, s') = s' > p'$. Define $\Gamma = (\gamma_k)_{k\in\mathbb{Z}}$ by $\gamma_k = k + \delta_k$, $k \in \mathbb{Z} \setminus \{0\}$, $\gamma_0 = 0$. Then as the example in [LS97, Theorem 2], the sequence Γ is not interpolating for PW_π^s. On the other hand, since $|\delta_k| = 1/2s' < 1/2p'$ we deduce from the sufficiency part of [LS97, Theorem 2] that Γ is complete interpolating for PW_π^p. ■

Remark 3.4. We have mentioned the translations Φ_a, $a \in \mathbb{R}$. These allow to translate the above example Γ to any line parallel to the real axis: $\Phi_a\Gamma$. By the properties of Φ_a, we keep the properties of uniform minimality and (non)-interpolation.

We now discuss the effect of increasing the size of the space in the Paley-Wiener case "in the direction of the inner function". More precisely we will consider the situation when we replace I by $I^{1+\varepsilon}$ on the K_I^p-side, which means on the Paley-Wiener side that we replace the type π by $\pi(1+\varepsilon) =: \pi + \eta$ for some $\eta > 0$. And for $p = 2$, on the Fourier side this means that we replace $[-\pi, \pi]$ by $[-(\pi + \eta), \pi + \eta]$.

We will use [Se95, Theorem 2.4] to prove the following result.

Proposition 3.5. *Let $\Gamma = \{\gamma_k\}_{k\in\mathbb{Z}}$ be defined by $\gamma_0 = 0$, $\gamma_k = k + \operatorname{sign}(k)/4$. Then $(k_\gamma)_{\gamma\in\Gamma}$ is uniformly minimal and not unconditional in PW_π^2, and for every $\eta > 0$, Γ is an unconditional sequence in $PW_{\pi+\eta}^2$.*

Proof of Proposition 3.5. The first part of the claim is established by Theorem 3.1.

We use [Se95, Theorem 2.4] for interpolation in the bigger space. Seip's theorem furnishes a sufficient density condition for unconditional sequences in Paley-Wiener spaces when $p = 2$ which makes this proof very easy. Recall that $n^+(r)$ denotes the largest number of points from a sequence of real numbers Λ to be found in an interval of length r. The upper uniform density is then defined as

$$D^+(\Lambda) := \lim_{r\to\infty} \frac{n^+(r)}{r}$$

(the limit exists by standard arguments on subadditivity of $n^+(r)$). [Se95, Theorem 2.4] states that when a sequence Λ, which is uniformly separated in the euclidean distance, satisfies $D^+(\Lambda) < \frac{\tau}{2\pi}$, then $(k_\lambda/\|k_\lambda\|_{PW^2_\tau})_{\lambda\in\Lambda}$ is an unconditional sequence in PW^2_τ (strictly speaking Seip's theorem yields the unconditionality for exponentials in $L^2([-\tau,\tau])$, but via the Fourier transform this is of course the same as for reproducing kernels). Our sequence Γ clearly satisfies $D^+(\Gamma) = 1$, and hence whenever $\tau > 2\pi$, then Γ is interpolating in PW^2_τ. ■

The proposition can also be shown by appealing to [SchS00, Theorem 3] which gives a kind of uniform non-uniqueness condition as sufficient condition for interpolation in Paley-Wiener spaces. It can in fact be shown using a perturbation result by Redheffer that the weak limits (in the sense of Beurling) of our sequence Γ have the same completeness radius (in the sense of Beurling-Malliavin) as Γ, i.e. π. So increasing the size of the interval makes these weak limits non-uniqueness in the bigger space (this is the most difficult condition of Schuster and Seip's result to be checked; concerning the other conditions appearing in their theorem, i.e. uniform separation and the two-sided Carleson condition, these are immediate).

Question. A natural question arises in the context of these results. Is it possible that the sequence Γ of Proposition 3.5 — which is dual bounded but not interpolating in PW^2_π — is interpolating in PW^p_π for some $p = 2-\varepsilon$ (or p in some intervalle $(2-\varepsilon, 2)$) for suitable small ε?

So this time we increase the size of the space in the direction p. Proposition 3.2 indicates that ε cannot be chosen arbitrarily big. This proposition also motivates another important remark. A sufficient condition for interpolation in terms of a suitable density and depending on the value of p, as encountered e.g. in the context of Bergman spaces where a sequence satisfying the criticial density is automatically interpolating in the bigger spaces, seems not expectable. This makes the question very delicate (note that the sequence Γ of Proposition 3.5 has the critical density for PW^2_π).

4. THE MAIN RESULT

Let I be an inner function, i.e. a function analytic on $\mathbb{D}$, bounded by 1, and such that $|I(\zeta)| = 1$ for a.e. $\zeta \in \mathbb{T}$. Such a function is called one-component when there exists an $\varepsilon \in (0,1)$ such that $L(I,\varepsilon) = \{z \in \mathbb{D} : |I(z)| < \varepsilon\}$ is connected. Simple examples of such functions are for example $I(z) = \exp((z+1)/(z-1))$ or Blaschke products with zeros not "too far" such as B_Λ associated with the interpolating sequence $\Lambda = \{1 - 1/2^n\}_n$. One-component inner functions appear for example in the context of embeddings for star invariant subspaces. For example, Treil and Volberg [TV96] discuss the embedding $K^p_I \subset L^p(\mu)$ when I is one-component.

The following result will be of interest for us

Theorem ([Al02]). *If I is a one-component inner function and $1 < p \leq \infty$, then*

$$C_1(I,p)\left(\frac{1-|I(a)|^2}{1-|a|^2}\right)^{1-1/p} \leq \left\|\frac{1-\overline{I(a)}I(z)}{1-\overline{a}z}\right\|_p \leq C_2(I,p)\left(\frac{1-|I(a)|^2}{1-|a|^2}\right)^{1-1/p} \tag{4.1}$$

for all $a \in \mathbb{D}$.

We will now discuss the principal results that lead to Theorem 1.1.

For a sequence S of points in $\mathbb{D}$, we introduce the related sequence $\{\epsilon_a\}_{a\in S}$ of independent Bernoulli variables.

We now increase K_I^p, when p is fixed, which means that we multiply a factor to the inner function I. More precisely let $J = IE$ where E is another inner function. Recall that $\overline{K_I^p + IK_E^p} = K_J^p$ (which gives an idea on the increase of the space; note that this identity can also be derived from a more general one in de Branges-Rovnyak spaces).

We first discuss when dual boundedness for $p > 1$ implies interpolation for $q = 1$.

Lemma 4.1. *Let $S \subset \mathbb{D}$ be dual bounded in K_I^p, $p > 1$, and let E be another inner function. If*

$$\|k_a^J\|_\infty \simeq \frac{\|k_a^I\|_{p'}\|k_a^E\|_2^2}{\|k_a^E\|_{p'}}, \tag{4.2}$$

then S is interpolating in K_J^1 with $J = IE$.

Proof. Let first $c_a = \dfrac{\|k_a^E\|_{p'}\|k_a^J\|_\infty}{\|k_a^I\|_{p'}\|k_a^E\|_2^2}$ which is comparable to a uniform constant.

Since S is dual bounded in K_I^p, the sequence $(k_a^I/\|k_a^I\|_{p'})_{a\in S}$ is uniformly minimal, so that there exists a dual sequence $(\rho_{p,a})_{a\in S}$ in K_I^p: $\langle \rho_{p,a}, k_{p',b}^I\rangle = \delta_{ab}$, i.e. $\rho_{p,a}(b) = \delta_{ab}\|k_b^I\|_{p'}$, and $\sup_{a\in S}\|\rho_{p,a}\|_p < \infty$. As in [Am08] the idea is now to take

$$\forall \lambda \in \ell^1,\ T(\lambda) := \sum_{a\in S}\lambda_a c_a \rho_{p,a}\frac{k_a^E}{\|k_a^E\|_{p'}}.$$

The sum defining T converges clearly under the assumption of the theorem since λ is summable. Also $k_a^E(a) = \|k_a^E\|_2^2$, and hence

$$T(\lambda)(a) = \lambda_a c_a \rho_{a,p}(a)\frac{k_a^E(a)}{\|k_a^E\|_{p'}} = \lambda_a c_a \frac{\|k_a^I\|_{p'}\|k_a^E\|_2^2}{\|k_a^E\|_{p'}} = \lambda_a\|k_a^J\|_\infty.$$

So, by equation (4.2), S is interpolating in K_J^1. ■

We shall now discuss the general situation.

Lemma 4.2. *Suppose that I and E are one-component inner functions. Let $S \subset \mathbb{D}$ be a dual bounded sequence in K_I^p; let $1 \leq s < p$ and q be such that $\dfrac{1}{s} = \dfrac{1}{p} + \dfrac{1}{q}$; suppose that the following conditions are satisfied.*

(i) $\|k_a^J\|_{s'} \simeq \dfrac{\|k_a^E\|_{s'}\|k_a^I\|_{p'}}{\|k_a^E\|_{p'}}$;

(ii) $\forall \lambda \in \ell^p(S),\ \mathbb{E}\left[\left\|\sum_{a\in S}\lambda_a\epsilon_a\rho_{p,a}\right\|_p^p\right] \lesssim \|\lambda\|_{\ell^p}^p$

(iii) *if* $q>2,\ S$ *is weakly* q*-Carleson in* K_E^q*,*

Then S *is* K_J^s *interpolating and moreover there exists a bounded linear interpolation operator* $T : l^s(S) \longrightarrow K_J^s$*,* $T(\nu)(a) = \nu_a\|k_a^J\|_{s'}$*.*

Observe that we do not need to require the Carleson condition on S when $q \le 2$.

Remark 4.3. Before proving the result, we discuss some special cases where the condition (i) is satisfied. Recall from (4.1) that for an arbitrary inner one-component function Θ we have

$$\|k_a^\Theta\|_{s'} \simeq \left(\frac{1-|\Theta(a)|^2}{1-|a|^2}\right)^{1/s}. \tag{4.3}$$

Hence when I, E are one-component, we get

$$\begin{aligned}\frac{\|k_a^E\|_{s'}\|k_a^I\|_{p'}}{\|k_a^E\|_{p'}} &\simeq \frac{\left(1-|E(a)|^2\right)^{1/s}\left(1-|I(a)|^2\right)^{1/p}}{\left(1-|E(a)|^2\right)^{1/p}\left(1-|a|^2\right)^{1/s}}\\ &= \frac{\left(1-|E(a)|^2\right)^{1/q}\left(1-|I(a)|^2\right)^{1/p}}{\left(1-|a|^2\right)^{1/s}}.\end{aligned} \tag{4.4}$$

From this we can deduce that (i) holds in the following cases.

(1) Suppose E, I are one-component and $\sup_{a\in S}|E(a)| \le \eta < 1$ and $\sup_{a\in S}|I(a)| \le \eta < 1$. Suppose also that $J = IE$ is one-component (it is not clear whether this follows from I and E being one-component). Clearly $\sup_{a\in S}|J(a)| < 1$, and (i) follows.

(2) $E = I$ and I is one-component, then $J = I^2$ (note that it is clear that when $L(I,\varepsilon)$ is connected then so is $L(I^2,\varepsilon^2)$); in this case we do not need the sup-condition, since

$$\left(1-|I(a)|^4\right)^{1/s} \simeq \left(1-|I(a)|^2\right)^{1/q}\left(1-|I(a)|^2\right)^{1/p},$$

which by (4.3) and (4.4) yields (i);

(3) I singular and $\forall\alpha>0,\ E = I^\alpha$ which implies $J = I^{1+\alpha}$.

Remark 4.4. If $p=1$ then dual boundedness of S in K_I^1 implies that S interpolating in K_I^1 (take the interpolation operator constructed in the proof of Lemma 4.1).

Proof of the Lemma. In view of Lemma 4.1 we can suppose $1<s<p$.

In order to prove the lemma we will construct a function f interpolating a sequence $\nu \in l^s$ weighted by the norm of the reproducing kernels. To do this, we will consider finitely supported sequences ν, say with only the first N components possibly different from zero, and check that the constants do not depend on $N \in \mathbb{N}$. So, for $1<s<p$ and $\nu \in \ell_N^s$ we shall build a function $h \in K_J^s$ such that:

$$\forall j = 0,\dots,N-1,\ h(a_j) = \nu_j\|k_{a_j}^J\|_{s'} \text{ and } \|h\|_{K_J^s} \le C\|\nu\|_{\ell_N^s}.$$

where the constant C is independent of N. The conclusion follows from a normal families argument (see also [Am08]).

We choose q such that $\frac{1}{s}=\frac{1}{p}+\frac{1}{q}$; then $q\in]p',\infty[$ with p' the conjugate exponent of p and we set $\nu_j=\lambda_j\mu_j$ with $\mu_j:=\left|\nu_j\right|^{s/q}\in\ell^q,\ \lambda_j:=\frac{\nu_j}{\left|\nu_j\right|}\left|\nu_j\right|^{s/p}\in\ell^p$ so that $\left\|\nu\right\|_s=\left\|\lambda\right\|_p\left\|\mu\right\|_q$.

Let now

$$c_a:=\frac{\|k_a^E\|_q\|k_a^J\|_{s'}}{\|k_a^I\|_{p'}k_a^E(a)}.$$

By (i), we have

$$c_a\simeq\frac{\|k_a^E\|_q\|k_a^E\|_{s'}}{\|k_a^E\|_{p'}k_a^E(a)}=\frac{\|k_a^E\|_q\|k_a^E\|_{s'}}{\|k_a^E\|_{p'}\|k_a^E\|_2^2}.$$

Since E is one-component we have (4.1), i.e. $\|k_a^E\|_r\simeq\left(\frac{1-\left|E(a)\right|^2}{1-\left|a\right|^2}\right)^{1/r'}$, where $\frac{1}{r}+\frac{1}{r'}=1$. Clearly $1/q'+1/s-1/p+2\times1/2=1/q'+1/q-1=0$, and hence $c_a\simeq C$, the constant being independent of $a\in S$.

Next set $h(z):=T(\nu)(z):=\sum_{a\in S}\nu_a c_a\rho_a k_{q,a}^E$. Then, because $\rho_a(b)=\delta_{ab}\|k_a^I\|_{p'}$:

$$\forall a\in S,\ h(a)=\nu_a c_a\|k_a^I\|_{p'}k_{q,a}^E(a).$$

Recall that $k_{q,a}^E(a)=k_a^E(a)/\|k_a^E\|_q$. Hence

$$h(a)=\nu_a c_a\|k_a^I\|_{p'}k_{q,a}^E(a)=\nu_a\times\frac{\|k_a^E\|_q\|k_a^J\|_{s'}}{\|k_a^I\|_{p'}k_a^E(a)}\times\|k_a^I\|_{p'}\times\frac{k_a^E(a)}{\|k_a^E\|_q}=\nu_a\|k_a^J\|_{s'}$$

and h satisfies the interpolation condition.

Let us now come to the estimate of the K_J^s norm of h.

Set

$$f(\epsilon,z):=\sum_{a\in S}\lambda_a c_a\epsilon_a\rho_a(z),\quad\text{and}\quad g(\epsilon,z):=\sum_{a\in S}\mu_a\epsilon_a k_{q,a}^E(z).$$

Then $h(z)=\mathbb{E}(f(\epsilon,z)g(\epsilon,z))$ because $\mathbb{E}(\epsilon_j\epsilon_k)=\delta_{jk}$.

So we get

$$|h(z)|^s=|\mathbb{E}(fg)|^s\leq(\mathbb{E}(|fg|))^s\leq\mathbb{E}(|fg|^s),$$

and hence

$$\|h\|_s=\left(\int_{\mathbb{T}}|h(z)|^s\,d\sigma(z)\right)^{1/s}\leq\left(\int_{\mathbb{T}}\mathbb{E}(|fg|^s)\,d\sigma(z)\right)^{1/s}.$$

By Hölder's inequality, we get

$$\int_{\mathbb{T}}\mathbb{E}(|fg|^s)\,d\sigma(z)=\mathbb{E}\left[\int_{\mathbb{T}}|fg|^s\ d\sigma(z)\right]\leq\left(\mathbb{E}\left[\int_{\mathbb{T}}|f|^p\ d\sigma\right]\right)^{s/p}\left(\mathbb{E}\left[\int_{\mathbb{T}}|g|^q\ d\sigma\right]\right)^{s/q}.\tag{4.5}$$

Now for $a \in S$, set $\tilde{\lambda}_a := c_a \lambda_a$. Then $\|\tilde{\lambda}\|_p \le C\|\lambda\|_p$ and the first factor in (4.5) is controlled by (ii) of the hypotheses of the Lemma:

$$\mathbb{E}\left[\int_{\mathbb{T}} |f|^p \, d\sigma\right] = \mathbb{E}\left[\left\|\sum_{a\in S} \lambda_a c_a \epsilon_a \rho_{p,a}\right\|_p^p\right] \lesssim \left\|\tilde{\lambda}\right\|_p^p \lesssim \|\lambda\|_{\ell^p}^p , \tag{4.6}$$

and the constants appearing here do not depend on N.

Consider the second factor in (4.5). Fubini's theorem gives:

$$\mathbb{E}\left[\int_{\mathbb{T}} |g|^q \, d\sigma\right] = \int_{\mathbb{T}} \mathbb{E}\left[|g|^q\right] \, d\sigma.$$

We apply Khinchin's inequalities to $\mathbb{E}\left[|g|^q\right]$:

$$\mathbb{E}\left[|g|^q\right] \simeq \left(\sum_{a\in S} |\mu_a|^2 \left|k_{q,a}^E\right|^2\right)^{q/2}.$$

If $q > 2$, then S weakly q-Carleson implies

$$\int_{\mathbb{T}} \mathbb{E}\left[|g|^q\right] \, d\sigma \lesssim \int_{\mathbb{T}} \left(\sum_{a\in S} |\mu_a|^2 \left|k_{q,a}^E\right|^2\right)^{q/2} d\sigma \lesssim \|\mu\|_{\ell^q}^q , \tag{4.7}$$

where, again, the constants do not depend on N.

If $q \le 2$ then $\left(\sum_{a\in S} |\mu_a|^2 \left|k_{q,a}^E\right|^2\right)^{q/2} \le \sum_{a\in S} |\mu_a|^q \left|k_{q,a}^E\right|^q$, and integrating over $\mathbb{T}$ we get:

$$\int_{\mathbb{T}} \mathbb{E}\left[|g|^q\right] \, d\sigma \le \int_{\mathbb{T}} \left(\sum_{a\in S} |\mu_a|^q \left|k_{q,a}^E\right|^q\right) d\sigma \le \sum_{a\in S} |\mu_a|^q \int_{\mathbb{T}} \left|k_{q,a}^E\right|^q \, d\sigma = \|\mu\|_{\ell^q} . \tag{4.8}$$

So putting (4.6) and (4.7) or (4.8) in (4.5) we get that S is an interpolating sequence for K_J^s. Clearly the operator T is a bounded linear interpolation operator. ■

We are now in a position to prove the main result of this paper.

Theorem 4.5. *Let* $1 < p \le 2,\ 1 \le s < p$ *and* q *such that* $\frac{1}{s} = \frac{1}{p} + \frac{1}{q}$. *Suppose that*

(i) *the dual sequence* $\{\rho_{p,a}\}_{a\in S}$ *exists and is norm bounded in* K_I^p,
(ii) $\|k_a^J\|_{s'} \simeq \dfrac{\|k_a^E\|_{s'}\|k_a^I\|_{p'}}{\|k_a^E\|_{p'}}$ *and*
(iii) S *is weakly* q*-Carleson in* K_E^q.

Then S *is* K_J^s*-interpolating and there exists a bounded linear interpolation operator.*

Before discussing special cases we mention a first consequence (using Proposition 2.3 and Fact 2.7) for the case of unconditionality.

Corollary 4.6. *Suppose the conditions of the preceding theorem fulfilled. Assume moreover that J is one-component and and that we have condition (iii) of Fact 2.7: the measure $\nu = \sum_{a\in S} \dfrac{1-|a|^2}{1-|J(a)|^2}\delta_a$ satisfies*

$$|\mu|(S(\zeta,r)) \leq Cr \tag{4.9}$$

for every Carleson window $S(\zeta = e^{it}, h)$ meeting the level set $L(J, 1/2)$. Then $(k_a^J/\|k_a^J\|_{s'})_{a\in S}$ is an unconditional sequence in $K_J^{s'}$.

As a corollary we obtain the first part of Theorem 1.1.

Corollary 4.7. *Let $1 < p \leq 2$. Let I be a one-component singular inner function and $S \subset \mathbb{D}$. Suppose that $\sup_{a\in S}|I(a)| < 1$. If $(k_a^I/\|k_a^I\|_{p'})_{a\in S}$ is uniformly minimal in $K_I^{p'}$, where $1/p + 1/p' = 1$ then for every $\varepsilon > 0$ and for every $1 \leq s < p$, S is an interpolating sequence in $K^s_{I^{1+\varepsilon}}$.*

Proof of Corollary 4.7. Condition (ii) of the theorem follows from the case (3) of Remark 4.3. The condition (i) of the theorem is fulfilled by the fact that $(k_a^I/\|k_a^I\|_{p'})_{a\in S}$ is uniformly minimal in $K_I^{p'}$. Let $(\rho_{p,a})_{a\in S}$ be the corresponding dual family in K_I^p. It remains to check the weak q-Carleson condition. In fact more is true: Since I is one-component and inner with $\sup_{a\in S}|I(a)| < 1$, we have for every $a \in S$, $1 < r < \infty$

$$\|k_a^I\|_r \simeq \left(\frac{1-|I(a)|^2}{1-|a|^2}\right)^{1-1/r} \simeq \left(\frac{1}{1-|a|^2}\right)^{1-1/r} \simeq \|k_a\|_r.$$

Hence, up to some constants c_a, $a \in S$, whose moduli are uniformly bounded above and below we get

$$\begin{aligned} \delta_{ab} &= \langle \rho_{p,a}, k_b^I/\|k_b^I\|_p\rangle = c_a\langle \rho_{p,a}, k_b^I/\|k_b\|_p\rangle = c_a\langle\rho_{p,a}, P_I(k_b/\|k_b\|_p)\rangle \\ &= c_a\langle P_I\rho_{p,a}, k_b/\|k_b\|_p\rangle \\ &= c_a\langle \rho_{p,a}, k_b/\|k_b\|_p\rangle. \end{aligned}$$

Hence $(k_a/\|k_a\|_{p'})_{a\in S}$ is a uniform minimal sequence in H^p which by the interpolation results is equivalent to $\Lambda \in (C)$. (We could also have shown this by using directly (2.1).) In particular, $(k_a/\|k_a\|_{p'})_{a\in S}$ is an unconditional sequence in any H^r, $1 < r < \infty$.

From this we can deduce that S is even r-Carleson for any $1 < r < \infty$: indeed, let $(\mu_a)_{a\in S} \in l^r$, then

$$\begin{aligned} \left\|\sum_{a\in S}\mu_a k_{a,r}^I\right\|_r^r &= \left\|P_I\sum_{a\in S}\mu_a\frac{k_a}{\|k_a^I\|_r}\right\|_r^r \leq c\left\|\sum_{a\in S}\mu_a\frac{k_a}{\|k_a^I\|_r}\right\|_r^r = c\left\|\sum_{a\in S}\mu_a\frac{\|k_a\|_r}{\|k_a^I\|_r}\frac{k_a}{\|k_a\|_r}\right\|_r^r \\ &\simeq \sum_{a\in S}|\mu_a|^r\left(\frac{\|k_a\|_q}{\|k_a^I\|_r}\right)^r \simeq \sum_{a\in S}|\mu_a|^r, \end{aligned} \tag{4.10}$$

where we have used that $\|k_a\|_r \simeq \|k_a^I\|_r$. This holds in particular for $r = q$, where $1/s = 1/p + 1/q$. ∎

We are now in a position to deduce also the second part of Theorem 1.1.

Corollary 4.8. *Let* $1 < p \leq 2$. *Let* I *be a one-component singular inner function and* $S \subset \mathbb{D}$. *Suppose that* $\sup_{a\in S}|I(a)| < 1$. *If* $(k_a^I/\|k_a^I\|_{p'})_{a\in S}$ *is uniformly minimal in* $K_I^{p'}$, *where* $1/p+1/p'=1$ *then for every* $\varepsilon > 0$ *and for every* $q < p$, $(k_a^I/\|k_a^I\|_{q'})_{a\in S}$ *is an unconditional basis in* $K^{q'}_{I^{1+\varepsilon}}$.

So in the present situation, we *increase* the space in the direction of the inner function and we *decrease* the space by increasing the power of integration to deduce unconditionality from uniform minimality.

Let us make another observation. In [Ni02, D4.4.9(5)] it is stated (in conjunction with [Ni02, Lemma D4.4.3]) that under the Carleson condition $S \in (C)$ the condition $\sup_{a\in S}|I(a)| < 1$ is equivalent to the existence of $N \in \mathbb{N}$ such that $(k_a^{I^N}/\|k_a^{I^N}\|_2)_{a\in S}$ is an unconditional sequence in $K^2_{I^N}$. In the present situation, when $(k_a^I/\|k_a^I\|_{p'})_{a\in S}$, $p' \geq 2$, is supposed uniformly minimal (which itself implies the Carleson condition under the assumptions on I and S; we do not know whether the Carleson condition could imply the uniform minimality in our context) then instead of taking I^N we can choose $I^{1+\varepsilon}$ for any $\varepsilon > 0$ (paying the price of replacing p' by $q' > p'$).

Proof of Corollary 4.8. In view of the preceding corollary and Corollary 2.5, it remains to check that S is $(l^s)^* = l^{s'}$-Carleson, which follows at once from (4.10) by taking $r = s'$. ■

Proof of the theorem. It remains to prove that the hypotheses of the theorem imply those of Lemma 4.2.We thus have to prove that

$$\mathbb{E}\left[\left\|\sum_{a\in S}\lambda_a\epsilon_a\rho_{p,a}\right\|_p^p\right] \lesssim \|\lambda\|^p_{\ell^p}.$$

under the assumption that the dual sequence $\{\rho_{p,a}\}_{a\in S}$ is uniformly bounded in K^p_I: $\sup_{a\in S}\|\rho_{p,a}\|_p \leq C$.

By Fubini's theorem

$$\mathbb{E}\left[\left\|\sum_{a\in S}\lambda_a\epsilon_a\rho_{p,a}\right\|_p^p\right] = \int_{\mathbb{T}}\mathbb{E}\left[\left|\sum_{a\in S}\lambda_a\epsilon_a\rho_{p,a}\right|^p\right]d\sigma,$$

and by Khinchin's inequalities we have

$$\mathbb{E}\left[\left|\sum_{a\in S}\lambda_a\epsilon_a\rho_{p,a}\right|^p\right] \simeq \left(\sum_{a\in S}|\lambda_a|^2|\rho_{p,a}|^2\right)^{p/2}.$$

Now, since $p \leq 2$, we have

$$\left(\sum_{a\in S}|\lambda_a|^2|\rho_{p,a}|^2\right)^{1/2} \leq \left(\sum_{a\in S}|\lambda_a|^p|\rho_{p,a}|^p\right)^{1/p},$$

and hence

$$\int_{\mathbb{T}}\mathbb{E}\left[\left|\sum_{a\in S}\lambda_a\epsilon_a\rho_{p,a}\right|^p\right]d\sigma \leq \int_{\mathbb{T}}\left(\sum_{a\in S}|\lambda_a|^p|\rho_{p,a}|^p\right)d\sigma = \sum_{a\in S}|\lambda_a|^p\|\rho_{p,a}\|_p^p.$$

So, finally

$$\mathbb{E}\left[\left\|\sum_{a\in S}\lambda_a\epsilon_a\rho_{p,a}\right\|_p^p\right]\lesssim\sup_{a\in S}\ \|\rho_{p,a}\|_p^p\,\|\lambda\|_p^p\,,$$

and consequently the theorem holds. ■

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