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Expansion of the propagation of chaos for Bird and Nanbu systems

Sylvain Rubenthaler*

October 15, 2010

Abstract

The Bird and Nanbu systems are particle systems used to approximate the solution of the mollified Boltzmann equation. In particular, they have the propagation of chaos property. Following [GM94, GM97, GM99], we use coupling techniques and results on branching processes to write an expansion of the error in the propagation of chaos in terms of the number of particles, for slightly more general systems than the ones cited above. This result leads to the proof of the a.s convergence and the central-limit theorem for these systems. In particular, we have a central-limit theorem for the empirical measure of the system under less assumptions than in [Mél98]. As in [GM94, GM97, GM99], these results apply to the trajectories of particles on an interval $[0; T]$.

Keywords: interacting particle systems, Boltzmann equation, nonlinear diffusion with jumps, random graphs and trees, coupling, propagation of chaos, Monte Carlo algorithms, U -statistics.

MSC 2010: 60J80, 60J85, 65M75, 82C22, 65M75, 65N75, 60C05, 60F17, 60J80, 60K35, 65C05, 75P05, 82C40, 82C80

1 Introduction

In a recent work ([DPR09b]), we showed an expansion of the propagation of chaos for a Feynman-Kac particle system. This particle system approximates a particular Feynman-Kac measure, in the sense that the empirical measure associated to the system converges to the Feynman-Kac measure when the number of particles N goes to ∞ . What is called propagation of chaos is the following double property of the particle system

- q particles, amongst the total of N particles, looked upon at a fixed time, are asymptotically independent when $N \rightarrow +\infty$ (q is fixed)
- and their law is converging to the Feynman-Kac law.

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In [DPR09b], we wrote an expansion in powers of N of the difference between the law of q independent particles, each of them of the Feynman-Kac law, and the law of q particles coming from the particle system. One can also call this expansion a functional representation like in [DPR09b]; in the present paper, we call it an expansion of the error in the propagation of chaos. In the setting of [DPR09b], the time is discrete. We showed there how to use this kind of expansion to derive a.s. convergence results (p. 824). In ([DPR09a]), we extend the result of [DPR09b] to the case where the time is continuous, still in the Feynman-Kac framework, and we show central-limit theorems for U -statistics of these systems of particles. The proof of the central-limit theorems for U -statistics relies only on the exploitation of the expansion described above.

We wish here to establish a similar expansion for a family of particles systems including Bird and Nanbu systems. We do not go as far as getting an expansion in the terms of Theorem 1.6 and Corollary 1.8 of [DPR09a], but what we have is enough for proving a central-limit theorem (Theorem 5.2). Bird and Nanbu systems are used to approximate the solution of the mollified Boltzmann equation. We refer mainly to [GM97] and take into account models described in (2.5), (2.6) of [GM97] (a similar description can be found in [GM99], Section 3). An other reference paper on the subject is [GM94]. The main points of interest of this paper are: it provides a sequel to the estimates on propagation of chaos of [GM97], [GM99] and it allows to apply the results of [DPR09b], [DPR09a] to Bird and Nanbu systems. In particular,

- we get a central-limit theorem for the empirical measure of the system (Th. 5.2) with less assumptions than in [Mél98] Th. 4.2, 4.3. (we suppose only the existence and unicity of a solution to (2.6))
- these results hold for trajectories of the particles on any interval $[0; T]$.

The proofs leading to the development in the propagation of chaos are radically different from those in [DPR09a] and this is why we decided to write them in a different paper.

In Section 2, we will recall the definitions of Bird and Nanbu models, as can be found in [GM97] and will give an equivalent definition, useful to our purposes. We will also introduce auxiliary particle systems which will be useful in the proofs. In Section 3, we will state and prove our first main theorem about the expansion of the error in propagation of chaos (Theorem 3.1). The proof relies on estimates on population growth found in [AN72] and on coupling ideas. In Section 4, we prove in what is called a Wick-type formula in [DPR09b] (see (3.6) p. 807 in [DPR09b] and [DPR09a], p.15 and Proposition 4.2). This formula (Corollary 4.6) and Proposition 4.2 are used in Section 5 to prove an a.s. convergence theorem for the empirical measure (Th. 5.1) and our second main theorem: a central-limit theorem for the empirical measure (Th. 5.2).

An important point is that we want to discuss here of the mathematical properties of a certain class of particle systems which includes Bird and Nanbu systems. This is why we will not discuss in details the physical models. Such a discussion can be found in [GM99].

2 Definition of the model

2.1 Mollified Boltzmann equation

Let $E = \mathbb{R}^6$ for the time being. We are given a Markov kernel L . The domain of L is denoted by $\mathcal{D}(L)$. We are interested in molecules in \mathbb{R}^3 which can diffuse according to the kernel L and collide with one another. The set \mathbb{R}^3 is partitioned in cells Δ of volume $|\Delta| = \delta^3$ and molecules in each cell interact as if in the same location. We set $(\forall x, y \in \mathbb{R}^3)$

$$I^\delta(x, y) = \frac{1}{\delta^3} \sum_{\Delta} \mathbf{1}_{x \in \Delta} \mathbf{1}_{y \in \Delta}.$$

When molecules collide, and if given their velocities v, w before the collision, the velocities after the collision depend on an impact factor $\nu \in \mathbb{S}^2$ (\mathbb{S}^2 is the unit sphere in \mathbb{R}^3). The collision transform v into $v^* = v + ((w - v) \cdot \nu)\nu$ and w into $w^* = w + ((v - w) \cdot \nu)\nu$. We introduce a nonnegative function B called the cross section such that $B(v - w, \nu)$ quantifies the likelihood of the interaction of molecules of velocities v, w with impact factor ν . To avoid difficulties, we suppose that

$$\sup_{x, y, v, w} \left(\int I^\delta(x, y) B(v - w, \nu) d\nu \right) < \infty. \quad (2.1)$$

We want to define a kernel $\mu(z, a, dh)$ ($z, a, h \in \mathbb{R}^6$). We write (x, v) , $x \in \mathbb{R}^3$, $v \in \mathbb{R}^3$, for the position and velocity of a molecule. We define μ by its action test functions f ,

$$\begin{aligned} & \int_{\mathbb{R}^6} f(x', v + v') \mu((x, v), (y, w), d(x', v')) \\ &= \int_{\mathbb{S}^2} f(x, v + ((w - v) \cdot \nu)\nu) I^\delta(x, y) B(v - w, \nu) d\nu. \end{aligned} \quad (2.2)$$

Let \tilde{P}_0 be a law on \mathbb{R}^3 . We are interested in finding $(\tilde{P}_t)_{t \geq 0}$ the solution of the following nonlinear martingale problem (with initial condition \tilde{P}_0 ,

$$\begin{aligned} & \forall \phi \in \mathcal{D}(L), \\ & \partial_t \langle \phi, \tilde{P}_t \rangle - \langle L\phi, \tilde{P}_t \rangle \\ &= \left\langle \int (\phi(z + h) - \phi(z)) \mu(z, a, dh), \tilde{P}_t(dz) \tilde{P}_t(da) \right\rangle. \end{aligned} \quad (2.3)$$

This equation is called the mollified Boltzmann equation with delocalized cross section (cf. [GM97]). Suppose we start at time 0 with a gas molecules distributed according to \tilde{P}_0 . The solution \tilde{P}_t is $\forall t$ the density of the molecules having undergone the diffusion L and the collision described above. This is only a short explanation of the physics behind this article. For a detailed study, the reader is referred to [GM99]. The purpose of the particle system introduced in [GM94, GM97, GM99] is to approximate the solution of (2.3) by an empirical measure.

2.2 Particles model

We will now introduce particle systems related to (2.3). These systems are used to approximate the solution of (2.3), (2.6), as it will be seen in (2.5). For the sake of generality, we use a more general setting than in the Subsection above. In all the following, we deal with particles evolving in $E := \mathbb{R}^d$. We set the mappings $e_i : h \in \mathbb{R}^d \mapsto e_i(h) = (0, \dots, 0, h, 0, \dots, 0) \in \mathbb{R}^{d \times N}$ (h at the i -th rank) ($1 \leq i \leq N$). We have a Markov generator L . We have a kernel $\widehat{\mu}(v, w, dh, dk)$ on \mathbb{R}^{2d} which is symmetrical (that is $\widehat{\mu}(v, w, dh, dk) = \widehat{\mu}(w, v, dk, dh)$). We set $\mu(v, w, dh)$ to be the marginal $\widehat{\mu}(v, w, dh \times \mathbb{R}^d)$. Our assumptions are the same as in [GM97]:

Hypothesis 1. 1. We suppose the equation (2.6) below has a unique solution (see [GM94, GM97, GM99] for more details). See [GM97] p. 119 for a discussion on $\mathcal{D}(L)$.

2. We suppose $\sup_{x,a} \widehat{\mu}(x, a, \mathbb{R}^d \times \mathbb{R}^d) \leq \Lambda < \infty$.

In Nanbu and Bird systems, the kernel $\widehat{\mu}$ and the generator L have specific features coming from physical considerations, like the ones explained above. In these systems, the coordinates in \mathbb{R}^d represent the position and speed of molecules but these considerations have no effect on our proof. This is why we claim to have a proof for systems more general than Bird and Nanbu systems.

The Nanbu and Bird systems are defined in (2.5) and (2.6) of [GM97], by the mean of integrals over Poisson processes. We give here an equivalent definition.

Definition 2.1. The particle system described in [GM97] is denoted by

$$(\overline{Z}_t)_{t \geq 0} = (\overline{Z}_t^i)_{t \geq 0, 1 \leq i \leq N} .$$

It is a process of N particles in \mathbb{R}^d and can be summarized by the following.

1. Particles $(\overline{Z}_0^i)_{1 \leq i \leq N}$ in \mathbb{R}^d are drawn i.i.d. at time 0 according to a law \widetilde{P}_0 .
2. Between jump times, the particles evolve independently from each other according to L .
3. We have a collection $(N_{i,j})_{1 \leq i < j \leq N}$ of independent Poisson processes of parameter $\Lambda/(N-1)$. For $i > j$, we set $N_{i,j} = N_{j,i}$. If $N_{i,j}$ has a jump at time t , we say there is an interaction between particles i and j and we take a uniform variable U on $[0, 1]$, independent of all the other variables, if $U \leq \frac{\widehat{\mu}(\overline{Z}_{t-}^i, \overline{Z}_{t-}^j, \mathbb{R}^{2d})}{\Lambda}$ then the system undergoes a jump:

(a) In the Bird system: $\overline{Z}_t = \overline{Z}_{t-} + e_i(H) + e_j(K)$ with

$$(H, K) \sim \frac{\widehat{\mu}(\overline{Z}_{t-}^i, \overline{Z}_{t-}^j, \dots)}{\widehat{\mu}(\overline{Z}_{t-}^{N,i}, \overline{Z}_{t-}^{N,j}, \mathbb{R}^{2d})} \quad (2.4)$$

(independently of all the other variables).

(b) If we replace $\widehat{\mu}$ by $\widehat{\mu}'(z, a, dh, dg) = \mu(z, a, dh) \otimes \delta_0(dg) + \delta_0(dh) \otimes \mu(a, z, dg)$ in (2.4), we obtain the Nanbu system (cf. Remark 2.6, p. 120, [GM97])

We will denote by $(\overline{Z}_{0:t}^i)_{1 \leq i \leq N}$ the system of the trajectories of particles on $[0, t]$ ($\forall t \geq 0$), that is $\forall i: \overline{Z}_{0:t}^i = (\overline{Z}_s^i)_{0 \leq s \leq t}$. We use again this notation $0 : t$ in the following for the same purpose.

We denote by $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ the Skorohod space of processes (in \mathbb{R}^d), embedded with the Skorohod topology. As in [GM94], we define the total variation norm by: \forall measure μ on a measurable space (S, \mathcal{S}) ,

$$\|\mu\|_{TV} = \sup \left\{ \int_S f(x) \mu(dx), \|f\|_\infty \leq 1 \right\}.$$

Theorem 3.1 of [GM97] implies that there is propagation of chaos for this system. This theorem says ($\forall q, t$):

$$\|\mathcal{L}(\overline{Z}_{0:t}^1, \dots, \overline{Z}_{0:t}^q) - \mathcal{L}(\overline{Z}_{0:t}^1)^{\otimes q}\|_{TV} \leq 2q(q-1) \frac{\Lambda t + \Lambda^2 t^2}{N-1},$$

and

$$\|\mathcal{L}(\overline{Z}_{0:t}^1) - \tilde{P}_{0:t}\|_{TV} \leq 6 \frac{e^{\Lambda t} - 1}{N+1}, \quad (2.5)$$

where $\tilde{P}_{0:t} \in \mathcal{P}(\mathbb{D}(\mathbb{R}^+, \mathbb{R}^d))$ is solution of (with \tilde{P}_0 fixed) the nonlinear martingale problem

$$\begin{aligned} & \forall \phi \in \mathcal{D}(L), \\ & \partial_t \langle \phi, \tilde{P}_t \rangle - \langle L\phi, \tilde{P}_t \rangle \\ & = \left\langle \int (\phi(z+h) - \phi(z)) \mu(z, a, dh), \tilde{P}_t(dz) \tilde{P}_t(da) \right\rangle. \end{aligned} \quad (2.6)$$

Recall that $\tilde{P}_{0:t} = (\tilde{P}_s)_{0 \leq s \leq t}$, $\forall t$.

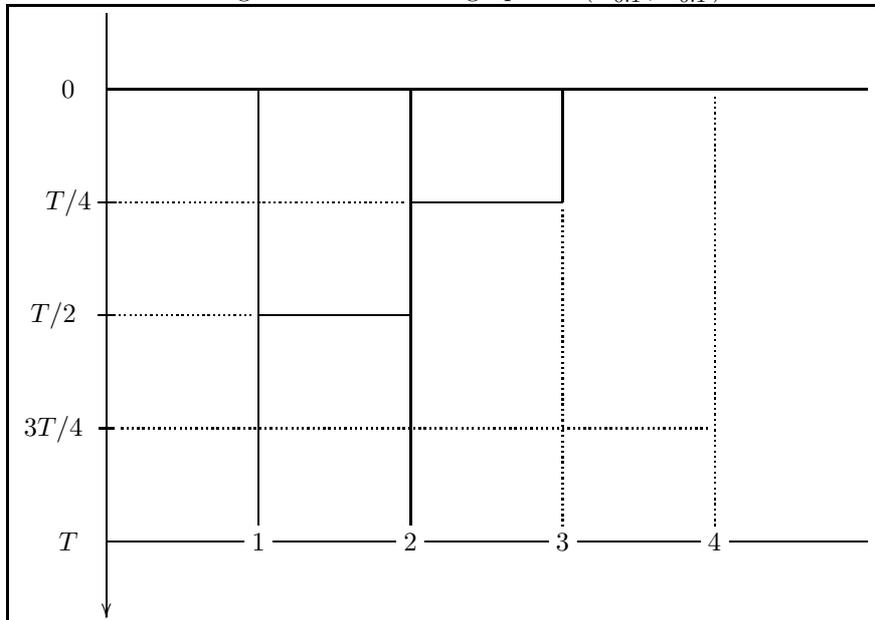
Remark 2.2. If μ is fixed, there exists different $\widehat{\mu}$ having the right marginal. It is in fact the choice of $\widehat{\mu}$ that leads to having different systems like, the Bird and Nanbu systems. We refer the reader to [GM94, GM97, GM99] for very good discussions on the difference between Bird and Nanbu models. What matters here is that our result apply to any systems satisfying Hypothesis 1 and having jumps of the form (2.4).

Remark 2.3. In the case $d = 6$ and μ defined by (2.2), the equations (2.3) and (2.6) coincide. If we have (2.1) in addition, then any $\widehat{\mu}$ such that $\widehat{\mu}(w, v, dh \times \mathbb{R}^d) = \mu(w, v, dh)$ satisfies Hyp. 1, 2.

We can deduce propagation of chaos from the previous results, that is $\forall t$, $\forall F$ bounded measurable,

$$|\mathcal{L}(\overline{Z}_{0:t}^1, \dots, \overline{Z}_{0:t}^q)(F) - \tilde{P}_{0:t}^{\otimes q}(F)| \leq \left(2q(q-1) \frac{\Lambda t + \Lambda^2 t^2}{N-1} + 6 \frac{e^{\Lambda t} - 1}{N+1} \right) \|F\|_\infty.$$

Figure 1: Interaction graph for $(\bar{Z}_{0:T}^1, \bar{Z}_{0:T}^2)$



In Theorem 3.1, we will go further than the above bound by writing an expansion of the left hand side term above in powers of N . We will use techniques introduced in [GM97]. The main point is that one looks at the processes backward in time.

Example 2.4. Suppose for example that $N = 4$, that $N_{1,2}$ has only one jump time in $T/2$, $N_{1,3}$ does not jump on $[0; T]$, $N_{2,3}$ has one jump time $= T/4$ on the interval $[0; T]$, $N_{3,4}$ has one jump in $3T/4$ on the interval $[0; T]$, and there is no other jump of the $N_{i,j}$'s.

Figure 1 is a pictorial representation of the example above. On the left is the time arrow for the particles. What we draw here is called the graph of interactions (for $\bar{Z}_{0:T}^1, \bar{Z}_{0:T}^2$) in [GM97, GM99]. Suppose we want to simulate $\bar{Z}_{0:T}^1, \bar{Z}_{0:T}^2$. We first simulate the interactions times of the system. Suppose these are the ones of the example above. In Figure 1, we represent by solid vertical lines the trajectories we have to simulate to get $\bar{Z}_{0:T}^1, \bar{Z}_{0:T}^2$. The particles numbers are to be found at the bottom of the graph. The horizontal solid lines represent the interaction we have to simulate in order to get $\bar{Z}_{0:T}^1, \bar{Z}_{0:T}^2$ (they may or may not induce jumps for the particles). For example, a horizontal solid line between the vertical solid lines representing the trajectories of $\bar{Z}_{0:T}^1, \bar{Z}_{0:T}^2$ represent an interaction between particle 1 and particle 2. The interactions are simulated following Definition 2.5. The trajectory $\bar{Z}_{0:T}^3$ is represented by a solid line between the times 0 and $T/4$ and by a dashed line between the times $T/4$

and T , with the number 3 at the bottom. As we want to simulate $\overline{Z}_{0:T}^1$ and $\overline{Z}_{0:T}^2$ and we have simulated the jumps as in the example above, then we are not interested in \overline{Z}_t^3 for $t > T/4$. In the same way, we draw dotted lines for the trajectory $\overline{Z}_{0:T}^4$ and the interaction related to this particle.

2.3 Other systems of particles

2.3.1 Backward point of view

From now on, we will work with a fixed time horizon $T > 0$ and a fixed $q \in \mathbb{N}^*$. For any $j \in \mathbb{N}^*$, we set $[j] = \{1, \dots, j\}$. Recall that \mathbb{N}^* is the set of positive integers. For $\lambda > 0$, we call $\mathcal{E}(\lambda)$ the exponential law of parameter λ .

We want to construct a system of particles $(Z_{0:T}^i)_{1 \leq i \leq N}$ such that the first q particles have the same law as $(\overline{Z}_{0:T}^1, \dots, \overline{Z}_{0:T}^q)$ (see Lemma 2.9). We take advantage of the fact that the processes $(N_{i,j}(T-t))_{0 \leq t \leq T}$ are Poisson processes to construct the interaction graph for the first q particles moving backward in time. This system of particles $(Z_{0:T}^i)_{1 \leq i \leq N}$ is in fact the central system in our paper, all other systems will be compared to it.

We start at $s = 0$ with $C_0^i = \{i\}$, $\forall i \in [q]$. For $i \in [q]$, we define $(C_s^i)_{s \geq 0}$, $(K_s^i)_{s \geq 0}$ (respectively taking values in $\mathcal{P}(\mathbb{N})$, \mathbb{N}^*) by the following. We take $(U_k)_{1 \leq i \leq q, 1 \leq k}$, $(V_k)_{1 \leq i \leq q, 1 \leq k}$ i.i.d. $\sim \mathcal{E}(1)$. In all the following, we will use the conventions: $\inf \emptyset = +\infty$ and $(\dots)_+$ is the nonnegative part. The processes (C^i) , (K^i) are piecewise constant and make jumps. At any time t , we set $K_t^i = \#C_t^i$. We define the jump times recursively by $T_0 = 0$ and

$$\begin{aligned} T_k' &= \inf \left\{ T_{k-1} \leq s \leq T : (s - T_{k-1}) \times \frac{\Lambda K_{T_{k-1}}(N - K_{T_{k-1}})_+}{N - 1} \geq U_k \right\} \\ T_k'' &= \left\{ T_{k-1} \leq s \leq T : (s - T_{k-1}) \times \frac{\Lambda K_{T_{k-1}}(K_{T_{k-1}} - 1)}{2(N - 1)} \geq V_k \right\} \\ T_k &= \inf(T_k', T_k''). \end{aligned}$$

At T_k :

- At $T_k = T_k'$, we take

$$\begin{aligned} r(k) &\text{ uniformly in } C_{T_k-}^1 \cup \dots \cup C_{T_k-}^q, \\ j(k) &\text{ uniformly in } [N] \setminus (C_{T_k-}^1 \cup \dots \cup C_{T_k-}^q). \end{aligned} \quad (2.7)$$

For any i such that that $r(k) \in C_{T_k-}^i$, we then perform the jumps: $C_{T_k}^i = C_{T_k-}^i \cup \{j(k)\}$.

Note that the $(\dots)_+$ in the definition of T_k' above forbids to be in the situation where we would be looking for $j(k)$ in \emptyset .

- If $T_k = T_k''$, we take

$$\begin{aligned} r(k) &\text{ uniformly in } C_{T_k-}^1 \cup \dots \cup C_{T_k-}^q, \\ j(k) &\text{ uniformly in } C_{T_k-}^1 \cup \dots \cup C_{T_k-}^q \setminus \{r(k)\}. \end{aligned} \quad (2.8)$$

This whole construction is analogous to the construction of the interaction graph found in [GM97], p. 122. For all $s \in [0, T]$, we set $K_t = \#(C_t^1 \cup \dots \cup C_t^q)$. We set $\forall t \leq T$,

$$\mathcal{K}_t = (K_s^i)_{1 \leq i \leq q, 0 \leq s \leq t}.$$

We now define an auxiliary process $(Z_s)_{0 \leq s \leq T} = (Z_s^i)_{0 \leq s \leq T, 1 \leq i \leq N}$ of N particles in \mathbb{R}^d .

Definition 2.5. Let $k' = \sup \{k, T_k < \infty\}$. The interaction times of the $(Z_s^i)_{1 \leq s \leq T, 1 \leq i \leq N}$ are $T - T_{k'} \leq T - T_{k'-1} \leq \dots \leq T - T_1$. (We say that the interaction times are defined backward in time.)

- Z_0^1, \dots, Z_0^N are i.i.d. $\sim \tilde{P}_0$
- Between the times $(T - T_k)_{k \geq 1}$, the Z^i 's evolve independently from each other according to the Markov generator L .
- At a jump time $T - T_k$ where T_k is a jump time of $C^{r(k), j(k)}$, $(Z^i)_{1 \leq i \leq N}$ undergoes an interaction having the same law as in Definition 2.1, (3a), with (i, j) replaced by $(r(k), j(k))$, in the case of the Bird system (use Definition 2.1 (3b) in the case of the Nanbu system).

Definition 2.6. For all $0 \leq t \leq T$, we set

$$L_t = \#\{k \in \mathbb{N} : T_k \leq t, T_k = T_k''\}.$$

We call this quantity the number of loops on $[0, t]$.

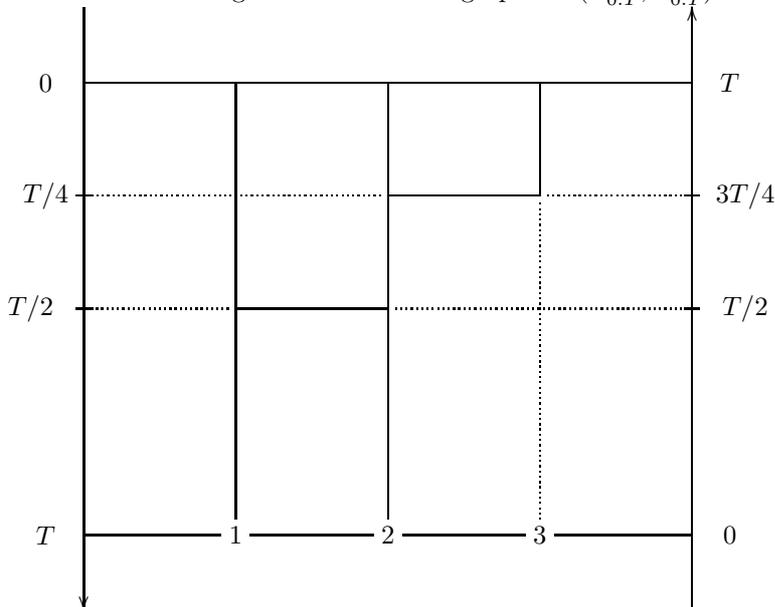
Example 2.7. We look now at a situation similar to Example 2.4. Take $q = 2$. Suppose for example, that $T_0 = 0, T_1 = T/2, T_2 = 3T/4, T_3 = +\infty, r(1) = 1, j(1) = 1, r(2) = 2, j(2) = 3$.

Then

- for $s \in [0, T/2[$, $K_s = 2, L_s = 0, K_s^1 = K_s^2 = 1$,
- for $s \in [T/2, 3T/4[$, $K_s = 2, L_s = 1, K_s^1 = K_s^2 = 1$,
- for $s \in [3T/4, T]$, $K_s = 3, L_s = 1, K_s^1 = 1, K_s^2 = 2$.

Figure 2 is a pictorial representation of the example above. On the left is the time arrow for the particles. On the right is the time arrow for the processes $(C^i), (K^i)$. What we draw here is called the graph of interactions (for $Z_{0:T}^1, Z_{0:T}^2$) in [GM97, GM99]. Suppose we want to simulate $Z_{0:T}^1, Z_{0:T}^2$. We first simulate the interactions times of the system. Suppose these are the ones of the

Figure 2: Interaction graph for $(Z_{0:T}^1, Z_{0:T}^2)$



example above. In Figure 2, we represent by solid vertical lines the trajectories we have to simulate to get $Z_{0:T}^1, Z_{0:T}^2$. The particles numbers are to be found at the bottom of the graph. The horizontal solid lines represent the interaction we have to simulate in order to get $Z_{0:T}^1, Z_{0:T}^2$ (they may or may not induce jumps for the particles). For example, a horizontal solid line between the vertical solid lines representing the trajectories of $Z_{0:T}^1, Z_{0:T}^2$ represent an interaction between particle 1 and particle 2. The interactions are simulated following Definition 2.5. The trajectory $Z_{0:T}^3$ is represented by a solid line between the times 0 and $T/4$ and by a dashed line between the times $T/4$ and T , with the number 3 at the bottom. As we want to simulate $Z_{0:T}^1$ and $Z_{0:T}^2$ and we have simulated the jumps as in the example above, then we are not interested in Z_t^3 for $t > T/4$ and we are not interested in any $Z_{0:T}^i$ with $i \geq 4$. Again, the time for the particles should be read on the left.

We have to keep in mind the following lemma throughout the whole paper.

Lemma 2.8. *Let us denote by $(N_t^\lambda)_{t \geq 0}$ an inhomogeneous Poisson process of rate $(\lambda_t)_{t \geq 0}$ (λ is supposed to be a deterministic càdlàg nonnegative process)*

1. *Let us denote the jumps of N^λ by $\tau_1 < \tau_2 < \dots$. Then $\forall k \in \mathbb{N}^*$, $\mathcal{L}(\tau_1, \tau_2, \dots, \tau_k | \tau_k \leq T < \tau_{k+1})$ is the law of the order statistics of k independent variables of law of density $t \mapsto \lambda_t / \int_0^T \lambda_s ds$ on $[0; T]$.*
2. *For any $j \in \mathbb{N}^*$, take càdlàg processes $(\alpha_t^j)_{t \geq 0}$ such that $\forall t, 0 \leq \alpha_t^1 \leq \alpha_t^1 + \alpha_t^2 \leq \dots \leq \alpha_t^1 + \dots + \alpha_t^j \leq 1$. Suppose we take $(W_k)_{k \geq 0}$ i.i.d. random*

variables of uniform law in $[0; 1]$ independent of N and set j processes $(N_t^i)_{t \geq 0, 1 \leq i \leq j}$ such that $N_0^i = 0, \forall i$, the processes N^i 's are a.s. piecewise constant and may jump at the jump times of N following this rule: $\Delta N_t^i = 1$ if and only if $\Delta N_t = 1$ and $\alpha_t^1 + \dots + \alpha_t^{i-1} \leq W_{N_t-} < \alpha_t^1 + \dots + \alpha_t^i$. Then the N^i 's are j independent inhomogeneous Poisson processes such that N^i has rate $(\alpha_t^i \times \lambda_t)_{t \geq 0}, \forall i$.

3. Take $j \in \mathbb{N}^*$, j independent inhomogeneous Poisson processes $(N^i)_{1 \leq i \leq j}$ respectively of rate $(\lambda_t^i)_{t \geq 0}$. Then $N_t = N_t^1 + \dots + N_t^j$ is an inhomogeneous Poisson process of rate $(\lambda_t^1 + \dots + \lambda_t^j)_{t \geq 0}$ and $\forall i, \forall s, \mathbb{P}(\Delta N_t^i = \Delta N_t | \Delta N_t = 1) = \mathbb{P}(\Delta N_t^i = \Delta N_t | \Delta N_t = 1, (N_s^k)_{1 \leq k \leq j, 0 \leq s < t}) = \frac{\lambda_t^i}{\lambda_t^1 + \dots + \lambda_t^j}$.

Proof. We write here a brief proof. Recall that the definition of an inhomogeneous Poisson process of rate $(\lambda_t)_{t \geq 0}$ is the following. We take Π a Poisson point process on $\mathbb{R}^+ \times \mathbb{R}^+$ with intensity $\mu = dx \times dy$ (see Chapter 2 of [Kin93] for the definition of a Poisson point process). Then $(N_t^\lambda)_{t \geq 0} = (\#\{\Pi \cap \{(x, y), x \leq t, y \leq \lambda_x\}\})_{t \geq 0}$ is an inhomogeneous Poisson process of rate $(\lambda_t)_{t \geq 0}$. Set $\forall x \geq 0, F(x) = \int_0^x \lambda_u du$ and $\forall y \geq 0, F^{-1}(y) = \inf\{x, F(x) > F(y)\}$. Set $G : (x, y) \in \mathbb{R}^+ \times \mathbb{R}^+ \mapsto (F(x), y) \in \mathbb{R}^+ \times \mathbb{R}^+$. The Mapping Theorem ([Kin93], p. 18) says that $G(\Pi)$ is a Poisson process with intensity μ^* defined by $\mu^*(B) = \mu(G^{-1}(B))$. In particular, $\forall t \geq 0$,

$$\begin{aligned} \mu^* (\{(x, y) \in (\mathbb{R}^+)^2, x \leq t, y \leq \lambda_{F^{-1}(x)}\}) \\ = \mu (\{(x, y) \in (\mathbb{R}^+)^2, x \leq F^{-1}(t), y \leq \lambda_x\}) \\ = \int_0^{F^{-1}(t)} \lambda_u du = t. \end{aligned}$$

This shows that $N_t' = (\#\{G(\Pi) \cap \{(x, y) \in \mathbb{R}^+ \times \mathbb{R}^+, x \leq t, y \leq \lambda_{F^{-1}(x)}\}\})_{t \geq 0}$ is a Poisson process of constant rate 1. Let us write $\tau'_1 \leq \tau'_2 \leq \dots$ for the ordered jumps of N_t' . It can be shown by direct calculation that $\forall k, t, \mathcal{L}((\tau'_1, \dots, \tau'_k) | \tau'_k \leq t < \tau'_{k+1})$ is the ordered statistics of k independent variables of uniform law on $[0; t]$. We have $N_t' = N_{F^{-1}(t)}^\lambda, \forall t$ so $\tau_i = F^{-1}(\tau'_i), \forall i$. This proves the point 1 of the Proposition.

We take now processes $(\lambda_t^i)_{t \geq 0}$ as in the point 2 of the Proposition. The Mapping Theorem implies that $\forall i$, if we define $\forall t$,

$$N_t^i = \#\{\Pi \cap \{(x, y) \in (\mathbb{R}^+)^2, x \leq t, \lambda_x^1 + \dots + \lambda_x^{i-1} \leq y \leq \lambda_x^1 + \dots + \lambda_x^i\}\},$$

then the (N_t^i) are inhomogeneous Poisson processes of rate $(\lambda_t^i)_{t \geq 0}$ respectively. This can be seen for any i by using the mapping $(x, y) \mapsto (x, y - (\lambda_x^1 + \dots + \lambda_x^{i-1}))$. These processes (N^i) are independent due to the definition of a Poisson point process. The sum process $N_t = N_t^1 + \dots + N_t^j$ is an inhomogeneous Poisson process of rate $(\lambda^1 + \dots + \lambda^j)$. As $\forall i, \mathbb{P}(\Delta N_t^i = 1 | \Delta N_t = 1, (N_s^j)_{0 \leq s < t}) =$

$\frac{\lambda_t^i}{\lambda_t^1 + \dots + \lambda_t^j}$ (this can be shown by direct calculation on Π), we get the point 3 of the Proposition. The idea here is that whatever structure we add, the law of the sum of j independent inhomogeneous Poisson processes of rates $\lambda^1, \dots, \lambda^j$ is unique.

We take now $(\alpha_t^i)_{1 \leq i \leq j, t \geq 0}$ as in the point 2. We set $N_t = \#(\Pi \cap \{(x, y) \in \mathbb{R}^+ \times \mathbb{R}^+, x \leq t, y \leq \lambda_x\})$ and $\forall i, N_t^i = \#(\Pi \cap \{(x, y) \in \mathbb{R}^+ \times \mathbb{R}^+, x \leq t, (\alpha_x^1 + \dots + \alpha_x^{i-1})\lambda_x \leq y \leq (\alpha_x^1 + \dots + \alpha_x^i)\lambda_x\})$. The process (N_t) is an inhomogeneous Poisson process of rate λ . Once again, the processes N^i are independent inhomogeneous Poisson processes of rates, respectively, $\alpha^i \lambda$. We have that $\mathbb{P}(\Delta N_t^i = \Delta N_t | \Delta N_t = 1) = \mathbb{P}(\Delta N_t^i = \Delta N_t | \Delta N_t = 1, (N_s)_{s \geq 0}) = \alpha_t^i$. This proves the point 2. Again, whatever structure we add, the law of a processes N^i , built from N by having $\Delta N_t^i = \Delta N_t$ with proba α_t^i at any jump point of N and independently of the trajectory of N , is unique. \square

We then get the following Lemma.

Lemma 2.9. *For all $T \geq 0$, $(Z_{0:T}^1, \dots, Z_{0:T}^q) \stackrel{law}{=} (\overline{Z}_{0:T}^1, \dots, \overline{Z}_{0:T}^q)$.*

Proof. We give here a brief explanation of why our construction is equivalent to the construction with an interaction graph in [GM97] Section 3.1 p. 122-123. We start the construction of the interaction times at time 0. We first look at the times $\{T_k, k \geq 1\} \cap \{T'_l, l \geq 1\}$. The law of $\tau = \inf\{T_k, k \geq 1\} \cap \{T'_l, l \geq 1\}$ is $\mathcal{E}(\frac{\Delta q(N-q)_+}{N-1})$. At this time τ , we choose $r(k)$ in $C_{\tau-}^1 \cup \dots \cup C_{\tau-}^q$ and $j(k) \in [N] \setminus C_{\tau-}^1 \cup \dots \cup C_{\tau-}^q$ and we perform the jump $C_{\tau}^{r(k)} = C_{\tau-}^{r(k)} \cup \{j(k)\}$. For example, in the Figure 2, we wait $T/2$ and then we add 3 to the set $C_{(T/2)-}^2 = \{2\}$.

The situation in [GM94, GM97, GM99] is the following. We have Poisson processes $N_{i,j}$ like in Subsection 2.1. We start at the bottom of the interaction graph and we move upward. There, we mean we follow the time arrow for the processes C^i, K^i . As the processes $(N_{i,j}(T-t))_{0 \leq t \leq T}$ are Poisson processes, we wait for $\tau' = \inf\{t : \text{jump time of } N_{i,j}(T-t), i \in [q], j \notin [q]\}$. And then, if τ' is a jump time for $N_{r',j'}$ with $r' \in [q]$, we add a branch corresponding to j' to the branch corresponding to r' (like in Figure 1 we add the branch with the label 3 to the branch with the label 2). The random times τ and τ' have the same law due to Lemma 2.8, 3. The random couple of indexes $(r, j), (r, j')$ have the same law due to Lemma 2.8, 2. One can go on and show that the way we add branches is exactly the same as in [GM97] Section 3.1 p. 122-123.

Now we look at the horizontal lines between existing branches (such at the line between 1 and 2 in Figure 2). Let $0 \leq t \leq T$. We set $j = \#\{k, T_k \leq t\}$. We

compute

$$\begin{aligned}
& \mathbb{P}(\forall k \leq j, T_k \neq T_k'' | j, T_1', T_2', \dots, T_j', (K_u)_{0 \leq u \leq t}) \\
&= \mathbb{P}(T_1' \frac{\Lambda q(q-1)}{2(N-1)} > V_1, \dots \\
&\dots, (t - T_k') \frac{\Lambda(q+j)(q+j-1)}{2(N-1)} > V_{j+1} | j, T_1', T_2', \dots, T_j', (K_u)_{0 \leq u \leq t}) \\
&= \exp\left(-\int_0^t \frac{\Lambda K_u(K_u-1)}{2(N-1)} du\right).
\end{aligned}$$

So, conditionnaly to $(K_u)_{0 \leq u \leq T}$, $(\#\{T_k = T_k'', T_k \leq t\})_{t \geq 0}$ is an inhomogeneous Poisson process of rate $\left(\frac{\Lambda K_u(K_u-1)}{2(N-1)}\right)_{0 \leq u \leq t}$. When a jump time of the form $T_k = T_k''$ occurs, we choose $r(k)$ uniformly in $C_{T_k-}^1 \cup \dots \cup C_{T_k-}^q$ and $j(k)$ uniformly in $C_{T_k-}^1 \cup \dots \cup C_{T_k-}^q \setminus \{r(k)\}$. We then add an horizontal line between branches $r(k)$ and $j(k)$. Due to Lemma 2.8, 2, this is the same as horizontal branches are added to existing vertical branches in [GM97]. \square

2.3.2 Auxiliary systems

We now define an auxiliary system $(\tilde{Z}_{0:T}^i)_{i \geq 1}$ with an infinite number of particles. We start at $s = 0$ with $\tilde{C}_0^i = \{i\}$, $\forall i \in [q]$. For $1 \leq i \leq N$, we define $(\tilde{C}_s^i)_{s \geq 0, 1 \leq i \leq q}$, $(\tilde{K}_s^i)_{s \geq 0, 1 \leq i \leq q}$ (respectively taking values in $\mathcal{P}(\mathbb{N}), \mathbb{N}$) by the following. The processes $(\tilde{C}^i), (\tilde{K}^i)$ are piecewise constant. At any time t , $\tilde{K}_t^i = \#\tilde{C}_t^i$. We take $(\tilde{U}_k)_{k \geq 1}, (\tilde{U}'_k)_{k \geq 1}$ i.i.d. $\sim \mathcal{E}(1)$. We define the jump times recursively by $\tilde{T}_0 = 0$ and

$$\begin{aligned}
\tilde{T}'_k &= \inf\{\tilde{T}_{k-1} \leq s \leq T, \\
&\quad (s - \tilde{T}_{k-1}) \left(\Lambda \tilde{K}_{\tilde{T}_{k-1}} - \frac{\Lambda K_{\tilde{T}_{k-1}}(N - K_{\tilde{T}_{k-1}})_+}{N-1} \right) \geq \tilde{U}_k \} \\
\tilde{T}''_k &= \inf\{\tilde{T}_{k-1} \leq s \leq T, \\
&\quad (s - \tilde{T}_{k-1}) \times \frac{\Lambda \tilde{K}_{\tilde{T}_{k-1}}(\tilde{K}_{\tilde{T}_{k-1}} - 1) - \Lambda K_{\tilde{T}_{k-1}}(K_{\tilde{T}_{k-1}} - 1)}{2(N-1)} \geq \tilde{U}'_k \} \\
\tilde{T}_k &= \inf(\tilde{T}'_k, \tilde{T}''_k, \inf\{T_l : T_l > \tilde{T}_{k-1}\})
\end{aligned}$$

(recall that the process (K_t) and the T_k 's are defined in Subsection 2.3.1). Note that $\{T_k, k \geq 0\} \subset \{\tilde{T}_k, k \geq 0\}$. At \tilde{T}_k :

- If $\tilde{T}_k = \tilde{T}'_k$ (note that it implies that $\tilde{K}_{\tilde{T}_k-} - \frac{K_{\tilde{T}_k-}(N - K_{\tilde{T}_k-})_+}{N-1} > 0$):

– With probability

$$\frac{K_{\tilde{T}_k-} - \frac{K_{\tilde{T}_k-}(N-K_{\tilde{T}_k-})_+}{N-1}}{\tilde{K}_{\tilde{T}_k-} - \frac{K_{\tilde{T}_k-}(N-K_{\tilde{T}_k-})_+}{N-1}}, \quad (2.9)$$

we take $\tilde{r}(k)$ uniformly in $C_{\tilde{T}_k-}^1 \cup \dots \cup C_{\tilde{T}_k-}^q$ and

$$\tilde{j}(k) = \min \left\{ \mathbb{N}^* \setminus (\tilde{C}_{\tilde{T}_k-}^1 \cup \dots \cup \tilde{C}_{\tilde{T}_k-}^q \cup [N]) \right\}.$$

We perform the jumps: $\tilde{C}_{\tilde{T}_k}^{\tilde{r}(k)} = \tilde{C}_{\tilde{T}_k-}^{\tilde{r}(k)} \cup \{\tilde{j}(k)\}$.

– With probability

$$\frac{\tilde{K}_{\tilde{T}_k} - K_{\tilde{T}_k-}}{\tilde{K}_{\tilde{T}_k-} - \frac{K_{\tilde{T}_k-}(N-K_{\tilde{T}_k-})_+}{N-1}}, \quad (2.10)$$

we take $\tilde{r}(k)$ uniformly in $(\tilde{C}_{\tilde{T}_k-}^1 \setminus C_{\tilde{T}_k-}^1) \cup \dots \cup (\tilde{C}_{\tilde{T}_k-}^q \setminus C_{\tilde{T}_k-}^q)$ and $\tilde{j}(k) = \min \left\{ \mathbb{N}^* \setminus (\tilde{C}_{\tilde{T}_k-}^1 \cup \dots \cup \tilde{C}_{\tilde{T}_k-}^q \cup [N]) \right\}$. We perform the jumps: $\tilde{C}_{\tilde{T}_k}^{\tilde{r}(k)} = \tilde{C}_{\tilde{T}_k-}^{\tilde{r}(k)} \cup \{\tilde{j}(k)\}$.

- If $\tilde{T}_k = \tilde{T}_k''$ (note that it implies $\tilde{K}_{\tilde{T}_k-} > K_{\tilde{T}_k-}$):

– With probability

$$\frac{(\tilde{K}_{\tilde{T}_k-} - K_{\tilde{T}_k-})K_{\tilde{T}_k-} + (\tilde{K}_{\tilde{T}_k-} - K_{\tilde{T}_k-})(\tilde{K}_{\tilde{T}_k-} - K_{\tilde{T}_k-} - 1)}{\tilde{K}_{\tilde{T}_k-}(\tilde{K}_{\tilde{T}_k-} - 1) - K_{\tilde{T}_k-}(K_{\tilde{T}_k-} - 1)}, \quad (2.11)$$

we take $\tilde{r}(k)$ uniformly in $(\tilde{C}_{\tilde{T}_k-}^1 \setminus C_{\tilde{T}_k-}^1) \cup \dots \cup (\tilde{C}_{\tilde{T}_k-}^q \setminus C_{\tilde{T}_k-}^q)$ and $\tilde{j}(k)$ uniformly in $(\tilde{C}_{\tilde{T}_k-}^1 \cup \dots \cup \tilde{C}_{\tilde{T}_k-}^q) \setminus \{\tilde{r}(k)\}$.

– With probability

$$\frac{(\tilde{K}_{\tilde{T}_k-} - K_{\tilde{T}_k-})K_{\tilde{T}_k-}}{\tilde{K}_{\tilde{T}_k-}(\tilde{K}_{\tilde{T}_k-} - 1) - K_{\tilde{T}_k-}(K_{\tilde{T}_k-} - 1)}, \quad (2.12)$$

we take $\tilde{r}(k)$ uniformly in $(\tilde{C}_{\tilde{T}_k-}^1 \setminus C_{\tilde{T}_k-}^1) \cup \dots \cup (\tilde{C}_{\tilde{T}_k-}^q \setminus C_{\tilde{T}_k-}^q)$ and $\tilde{j}(k)$ uniformly in $C_{\tilde{T}_k-}^1 \cup \dots \cup C_{\tilde{T}_k-}^q$.

- If $\tilde{T}_k = T_l'$ for some l , we take $\tilde{r}(k) = r(l)$, $\tilde{j}(k) = j(l)$ as in 2.3.1, (2.7). We perform the jumps: $\tilde{C}_{\tilde{T}_k}^{\tilde{r}(k)} = \tilde{C}_{\tilde{T}_k-}^{\tilde{r}(k)} \cup \{\tilde{j}(k)\}$.
- If $\tilde{T}_k \in \{T_l'', l \geq 1\}$, we take $\tilde{r}(k) = r(k)$, $\tilde{j}(k) = j(k)$ as in 2.3.1, (2.8).

We define

$$\tilde{L}_t = \# \left\{ k : \tilde{T}_k \in \left\{ T_l'', \tilde{T}_l'', l \geq 1 \right\}, \tilde{T}_k \leq T \right\}.$$

We set $\forall s, t \leq T, i \in [q]$,

$$\begin{aligned} \tilde{K}_s &= \#(\tilde{C}_s^1 + \dots + \tilde{C}_s^q), \\ \tilde{\mathcal{K}}_t &= (\tilde{K}_s^j)_{1 \leq j \leq q, 0 \leq s \leq t}. \end{aligned}$$

Notation 2.10. We write $\mathbb{E}_{\mathcal{K}_t}(\dots) = \mathbb{E}(\dots | \mathcal{K}_t)$, $\mathbb{P}_{\mathcal{K}_t}(\dots) = \mathbb{P}(\dots | \mathcal{K}_t)$, $\mathbb{E}_{\tilde{\mathcal{K}}_t}(\dots) = \mathbb{E}(\dots | \tilde{\mathcal{K}}_t)$, $\mathbb{P}_{\tilde{\mathcal{K}}_t}(\dots) = \mathbb{P}(\dots | \tilde{\mathcal{K}}_t)$, $\mathbb{E}_{\mathcal{K}_t, \tilde{\mathcal{K}}_t}(\dots) = \mathbb{E}(\dots | \mathcal{K}_t, \tilde{\mathcal{K}}_t)$.

We define $\forall 1 \leq i < j \leq q$

$$\begin{aligned} \mathcal{T}^{i,j} &= \\ &\left\{ \tilde{T}_k \leq T, k \geq 0, \tilde{T}_k \in \left\{ T_l', \tilde{T}_l', l \geq 1 \right\}, \tilde{r}(k) \text{ or } \tilde{j}(k) \in \left\{ \tilde{C}_{\tilde{T}_k}^l, i \leq l \leq j \right\} \right\} \\ &\cap \left\{ \tilde{T}_k \leq T, k \geq 0, \tilde{T}_k \in \left\{ T_l'', \tilde{T}_l'', l \geq 1 \right\}, \tilde{r}(k) \text{ and } \tilde{j}(k) \in \left\{ \tilde{C}_{\tilde{T}_k}^l, i \leq l \leq j \right\} \right\}. \end{aligned} \quad (2.13)$$

We define $\forall t, j$,

$$\tilde{L}_t^{j-1,j} = \# \left\{ \tilde{T}_k^{j-1,j} \leq t, \tilde{T}_k^{j-1,j} \in \left\{ T_l'', \tilde{T}_l'', l \geq 1 \right\} \right\}. \quad (2.14)$$

The following lemma is a consequence of Lemma 2.8.

Lemma 2.11. 1. The process $(\tilde{K}_s)_{s \geq 0}$ is piecewise constant, has jumps of size 1 and satisfies $\forall 0 \leq s \leq t$

$$\mathbb{P}(\tilde{K}_t = \tilde{K}_s | \tilde{K}_s) = \exp(-\Lambda(t-s)\tilde{K}_s).$$

The process $(\tilde{L}_s)_{s \geq 0}$ is piecewise constant, has jumps of size 1 and satisfies $\forall 0 \leq s \leq t$

$$\mathbb{P}(\tilde{L}_t = \tilde{L}_s | \tilde{L}_s, (\tilde{K}_u)_{0 \leq u \leq t}) = \exp \left(- \int_s^t \frac{\Lambda \tilde{K}_u (\tilde{K}_u - 1)}{N - 1} du \right).$$

It means that, knowing $(\tilde{K}_u)_{0 \leq u \leq T}$, $(\tilde{L}_t)_{t \geq 0}$ is an inhomogeneous Poisson process of rate $(\Lambda \tilde{K}_u (\tilde{K}_u - 1) / (N - 1))_{0 \leq u \leq T}$.

2. For all t , $\tilde{K}_t \geq K_t$ and $\tilde{L}_t \geq L_t$ a.s. .

3. If $T_1 = \tilde{T}_1, \dots, T_k = \tilde{T}_k$ then $\tilde{K}_{T_k} = K_{T_k}, \tilde{L}_{T_k} = L_{T_k}$.

4. The processes $(\tilde{K}_t^i)_{t \geq 0}$ are independent. They are piecewise constant, have jumps of size 1 and satisfy $\forall 0 \leq s \leq t, \forall 1 \leq i \leq q$,

$$\mathbb{P}(\tilde{K}_t^i = \tilde{K}_s^i | \tilde{K}_s^i) = \exp(-\Lambda(t-s)\tilde{K}_s^i).$$

These processes are thus q independent Yule processes (see [AN72], p. 102-109, p. 109 for the law of the Yule process).

5. Conditionnaly to \tilde{K}_T , for $i \in \llbracket q/2 \rrbracket$, the processes $(\tilde{L}_t^{2i-1, 2i})_{0 \leq t \leq T}$ are independent non homogeneous Poisson processes of rates, respectively, $\left(\frac{\tilde{K}_t^{2i-1} \tilde{K}_t^{2i}}{N-1}\right)_{0 \leq t \leq T}$.

Proof. The process $(\tilde{K}_t)_{0 \leq t \leq T}$ is piecewise constant and has jumps of size 1. The jump times of \tilde{K}_t belong to $\{T'_k, k \geq 1\}$ or to $\{\tilde{T}'_k, k \geq 1\}$. The jump times of K_t belong to $\{T'_k, k \geq 1\}$. Suppose we are at time s , and we know K_s . We define times \hat{T}_k by recurrence :

$$\hat{T}_0 = 0, \hat{T}_k = \inf \left\{ \left\{ T''_i, \hat{T}_{k-1} \leq T''_i \leq T \right\} \cup \left\{ \tilde{T}''_i, \hat{T}_{k-1} \leq \tilde{T}''_i \leq T \right\} \right\}.$$

We set $j = \#\{\hat{T}_k, s \leq \hat{T}_k \leq t\}$, $\{\hat{T}_{k_1} \leq \dots \leq \hat{T}_{k_j}\} = \{\hat{T}_k, s \leq \hat{T}_k \leq t\}$. We compute (with the convention $\hat{T}_{k_0} = s, \hat{T}_{k_{j+1}} = t$)

$$\begin{aligned} \mathbb{E}(\mathbf{1}_{\tilde{K}_t = \tilde{K}_s} | \tilde{K}_s) &= \\ &= \mathbb{E}(\mathbb{E}(\mathbf{1}_{\tilde{K}_t = \tilde{K}_s} | \tilde{K}_s, K_s, \hat{T}_{k_1}, \dots, \hat{T}_{k_j}) | \tilde{K}_s) = \\ &= \mathbb{E}(\mathbb{E}(\prod_{i=1}^j \mathbf{1}_{U_i \geq (\hat{T}_i - \hat{T}_{i-1}) \frac{\Lambda K_{\hat{T}_{i-1}} (N - K_{\hat{T}_{i-1}})_+}{N-1}} du \mathbf{1}_{\tilde{U}_i \geq (\hat{T}_i - \hat{T}_{i-1}) \left(\Lambda \tilde{K}_{\hat{T}_{i-1}} - \frac{\Lambda K_{\hat{T}_{i-1}} (N - K_{\hat{T}_{i-1}})_+}{N-1} \right)} | \tilde{K}_s, K_s, \hat{T}_{k_1}, \dots, \hat{T}_{k_j}) | \tilde{K}_s) = \\ &= \exp(-(t-s)\Lambda \tilde{K}_s). \end{aligned}$$

The process $(\tilde{L}_t)_{0 \leq t \leq T}$ is piecewise constant and has jumps of size 1. The jump times of this process belong to $\{T''_k, k \geq 1\}$ or to $\{\tilde{T}''_k, k \geq 1\}$. Suppose we are at time s and we know $\tilde{L}_s, (\tilde{K}_u)_{0 \leq u \leq T}$. Let $t \geq s$. We denote by $\tilde{T}_{k_1} \leq \dots \leq \tilde{T}_{k_j}$ the jumps of \tilde{K}_u in $[s; t]$. Due to the properties of the exponential law, the probability $\mathbb{P}(\tilde{L}_t = \tilde{L}_s | \tilde{L}_s, (\tilde{K}_u)_{0 \leq u \leq T})$ is equal to

$$\begin{aligned} \mathbb{E}(\mathbb{P}((\tilde{T}_{k_1} - s) \frac{\Lambda K_s (K_s - 1)}{N-1} \geq V'_1, \dots, (t - \tilde{T}_{k_j}) \frac{\Lambda K_{\tilde{T}_{k_j}} (K_{\tilde{T}_{k_j}} - 1)}{N-1} \geq V'_j, \\ (\tilde{T}_{k_1} - s) \left(\frac{\Lambda \tilde{K}_s (\tilde{K}_s - 1)}{N-1} - \frac{\Lambda K_s (K_s - 1)}{N-1} \right) \geq V''_1, \dots \\ \dots, (t - \tilde{T}_{k_j}) \left(\frac{\Lambda \tilde{K}_{\tilde{T}_{k_j}} (\tilde{K}_{\tilde{T}_{k_j}} - 1)}{N-1} - \frac{\Lambda K_{\tilde{T}_{k_j}} (K_{\tilde{T}_{k_j}} - 1)}{N-1} \right) \geq V''_j \\ | (K_u)_{0 \leq u \leq T}, (\tilde{K}_u)_{0 \leq u \leq T}, \tilde{L}_u) | (\tilde{K}_u)_{0 \leq u \leq T}, \tilde{L}_u), \end{aligned}$$

for some independent V'_i, V''_i of law $\mathcal{E}(1)$. This last probability is equal to $\mathbb{P}\left(\int_s^t \frac{\Lambda \tilde{K}_u (\tilde{K}_u - 1)}{N-1} du \geq V'_1 | (\tilde{K}_u)_{0 \leq u \leq T}\right)$. This proves the point 1 of the Lemma.

We have $\forall \omega, t, \Delta K_t(\omega) = 1 \Rightarrow \Delta \tilde{K}_t(\omega) = 1$ and $\Delta L_t(\omega) = 1 \Rightarrow \Delta \tilde{L}_t(\omega) = 1$, so we have the point 2 of the Lemma. The point 3 of the Lemma is immediate.

Let $k \geq 1$. Suppose we are at time \tilde{T}_{k-1} , with $\tilde{T}_{k-1} < T$. Using the properties of the exponential law we can say that

$$\inf\{T'_l, T'_l \geq \tilde{T}_{k-1}\} - \tilde{T}_{k-1} \sim \inf\left(T, \mathcal{E}\left(\frac{\Lambda K_{\tilde{T}_{k-1}}(N - K_{\tilde{T}_{k-1}})_+}{N-1}\right)\right),$$

$$\inf\{T''_l, T''_l \geq \tilde{T}_{k-1}\} - \tilde{T}_{k-1} \sim \inf\left(T, \mathcal{E}\left(\frac{\Lambda K_{\tilde{T}_{k-1}}(K_{\tilde{T}_{k-1}} - 1)}{2(N-1)}\right)\right).$$

By direct computation, one can show that the infimum of four independent exponential variables E_1, \dots, E_4 of parameters, respectively, $\lambda_1, \dots, \lambda_4$ satisfies $\mathbb{P}(E_1 = \inf(E_1, \dots, E_4) | \inf(E_1, \dots, E_4) < t) = \frac{\lambda_1}{\lambda_1 + \dots + \lambda_4} (\forall t > 0)$. So

$$\mathbb{P}(\tilde{T}_k \in \{\tilde{T}'_l, l \geq 1\} | \mathcal{K}_{T_{k-1}}, \tilde{\mathcal{K}}_{T_{k-1}}, \tilde{T}_k < T) = \frac{\Lambda \tilde{K}_{\tilde{T}_{k-1}} - \frac{\Lambda K_{\tilde{T}_{k-1}}(N - K_{\tilde{T}_{k-1}})_+}{N-1}}{\Lambda \tilde{K}_{\tilde{T}_{k-1}} + \frac{\Lambda \tilde{K}_{\tilde{T}_{k-1}}(K_{\tilde{T}_{k-1}} - 1)}{2(N-1)}},$$

$$\mathbb{P}(\tilde{T}_k \in \{T'_l, l \geq 1\} | \mathcal{K}_{T_{k-1}}, \tilde{\mathcal{K}}_{T_{k-1}}, \tilde{T}_k < T) = \frac{\frac{\Lambda K_{\tilde{T}_{k-1}}(N - K_{\tilde{T}_{k-1}})_+}{N-1}}{\Lambda \tilde{K}_{\tilde{T}_{k-1}} + \frac{\Lambda \tilde{K}_{\tilde{T}_{k-1}}(K_{\tilde{T}_{k-1}} - 1)}{2(N-1)}},$$

so, recalling (2.7), (2.9), (2.10),

$$\begin{aligned} \mathbb{P}(\Delta \tilde{K}_{\tilde{T}_k}^i = 1 | \tilde{T}_k \in \{T'_l, \tilde{T}'_l, l \geq 1\}, \mathcal{K}_{T_{k-1}}, \tilde{\mathcal{K}}_{T_{k-1}}, \tilde{T}_k < T) = \\ \left(\frac{\tilde{K}_{\tilde{T}_{k-1}} - \frac{\Lambda K_{\tilde{T}_{k-1}}(N - K_{\tilde{T}_{k-1}})_+}{N-1}}{\tilde{K}_{\tilde{T}_{k-1}}} \right) \\ \times \left[\frac{K_{\tilde{T}_{k-1}} - \frac{K_{\tilde{T}_{k-1}}(N - K_{\tilde{T}_{k-1}})_+}{N-1}}{\tilde{K}_{\tilde{T}_{k-1}} - \frac{K_{\tilde{T}_{k-1}}(N - K_{\tilde{T}_{k-1}})_+}{N-1}} \times \frac{K_{\tilde{T}_{k-1}}^i}{K_{\tilde{T}_{k-1}}} \right. \\ \left. + \frac{\tilde{K}_{\tilde{T}_{k-1}} - K_{\tilde{T}_{k-1}}}{\tilde{K}_{\tilde{T}_{k-1}} - \frac{K_{\tilde{T}_{k-1}}(N - K_{\tilde{T}_{k-1}})_+}{N-1}} \times \frac{\tilde{K}_{\tilde{T}_{k-1}}^i - K_{\tilde{T}_{k-1}}^i}{\tilde{K}_{\tilde{T}_{k-1}} - K_{\tilde{T}_{k-1}}} \right] \\ + \frac{\left(\frac{K_{\tilde{T}_{k-1}}(N - K_{\tilde{T}_{k-1}})_+}{N-1} \right)}{\tilde{K}_{\tilde{T}_{k-1}}} \times \frac{K_{\tilde{T}_{k-1}}^i}{K_{\tilde{T}_{k-1}}} = \frac{\tilde{K}_{\tilde{T}_{k-1}}^i}{\tilde{K}_{\tilde{T}_{k-1}}}, \end{aligned}$$

and this late expression does not depend on \tilde{T}_k or \mathcal{K}_T . By point 1, the process $(\tilde{K}_s)_{s \geq 0}$ is equal in law to the sum of q in dependant Yule processes $Y_s^{(1)}, \dots, Y_s^{(q)}$ and its law is thus independent of N (see [AN72], p. 102-109, p. 109 for the law of the Yule process). We have

$$\mathbb{P}(Y_s^{(1)} = k) = e^{-s\Lambda}(1 - e^{-s\Lambda})^{k-1} \quad (2.15)$$

and so:

$$\begin{aligned}
\mathbb{P}(\tilde{K}_t = k) &= \mathbb{P}(Y_t^{(1)} + \dots + Y_t^{(q)} = k) \\
&\leq \sum_{i=1}^q \mathbb{P}(Y_t^{(i)} \geq \lceil \frac{k}{q} \rceil) \\
&\leq q(1 - e^{-t\Lambda})^{k/q-1}. \tag{2.16}
\end{aligned}$$

We have $\mathbb{P}(\tilde{K}_{t+\varepsilon} \geq \tilde{K}_t + 2 | \tilde{\mathcal{K}}_t) \leq (1 - e^{-\Lambda \tilde{K}_t \varepsilon})(1 - e^{-\Lambda(\tilde{K}_t + 1)\varepsilon}) \leq \Lambda^2(\tilde{K}_t + 1)^2 \varepsilon^2 = o(\varepsilon)$. This $o(\varepsilon)$ is a quantity bounded by $C(\tilde{K}_t + 1)^2 \varepsilon^2$, it will be the same in the rest of the proof for the other $o(\varepsilon)$. We also have

$$\begin{aligned}
\mathbb{P}(\tilde{K}_{t+\varepsilon} \geq \tilde{K}_t + 1 | \tilde{\mathcal{K}}_t) &= 1 - e^{-\Lambda \tilde{K}_t \varepsilon} \\
&= \Lambda \tilde{K}_t \varepsilon + o(\varepsilon).
\end{aligned}$$

The process $((\tilde{K}_t^1, \dots, \tilde{K}_t^q))_{0 \leq t \leq T}$ is Markov. For any bounded F , $0 \leq t < T$,

$$\begin{aligned}
&\mathbb{E}(F(\tilde{K}_{t+\varepsilon}^1, \dots, \tilde{K}_{t+\varepsilon}^q) - F(\tilde{K}_t^1, \dots, \tilde{K}_t^q) | \tilde{\mathcal{K}}_t) = \\
&\mathbb{E}(F(\tilde{K}_{t+\varepsilon}^1, \dots, \tilde{K}_{t+\varepsilon}^q) - F(\tilde{K}_t^1, \dots, \tilde{K}_t^q) | \tilde{\mathcal{K}}_t, \tilde{K}_{t+\varepsilon} = \tilde{K}_t + 1) \mathbb{P}(\tilde{K}_{t+\varepsilon} = \tilde{K}_t + 1 | \tilde{\mathcal{K}}_t) \\
&+ \mathbb{E}(F(\tilde{K}_{t+\varepsilon}^1, \dots, \tilde{K}_{t+\varepsilon}^q) - F(\tilde{K}_t^1, \dots, \tilde{K}_t^q) | \tilde{\mathcal{K}}_t, \tilde{K}_{t+\varepsilon} \geq \tilde{K}_t + 2) \mathbb{P}(\tilde{K}_{t+\varepsilon} \geq \tilde{K}_t + 2 | \tilde{\mathcal{K}}_t) = \\
&\sum_{i=1}^q [(F(\tilde{K}_t^1, \dots, \tilde{K}_t^i + 1, \dots, \tilde{K}_t^q) - F(\tilde{K}_t^1, \dots, \tilde{K}_t^q)) \frac{\tilde{K}_t^i}{\tilde{K}_t}] (\mathbb{P}(\tilde{K}_{t+\varepsilon} \geq \tilde{K}_t + 1 | \tilde{\mathcal{K}}_t) + o(\varepsilon^2)) + o(\varepsilon^2) = \\
&\sum_{i=1}^q (F(\tilde{K}_t^1, \dots, \tilde{K}_t^i + 1, \dots, \tilde{K}_t^q) - F(\tilde{K}_t^1, \dots, \tilde{K}_t^q)) \Lambda \tilde{K}_t^i + o(\varepsilon^2).
\end{aligned}$$

And this proves the point 4 of the Lemma.

Reasoning as above, we can show that, conditionnaly to $\mathcal{K}_T, \tilde{\mathcal{K}}_T$, the process $(\#\{T_k, T_k = T_k'', T_k \leq t\})_{0 \leq t \leq T}$ is an homogeneous Poisson process of rate $(\frac{\Lambda K_t(K_t - 1)}{2(N-1)})_{0 \leq t \leq T}$ and the process $(\#\{\tilde{T}_k, \tilde{T}_k = \tilde{T}_k'', \tilde{T}_k \leq t\})_{0 \leq t \leq T}$ is an homogeneous Poisson process of rate $(\frac{\Lambda \tilde{K}_t(\tilde{K}_t - 1) - \Lambda K_t(K_t - 1)}{2(N-1)})_{0 \leq t \leq T}$. So, by Lemma 2.8, 3,

$$\mathbb{P}(t \in \{T_t'', l \geq 1\} | \mathcal{K}_T, \tilde{\mathcal{K}}_T, \Delta \tilde{L}_t = 1) = \frac{\Lambda K_t(K_t - 1)}{\Lambda \tilde{K}_t(\tilde{K}_t - 1)}.$$

We have for all i (recalling (2.8))

$$\mathbb{P}(\Delta \tilde{L}_t^{i, i+1} = 1 | \mathcal{K}_T, \tilde{\mathcal{K}}_T, t \in \{T_t'', l \geq 1\}) = \frac{2K_t^i K_t^{i+1}}{K_t(K_t - 1)},$$

and (recalling (2.11), (2.12))

$$\begin{aligned}
\mathbb{P}(\Delta\tilde{L}_t^{i,i+1} = 1 | \mathcal{K}_T, \tilde{\mathcal{K}}_T, t \in \{\tilde{T}_l'', l \geq 1\}) &= \\
&\frac{(\tilde{K}_t - K_t)K_t + (\tilde{K}_t - K_t)(\tilde{K}_t - K_t - 1)}{\tilde{K}_t(\tilde{K}_t - 1) - K_t(K_t - 1)} \\
&\times \left(\frac{(\tilde{K}_t^i - K_t^i)\tilde{K}_t^{i+1}}{(\tilde{K}_t - K_t)(\tilde{K}_t - 1)} + \frac{(\tilde{K}_t^{i+1} - K_t^i)\tilde{K}_t^i}{(\tilde{K}_t - K_t)(\tilde{K}_t - 1)} \right) \\
&+ \frac{(\tilde{K}_t - K_t)}{\tilde{K}_t(\tilde{K}_t - 1) - K_t(K_t - 1)} \times \left(\frac{(\tilde{K}_t^i - K_t^i)\tilde{K}_t^{i+1}}{(\tilde{K}_t - K_t)K_t} + \frac{(\tilde{K}_t^{i+1} - K_t^{i+1})K_t^i}{(\tilde{K}_t - K_t)K_t} \right) = \\
&\frac{K_t - 1}{\tilde{K}_t(\tilde{K}_t - 1) - K_t(K_t - 1)} \times \left(\frac{(\tilde{K}_t^i - K_t^i)\tilde{K}_t^i + (\tilde{K}_t^{i+1} - K_t^{i+1})\tilde{K}_t^{i+1}}{K_t - 1} \right) \\
&+ \frac{1}{\tilde{K}_t(\tilde{K}_t - 1) - K_t(K_t - 1)} \times \left((\tilde{K}_t^i - K_t^i)K_t^{i+1} + (\tilde{K}_t^i - K_t^{i+1})K_t^i \right) = \\
&\frac{(\tilde{K}_t^i - K_t^i)(\tilde{K}_t^{i+1} + K_t^{i+1}) + (\tilde{K}_t^{i+1} - K_t^{i+1})(\tilde{K}_t^i + K_t^i)}{\tilde{K}_t(\tilde{K}_t - 1) - K_t(K_t - 1)}.
\end{aligned}$$

So

$$\begin{aligned}
\mathbb{P}(\Delta\tilde{L}_t^{i,i+1} = 1 | \mathcal{K}_T, \tilde{\mathcal{K}}_t, \Delta\tilde{L}_t = 1) &= \\
&\left(\frac{\tilde{K}_t(\tilde{K}_t - 1) - K_t(K_t - 1)}{\tilde{K}_t(\tilde{K}_t - 1)} \right) \\
&\times \left(\frac{(\tilde{K}_t^i - K_t^i)(\tilde{K}_t^{i+1} + K_t^{i+1}) + (\tilde{K}_t^{i+1} - K_t^{i+1})(\tilde{K}_t^i + K_t^i)}{\tilde{K}_t(\tilde{K}_t - 1) - K_t(K_t - 1)} \right) \\
&+ \left(\frac{K_t(K_t - 1)}{\tilde{K}_t(\tilde{K}_t - 1)} \right) \left(\frac{2K_t^i K_t^{i+1}}{K_t(K_t - 1)} \right) = \\
&\frac{2\tilde{K}_t^i \tilde{K}_t^{i+1}}{\tilde{K}_t(\tilde{K}_t - 1)},
\end{aligned}$$

so

$$\mathbb{P}(\Delta\tilde{L}_t^{i,i+1} = 1 | \tilde{\mathcal{K}}_t, \Delta\tilde{L}_t = 1) = \frac{2\tilde{K}_t^i \tilde{K}_t^{i+1}}{\tilde{K}_t(\tilde{K}_t - 1)}.$$

So, using Lemma 2.8, 3 we have the point 5 of the Lemma. \square

We carry on with the definition of (\tilde{Z}) .

Definition 2.12. Let $\tilde{k}' = \sup \{k, \tilde{T}_k < \infty\}$. The interaction times of the \tilde{Z}^i are $T - \tilde{T}_{\tilde{k}'} \leq T - \tilde{T}_{\tilde{k}'-1} \leq \dots \leq T - \tilde{T}_1$.

- The (\tilde{Z}_0^i) are i.i.d. $\sim \tilde{P}_0$.
- Between the jump times, the \tilde{Z}^i evolve independently from each other according to the Markov generator L .
- At a jump time $T - \tilde{T}_k$, (\tilde{Z}) undergo a jump like in Definition 2.1, (3), with i, j replaced by $\tilde{r}(k), \tilde{j}(k)$.

By doing this, we have coupled the interaction times of the systems $(Z_{0:T}^i)_{i \geq 0}$, $(\tilde{Z}_{0:T}^i)_{i \geq 0}$. We can couple further and assume that $\forall i$, $\tilde{Z}_{0:T}^i$ and $Z_{0:T}^i$ coincide on the event $\{\tilde{T}_k, k \geq 1\} \cap \{\tilde{T}'_k, k \geq 1\} = \emptyset$.

Definition 2.13. We define the auxiliary system $(\tilde{Z}_{0:T}^i)_{i \geq 0}$ such that

- it has interactions at times $\{T - \tilde{T}_k, k \geq 1\} \setminus \{T - T''_k, T - \tilde{T}''_k, k \geq 1\}$
- the rest of the definition is the same as for $(\tilde{Z}_{0:T}^i)_{i \geq 0}$.

By doing this, we have coupled the interaction times of the systems $(\tilde{Z}_{0:T}^i)_{i \geq 0}$, $(\tilde{Z}_{0:T}^i)_{i \geq 0}$. We can couple further and assume that $\forall i$, $\tilde{Z}_{0:T}^i$ and $\tilde{Z}_{0:T}^i$ coincide on the event $\{\tilde{T}_k, k \geq 1\} \cap \{T''_k, \tilde{T}''_k, k \geq 1\} = \emptyset$.

Theorem 2.14 (Graham & Méléard). According to [GM97] (Section 3.4, p. 124) or, equivalently [GM94] (Section 5), $(\tilde{Z}_{0:T}^1, \dots, \tilde{Z}_{0:T}^q)$ has the law $\tilde{P}_{0:T}^{\otimes q}$.

Suppose $q = 2$. The figures 3, 4, 5 are realizations of the interaction graphs for $(Z_{0:T}^1, Z_{0:T}^2)$, $(\tilde{Z}_{0:T}^1, \tilde{Z}_{0:T}^2)$, $(\tilde{Z}_{0:T}^1, \tilde{Z}_{0:T}^2)$, $N = 10$.

3 Expansion of the propagation of chaos

We define for any $n, j \in \mathbb{N}^*$, $j \leq n$:

$$\langle j, n \rangle = \{a : [j] \rightarrow [n], a \text{ injective}\}, (n)_j = \#\langle j, n \rangle = \frac{n!}{(n-j)!}.$$

Let us set

$$\eta_{0:t}^N = \frac{1}{N} \sum_{1 \leq i \leq N} \delta_{\tilde{Z}_{0:t}^i},$$

Figure 3: Interaction graph for $(Z_{0:T}^{N,1}, Z_{0:T}^{N,2})$

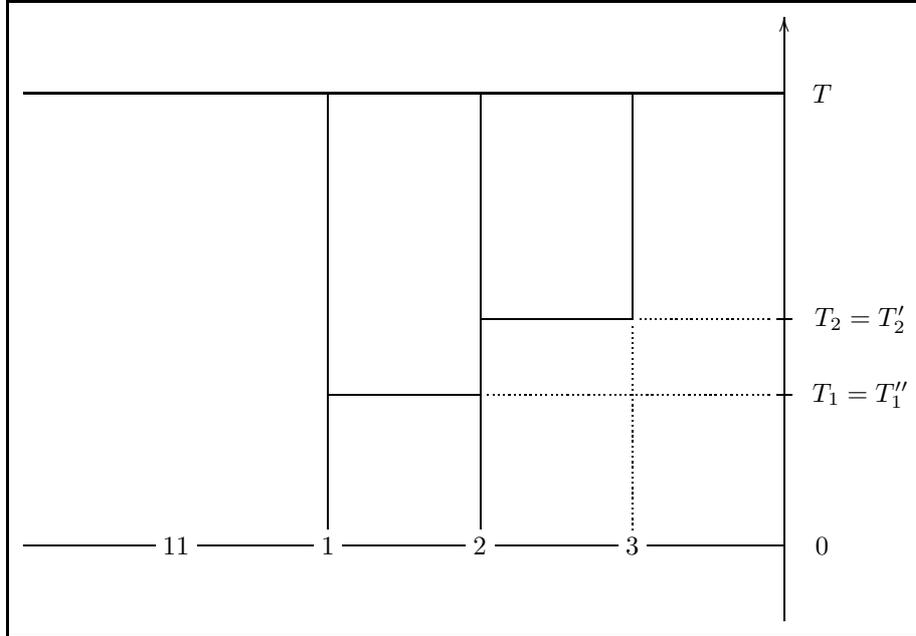


Figure 4: Interaction graph for $(\tilde{Z}_{0:T}^{N,1}, \tilde{Z}_{0:T}^{N,2})$

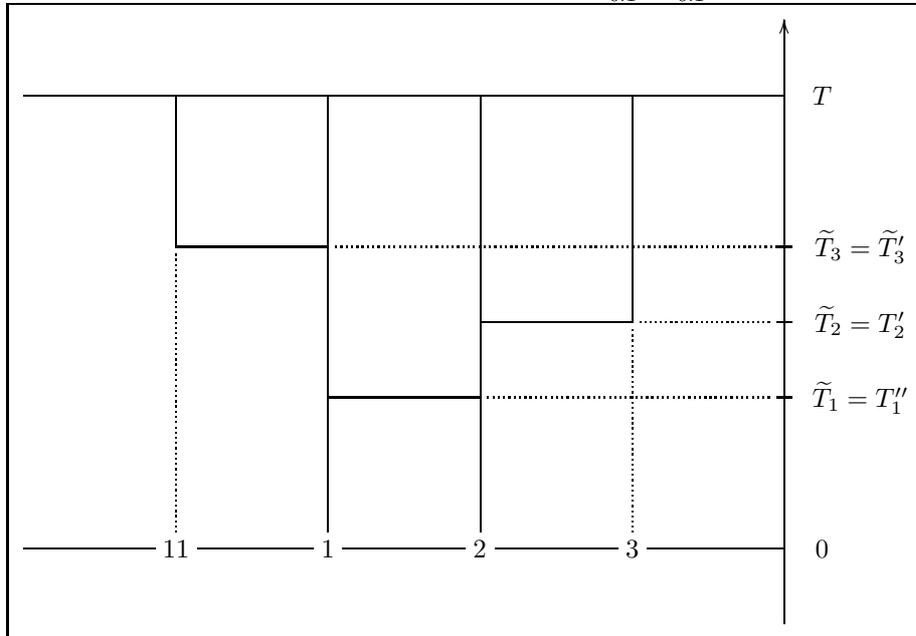
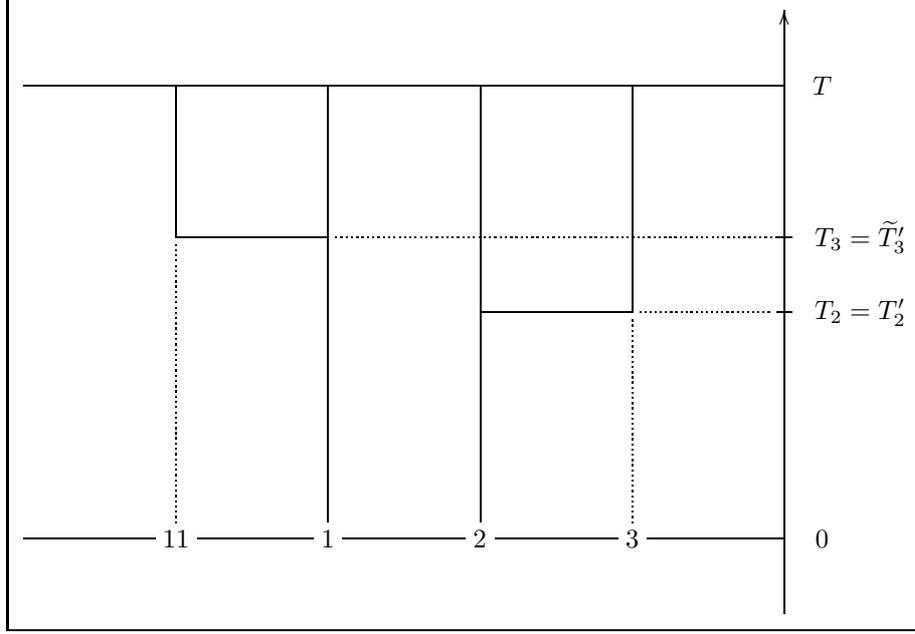


Figure 5: Interaction graph for $(\tilde{Z}_{0:T}^{N,1}, \tilde{Z}_{0:T}^{N,2})$



$$(\eta_{0:t}^N)^{\odot q} = \frac{1}{(N)_q} \sum_{a \in \langle q, N \rangle} \delta_{(\bar{Z}_{0:t}^{a(1)}, \dots, \bar{Z}_{0:t}^{a(q)})}.$$

For any function $F : \mathbb{D}([0; t], \mathbb{R}^d)^q \rightarrow \mathbb{R}$, we call $(\eta_{0:t}^N)^{\odot q}(F)$ a U -statistics. Note that for all functions F ,

$$\mathbb{E}(F(\bar{Z}_{0:t}^1, \dots, \bar{Z}_{0:t}^q)) = \mathbb{E}((\eta_{0:t}^N)^{\odot q}(F)). \quad (3.1)$$

We define

$$F_{\text{sym}}(x^1, \dots, x^q) = \frac{1}{q!} \sum_{\sigma \in \mathcal{S}_q} F(x^{\sigma(1)}, \dots, x^{\sigma(q)}),$$

where the sum is taken over the set \mathcal{S}_q of the permutations of $[q]$. We say that $F : \mathbb{D}([0; t], \mathbb{R}^{qd}) \rightarrow \mathbb{R}$ is symmetric if for all σ in the set of permutation of $[q]$ (denoted by \mathcal{S}_q), $\forall x_1, \dots, x_q \in \mathbb{D}([0; t], \mathbb{R}^{qd})$, $F(x_{\sigma(1)}, \dots, x_{\sigma(q)}) = F(x_1, \dots, x_q)$. If F is symmetric then $F_{\text{sym}} = F$. Note that $\forall F$

$$(\eta_{0:t}^N)^{\odot q}(F) = (\eta_{0:t}^N)^{\odot q}(F_{\text{sym}}).$$

Theorem 3.1. Set $\alpha = e^{-\Lambda T}$. For all $q \geq 1$, for any bounded measurable symmetric F , $\forall T \geq 0$, $\forall l_0 \geq 1$,

$$\mathbb{E}((\eta_{0:T}^N)^{\odot q}(F)) = \sum_{0 \leq l \leq l_0} \left[\frac{1}{(N-1)^l} \Delta_{q,T}^{N,l}(F) \right] + \frac{1}{(N-1)^{l_0+1}} \bar{\Delta}_{q,T}^{N,l_0+1}(F) \quad (3.2)$$

where the $\Delta_{q,T}^{N,l}$, $\overline{\Delta}_{q,T}^{N,l_0+1}$ are nonnegative measures uniformly bounded in N defined by, for any bounded measurable F ,

$$\Delta_{q,T}^{N,l}(F) = \mathbb{E}(F(Z_{0:T}^1, \dots, Z_{0:T}^q) | L_T = l) \mathbb{P}(L_T = l) N^l$$

$$\overline{\Delta}_{q,T}^{N,l}(F) = \mathbb{E}(F(Z_{0:T}^1, \dots, Z_{0:T}^q) | L_T \geq l) \mathbb{P}(L_T \geq l) N^l.$$

We further have the following bounds ($\forall F \in C_b^+(\mathbb{R}^{qd})$)

$$\sup(\Delta_{q,T}^{N,l}(F), \overline{\Delta}_{q,T}^{N,l}(F)) \leq \frac{\Lambda^l}{l!} \sup_{N \geq 1} \left(\left(\frac{N-1}{N} \right)^l \right) \mathbb{E}(\tilde{K}_T^{2l}) \|F\|_\infty < \infty.$$

Let us define $\mathbb{P}_{T,q}^N(F) = \mathbb{E}((\eta_T^N)^{\odot q}(F))$. According to the terminology of [DPR09b], p. 782, we cannot say that the sequence of measure $(\mathbb{P}_{T,q}^N)_{N \geq 1}$ is differentiable up to any order because the $\Delta_{q,T}^{N,l}$ appearing in the development depend on N . We will get enough results though in the following of this Section to prove the convergence results of Section 5.

Proof. We have by Lemma 2.9, $\forall l_0$:

$$\begin{aligned} \mathbb{E}(F(\overline{Z}_T^1, \dots, \overline{Z}_T^q)) &= \sum [\mathbb{E}(F(Z_T^1, \dots, Z_T^q) | L_T = l) \mathbb{P}(L_T = l)] \\ &\quad + \mathbb{E}(F(Z_T^1, \dots, Z_T^q) | L_T \geq l_0 + 1) \\ &\quad \times \mathbb{P}(L_T \geq l_0 + 1). \end{aligned}$$

It is sufficient for the proof to show that $\mathbb{P}(L_T \geq l)$ is of order $\leq 1/N^l$, $\forall l \in \mathbb{N}^*$. We have (using the notation N^{\cdots} of Lemma 2.8)

$$\begin{aligned} \mathbb{P}(\tilde{L}_T \geq l) &= \mathbb{E}(\mathbb{E}_{\tilde{K}_T}(\mathbf{1}_{\tilde{L}_T \geq l})) \\ \text{(Lemma 2.11)} &= \mathbb{E}(\mathbb{P}_{\tilde{K}_t}(N^{\Lambda \tilde{K}_t (\tilde{K}_t - 1) / (N-1)} \geq l)) \\ &\leq \mathbb{E}(\mathbb{P}_{\tilde{K}_t}(N^{\Lambda \tilde{K}_T (\tilde{K}_T - 1) / (N-1)} \geq l)) \\ &\leq \frac{1}{l!} \left(\frac{\Lambda \tilde{K}_T (\tilde{K}_T - 1)}{N-1} \right)^l. \end{aligned} \tag{3.3}$$

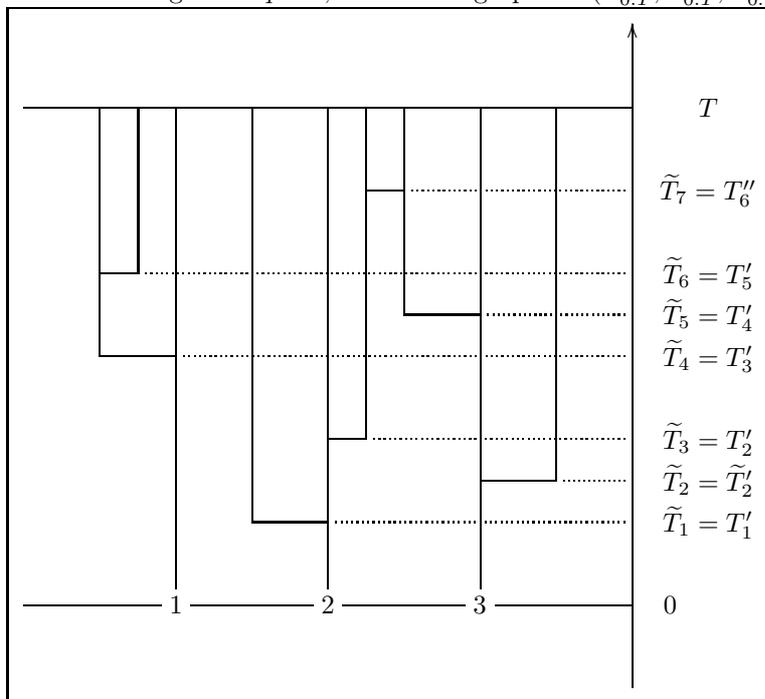
And $\mathbb{E}((\tilde{K}_T)^{2l}) < \infty$ by (2.16). □

A consequence of equations (3.3) is the following.

Lemma 3.2. *For all $l \in \mathbb{N}^*$,*

$$N^l \mathbb{P}(\tilde{L}_T > l) \xrightarrow{N \rightarrow +\infty} 0.$$

Figure 6: $q = 3$, interaction graph for $(\tilde{Z}_{0:T}^1, \tilde{Z}_{0:T}^2, \tilde{Z}_{0:T}^3)$



4 Rate of convergence for centered functions

4.1 Definitions and result

For $i \in [q]$, we define the event

$$A_i = \left\{ \left\{ \tilde{T}_k : k \geq 0, \tilde{T}_k \leq T, \tilde{r}(k) \text{ or } \tilde{j}(k) \in \tilde{C}_{\tilde{T}_k-}^i \right\} \subset \{T'_k, k \geq 1\} \right\}$$

Recall that the definition of the $\mathcal{T}^{i,j}$ is in (2.13). We order $\mathcal{T}^{i,j}$ by writing $\mathcal{T}^{i,j} = \left\{ \tilde{T}_1^{i,j} \leq \tilde{T}_2^{i,j} \leq \dots \right\}$. For any i, j, k , we set $\tilde{T}_k^{i,j} = \tilde{T}_{\sigma_{i,j}(k)}$ ($\sigma_{i,j} : \mathbb{N} \rightarrow \mathbb{N}$). We set $\forall 1 \leq i < j \leq q, 1 \leq l \leq q$,

$$\begin{aligned} A_l^{i,j} &= \left\{ \left\{ \tilde{T}_k^{i,j} : k \geq 0, \tilde{r}(\sigma_{i,j}(k)) \text{ or } \tilde{j}(\sigma_{i,j}(k)) \in \tilde{C}_{\tilde{T}_k^{i,j}-}^l \right\} \subset \{T'_k, k \geq 1\} \right\}, \\ B_{i,j} &= \left(A_i^{i,j} \right)^c \cap \left(A_{i+1}^{i,j} \right)^c \cap \dots \cap \left(A_j^{i,j} \right)^c, \\ B'_{i,j} &= \left\{ \left\{ \tilde{T}_k^{i,j}, k \geq 0 \right\} \cap \left\{ \tilde{T}'_l, l \geq 1 \right\} \neq \emptyset \right\}, \\ \tilde{L}_{i,i+i} &= \left\{ \# \left(\left\{ \tilde{T}_k^{i,i+1}, k \geq 0 \right\} \cap \left\{ T'_l, \tilde{T}'_l, l \geq 1 \right\} \right) = 1 \right\}, \\ \tilde{L}_{1,q} &= \tilde{L}_{1,2} \cap \dots \cap \tilde{L}_{q-1,q} \cap \left\{ \# \left(\left\{ \tilde{T}_k, k \geq 1 \right\} \cap \left\{ T'_l, \tilde{T}'_l, l \geq 1 \right\} \right) = q/2 \right\}, \end{aligned}$$

and

$$\tilde{A}_i = \left\{ \left\{ \tilde{T}_k : k \geq 0, \tilde{T}_k \leq T, \tilde{r}(k) \text{ or } \tilde{j}(k) \in \tilde{C}_{\tilde{T}_k-}^i \right\} \subset \left\{ T'_k, \tilde{T}'_k, k \geq 1 \right\} \right\},$$

$$\begin{aligned} \tilde{A}_l^{i,j} &= \\ \left\{ \left\{ \tilde{T}_k^{i,j} : k \geq 0, \tilde{T}_k^{i,j} \leq T, \tilde{r}(\sigma_{i,j}(k)) \text{ or } \tilde{j}(\sigma_{i,j}(k)) \in \tilde{C}_{\tilde{T}_k^{i,j}-}^l \right\} \subset \left\{ T'_k, \tilde{T}'_k, k \geq 1 \right\} \right\}, \\ \tilde{B}_{i,j} &= \left(\tilde{A}_i^{i,j} \right)^c \cap \left(\tilde{A}_{i+1}^{i,j} \right)^c \cap \dots \cap \left(\tilde{A}_j^{i,j} \right)^c. \end{aligned}$$

Let us have a look at Figure 6 to clarify the notions above. Suppose $\omega \in \Omega$ is such that the graph above occurs. Note that: $\omega \in A_1$, $\omega \in A_2^c$, $\omega \in A_3^c$, $\omega \in A_{1,2}^{1,2}$.

Definition 4.1. We define a set of “centered” functions:

$$\begin{aligned} \mathcal{B}_0^{sym}(q) &= \left\{ F : \mathbb{D}([0; T], \mathbb{R}^{qd}) \rightarrow \mathbb{R}^+, F \text{ measurable, symmetric, bounded,} \right. \\ &\quad \left. \int_{x_1, \dots, x_q \in \mathbb{R}^d} F(x_1, \dots, x_q) \tilde{P}_{0:T}(dx_q) = 0 \right\}. \end{aligned}$$

Proposition 4.2. For $F \in \mathcal{B}_0^{sym}(q)$, we have:

1. for q odd, $N^{q/2} \mathbb{E}((\eta_T^N)^{\odot q}(F)) \xrightarrow{N \rightarrow +\infty} 0$,

2. for q even,

$$\begin{aligned}
& N^{q/2} \mathbb{E}((\eta_T^N)^{\odot q}(F)) \\
& \xrightarrow{N \rightarrow +\infty} \sum_{1 \leq k \leq q/2} I_q C_{q/2}^k (-1)^{(q-k)/2} \\
& \times \mathbb{E}[\mathbb{E}(F(\tilde{Z}_{0:T}^1, \dots, \tilde{Z}_{0:T}^{2k}, \tilde{Z}_{0:T}^{k+1}, \dots, \tilde{Z}_{0:T}^q) - F(\tilde{Z}_{0:T}^1, \dots, \tilde{Z}_{0:T}^q) | \tilde{L}_{1,q}) \\
& \quad \times \prod_{1 \leq i \leq q/2} \int_0^T \Lambda \tilde{K}_s^{2i-1} \tilde{K}_s^{2i} ds] \quad (4.1)
\end{aligned}$$

where the limit indeed does not depend on N .

4.2 Technical lemmas

Before going into the proof of the above proposition, we need some technical lemmas.

Lemma 4.3. We define $\forall j, \forall k_0 = 0 < k_1 < k_2 < \dots < k_{j-1} < k_j = q, \forall 1 \leq r \leq j,$

$$\begin{aligned}
\overline{B}_{k_1, \dots, k_j}^{j,r} &= \cap_{1 \leq i \leq r} (B_{k_{i-1}+1, k_i} \cap B'_{k_{i-1}+1, k_i}) \cap_{r+1 \leq i \leq j} (B_{k_{i-1}+1, k_i} \setminus B'_{k_{i-1}+1, k_i}), \\
\tilde{\overline{B}}_{k_1, \dots, k_j}^{j,r} &= \cap_{1 \leq i \leq r} (\tilde{B}_{k_{i-1}+1, k_i} \cap B'_{k_{i-1}+1, k_i}) \cap_{r+1 \leq i \leq j} (\tilde{B}_{k_{i-1}+1, k_i} \setminus B'_{k_{i-1}+1, k_i}).
\end{aligned}$$

We have

$$N^{q/2} \mathbb{P}(\overline{B}_{k_1, \dots, k_j}^{j,r}) \xrightarrow{N \rightarrow +\infty} 0, \quad (4.2)$$

$$N^{q/2} \mathbb{P}(\tilde{\overline{B}}_{k_1, \dots, k_j}^{j,r}) \xrightarrow{N \rightarrow +\infty} 0. \quad (4.3)$$

Proof. Let us fix $1 \leq s \leq j$. We set $\forall 0 \leq t \leq T,$

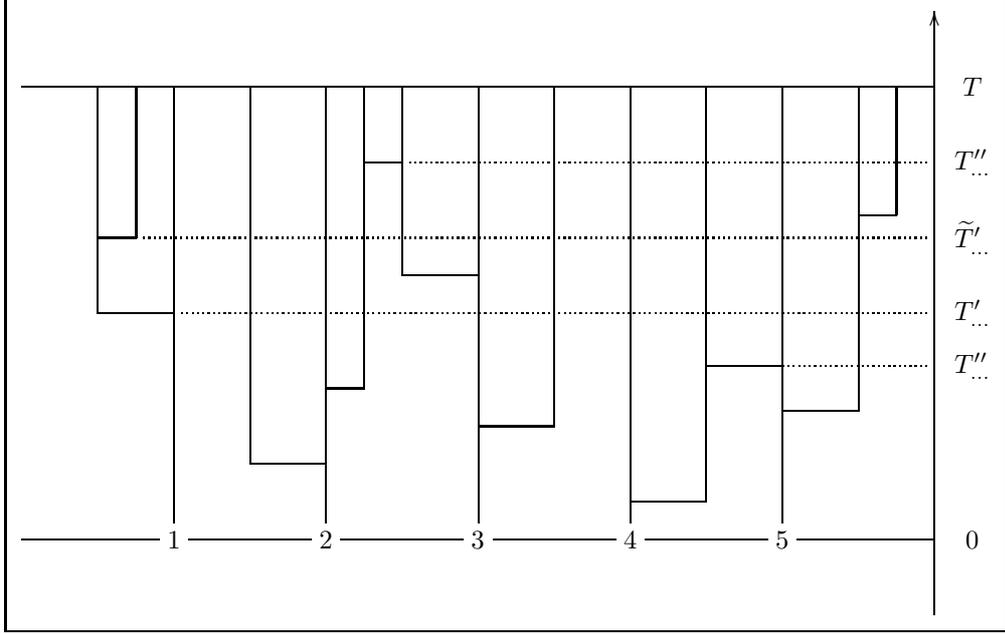
$$L_t^{k_{s-1}+1, k_s} = \#\left\{ \tilde{T}_l^{k_{s-1}+1, k_s} : l \geq 1, \tilde{T}_l^{k_{s-1}+1, k_s} \leq t, \tilde{T}_l^{k_{s-1}+1, k_s} = T_i'', i \geq 1 \right\} \quad (4.4)$$

$$\begin{aligned}
E_t^{k_{s-1}+1, k_s} &= \#\{T_l^{k_{s-1}+1, k_s} : l \geq 1, T_l^{k_{s-1}+1, k_s} \leq t, T_l^{k_{s-1}+1, k_s} = \tilde{T}_i', i \geq 1, \\
& \quad \tilde{r}(i) \in C_{\tilde{T}_i'}^j, k_{s-1} + 1 \leq j \leq k_s\}. \quad (4.5)
\end{aligned}$$

We have

$$\begin{aligned}
& B_{k_{s-1}+1, k_s} \cap B'_{k_{s-1}+1, k_s} \\
& \subset \cup_{1 \leq i \leq k_s - k_{s-1}} \left\{ L_T^{k_{s-1}+1, k_s} \geq \lceil \frac{k_s - k_{s-1} - i}{2} \rceil, E_T^{k_{s-1}+1, k_s} \geq i \right\}, \quad (4.6)
\end{aligned}$$

Figure 7: $q = 5$, interaction graph for $(\tilde{Z}_{0:T}^1, \dots, \tilde{Z}_{0:T}^5)$



$$B_{k_{s-1}+1, k_s} \setminus B'_{k_{s-1}+1, k_s} \subset \left\{ L_T^{k_{s-1}+1, k_s} \geq \lceil \frac{k_s - k_{s-1}}{2} \rceil \right\}.$$

The Figure 7 illustrates (4.6). In this case, we are in the event $\{L_T^{1,5} \geq 2\} \cap \{E_T^{1,5} \geq 1\}$, which contains $B_{1,5} \cap B'_{1,5}$.

We set $E_T = \# \left\{ \tilde{T}_k : \tilde{T}_k = \tilde{T}'_k, \tilde{r}(k) \in C_{\tilde{T}'_k}^1 \cup \dots \cup C_{\tilde{T}'_k}^q \right\}$. We have:

$$\sum_{s=1}^j L_T^{k_{s-1}+1, k_s} \leq \tilde{L}_T,$$

$$\sum_{s=1}^j E_T^{k_{s-1}+1, k_s} \leq E_T.$$

Knowing \mathcal{K}_T and $\tilde{\mathcal{K}}_T$, \tilde{L}_T and E_T are independent. Knowing \mathcal{K}_T , $\tilde{\mathcal{K}}_T$, $\left\{ \tilde{T}_k : \tilde{T}_k = \tilde{T}'_k \right\}$

is fixed and $\#\{\tilde{T}_k : \tilde{T}_k = \tilde{T}'_k\} = \tilde{K}_T - K_T$. So, using (2.9), if $\tilde{T}_k = \tilde{T}'_k$,

$$\begin{aligned} \mathbb{P}_{\mathcal{K}_T, \tilde{\mathcal{K}}_T}(\tilde{r}(k) \in C_{\tilde{T}_{k-}}^1 \cup \dots \cup C_{\tilde{T}_{k-}}^q) &= \frac{K_{\tilde{T}_{k-}} - \frac{K_{\tilde{T}_{k-}}(N - K_{\tilde{T}_{k-}})_+}{N-1}}{\tilde{K}_{\tilde{T}_{k-}} - \frac{K_{\tilde{T}_{k-}}(N - K_{\tilde{T}_{k-}})_+}{N-1}} \\ &\leq K_{\tilde{T}_{k-}} - \frac{K_{\tilde{T}_{k-}}(N - K_{\tilde{T}_{k-}})_+}{N-1} \leq \frac{K_{\tilde{T}_{k-}}(K_{\tilde{T}_{k-}} - 1)}{N-1} \leq \frac{K_{\tilde{T}_{k-}}^2}{N} \leq \frac{(\tilde{K}_T)^2}{N}, \end{aligned}$$

and, $\forall m$, if $\tilde{K}_T \leq N^{1/3}$,

$$\begin{aligned} \mathbb{P}_{\mathcal{K}_T, \tilde{\mathcal{K}}_T}(E_T = m) &\leq C_{\tilde{K}_T - K_T}^m \left(\frac{\tilde{K}_T^2}{N}\right)^m \mathbf{1}_{\tilde{K}_T - K_T \geq m} \\ &\leq \frac{(\tilde{K}_T)^{3m}}{m!N^m} \end{aligned}$$

so, $\forall n$,

$$\begin{aligned} \mathbb{P}_{\mathcal{K}_T, \tilde{\mathcal{K}}_T}(E_T \geq n) &\leq \sum_{m \geq n} \frac{(\tilde{K}_T)^{3m}}{m!N^m} \\ &\leq \frac{(\tilde{K}_T)^{3n}}{n!N^n} \exp\left(\frac{\tilde{K}_T^3}{N}\right) \\ (\text{because } \tilde{K}_T \leq N^{1/3}) &\leq \frac{(\tilde{K}_T)^{3n}}{n!N^n} e^1. \end{aligned} \tag{4.7}$$

We use (2.16), to get $\forall l$

$$\begin{aligned} \mathbb{P}(\tilde{K}_T \geq N^{1/3}) &\leq \sum_{k \geq N^{1/3}} q(1-\alpha)^{k/q-1} \\ &= \frac{q(1-\alpha)^{\lfloor N^{1/3} \rfloor / q - 1}}{(1 - (1-\alpha)^{1/q})} \\ &\leq \frac{q}{(1 - (1-\alpha)^{1/q})} \frac{1}{N^l} \sup_{N \geq 1} \left(N^l (1-\alpha)^{\lfloor N^{1/3} \rfloor / q - 1} \right). \end{aligned}$$

So one can get $\mathbb{P}(\tilde{K}_T \geq N^{1/3}) \leq C/N^{q/2+1}$ for some constant C . Using Lemma 2.11 and using the notation of Lemma 2.8, we get $\forall m$

$$\begin{aligned} \mathbb{P}_{\tilde{\mathcal{K}}_T, \mathcal{K}_T}(\tilde{L}_T \geq m) &\leq \mathbb{P}_{\tilde{\mathcal{K}}_T, \mathcal{K}_T}(N^{\Lambda \tilde{K}_T} (\tilde{K}_T - 1)^{1/N} \geq m) \\ &\leq \left(\frac{\Lambda(\tilde{K}_T)^2}{N-1}\right)^m. \end{aligned} \tag{4.8}$$

So $\forall m_1, m_2$,

$$\begin{aligned} & \mathbb{P}_{\tilde{\mathcal{K}}_T, \mathcal{K}_T}(\tilde{L}_T \geq m_1, E_T \geq m_2) \\ & \leq \mathbf{1}_{\tilde{K}_T \geq N^{1/3}} + \mathbf{1}_{\tilde{K}_T \leq N^{1/3}} \frac{\Lambda^{m_1} (\tilde{K}_T)^{2m_1+3m_2} e^1}{(N-1)^{m_1+m_2}}. \end{aligned} \quad (4.9)$$

So (for some constant C)

$$\begin{aligned} & \mathbb{P}_{\tilde{\mathcal{K}}_T, \mathcal{K}_T}(\overline{B}_{k_1, \dots, k_j}^{j,r}) \\ & \leq \mathbf{1}_{\tilde{K}_T \geq N^{1/3}} + \mathbf{1}_{\tilde{K}_T \leq N^{1/3}} \sum_{1 \leq i_1 \leq k_1 - k_0} \cdots \sum_{1 \leq i_r \leq k_r - k_{r-1}} \\ & \quad \times \mathbb{P}(\tilde{L}_T \geq \sum_{s=1}^r \lceil \frac{k_s - k_{s-1} - i_s}{2} \rceil + \sum_{s=r+1}^j \lceil \frac{k_s - k_{s-1}}{2} \rceil, \\ & \quad \quad \quad E_T \geq \sum_{s=1}^r i_s) \quad (4.10) \\ & \leq \mathbf{1}_{\tilde{K}_T \geq N^{1/3}} + \mathbf{1}_{\tilde{K}_T \leq N^{1/3}} C \frac{(\tilde{K}_T)^{5q/2}}{(N-1)^{(q+1)/2}} \end{aligned}$$

because we have the following inequalities $\forall j, k_1, \dots, k_j, r, i_1, \dots, i_r$

$$\begin{aligned} \sum_{s=1}^r \lceil \frac{k_s - k_{s-1} - i_s}{2} \rceil + \sum_{s=r+1}^j \lceil \frac{k_s - k_{s-1}}{2} \rceil + \sum_{s=1}^r i_s & \geq \frac{q}{2} + \sum_{s=1}^r \frac{i_s}{2} \\ & \geq \frac{q+1}{2}, \end{aligned}$$

$$\begin{aligned} & \sum_{s=1}^r \lceil \frac{k_s - k_{s-1} - i_s}{2} \rceil + \sum_{s=r+1}^j \lceil \frac{k_s - k_{s-1}}{2} \rceil + \sum_{s=1}^r i_s \\ & \leq \sum_{s=1}^r \left(\frac{k_s - k_{s-1} - i_s}{2} + 1 \right) + \sum_{s=r+1}^j \left(\frac{k_s - k_{s-1}}{2} + 1 \right) + j \\ & \leq \frac{q}{2} + 2j \leq \frac{5q}{2}. \end{aligned}$$

By (2.16), $\mathbb{E}(\tilde{K}_T^k) < \infty, \forall k$. So, by the dominated convergence theorem, we get

$$N^{q/2} \mathbb{P}(\overline{B}_{k_1, \dots, k_j}^{j,r}) \xrightarrow{N \rightarrow +\infty} 0.$$

So we have (4.2).

We set $\forall 0 \leq t \leq T, \forall s$,

$$\begin{aligned} \tilde{L}_t^{k_{s-1}+1, k_s} & = \#\{\tilde{T} : l \geq 1, \tilde{T}_l^{k_{s-1}+1, k_s} \leq t, \\ & \quad \tilde{T}_l^{k_{s-1}+1, k_s} = T_i'' \text{ or } \tilde{T}_l^{k_{s-1}+1, k_s} = \tilde{T}_i'', i \geq 1\}. \end{aligned}$$

We have a.s. $\sum_{s=1}^j \tilde{L}_T^{k_{s-1}+1, k_s} \leq \tilde{L}_T$. We have

$$\begin{aligned} & \tilde{B}_{k_{s-1}+1, k_s} \cap B'_{k_{s-1}+1, k_s} \\ & \subset \cup_{1 \leq i \leq k_s - k_{s-1} + 1} \left\{ \tilde{L}_T^{k_{s-1}+1, k_s} \geq \lceil \frac{k_s - k_{s-1} - i}{2} \rceil, E_T^{k_{s-1}+1, k_s} \geq i \right\}, \\ & \tilde{B}_{k_{s-1}+1, k_s} \setminus B'_{k_{s-1}+1, k_s} \subset \left\{ \tilde{L}_T^{k_{s-1}+1, k_s} \geq \lceil \frac{k_s - k_{s-1}}{2} \rceil \right\}. \end{aligned}$$

Reasoning as above, we get (4.3). □

Lemma 4.4. *For $j \in \{1, \dots, q\}$, $k_0 = 1 \leq k_1 \leq k_2 \leq \dots \leq k_{j-1} \leq k_j = q$, if $\exists i$ such that $k_i - k_{i-1}$ is odd then*

$$N^{q/2} \mathbb{P}(B_{1, k_1}, \dots, B_{k_{j-1}+1, k_j}) \xrightarrow{N \rightarrow +\infty} 0.$$

In particular, if q is odd, then $\forall j \in \{1, \dots, q\}$, $\forall 1 \leq k_1 \leq \dots \leq k_j = q$, we have the above convergence towards 0.

Proof. We decompose

$$\begin{aligned} & B_{k_0+1, k_1} \cup \dots \cup B_{k_{j-1}+1, k_j} \\ & = \cup_{r=0}^j \cup_{I \subset [j], \#I=r} [\cup_{i \in I} (B_{k_{i-1}+1, k_i} \cap B'_{k_{i-1}+1, k_i}) \cup_{i \notin I} \\ & \qquad \qquad \qquad (B_{k_{i-1}+1, k_i} \setminus B'_{k_{i-1}+1, k_i})]. \end{aligned}$$

Using Lemma 4.3 and the symmetry of the problem, we get $\forall I \subset [j], \#I \geq 1$,

$$N^{q/2} \mathbb{P} \left(\cup_{i \in I} (B_{k_{i-1}+1, k_i} \cap B'_{k_{i-1}+1, k_i}) \cup_{i \notin I} (B_{k_{i-1}+1, k_i} \setminus B'_{k_{i-1}+1, k_i}) \right) \xrightarrow{N \rightarrow +\infty} 0.$$

Now, we still have to look at the limit of

$$N^{q/2} \mathbb{P}((B_{1, k_1} \setminus B'_{1, k_1}) \cap \dots \cap (B_{k_{j-1}+1, k_j} \setminus B'_{k_{j-1}+1, k_j})).$$

Note that Equation (4.10) is still valid for $r = 0$. As $\exists i$ such that $k_i - k_{i-1}$ is odd, we have

$$\sum_{s=1}^j \lceil \frac{k_s - k_{s-1}}{2} \rceil \geq \frac{q+1}{2}.$$

So, using (4.9) and following the same reasoning as in the proof of Lemma 4.2, we get the result. □

We set (for k even)

$$I_k = \frac{k!}{2^{k/2}(k/2)!}$$

(this the number of partitions of $[k]$ into $k/2$ pairs).

We define the auxiliary particle system $\forall 1 \leq k \leq q$, $(\check{Z}_{0:T}^{1,k,1}, \check{Z}_{0:T}^{1,k,2}, \dots)$ by saying

- it has interactions at times $\{T - T_k, k \geq 1\} \cap \{T - t, t \in \mathcal{T}_{1,k}\}$,
- the rest of the definition is the same as for $(\tilde{Z}_{0:T}^i)$.

By doing this, we have coupled the interaction times of $(\check{Z}_{0:T})$ with the interaction times of the other systems. We can couple further and assume that $(Z_{0:T}^1, \dots, Z_{0:T}^k)$ and $(\check{Z}_{0:T}^{1,k,1}, \dots, \check{Z}_{0:T}^{1,k,k})$ coincide on the event $\{T - T_k, k \geq 1\} \cap \{T - t, t \in \mathcal{T}_{1,k}\} = \{T - T_k, k \geq 1\}$.

Lemma 4.5. *If q is an even integer, $j \in \{1, \dots, q\}$, $k_0 = 1 \leq k_1 \leq k_2 \leq \dots \leq k_{j-1} \leq k_j = q$ with $\forall i, k_i - k_{i-1}$ even, $F \in \mathcal{B}_0^{sym}(q)$,*

$$\begin{aligned} & N^{q/2} \mathbb{E}(F(\check{Z}_{0:T}^{1,k_1,1}, \dots, \check{Z}_{0:T}^{1,k_1,k_1}, \check{Z}_{0:T}^{\tilde{k}_1+1}, \dots, \check{Z}_{0:T}^q) \mathbf{1}_{B_{1,k_1}} \dots \mathbf{1}_{B_{k_{j-1}+1,k_j}}) \\ & \xrightarrow{N \rightarrow +\infty} I_{k_1} I_{k_2 - k_1} \dots I_{q - k_{j-1}} \\ & \times \mathbb{E} \left(\mathbb{E}_{\tilde{\mathcal{K}}_T} \left(F(\tilde{Z}_{0:T}^1, \dots, \tilde{Z}_{0:T}^{k_1}, \tilde{Z}_{0:T}^{\tilde{k}_1+1}, \dots, \tilde{Z}_{0:T}^q) | \tilde{L}_{1,q} \right) \prod_{i=1}^{q/2} \int_0^T \Lambda \tilde{K}_s^{2i-1} \tilde{K}_s^{2i} ds \right), \end{aligned}$$

where the last expectation indeed does not depend on N .

Proof. Note that $\forall i, \tilde{B}_{k_{i-1}+1,k_i} \setminus B'_{k_{i-1}+1,k_i} = B_{k_{i-1}+1,k_i} \setminus B'_{k_{i-1}+1,k_i}$. By Lemma 4.3, we get

$$\begin{aligned} & \lim_{N \rightarrow +\infty} N^{q/2} \mathbb{E}(F(\check{Z}_{0:T}^{1,k_1,1}, \dots, \check{Z}_{0:T}^q) \mathbf{1}_{B_{1,k_1}} \dots \mathbf{1}_{B_{k_{j-1}+1,k_j}}) \\ & = \lim_{N \rightarrow +\infty} N^{q/2} \mathbb{E}(F(\check{Z}_{0:T}^{1,k_1,1}, \dots, \check{Z}_{0:T}^{1,k_1,k_1}, \check{Z}_{0:T}^{\tilde{k}_1+1}, \dots, \check{Z}_{0:T}^q) \\ & \quad \mathbf{1}_{B_{1,k_1} \setminus B'_{1,k_1}} \dots \mathbf{1}_{B_{k_{j-1}+1,k_j} \setminus B'_{k_{j-1}+1,k_j}}) \end{aligned}$$

We have

$$\begin{aligned} & [(\check{Z}_{0:T}^{1,k_1,1}(\omega), \dots, \check{Z}_{0:T}^{1,k_1,k_1}(\omega)) \mathbf{1}_{B_{1,k_1} \setminus B'_{1,k_1}}(\omega) \dots \mathbf{1}_{B_{k_{j-1}+1,k_j} \setminus B'_{k_{j-1}+1,k_j}}(\omega) \\ & \quad \neq (Z_{0:T}^1(\omega), \dots, Z_{0:T}^q(\omega)) \mathbf{1}_{B_{1,k_1} \setminus B'_{1,k_1}}(\omega) \dots \mathbf{1}_{B_{k_{j-1}+1,k_j} \setminus B'_{k_{j-1}+1,k_j}}(\omega)] \\ & \quad \Rightarrow \tilde{L}_T(\omega) > \frac{q}{2}. \end{aligned}$$

So, by Lemma 3.2, the limit we are looking for is

$$\begin{aligned}
& \lim_{N \rightarrow +\infty} N^{q/2} \mathbb{E}(F(Z_{0:T}^1, \dots, Z_{0:T}^{k_1}, \tilde{Z}_{0:T}^{\tilde{k}_1+1}, \dots, \tilde{Z}_{0:T}^q) \\
& \quad \mathbf{1}_{B_{1,k_1} \setminus B'_{1,k_1}} \cdots \mathbf{1}_{B_{k_{j-1}+1,k_j} \setminus B'_{k_{j-1}+1,k_j}}) \\
& \quad \text{(by Lemma 4.3)} \\
& = \lim_{N \rightarrow q/2} N^{q/2} \mathbb{E}(F(Z_{0:T}^1, \dots, Z_{0:T}^{k_1}, \tilde{Z}_{0:T}^{\tilde{k}_1+1}, \dots, \tilde{Z}_{0:T}^q) \\
& \quad \mathbf{1}_{\tilde{B}_{1,k_1} \setminus B'_{1,k_1}} \cdots \mathbf{1}_{\tilde{B}_{k_{j-1}+1,k_j} \setminus B'_{k_{j-1}+1,k_j}})
\end{aligned}$$

As $\forall 1 \leq i \leq k_1$, $\tilde{Z}_{0:T}^i(\omega) = Z_{0:T}^i(\omega)$ if $\omega \in \tilde{B}_{1,k_1} \setminus B_{1,k_1}$, the last quantity is equal to

$$\begin{aligned}
& \lim_{N \rightarrow +\infty} N^{q/2} \mathbb{E}(F(\tilde{Z}_{0:T}^1, \dots, \tilde{Z}_{0:T}^{k_1}, \tilde{Z}_{0:T}^{\tilde{k}_1+1}, \dots, \tilde{Z}_{0:T}^q) \mathbf{1}_{\tilde{B}_{1,k_1} \setminus B'_{1,k_1}} \cdots \mathbf{1}_{\tilde{B}_{k_{j-1}+1,k_j} \setminus B'_{k_{j-1}+1,k_j}}) \\
& \quad \text{(by Lemma 4.3)} \\
& = \lim_{N \rightarrow +\infty} N^{q/2} \mathbb{E}(F(\tilde{Z}_{0:T}^1, \dots, \tilde{Z}_{0:T}^{k_1}, \tilde{Z}_{0:T}^{\tilde{k}_1+1}, \dots, \tilde{Z}_{0:T}^q) \mathbf{1}_{\tilde{B}_{k_0,k_1}} \cdots \mathbf{1}_{\tilde{B}_{k_{j-1}+1,k_j}})
\end{aligned}$$

We decompose

$$\begin{aligned}
& N^{q/2} \mathbb{E}(F(\tilde{Z}_{0:T}^1, \dots, \tilde{Z}_{0:T}^{k_1}, \tilde{Z}_{0:T}^{\tilde{k}_1+1}, \dots, \tilde{Z}_{0:T}^q) \mathbf{1}_{\tilde{B}_{k_0,k_1}} \cdots \mathbf{1}_{\tilde{B}_{k_{j-1}+1,k_j}}) = \\
& N^{q/2} \mathbb{E}(F(\tilde{Z}_{0:T}^1, \dots, \tilde{Z}_{0:T}^{k_1}, \tilde{Z}_{0:T}^{\tilde{k}_1+1}, \dots, \tilde{Z}_{0:T}^q) \mathbf{1}_{\tilde{B}_{k_0,k_1}} \cdots \mathbf{1}_{\tilde{B}_{k_{j-1}+1,k_j}} (\mathbf{1}_{\tilde{L}_T=q/2} + \mathbf{1}_{\tilde{L}_T>q/2})).
\end{aligned}$$

By Lemma 3.2, $N^{q/2} \mathbb{P}(\tilde{L}_T > q/2) \xrightarrow{N \rightarrow +\infty} 0$, so we look for the limit when $N \rightarrow +\infty$ of

$$\begin{aligned}
& N^{q/2} \mathbb{E}(F(\tilde{Z}_{0:T}^1, \dots, \tilde{Z}_{0:T}^{k_1}, \tilde{Z}_{0:T}^{\tilde{k}_1+1}, \dots, \tilde{Z}_{0:T}^q) \mathbf{1}_{\tilde{B}_{k_0,k_1}} \cdots \mathbf{1}_{\tilde{B}_{k_{j-1}+1,k_j}} \mathbf{1}_{\tilde{L}_T=q/2}) \\
& = \text{(using the symmetry of the problem)} \\
& N^{q/2} I_{k_1} I_{k_2-k_1} \cdots I_{k_j-k_{j-1}} N^{q/2} \mathbb{E}(F(\tilde{Z}_{0:T}^1, \dots, \tilde{Z}_{0:T}^{k_1}, \tilde{Z}_{0:T}^{\tilde{k}_1+1}, \dots, \tilde{Z}_{0:T}^q) \mathbf{1}_{\tilde{L}_{1,q}}).
\end{aligned}$$

So we look for the limit, when $N \rightarrow +\infty$ of

$$\begin{aligned}
& N^{q/2} \mathbb{E}(F(\tilde{Z}_{0:T}^1, \dots, \tilde{Z}_{0:T}^{k_1}, \tilde{Z}_{0:T}^{\tilde{k}_1+1}, \dots, \tilde{Z}_{0:T}^q) \mathbf{1}_{\tilde{L}_{1,q}}) \\
& = N^{q/2} \mathbb{E}(\mathbb{E}_{\tilde{\mathcal{K}}_T}(F(\tilde{Z}_{0:T}^1, \dots, \tilde{Z}_{0:T}^{k_1}, \tilde{Z}_{0:T}^{\tilde{k}_1+1}, \dots, \tilde{Z}_{0:T}^q) | \tilde{L}_{1,q}) | \tilde{\mathcal{K}}_T(\tilde{L}_{1,q})).
\end{aligned}$$

We set $\forall j, t$ (recall definition of $L^{j-1,j}$ in (2.14))

$$\tilde{L}'_{1,q} = \{L_T^{1,2} \geq 1\} \cap \cdots \cap \{L_T^{q-1,q} \geq 1\}.$$

We set $\forall j \in [q/2]$

$$\alpha(2j-1, 2j) = \exp\left(-\int_0^T \frac{\Lambda \tilde{K}_s^{2j-1} \tilde{K}_s^{2j}}{N-1} ds\right).$$

We have

$$\begin{aligned} N^{q/2} \mathbb{P}_{\tilde{\mathcal{K}}_T}(\tilde{L}'_{1,q}) &= \prod_{1 \leq j \leq q/2} (1 - \alpha(2j-1, 2j)) \\ &\xrightarrow[N \rightarrow +\infty]{\text{a.s.}} \prod_{1 \leq j \leq q/2} (T \Lambda \tilde{K}_t^{2j-1} \tilde{K}_T^{2j}). \end{aligned}$$

We have

$$\mathbb{P}_{\tilde{\mathcal{K}}_T}(\tilde{L}'_{1,q} \setminus \tilde{L}_{1,q}) \leq \mathbb{P}_{\tilde{\mathcal{K}}_T}(\tilde{L}_T > q/2),$$

so, by Lemma 3.2, $N^{q/2} \mathbb{P}_{\tilde{\mathcal{K}}_T}(\tilde{L}'_{1,q} \setminus \tilde{L}_{1,q}) \xrightarrow[N \rightarrow +\infty]{\text{a.s.}} 0$. And so :

$$N^{q/2} \mathbb{P}_{\tilde{\mathcal{K}}_T}(\tilde{L}_{1,q}) \xrightarrow[N \rightarrow +\infty]{\text{a.s.}} \prod_{1 \leq i \leq q/2} \int_0^T \Lambda \tilde{K}_s^{2i-1} \tilde{K}_s^{2i} ds.$$

Now,

$$\begin{aligned} N^{q/2} \mathbb{P}_{\tilde{\mathcal{K}}_T}(\tilde{L}_{1,q}) &\leq N^{q/2} \mathbb{P}_{\tilde{\mathcal{K}}_T}(\tilde{L}'_{1,q}) \\ &\leq \left(\frac{N}{N-1}\right)^{q/2} \prod_{1 \leq j \leq q/2} \int_0^T \Lambda \tilde{K}_s^{2i-1} \tilde{K}_s^{2i} ds \\ &\leq 2^{q/2} T^{q/2} \Lambda^{q/2} (\tilde{K}_T)^q. \end{aligned}$$

which is of finite expectation by (2.16) and Lemma 2.11. So, using the dominated convergence theorem, we get

$$\begin{aligned} &\lim_{N \rightarrow +\infty} \mathbb{E}(\mathbb{E}_{\tilde{\mathcal{K}}_T}(F(\tilde{Z}_{0:T}^1, \dots, \tilde{Z}_{0:T}^{k_1}, \tilde{Z}_{0:T}^{\tilde{k}_1+1}, \dots, \tilde{Z}_{0:T}^q) | \tilde{L}_{1,q}) N^{q/2} \mathbb{P}_{\tilde{\mathcal{K}}_T}(\tilde{L}_{1,q})) \\ &= \mathbb{E}(\mathbb{E}_{\tilde{\mathcal{K}}_T}(F(\tilde{Z}_{0:T}^1, \dots, \tilde{Z}_{0:T}^{k_1}, \tilde{Z}_{0:T}^{\tilde{k}_1+1}, \dots, \tilde{Z}_{0:T}^q) | \tilde{L}_{1,q}) \prod_{i=1}^{q/2} \int_0^T \Lambda \tilde{K}_s^{2i-1} \tilde{K}_s^{2i} ds). \end{aligned}$$

By Lemma 2.8, 1 and Lemma 2.11, 5, $\mathbb{E}_{\tilde{\mathcal{K}}_T}(F(\tilde{Z}_{0:T}^1, \dots, \tilde{Z}_{0:T}^{k_1}, \tilde{Z}_{0:T}^{\tilde{k}_1+1}, \dots, \tilde{Z}_{0:T}^q) | \tilde{L}_{1,q})$ is a random variable whose law does not depend on N . □

4.3 Proof of Proposition 4.2

Proof. Because $(Z_{0:T}^1, \dots, Z_{0:T}^q) = (\tilde{Z}_{0:T}^1, \dots, \tilde{Z}_{0:T}^q)$ on $A_1 \cap \dots \cap A_q$, we have

$$\begin{aligned}
& \mathbb{E}(F(Z_{0:T}^1, \dots, Z_{0:T}^q)) = \\
& \mathbb{E}(F(Z_{0:T}^1, \dots, Z_{0:T}^q) \mathbf{1}_{(A_1 \cap \dots \cap A_q)^c}) + \mathbb{E}(F(Z_{0:T}^1, \dots, Z_{0:T}^q) \mathbf{1}_{A_1 \cap \dots \cap A_q}) = \\
& \mathbb{E}(F(Z_{0:T}^1, \dots, Z_{0:T}^q) \mathbf{1}_{(A_1 \cap \dots \cap A_q)^c}) + \mathbb{E}(F(\tilde{Z}_{0:T}^1, \dots, \tilde{Z}_{0:T}^q)) \\
& \quad - \mathbb{E}(F(\tilde{Z}_{0:T}^1, \dots, \tilde{Z}_{0:T}^q) \mathbf{1}_{(A_1 \cap \dots \cap A_q)^c}) = \\
& \mathbb{E}(F(Z_{0:T}^1, \dots, Z_{0:T}^q) \mathbf{1}_{(A_1 \cap \dots \cap A_q)^c}) - \mathbb{E}(F(\tilde{Z}_{0:T}^1, \dots, \tilde{Z}_{0:T}^q) \mathbf{1}_{(A_1 \cap \dots \cap A_q)^c}). \quad (4.11)
\end{aligned}$$

We decompose (using the symmetry of the problem)

$$\begin{aligned}
& \mathbb{E}(F(Z_{0:T}^1, \dots, Z_{0:T}^q) \mathbf{1}_{(A_1 \cap \dots \cap A_q)^c}) = \\
& \sum_{k=1}^q C_q^k \mathbb{E}(F(Z_{0:T}^1, \dots, Z_{0:T}^q) \mathbf{1}_{A_1^c} \dots \mathbf{1}_{A_k^c} \mathbf{1}_{A_{k+1}} \dots \mathbf{1}_{A_q}) = \\
& \sum_{k=1}^q C_q^k \mathbb{E}(F(Z_{0:T}^1, \dots, Z_{0:T}^q) \mathbf{1}_{B_{1,k}} \mathbf{1}_{A_{k+1}} \dots \mathbf{1}_{A_q}) = \\
& \sum_{k=1}^q C_q^k \mathbb{E}(F(\check{Z}_{0:T}^{1,k,1}, \dots, \check{Z}_{0:T}^{1,k,k}, \tilde{Z}_{0:T}^{k+1}, \dots, \tilde{Z}_{0:T}^q) \mathbf{1}_{B_{1,k}} \mathbf{1}_{A_{k+1}} \dots \mathbf{1}_{A_q}) = \\
& \sum_{k=1}^q C_q^k \mathbb{E}(F(\check{Z}_{0:T}^{1,k,1}, \dots, \check{Z}_{0:T}^{1,k,k}, \tilde{Z}_{0:T}^{k+1}, \dots, \tilde{Z}_{0:T}^q) \mathbf{1}_{B_{1,k}}) \\
& \quad - C_q^k \mathbb{E}(F(\check{Z}_{0:T}^{1,k,1}, \dots, \check{Z}_{0:T}^{1,k,k}, \tilde{Z}_{0:T}^{k+1}, \dots, \tilde{Z}_{0:T}^q) \mathbf{1}_{B_{1,k}} \mathbf{1}_{(A_{k+1} \cap \dots \cap A_q)^c}) = \\
& \sum_{k=1}^q -C_q^k \mathbb{E}(F(\check{Z}_{0:T}^{1,k,1}, \dots, \check{Z}_{0:T}^{1,k,k}, \tilde{Z}_{0:T}^{k+1}, \dots, \tilde{Z}_{0:T}^q) \mathbf{1}_{B_{1,k}} \mathbf{1}_{(A_{k+1} \cap \dots \cap A_q)^c}).
\end{aligned}$$

because for all k , $(\tilde{Z}_{0:T}^{k+1}, \dots, \tilde{Z}_{0:T}^q)$ is independent of $(\check{Z}_{0:T}^{1,k,1}, \dots, \check{Z}_{0:T}^{1,k,k}, \mathbf{1}_{B_{1,k}})$ and has the law $\tilde{P}_{0:t}^{\otimes(q-k)}$. We also have that, $\forall k_2 > k_1$, $(\tilde{Z}_{0:T}^{k_2+1}, \dots, \tilde{Z}_{0:T}^q)$ is independent of $(\check{Z}_{0:T}^{1,k_1,1}, \dots, \check{Z}_{0:T}^{1,k_1,k_1}, \mathbf{1}_{B_{1,k_1}}, \tilde{Z}_{0:T}^{k_1+1}, \dots, \tilde{Z}_{0:T}^{k_2})$, so we get in the same way as above

$$\begin{aligned}
& \mathbb{E}(F(Z_{0:T}^{N,1}, \dots, Z_{0:T}^{N,q}) \mathbf{1}_{(A_1 \cap \dots \cap A_q)^c}) = \\
& \quad - \mathbb{E}(F(\check{Z}_{0:T}^{1,q,1}, \dots, \check{Z}_{0:T}^{1,q,q}) \mathbf{1}_{B_{1,q}}) \\
& \quad + \sum_{1 \leq k_1 \leq q-1} \sum_{k_1+1 \leq k_2 \leq q} \\
& \mathbb{E}(F(\check{Z}_{0:T}^{1,k_1,1}, \dots, \check{Z}_{0:T}^{1,k_1,k_1}, \tilde{Z}_{0:T}^{k_2+1}, \dots, \tilde{Z}_{0:T}^q) \mathbf{1}_{B_{1,k_1}} \mathbf{1}_{B_{k_1+1,k_2}} \mathbf{1}_{(A_{k_2+1} \cap \dots \cap A_q)^c}).
\end{aligned}$$

A recurrence then gives us

$$\begin{aligned} \mathbb{E}(F(Z_{0:T}^1, \dots, Z_{0:T}^q) \mathbf{1}_{(A_1 \cap \dots \cap A_q)^c}) &= \\ & \sum_{j=1}^q \sum_{1 \leq k_1 < \dots < k_{j-1} < k_j = q} C_q^{k_1} C_{q-k_1}^{q-k_2} \dots C_{q-k_{j-1}}^0 (-1)^{j+1} \\ & \times \mathbb{E}(F(\tilde{Z}_{0:T}^{1, k_1, 1}, \dots, \tilde{Z}_{0:T}^{1, k_1, k_1}, \tilde{Z}_{0:T}^{\tilde{k}_1+1}, \dots, \tilde{Z}_{0:T}^q) \mathbf{1}_{B_{1, k_1}} \dots \mathbf{1}_{B_{k_{j-1}+1, k_j}}) = \end{aligned}$$

And, in the same way (using at the first step of the recurrence the fact that $(\tilde{Z}_{0:T}^{\tilde{k}_1+1}, \dots, \tilde{Z}_{0:T}^q)$ is independent of $(\tilde{Z}_{0:T}^1, \dots, \tilde{Z}_{0:T}^{\tilde{k}_1}, \mathbf{1}_{B_{1, k_1}})$):

$$\begin{aligned} \mathbb{E}(F(\tilde{Z}_{0:T}^1, \dots, \tilde{Z}_{0:T}^q) \mathbf{1}_{(A_1 \cap \dots \cap A_q)^c}) &= \\ & \sum_{j=1}^q \sum_{1 \leq k_1 < \dots < k_{j-1} < k_j = q} C_q^{k_1} C_{q-k_1}^{q-k_2} \dots C_{q-k_{j-1}}^0 (-1)^{j+1} \\ & \times \mathbb{E}(F(\tilde{Z}_{0:T}^1, \dots, \tilde{Z}_{0:T}^q) \mathbf{1}_{B_{1, k_1}} \dots \mathbf{1}_{B_{k_{j-1}+1, k_j}}). \end{aligned}$$

By Lemma 4.4, for any j, k_1, \dots, k_j as in the sum above, if q is odd or if $\exists i$ such that $k_i - k_{i-1}$ is odd, we have

$$\begin{aligned} N^{q/2} \mathbb{E}(F(Z_{0:T}^1, \dots, Z_{0:T}^k, \tilde{Z}_{0:T}^{\tilde{k}_1+1}, \dots, \tilde{Z}_{0:T}^q) \mathbf{1}_{B_{1, k_1}} \dots \mathbf{1}_{B_{k_{j-1}+1, k_j}}) &\xrightarrow{N \rightarrow +\infty} 0 \\ N^{q/2} \mathbb{E}(F(\tilde{Z}_{0:T}^1, \dots, \tilde{Z}_{0:T}^q) \mathbf{1}_{B_{1, k_1}} \dots \mathbf{1}_{B_{k_{j-1}+1, k_j}}) &\xrightarrow{N \rightarrow +\infty} 0. \end{aligned}$$

So we have the point 1 of the Proposition. We suppose q even from now on. By Lemma 4.5:

$$\begin{aligned}
& N^{q/2} \mathbb{E}(F(Z_{0:T}^1, \dots, Z_{0:T}^q) \mathbf{1}_{(A_1 \cap \dots \cap A_q)^c}) \\
& \xrightarrow{N \rightarrow +\infty} \sum_{j=1}^{q/2} \sum_{1 \leq k_1 < \dots < k_j = q; k_1, \dots \text{ even}} C_q^{k_1} \dots C_{q-k_{j-1}}^0 (-1)^{j+1} \\
& \times I_{k_1} I_{k_2 - k_1} \dots I_{k_j - k_{j-1}} \mathbb{E}(\mathbb{E}_{\tilde{\mathcal{K}}_T} (F(\tilde{Z}_{0:T}^1, \dots, \tilde{Z}_{0:T}^{k_1}, \tilde{Z}_{0:T}^{k_1+1}, \dots, \tilde{Z}_{0:T}^q) | \tilde{L}_{1,q})) \\
& \quad \times \prod_{1 \leq i \leq q/2} \int_0^T \Lambda \tilde{K}_s^{2i-1} \tilde{K}_s^{2i} ds) = \\
& \sum_{2 \leq k_1 \leq q; k_1 \text{ even}} C_q^{k_1} I_{k_1} \sum_{i=1}^{(q-k_1)/2} \sum_{k_1+1 \leq k'_1 < \dots < k'_i; k'_1, \dots \text{ even}} C_{q-k_1}^{q-k'_1} \dots C_{q-k'_{i-1}}^{q-k'_i} \\
& \times I_{k'_1 - k_1} \dots I_{k'_i - k'_{i-1}} (-1)^i \mathbb{E}(\mathbb{E}_{\tilde{\mathcal{K}}_T} (F(\tilde{Z}_{0:T}^1, \dots, \tilde{Z}_{0:T}^{k_1}, \tilde{Z}_{0:T}^{k_1+1}, \dots, \tilde{Z}_{0:T}^q) | \tilde{L}_{1,q})) \\
& \quad \times \prod_{1 \leq i \leq q/2} \int_0^T \Lambda \tilde{K}_s^{2i-1} \tilde{K}_s^{2i} ds),
\end{aligned}$$

with the convention that for $k_1 = q$, the sum $\sum_{i=1}^{(q-k_1)/2} (\dots)$ above is equal to 1. For $k_1 < q$, we get by Lemma 6.1:

$$\begin{aligned}
& \sum_{i=1}^{(q-k_1)/2} \sum_{k_1+1 \leq k'_1 < \dots < k'_i = q; k'_1, \dots \text{ even}} C_{q-k_1}^{q-k'_1} \dots C_{q-k'_{i-1}}^{q-k'_i} I_{k'_1 - k_1} \dots I_{k'_i - k'_{i-1}} (-1)^i \\
& = I_{q-k_1} (-1)^{\frac{q-k_1}{2}}, \quad (4.12)
\end{aligned}$$

and So, we get (as $\forall k_1$ even, $k_1 \leq q$, $C_q^{k_1} I_{k_1} I_{q-k_1} = C_{q/2}^{k_1/2}$)

$$\begin{aligned}
& N^{q/2} \mathbb{E}(F(Z_{0:T}^1, \dots, Z_{0:T}^q) \mathbf{1}_{(A_1 \cap \dots \cap A_q)^c}) \xrightarrow{N \rightarrow +\infty} \\
& \sum_{2 \leq k_1 \leq q; k_1 \text{ even}} C_{q/2}^{k_1/2} I_q \mathbb{E}(F(\tilde{Z}_{0:T}^1, \dots, \tilde{Z}_{0:T}^{k_1}, \tilde{Z}_{0:T}^{k_1+1}, \dots, \tilde{Z}_{0:T}^q) | \tilde{L}_{1,q})) (-1)^{(q-k_1)/2}.
\end{aligned}$$

And in the same way:

$$\begin{aligned}
& N^{q/2} \mathbb{E}(\tilde{Z}_{0:T}^1, \dots, \tilde{Z}_{0:T}^q) \\
& \xrightarrow{N \rightarrow +\infty} \sum_{2 \leq k_1 \leq q; k_1 \text{ even}} C_{q/2}^{k_1/2} I_q \mathbb{E}(F(\tilde{Z}_{0:T}^1, \dots, \tilde{Z}_{0:T}^q) | \tilde{L}_{1,q})) (-1)^{(q-k_1)/2}.
\end{aligned}$$

And Equation (4.11) gives us the point 2 of the Proposition. \square

4.4 Wick formula

We suppose here that q is even. We introduce an auxiliary system of particles $(\tilde{Z}_{0:T}^{\tilde{1}}, \tilde{Z}_{0:T}^{\tilde{2}}, \dots)$ such that

- it has interaction times $\{T - \tilde{T}_k, k \geq 1\} \cap \{T - t, t \in \mathcal{T}^{1,2} \cap \dots \cap \mathcal{T}^{q-1,q}\}$
- the rest of the definition is the same as for $(\tilde{Z}_{0:T}^i)_{i \geq 1}$.

By doing this, we have coupled the interaction times of the system $(\tilde{Z}_{0:T}^i)_{i \geq 1}$ and of the system $(\tilde{Z}_{0:T}^i)_{i \geq 1}$. We can couple further and assume that $(\tilde{Z}_{0:T}^{\tilde{1}}, \dots, \tilde{Z}_{0:T}^{\tilde{q}})$ and $(\tilde{Z}_{0:T}^{\tilde{1}}, \dots, \tilde{Z}_{0:T}^{\tilde{q}})$ coincide on the event $\{\tilde{T}_k, k \geq 1\} \cap \mathcal{T}^{1,2} \cap \dots \cap \mathcal{T}^{q-1,q} = \{\tilde{T}_k, k \geq 1\}$. We set $\forall f, g$ bounded $E \rightarrow \mathbb{R}$

$$V_{0:T}(f, g) = \mathbb{E} \left(\mathbb{E}_{\tilde{K}_{0:T}^1, \tilde{K}_{0:T}^2} (f(\tilde{Z}_{0:T}^{\tilde{1}})g(\tilde{Z}_{0:T}^{\tilde{2}}) - f(\tilde{Z}_{0:T}^{\tilde{1}})g(\tilde{Z}_{0:T}^{\tilde{2}}) | \tilde{L}_{1,2}) \int_0^T \Lambda \tilde{K}_s^1 \tilde{K}_s^2 ds \right).$$

Note that $\forall f, g$, $V_{0:T}(f, g) = V_{0:T}(g, f)$.

Corollary 4.6. [Wick formula] For $F \in \mathcal{B}_0^{sym}(q)$ of the form $(f_1 \otimes \dots \otimes f_q)_{sym}$ and q even,

$$N^{q/2} \mathbb{E}(F(Z_{0:T}^1, \dots, Z_{0:T}^q)) \xrightarrow{N \rightarrow +\infty} \sum_{J \in \mathcal{I}_q} \prod_{\{a,b\} \in J} V_{0:T}(f_a, f_b).$$

The name ‘‘Wick formula’’ comes from the Wick formula on the expectation of a product of Gaussians. In this formula, there is a sum over pairings, just as in the above Corollary. See Theorem 22.3, p. 360 in [NS06] for the Wick formula.

Proof. To shorten the notations, we will write

$$\prod_{i=1}^{q/2} \int_0^T \Lambda \tilde{K}_s^{2i-1} \tilde{K}_s^{2i} ds = p.$$

With this particular form for F , the limit in (4.1) of Proposition 4.2 becomes

$$\begin{aligned} & \frac{1}{q!} \sum_{\sigma \in \mathcal{S}_q} \sum_{k=1}^{q/2} I_q C_{q/2}^k (-1)^{(q-k)/2} \\ & \mathbb{E}(\mathbb{E}_{\tilde{K}_T} (f_{\sigma(1)}(\tilde{Z}_{0:T}^{\tilde{1}}) \dots f_{\sigma(2k)}(\tilde{Z}_{0:T}^{\tilde{2k}}) f_{\sigma(2k+1)}(\tilde{Z}_{0:T}^{\tilde{2k+1}}) \dots f_{\sigma(q)}(\tilde{Z}_{0:T}^{\tilde{q}}) \\ & \quad - f_{\sigma(1)}(\tilde{Z}_{0:T}^{\tilde{1}}) \dots f_{\sigma(q)}(\tilde{Z}_{0:T}^{\tilde{q}}) | \tilde{L}_{1,q}) p). \end{aligned}$$

For any $s \in \mathcal{S}_q$ such that $\forall i \in [q/2]$, $s(2i) - s(2i - 1) = 1$ and $s(2i)$ is even, $s(2i - 1)$ is odd, $\forall \sigma$, we have the following due to the symmetry of the problem

$$\begin{aligned} & \mathbb{E}(\mathbb{E}_{\tilde{\mathcal{K}}_T}(f_{\sigma(1)}(\tilde{Z}_{0:T}^1) \cdots f_{\sigma(2k)}(\tilde{Z}_{0:T}^{2k}) f_{\sigma(2k+1)}(\tilde{Z}_{0:T}^{2k+1}) \cdots f_{\sigma(q)}(\tilde{Z}_{0:T}^q) \\ & \quad - f_{\sigma(1)}(\tilde{Z}_{0:T}^1) \cdots f_{\sigma(q)}(\tilde{Z}_{0:T}^q) | \tilde{L}_{1,q}) p) \\ & = \mathbb{E}(\mathbb{E}_{\tilde{\mathcal{K}}_T}(f_{\sigma(1)}(\tilde{Z}_{0:T}^{s(1)}) \cdots f_{\sigma(2k)}(\tilde{Z}_{0:T}^{s(2k)}) f_{\sigma(2k+1)}(\tilde{Z}_{0:T}^{s(2k+1)}) \cdots f_{\sigma(q)}(\tilde{Z}_{0:T}^{s(q)}) \\ & \quad - f_{\sigma(1)}(\tilde{Z}_{0:T}^{s(1)}) \cdots f_{\sigma(q)}(\tilde{Z}_{0:T}^{s(q)}) | \tilde{L}_{1,q}) p) \end{aligned}$$

So the limit becomes

$$\begin{aligned} & \frac{1}{q!} \sum_{\sigma \in \mathcal{S}_q} \sum_{k=1}^{q/2} I_q (-1)^{(q-k)/2} \sum_{I \subset [q/2], \#I=k/2} \mathbb{E}(\mathbb{E}_{\tilde{\mathcal{K}}_T}(\prod_{i \in I} f_{\sigma(2i-1)}(\tilde{Z}_{0:T}^{2i-1}) f_{\sigma(2i)}(\tilde{Z}_{0:T}^{2i}) \\ & \quad \prod_{i \notin I} f_{\sigma(2i-1)}(\tilde{Z}_{0:T}^{2i-1}) f_{\sigma(2i)}(\tilde{Z}_{0:T}^{2i}) | \tilde{L}_{1,q}) p) = \\ & \quad \frac{1}{q!} \sum_{\sigma \in \mathcal{S}_q} I_q \mathbb{E}(\mathbb{E}_{\tilde{\mathcal{K}}_T} (\\ & \quad \prod_{i=1}^{q/2} (f_{\sigma(2i-1)}(\tilde{Z}_{0:T}^{2i-1}) f_{\sigma(2i)}(\tilde{Z}_{0:T}^{2i}) - f_{\sigma(2i-1)}(\tilde{Z}_{0:T}^{2i-1}) f_{\sigma(2i)}(\tilde{Z}_{0:T}^{2i})) | \tilde{L}_{1,q}) p) = \\ & \quad \frac{1}{q!} \sum_{\sigma \in \mathcal{S}_q} I_q \mathbb{E}(\mathbb{E}_{\tilde{\mathcal{K}}_T} (\\ & \quad \prod_{i=1}^{q/2} (f_{\sigma(2i-1)}(\tilde{Z}_{0:T}^{2i-1}) f_{\sigma(2i)}(\tilde{Z}_{0:T}^{2i}) - f_{\sigma(2i-1)}(\tilde{Z}_{0:T}^{2i-1}) f_{\sigma(2i)}(\tilde{Z}_{0:T}^{2i})) | \tilde{L}_{1,2}, \dots, \tilde{L}_{q-1,q}) p). \end{aligned} \tag{4.13}$$

By Lemma 2.11, 5, the processes $L^{1,2}, \dots, L^{q,q-1}$ defined in (4.4) are independent conditionnaly to $\tilde{\mathcal{K}}_T$. And for all $i \in [q/2]$,

$$\begin{aligned} & \mathbb{E}_{\tilde{\mathcal{K}}_T}(((f_{\sigma(2i-1)}(\tilde{Z}_{0:T}^{2i-1}) f_{\sigma(2i)}(\tilde{Z}_{0:T}^{2i}) - f_{\sigma(2i-1)}(\tilde{Z}_{0:T}^{2i-1}) f_{\sigma(2i)}(\tilde{Z}_{0:T}^{2i})) | \tilde{L}_{1,2}, \dots, \tilde{L}_{q-1,q}) \\ & = \mathbb{E}(\mathbb{E}_{\tilde{K}_{0:T}^{2i-1}, \tilde{K}_{0:T}^{2i}}((f_{\sigma(2i-1)}(\tilde{Z}_{0:T}^{2i-1}) f_{\sigma(2i)}(\tilde{Z}_{0:T}^{2i}) - f_{\sigma(2i-1)}(\tilde{Z}_{0:T}^{2i-1}) f_{\sigma(2i)}(\tilde{Z}_{0:T}^{2i})) | \tilde{L}_{2i-1,2i})) \end{aligned}$$

So the quantity in (4.13) is equal to

$$\begin{aligned}
& \frac{1}{q!} \sum_{\sigma \in \mathcal{S}_q} I_q \mathbb{E} \left(\prod_{i=1}^{q/2} \left[\mathbb{E}_{\tilde{K}_{0:T}^{2i-1}, \tilde{K}_{0:T}^{2i}} \left((f_{\sigma(2i-1)}(\tilde{Z}_{0:T}^{\approx 2i-1}) f_{\sigma(2i)}(\tilde{Z}_{0:T}^{\approx 2i}) \right. \right. \right. \\
& \quad \left. \left. \left. - f_{\sigma(2i-1)}(\tilde{Z}_{0:T}^{\approx 2i-1}) f_{\sigma(2i)}(\tilde{Z}_{0:T}^{\approx 2i}) \right) | \tilde{L}_{2i-1, 2i} \right) \right] \int_0^T \Lambda \tilde{K}_s^{2i-1} \tilde{K}_s^{2i-1} ds \Big) = \\
& \quad \text{(by Lemma 2.11, 4)} \\
& \frac{1}{q!} \sum_{\sigma \in \mathcal{S}_q} I_q \prod_{i=1}^{q/2} \mathbb{E} \left(\mathbb{E}_{\tilde{K}_{0:T}^{2i-1}, \tilde{K}_{0:T}^{2i}} \left((f_{\sigma(2i-1)}(\tilde{Z}_{0:T}^{\approx 2i-1}) f_{\sigma(2i)}(\tilde{Z}_{0:T}^{\approx 2i}) \right. \right. \right. \\
& \quad \left. \left. \left. - f_{\sigma(2i-1)}(\tilde{Z}_{0:T}^{\approx 2i-1}) f_{\sigma(2i)}(\tilde{Z}_{0:T}^{\approx 2i}) \right) | \tilde{L}_{2i-1, 2i} \right) \int_0^T \Lambda \tilde{K}_s^{2i-1} \tilde{K}_s^{2i-1} ds = \\
& \quad \sum_{J \in \mathcal{I}_q} \prod_{\{a, b\} \in J} V_{0:T}(f_a, f_b).
\end{aligned}$$

□

5 Convergence theorems

All the theorems of this section are valid under Hypothesis 1.

5.1 Almost sure convergence

Theorem 5.1. *For any measurable bounded f , $T \geq 0$,*

$$\eta_{0:T}^N(f) \xrightarrow[N \rightarrow +\infty]{p.s.} \tilde{P}_{0:T}(f).$$

Proof. We recall the notations of [DPR09b]. For any empirical measure $m(x) = \frac{1}{N} \sum_{i=1}^N \delta_{x^i}$ (based on N points x^1, x^2, \dots, x^N), any q ,

$$m(x)^{\otimes q} := \frac{1}{N^q} \sum_{a \in [N]^{[q]}} \delta_{(x^{a(1)}, \dots, x^{a(q)})},$$

where $[N]^{[q]} = a : [q] \rightarrow [N]$. Note that for any F

$$m(x)^{\otimes q}(F) = m(x)^{\otimes q}(F_{\text{sym}}).$$

We define, $\forall 1 \leq p \leq q$, $[q]_p^{[q]} := \{a \in [q]^{[q]}, \#\text{Im}(a) = p\}$ and $(\forall k \leq q)$

$$\partial^k L_q = \sum_{q-k \leq p \leq q} s(p, q-k) \frac{1}{(q)_p} \sum_{a \in [q]_p^{[q]}} a$$

(the $s(\cdot, \cdot)$ are the Stirling numbers of the first kind) and $\forall F$ (of q variables), $\forall b \in [q]^{[q]}$,

$$D_b(F)(x^1, \dots, x^q) = F(x^{b(1)}, \dots, x^{b(q)}),$$

$$D_{\partial^k L_q}(F) = \sum_{q-k \leq p \leq q} s(p, q-k) \frac{1}{(q)_p} \sum_{a \in [q]_p^{[q]}} D_a(F).$$

The derivative like notation $\partial^k L$ comes from [DPR09b], where it makes sense to think of a derivative at this point. We keep the same notation in order to be consistent but it has no particular meaning in our setting. We then have, by Corollary 2.3 p. 789 of [DPR09b], for any empirical measure $m(x)$ (based on N points), any F of q variables,

$$m(x)^{\otimes q}(F) = m(x)^{\odot q} \left(\sum_{0 \leq k < q} \frac{1}{N^k} D_{\partial^k L_q}(F) \right).$$

Suppose $F \in \mathcal{B}_0^{\text{sym}}(q)$,

$$\mathbb{E}((\eta_{0:T}^N)^{\otimes q}(F)) = \sum_{0 \leq k < q} \frac{1}{N^k} \sum_{q-k \leq p \leq q} s(p, q-k) \sum_{a \in [q]_p^{[q]}} \mathbb{E}((\eta_{0:T}^N)^{\odot q}(D_a(F))). \quad (5.1)$$

We take $a \in [q]^{[q]}$ with $p = \#\text{Im}(a) \geq q-k$ and $k < q/2$. Note that $\#\{i \in [q], \#a(\{i\}) = 1\} \geq q-2k > 0$.

We have now to use the Hoeffding's decomposition (see [dlPG99, Lee90], or [DPR09a], for the details). For any symmetrical $G : \mathbb{D}([0; T], \mathbb{R}^{q^d}) \rightarrow \mathbb{R}$, we define

$$\theta = \int G(x_1, \dots, x_q) \tilde{P}_{0:T}(dx_1, \dots, dx_q),$$

$$G^{(j)}(x_1, \dots, x_j) = \int G(x_1, \dots, x_q) \tilde{P}_{0:T}^{\otimes(q-j)}(dx_{j+1}, \dots, dx_q),$$

and recursively

$$h^{(1)}(x_1) = G^{(1)}(x_1) - \theta,$$

$$h^{(k)}(x_1, \dots, x_k) = G^{(j)}(x_1, \dots, x_k) - \sum_{i=1}^{j-1} \sum_{(j,i)} h^{(i)} - \theta,$$

where $\sum_{(j,i)} h^{(i)}$ is an abbreviation for the function

$$(x_1, \dots, x_j) \mapsto \sum_{1 \leq r_1 < \dots < r_i \leq j} h^{(i)}(x_{r_1}, \dots, x_{r_i}).$$

For all j , $h^{(j)} \in \mathcal{B}_0^{\text{sym}}(j)$. We have the formula

$$G(x_1, \dots, x_q) = h^{(q)}(x_1, \dots, x_q) + \sum_{j=1}^{q-1} \sum_{(q,j)} h^{(j)}.$$

We take now $G(x_1, \dots, x_q) = D_a(F)$. For $j < q - 2k$, $G^{(j)} = 0$. So we can show by recurrence that $h^{(j)} = 0$ for $j < q - 2k$. So

$$G(x_1, \dots, x_q) = h^{(q)}(x_1, \dots, x_q) + \sum_{j=q-2k}^{q-1} \sum_{(q,j)} h^{(j)}.$$

So, by Proposition 4.2, we have for some constant C

$$\mathbb{E}(D_a F(Z_{0:T}^1, \dots, Z_{0:T}^q)) \leq \frac{C}{N^{(q-2k)/2}}.$$

And so, by (5.1),

$$\mathbb{E}((\eta_{0:T}^N)^{\otimes q}(F)) \leq \frac{C}{N^{\frac{q}{2}}}.$$

Suppose we take a bounded function $f : E \rightarrow \mathbb{R}$. We set $\bar{f} = f - \tilde{P}_T(f)$. We then have (with the notation $\bar{f}^{\otimes q}(x^1, \dots, x^q) := \bar{f}(x^1) \times \dots \times \bar{f}(x^q)$)

$$\begin{aligned} \mathbb{E}((\eta_{0:T}^N(f) - \tilde{P}_{0:T}(f))^q) &= \mathbb{E}((\eta_{0:T}^N(\bar{f}))^q) \\ &= \mathbb{E}((\eta_{0:T}^N)^{\otimes q}(\bar{f}^{\otimes q})) \\ &= \mathbb{E}((\eta_{0:T}^N)^{\otimes q}(\bar{f}^{\otimes q})_{\text{sym}}) \\ &\leq \frac{C}{N^{\frac{q}{2}}}. \end{aligned} \tag{5.2}$$

Provided we take $q = 4$, we can apply Borel-Cantelli Lemma to finish the proof. \square

5.2 Central-limit theorem

Theorem 5.2. *Suppose q even. For all $f_1, \dots, f_q \in \mathcal{B}_0^{\text{sym}}(1)$, $\forall T \geq 0$,*

$$N^{q/2}(\eta_{0:T}^N(f_1), \dots, \eta_{0:T}^N(f_q)) \xrightarrow[N \rightarrow +\infty]{law} \mathcal{N}(0, K),$$

with $K(i, j) = \tilde{P}_{0:T}(f_i f_j) + V_{0:T}(f_i, f_j)$, $\forall i, j$.

Note that we can bound the component of the variance K . Take f_1, \dots, f_j as above. For all i, j :

$$|\tilde{P}_{0:T}(f_i f_j)| \leq \|f_i\|_\infty \|f_j\|_\infty,$$

$$\begin{aligned} |V_{0:T}(f_i, f_j)| &\leq 2\|f_i\|_\infty \|f_j\|_\infty \mathbb{E}(T \Lambda \tilde{K}_T^1 \tilde{K}_T^2) \\ \text{(by Lemma 2.11 and (2.15))} &\leq 2\|f_i\|_\infty \|f_j\|_\infty T \Lambda e^{2T\Lambda}. \end{aligned}$$

Proof. For any u_1, \dots, u_q , we have:

$$\begin{aligned}
& \mathbb{E} \left(\exp \left(N \eta_{0:T}^N \left(\log \left(1 + \frac{i u_1 f_1 + \dots + i u_q f_q}{\sqrt{N}} \right) \right) \right) \right) \\
&= \mathbb{E} \left(\prod_{j=1}^N \left(1 + \frac{i u_1 f_1(Z_{0:T}^j) + \dots + i u_q f_q(Z_{0:T}^j)}{\sqrt{N}} \right) \right) \\
&= \mathbb{E} \left(\sum_{0 \leq k \leq N} \frac{1}{N^{k/2}} \sum_{1 \leq j_1, \dots, j_k \leq q} i^k u_{j_1} \dots u_{j_k} \right. \\
&\quad \times \sum_{1 \leq i_1 < \dots < i_k \leq N} f_{j_1}(Z_{0:T}^{i_1}) \dots f_{j_k}(Z_{0:T}^{i_k}) \left. \right) \\
&= \mathbb{E} \left(\sum_{0 \leq k \leq N} \frac{\binom{N}{k}}{N^{k/2}} \sum_{1 \leq j_1, \dots, j_k \leq q} i^k u_{j_1} \dots u_{j_k} \frac{1}{k!} (\eta_{0:T}^N)^{\odot k} (f_{j_1} \otimes \dots \otimes f_{j_k}) \right) \\
&= \mathbb{E} \left(\sum_{0 \leq k \leq N} \frac{\binom{N}{k}}{N^{k/2}} \sum_{1 \leq j_1, \dots, j_k \leq q} i^k u_{j_1} \dots u_{j_k} \frac{1}{k!} (\eta_{0:T}^N)^{\odot k} (f_{j_1} \otimes \dots \otimes f_{j_k})_{\text{sym}} \right) \\
&\text{(using Cor. 4.6)} \\
&\xrightarrow{N \rightarrow +\infty} \sum_{k \geq 0, k \text{ even}} (-1)^{k/2} \sum_{1 \leq j_1, \dots, j_k \leq q} \frac{u_{j_1} \dots u_{j_k}}{k!} \sum_{I_k \in \mathcal{I}_k} \prod_{\{a, b\} \in I_k} V_{0:T}(f_a, f_b) \\
&= \sum_{k \geq 0, k \text{ even}} \frac{(-1)^{k/2}}{2^{k/2} (k/2)!} \\
&\quad \times \sum_{1 \leq j_1, \dots, j_k \leq q} u_{j_1} \dots u_{j_k} V_{0:T}(f_{j_1}, f_{j_2}) \dots V_{0:T}(f_{j_{k-1}}, f_{j_k}) \\
&= \sum_{k \geq 0, k \text{ even}} \frac{(-1)^{k/2}}{2^{k/2} (k/2)!} \left(\sum_{1 \leq j_1, j_2 \leq q} u_{j_1} u_{j_2} V_{0:T}(f_{j_1}, f_{j_2}) \right)^{k/2} \\
&= \exp \left(-\frac{1}{2} \sum_{1 \leq j_1, j_2 \leq q} u_{j_1} u_{j_2} V_{0:T}(f_{j_1}, f_{j_2}) \right). \tag{5.3}
\end{aligned}$$

We can also a series development of the log in (??) and write:

$$\begin{aligned}
& \mathbb{E} \left(\exp \left(N \eta_{0:T}^N \left(\log \left(1 + \frac{i u_1 f_1 + \dots + i u_q f_q}{\sqrt{N}} \right) \right) \right) \right) \\
&= \mathbb{E} \left(\exp \left(\sum_{k \geq 1} \frac{(-1)^{k+1}}{k} N^{1-k/2} \eta_{0:T}^N \left((i u_1 f_1 + \dots + i u_q f_q)^k \right) \right) \right). \tag{5.4}
\end{aligned}$$

Let us set $f = \frac{i u_1 f_1 + \dots + i u_q f_q}{\sqrt{N}}$. We have $\|f\|_\infty \leq C/N^{k/2}$ for some constant C . So, we can bound the remaining term in the series development of the log can

be bounded by

$$\left\| \sum_{k \geq 3} \frac{(-1)^{k+1}}{k} N^{1-k/2} (iu_1 f_1 + \dots + iu_q f_q)^k \right\|_\infty \leq \frac{C}{N},$$

for some constant C . So the limit when $N \rightarrow +\infty$ of (5.4) is the same as the limit of

$$\begin{aligned} & \mathbb{E}(\exp(\sqrt{N}(iu_1 \eta_{0:T}^N(f_1) + \dots + iu_q \eta_{0:T}^N(f_q))) \\ & \quad \times \exp\left(\frac{1}{2} \eta_{0:T}^N((u_1 f_1 + \dots + u_q f_q)^2)\right)). \end{aligned}$$

We have

$$\begin{aligned} & |\mathbb{E}(e^{\sqrt{N} \eta_{0:T}^N(f)} e^{-\frac{1}{2} \eta_{0:T}^N(f^2)}) - \mathbb{E}(e^{\sqrt{N} \eta_{0:T}^N(f)} e^{-\frac{1}{2} \tilde{P}_{0:T}(f^2)})| \\ & \leq \left(\mathbb{E}(e^{2\sqrt{N} \eta_{0:T}^N(f)} e^{-\frac{4}{2} \eta_{0:T}^N(f^2)}) \right)^{1/2} \left(\mathbb{E}(e^{\frac{2}{2} \eta_{0:T}^N(f)} (1 - e^{-\frac{1}{2}(\tilde{P}_{0:T}(f^2) - \eta_{0:T}^N(f^2))})^2) \right)^{1/2} \end{aligned} \quad (5.5)$$

and (for some constant which might change from line to line)

$$\begin{aligned} & \mathbb{E}(e^{\frac{2}{2} \eta_{0:T}^N(f)} (1 - e^{-\frac{1}{2}(\tilde{P}_{0:T}(f^2) - \eta_{0:T}^N(f^2))})^2) \\ & \leq C \mathbb{E}((\tilde{P}_{0:T}(f^2) - \eta_{0:T}^N(f^2))^2) \\ & \quad \text{(Equation (5.2))} \xrightarrow{N \rightarrow +\infty} 0. \end{aligned}$$

So, by (5.3), the left-hand side of (5.5) goes to 0 as $N \rightarrow +\infty$. So

$$\begin{aligned} & \lim_{N \rightarrow 0} \exp\left(\sqrt{N} \eta_{0:T}^N \left(\log\left(1 + \frac{iu_1 f_1 + \dots + iu_q f_q}{\sqrt{N}}\right) \right)\right) \\ & = \lim_{N \rightarrow 0} \mathbb{E} \left(e^{\sqrt{N} \eta_{0:T}^N(iu_1 f_1 + \dots + iu_q f_q)} e^{\frac{1}{2} \tilde{P}_{0:T}((u_1 f_1 + \dots + iu_q f_1)^2)} \right), \end{aligned}$$

(meaning that if these limits exist, they are equal) which concludes the proof. \square

6 Technical lemma

For q even, we define

$$N_q = \sum_{j=1}^{q/2} \sum_{k_0=0 < k_1 < \dots < k_j=q, k_1, \dots, k_j \text{ even}} C_q^{k_1} C_{q-k_1}^{q-k_2} \dots C_{q-k_{j-1}}^{q-k_j} I_{k_2-k_1} \dots I_{k_j-k_{j-1}} (-1)^{j+1}.$$

Lemma 6.1. *We have $\forall q \geq 2$ even*

$$N_q = (-1)^{q/2+1} I_q.$$

Proof. We prove the result by recurrence.

- For $q = 2$: $N_2 = -I_2 = (-1)^{q/2+1} I_2$.
- If the result is true $\forall j$ even $\leq q - 2$. We have

$$\begin{aligned} N_q &= I_q - \sum_{2 \leq k_1 \leq q-2, k_1 \text{ even}} C_q^{k_1} I_{k_1} N_{q-k_1} \\ &= I_q + \sum_{2 \leq k_1 \leq q-2, k_1 \text{ even}} (-1)^{\frac{q-k_1}{2}} C_q^{k_1} I_{k_1} I_{q-k_1} \\ &= I_q + \sum_{2 \leq k_1 \leq q, k_1 \text{ even}} (-1)^{\frac{q-k_1}{2}} I_q C_{q/2}^{k_1/2} \\ &= (-1)^{q/2+1} I_q. \end{aligned}$$

□

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