Expansion of the propagation of chaos for Bird and Nanbu systems.

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Abstract

The Bird and Nanbu systems are particle systems used to approximate the solution of the mollified Boltzmann equation. In particular, they have the propagation of chaos property. Following [GM94], we use coupling techniques and results on branching processes to write an expansion of the error in the propagation of chaos in terms of the number of particles, for slightly more general systems than the ones cited above. This result leads to the proof of the a.s convergence and the central-limit theorem for these systems. In particular, we have a central-limit theorem for the empirical measure of the system at a fixed time t under far less stringent assumptions then in [Mel98].

Keywords: interacting particle systems, Boltzmann equation, nonlinear diffusion with jumps, random graphs and trees, coupling, propagation of chaos, Monte Carlo algorithms.

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1 Introduction

In a recent work ([DPR09a]), we showed a expansion of the propagation of chaos for a Feynman-Kac particle system. This particle system approximates a particular Feynman-Kac measure, in the sense that the empirical measure associated to the system converges to the Feynman-Kac measure when the number of particles N goes to ∞. What is called propagation of chaos is the property of the particle system that q particles, amongst the total of N particles, looked upon at a fixed time, are asymptotically independent when $N \to +\infty$ (q is fixed) and their law is converging to the Feynman-Kac law. In [DPR09a], we wrote an expansion in powers of N of the difference between the law of q independent particles, each of them of the Feynman-Kac law, and the law of q particles coming from the particle system. One can also call this expansion a

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functional representation like in [DPR09a]; in the present paper, we call it an expansion of the error in the propagation of chaos. In the setting of [DPR09a], the time is discrete. We showed there how to use this kind of expansion to derive a.s. convergence results (p. 824). In ([DPR09b]), we extend the result of [DPR09a] to the case where the time is continuous, still in the Feynman-Kac framework, and we show central-limit theorems for U-statistics of these systems of particles. The proof of the central-limit theorems for U-statistics relies only on the exploitation of the expansion described above.

We wish here to establish a similar expansion for a family of particles systems including Bird and Nanbu systems (that is, the equivalent of Theorem 1.6 and Corollary 1.8 of [DPR09b]). Bird and Nanbu systems are used to approximate the solution of the mollified Boltzmann equation. We refer mainly to [GM97] and take into account models described in (2.5), (2.6) of [GM97] (a similar description can be found in [GM99], Section 3). An other reference paper on the subject is [GM94]. The two main points of interest of this paper are: it provides a sequel to the estimates on propagation of chaos of [GM97], [GM99] and it allows to apply the results of [DPR09a], [DPR09b] to Bird and Nanbu systems. In particular,

- we get a.s. sure convergence (Th. 5.1) (completely new, based on our knowledge)
- and we get a central-limit theorem for the empirical measure of the system (Th. 5.2) with far less stringent assumptions then in [Mel98] Th. 4.2, 4.3.

The proofs leading to the development in the propagation of chaos are radically different from those in [DPR09b] and this is why we decided to write them in a different paper.

In Section 2, we will recall the definitions of Bird and Nanbu models, as can be found in [GM97] and will give an equivalent definition, useful to our purposes. In Section 3, we will state and prove our main theorem about the expansion of the error in propagation of chaos (Theorem 3.1). The proof relies on estimates on population growth found in [AN72] and on coupling ideas. In Section 4, we prove in what is called a Wick-type formula in [DPR09a] (see (3.6) p. 807 in [DPR09a] and [DPR09b], p.15 and Proposition 4.3), this formula (Corollary 4.4) and Proposition 4.3 are used in Section 5 to prove an a.s. convergence theorem for the empirical measure (Th. 5.1) and central-limit theorem for the emprirical measure (Th. 5.2).

2 Definition of the model

2.1 Bird and Nanbu models

In all the following, we deal with particles evolving in $E := \mathbb{R}^d$. We set the mappings $e_i : h \in \mathbb{R}^d \mapsto e_i(h) = (0, \ldots, 0, h, 0, \ldots, 0) \in \mathbb{R}^{d\times N}$ (h at the i-th rank) ($1 \leq i \leq N$). We have a kernel $\widehat{\mu}(v, w, dh, dk)$ on $\mathbb{R}^{2d}$ which is symmetrical

(that is $\widehat{\mu}(v,w,dh,dk)=\widehat{\mu}(w,v,dk,dh)$). We set $\mu(v,w,dh)$ to be the marginal $\widehat{\mu}(v,w,dh\times\mathbb{R}^d)$. Our assumptions are the same as in [GM97]:

(H1) We are given a Markov generator L on $\mathbb{R}^d$ acting on a sufficiently large domain $\mathcal{D}(L)$ of $L^\infty(\mathbb{R}^d)$. (See [GM97] p. 119 for a discussion on $\mathcal{D}(L)$).

(H2) We suppose $\sup_{x,a}\widehat{\mu}(x,a,\mathbb{R}^d\times\mathbb{R}^d)\leq\Lambda<\infty$.

In Nanbu and Bird systems, he kernel $\widehat{\mu}$ and the generator L have specific features coming from physical considerations, the coordinates in $\mathbb{R}^d$ represent the position and speed of molecules but these considerations have no effect on our proof. This is why we claim to have a proof for systems more general than Bird and Nanbu systems.

The Nanbu and Bird systems are defined in (2.5) and (2.6) of [GM97], by the mean of integrals over Poisson processes. We give here an equivalent definition.

Definition 2.1. *The particle system described in [GM97] is denoted by*

$$(\overline{Z}_t^N)_{t\geq 0}=(\overline{Z}_t^{N,i})_{t\geq 0,1\leq i\leq N}\ .$$

It is a process of N particles in $\mathbb{R}^d$ and can be summarized by the following.

1. *Particles $(\overline{Z}_0^{N,i})_{1\leq i\leq N}$ in $\mathbb{R}^d$ are drawn i.i.d. at time 0 according to a law $\widetilde{P_0}$.*
2. *Between jump times, the particles evolve independently from each other according to L.*
3. *We have a collection $(N_{i,j})_{1\leq i<j\leq N}$ of independent Poisson processes of parameter $\Lambda/(N-1)$. For $i>j$, we set $N_{i,j}=N_{j,i}$. If $N_{i,j}$ has a jump at time t, we say there is an <u>interaction</u> between particles i and j and we take a uniform variable U on $[0,1]$, in dependant of all the other variables, if $U\leq\frac{\widehat{\mu}(\overline{Z}_{t-}^{N,i},\overline{Z}_{t-}^{N,j},\mathbb{R}^{2d})}{\Lambda}$ then the system undergoes a <u>jump</u>:*
   - *In the Bird system: $\overline{Z}_t^N=\overline{Z}_{t-}^N+e_i(H)+e_j(K)$ with*
   $$(H,K)\sim\frac{\widehat{\mu}(\overline{Z}_{t-}^{N,i},\overline{Z}_{t-}^{N,j},.,.)}{\widehat{\mu}(\overline{Z}_{t-}^{N,i},\overline{Z}_{t-}^{N,j},\mathbb{R}^{2d})}\qquad(2.1)$$
   (independently of all the other variables).
   - *If we replace $\widehat{\mu}$ by $\widehat{\mu'}(z,a,dh,dg)=\mu(z,a,dh)\otimes\delta_0(dg)+\delta_0(dh)\otimes\mu(a,z,dg)$ in (2.1), we obtain the Nanbu system (cf. Remark 2.6, p. 120, [GM97])*

Theorem 3.1 of [GM97] implies that there is propagation of chaos for this system. This theorem says $(\forall q,t)$:

$$\|\mathcal{L}(\overline{Z}_t^{N,1},\dots,\overline{Z}_t^{N,q})-\mathcal{L}(\overline{Z}_t^{N,1})^{\otimes q}\|_{TV}\leq 2q(q-1)\frac{\Lambda t+\Lambda^2t^2}{N-1}\ ,$$

and

$$\|\mathcal{L}(\overline{Z}_t^{N,1}) - \widetilde{P}_t\|_{TV} \leq 6\frac{e^{\Lambda t}-1}{N+1} \ ,$$

where $(\widetilde{P}_t)$ is solution of (with $\widetilde{P}_0$ fixed)

$$\begin{aligned}
&\forall \phi \in \mathcal{D}(L), \\
&\partial_t \langle \phi, \widetilde{P}_t \rangle - \langle L\phi, \widetilde{P}_t \rangle \\
&= \left\langle \int \frac{1}{2}(\phi(z+h) - \phi(z) + \phi(a+k) - \phi(a))\widehat{\mu}(z,a,dh,dk), \widetilde{P}_t(dz)\widetilde{P}_t(da) \right\rangle
\end{aligned}$$

We can deduce propagation of chaos from the previous results, that is $\forall t$, $\forall F$ bounded measurable,

$$|\mathcal{L}(\overline{Z}_t^{N,1}, \ldots, \overline{Z}_t^{N,q})(F) - \widetilde{P}_t^{\otimes q}(F)| \leq \left(2q(q-1)\frac{\Lambda t + \Lambda^2 t^2}{N-1} + 6\frac{e^{\Lambda t}-1}{N+1} \right) \|F\|_\infty \ .$$

In Theorem 3.1, we will go further than the above bound by writing an expansion of the left hand side term above in powers of N. We will use techniques introduced in [GM97]. The main point is that we look at the processes backward in time.

2.2 Backward point of view

From now on, we will work with a fixed time horizon $T > 0$. For any $j \in \mathbb{N}^*$, we set $[j] = \{1, \ldots, j\}$. For $\lambda > 0$, we call $\mathcal{E}(\lambda)$ the exponential law of parameter λ.

We fix $q \in \mathbb{N}^*$. We start in $s = 0$ with $C_0^i = \{i\}$, $\forall i \in [q]$. We set $\forall i$, $K_0^i = \#C_0^i$. For $1 \leq i < j \leq N$, we define processes $(N_s^{i,j} = N_s^{j,i})_{s \geq 0}$ and for $i \in [q]$, we define $(C_s^i)_{s \geq 0}$, $(K_s^i)_{s \geq 0}$ (respectively in $\mathbb{N}$, $\mathcal{P}(\mathbb{N})$, $\mathbb{N}^*$) by the following. For all $s \in [0,T]$, we set

$$K_s = \#(C_s^1 \cup \cdots \cup C_s^q) \ .$$

The processes $(N^{i,j}), (C^i), (K^i)$ are piecewise constant and make jumps. We define the jump times recursively by (taking $(U_k)_{1 \leq i \leq q, 1 \leq k}, (V_k)_{1 \leq i \leq q, 1 \leq k}$ i.i.d. $\sim \mathcal{E}(1)$) $T_0 = 0$ and (always with the convention $\inf \emptyset = +\infty$ and $(\ldots)_+$ standing for the nonnegative part).

$$\begin{aligned}
T_k' &= \inf \left\{ T_{k-1} \leq s \leq T : \int_{T_{k-1}}^s \frac{\Lambda K_u (N - K_u)_+}{N-1} du \geq U_k \right\} \\
T_k'' &= \inf \left\{ T_{k-1} \leq s \leq T : \int_{T_{k-1}}^s \frac{\Lambda K_u (K_u - 1)}{2(N-1)} du \geq V_k \right\} \\
T_k &= \inf(T_k', T_k'') \ .
\end{aligned}$$

In T_k:

- If $T_k = T'_k$, we take $r(k)$ uniformly in $C^1_{T_k-} \cup \cdots \cup C^q_{T_k-}$ and $j(k)$ uniformly in $[N]\backslash(C^1_{T_k-} \cup \cdots \cup C^q_{T_k-})$. For any i such that that $r(k) \in C^i_{T_k-}$, we then perform the jumps: $C^i_{T_k} = C^i_{T_k-} \cup \{j(k)\}$ and $N^{r(k),j(k)}(T_k) = N^{r(k),j(k)}(T_k-) + 1$.

  Notice that the $(\dots)_+$ in the definition of T'_k above forbids to be in the situation where we would be looking for $j(k)$ in $\emptyset$.

- If $T_k = T''_k$, we take $r(k)$ uniformly in $C^1_{T_k-} \cup \cdots \cup C^q_{T_k-}$ and $j(k)$ uniformly in $C^1_{T_k-} \cup \cdots \cup C^q_{T_k-} \backslash \{r(k)\}$. For any i, i' such that $r(k) \in C^i_{T_k-}$ and $j(k) \in C^{i'}_{T_k-}$, we perform the jumps: $C^i_{T_k} = C^i_{T_k-} \cup \{j(k)\}$, $C^{i'}_{T_k} = C^{i'}_{T_k-} \cup \{r(k)\}$ and $N^{r(k),j(k)}(T_k) = N^{r(k),j(k)}(T_k-) + 1$.

For all $i,\ t$, we set $K^i_t = \#C^i_t$. This whole construction is analogous to the construction of the interaction graph found in [GM97], p. 122.

We now define an auxiliary process $(Z^N_s)_{0\le s\le T} = (Z^{N,i}_s)_{0\le s\le T, 1\le i\le N}$ of N particles in $\mathbb{R}^d$.

Definition 2.2. *The interaction times of the $(Z^{N,i}_s)_{1\le s\le T, 1\le i\le N}$ are $\{T - T_k, k \ge 1, T_k \le T\}$. (We say that the times (T_k) are defined backward in time.)*

- *$Z^{N,1}_0, \dots, Z^{N,N}_0$ are i.i.d. $\sim \widetilde{P}_0$*
- *Between the times $(T - T_k)_{k\ge 1}$, the $Z^{N,i}$'s evolve independently from each other according to the Markov generator L.*
- *At a jump time $T - T_k$ where T_k is a jump time of $N^{i,j}$, (Z^N) undergoes an interaction having the same law as in Definition 2.1, (3.).*

Definition 2.3. *For all $t \ge 0$, we set*

$$L_t = \#\{k \in \mathbb{N} : T_k \le t, T_k = T''_k\} \ .$$

We call this quantity the number of loops on $[0, t]$.

Example 2.4. *Take $q = 2$. Suppose for example, that the only jumps of the $N^{i,j}$'s occurring in $[0, T]$ are*

$$\Delta N^{2,3}(2T/3) = 1 \ , \ \Delta N^{1,2}(T/3) = 1$$

then

- *for $s \in [0, T/3[$, $K_s = 2$, $L_s = 0$, $K^1_s = K^2_s = 1$,*
- *for $s \in [T/3, 2T/3[$, $K_s = 2$, $L_s = 1$, $K^1_s = K^2_s = 2$,*
- *for $s \in [2T/3, T]$, $K_s = 3$, $L_s = 1$, $K^1_s = 2$, $K^2_s = 3$.*

We have to keep in mind the following lemma throughout the whole paper.

Lemma 2.5. *1. If $Y_1, \dots, Y_k$ are independent and respectively of law $\mathcal{E}(\lambda_1)$, $\dots$,$\mathcal{E}(\lambda_k)$ (λ_1 , $\dots$, $\lambda_k > 0$, $k \in \mathbb{N}^*$) then*

$$\inf(Y_1, \dots, Y_k) \sim \mathcal{E}(\lambda_1 + \dots + \lambda_k) \ ,$$

$$\mathbb{P}(Y_1 = \inf(Y_1, \dots, Y_k)) = \frac{\lambda_1}{\lambda_1 + \dots + \lambda_k} \ .$$

2. For $Y \sim \mathcal{E}(\lambda)$ ($\lambda > 0$), $\mathcal{L}(Y - t | Y \geq t) = \mathcal{E}(\lambda)$ (for any $t \geq 0$).

3. If $Y_1, Y_2, \dots$ i.i.d. $\sim \mathcal{E}(\lambda)$ ($\lambda > 0$), $k \in \mathbb{N}^*$, $U_1, U_2, \dots, U_k$ is the order statistics of k uniform random variables on $[0, T]$ then

$$\begin{aligned} &\mathcal{L}\left((Y_1, Y_1 + Y_2, \dots, Y_1 + \dots + Y_k)\right. \\ &\left.|Y_1 + \dots + Y_k < T \leq Y_1 + \dots + Y_{k+1}\right) \\ &= \mathcal{L}((U_1, \dots, U_k)) \end{aligned}$$

We then have:

Lemma 2.6. *For all $T \geq 0$, $(Z_T^{N,1}, \dots, Z_T^{N,q}) \overset{law}{=} (\overline{Z}_T^{N,1}, \dots, \overline{Z}_T^{N,q})$.*

The system $(Z_s^N)_{0 \leq s \leq Y}$ is of use in Section 4 but is also useful to understand the next auxiliary process, which we use in Section 3. We now define, for a fixed time horizon $T \geq 0$, the auxiliary process $(\widehat{Z}_s^N)_{0 \leq s \leq T} = (\widehat{Z}_{0 \leq s \leq T}^{N,i})_{1 \leq i \leq N, 0 \leq s \leq T}$.

We start in $s = 0$, with $\widehat{C}_0 = [q]$, $\widehat{K}_0 = q$, $\widehat{L}_0 = 0$. We define processes $(\widehat{C}_s)_{s \geq 0}$, $(\widehat{K})_{s \geq 0}$, $(\widehat{L}_s)_{s \geq 0}$ (respectively in $\mathcal{P}(\mathbb{N})$, $\mathbb{N}^*$, $\mathbb{N}^*$). These processes are piecewise constant and make jumps. We take $(\widehat{U}_k)_{k \geq 0}$ to be i.i.d. $\sim \mathcal{E}(1)$ and $(\widehat{A}_k)_{k \geq 0}$ to be i.i.d. $\sim \mathcal{U}([0,1])$ (these variables are independent of all the other variables). We define recursively the jump times $(\widehat{T}_k)_{k \geq 0}$ by $\widehat{T}_0 = 0$ and

$$\widehat{T}_{k+1} = \inf \left\{ s \geq \widehat{T}_k : \int_{\widehat{T}_k}^s \frac{(2(N - \widehat{K}_u)_+ + \widehat{K}_u - 1)}{2(N-1)} \Lambda \widehat{K}_u du \geq \widehat{U}_k \right\}$$

In $\widehat{T}_k$:

- If $\widehat{A}_k \leq \frac{2(N - \widehat{K}_u)_+}{2(N - \widehat{K}_u)_+ + \widehat{K}_u - 1}$ then we perform the following jump: $\widehat{K}_{\widehat{T}_k} = \widehat{K}_{\widehat{T}_k -} + 1$, $\widehat{L}_{\widehat{T}_k} = \widehat{L}_{\widehat{T}_k -}$, we choose $\widehat{i}(k)$ uniformly in $[N] \backslash \widehat{C}_{\widehat{T}_k -}$ and $\widehat{C}_{\widehat{T}_k} = \widehat{C}_{\widehat{T}_k -} \cup \{\widehat{i}(k)\}$. We choose $\widehat{j}(k)$ uniformly in $\widehat{C}_{\widehat{T}_k -}$.

- If $\widehat{A}_k > \frac{2(N - \widehat{K}_u)_+}{2(N - \widehat{K}_u)_+ + \widehat{K}_u - 1}$ then we perform the following jump: $\widehat{K}_{\widehat{T}_k} = \widehat{K}_{\widehat{T}_k -}$, $\widehat{L}_{\widehat{T}_k} = \widehat{L}_{\widehat{T}_k -} + 1$, $\widehat{C}_{\widehat{T}_k} = \widehat{C}_{\widehat{T}_k -}$. We choose $\widehat{i}(k) \neq \widehat{j}(k)$ uniformly in $\widehat{C}_{\widehat{T}_k -}$.

Definition 2.7. *The interaction times of the $(\widehat{Z}_s^{N,i})_{1 \leq s \leq T, 1 \leq i \leq N}$ are $\{T - \widehat{T}_k, k \geq 1, \widehat{T}_k \leq T\}$. (This is why we say that the times $(\widehat{T}_k)$ are defined backward in time.)*

- $\widehat{Z}_0^{N,1}, \dots, \widehat{Z}_0^{N,N}$ *are i.i.d.* $\sim \widetilde{P}_0$
- *Between the interaction times* $(\widehat{T}_k)_{k\geq 1}$*, the* $\widehat{Z}^{N,i}$ *'s evolve independently from each other according to the Markov generator* L*.*
- *At an interaction time* $T - \widehat{T}_k$*,* $(\widehat{Z}^N)$ *undergoes an interaction having the same law as in Definition 2.1, (3.), with* i, j *replaced by* $\widehat{i}(k), \widehat{j}(k)$*.*

Keeping in mind Lemma 2.5, we get:

Lemma 2.8. *For all* $T \geq 0$*,*

$$(Z_T^1, \dots, Z_T^q) \overset{law}{=} (\widehat{Z}_T^1, \dots, \widehat{Z}_T^q) \ ,$$

$$(K_t, L_t)_{0\leq t\leq T} \overset{law}{=} (\widehat{K}_t, \widehat{L}_t)_{0\leq t\leq T} \ .$$

3 Expansion of the propagation of chaos

We define for any $N, q \in \mathbb{N}^*$, $q \leq N$:

$$\langle q, N\rangle = \{a : [q] \to [N], a \text{ injective } \} \ , (N)_q = \#\langle q, N\rangle = \frac{N!}{(N-q)!} \ .$$

Let us set

$$\eta_t^N = \frac{1}{N} \sum_{1\leq i\leq N} \delta_{\overline{Z}_t^{N,i}} \ ,$$

$$(\eta_t^N)^{\odot q} = \frac{1}{(N)_q} \sum_{a\in\langle q,N\rangle} \delta_{(\overline{Z}_t^{N,a(1)}, \dots, \overline{Z}_t^{N,a(q)})} \ .$$

For any function $F : \mathbb{R}^{qd} \to \mathbb{R}$, we call $(\eta_t^N)^{\odot q}(F)$ a U-statistics. Notice that for all function F,

$$\mathbb{E}(F(\overline{Z}_t^{N,1}, \dots, \overline{Z}_t^{N,q})) = \mathbb{E}((\eta_t^N)^{\odot q}(F)) \ . \tag{3.1}$$

We define

$$F_{\text{sym}}(x^1, \dots, x^q) = \frac{1}{q!} \sum_{\sigma\in\mathcal{S}_q} F(x^{\sigma(1)}, \dots, x^{\sigma(q)}) \ ,$$

where the sum is taken over the set $\mathcal{S}_q$ of the permutations of $[q]$. Notice that

$$(\eta_t^N)^{\odot q}(F) = (\eta_t^N)^{\odot q}(F_{\text{sym}}) \ .$$

Theorem 3.1. *Set $\alpha = e^{-\Lambda T}$. For all $q \geq 1$, for any bounded measurable F, $\forall T \geq 0$, $\forall l_0 \geq 1$,*

$$\begin{aligned}\mathbb{E}((\eta_T^N)^{\odot q}(F)) &= \widetilde{P}_T^{\otimes q}(F)\mathbb{P}(L_T = 0) \\ &\quad + \sum_{1 \leq l \leq l_0} \left[\frac{1}{(N-1)^l}\Delta_{q,T}^{N,l}(F)\right] + \frac{1}{(N-1)^{l_0+1}}\overline{\Delta}_{q,T}^{N,l_0+1}(F),\end{aligned} \tag{3.2}$$

where the $\Delta_{q,T}^{N,l}$, $\overline{\Delta}_{q,T}^{N,l_0+1}$ are nonnegative measures uniformly bounded in N defined by, for any bounded measurable F,

$$\Delta_{q,T}^{N,l}(F) = \mathbb{E}(F(Z_T^{N,1}, \ldots, Z_T^{N,q})|L_T = l)\mathbb{P}(L_T = l)(N-1)^l$$

$$\overline{\Delta}_{q,T}^{N,l}(F) = \mathbb{E}(F(Z_T^{N,1}, \ldots, Z_T^{N,q})|L_T \geq l)\mathbb{P}(L_T \geq l)(N-1)^l \ .$$

We further have the following bounds ($\forall F \in C_b^+(\mathbb{R}^{qd})$)

$$\begin{aligned}&\sup(\Delta_{q,T}^{N,l}(F), \overline{\Delta}_{q,T}^{N,l}(F)) \\ &\quad \leq \frac{q(1-\alpha)^{1/q-1}}{l!}\frac{(2l+1)!}{(1-(1-\alpha)^{1/q})^{2l+2}} \times \|F\|_\infty \\ &\qquad + \frac{q}{(1-(1-\alpha)^{1/q})}\sup_{N \geq 1}\left((1-\alpha)^{\lfloor\sqrt{N-1}\rfloor/q-1}(N-1)^l\right)\|F\|_\infty \ .\end{aligned}$$

Let us define $\mathbb{P}_{T,q}^N(F) = \mathbb{E}((\eta_T^N)^{\odot q}(F))$. Using the terminology of [DPR09a], p. 782, we cannot say that the sequence of measure $(\mathbb{P}_{T,q}^N)_{N \geq 1}$ is differentiable up to any order because the $\Delta_{q,T}^{N,l}$ appearing in the development depend on N.

For a specific class of functions F, we will compute what is the order of $\Delta_{q,T}^{N,\lfloor q/2 \rfloor}(F)$ (Proposition 4.3, Corollary 4.4). This last result, together with the boundedness of the $\Delta_{q,T}^{N,l}$, is what we need on the proofs of Section 5.

Proof. According to [GM97] (section 3.4, p. 124) or, equivalently [GM94] (section 5),

$$\mathbb{E}(F(Z_T^{N,1}, \ldots, Z_T^{N,q})|L_T = 0) = \widetilde{P}_T(F) \ .$$

We have by Lemma 2.6, $\forall l_0$:

$$\begin{aligned}\mathbb{E}(F(\overline{Z}_T^{N,1}, \ldots, \overline{Z}_T^{N,q})) &= \mathbb{E}(F(Z_T^{N,1}, \ldots, Z_T^{N,q})|L_T = 0)\mathbb{P}(L_T = 0) \\ &\quad + \sum_{l=1}^{l_0}[\mathbb{E}(F(Z_T^{N,1}, \ldots, Z_T^{N,q})|L_T = l)\mathbb{P}(L_T = l)] \\ &\quad + \mathbb{E}(F(Z_T^{N,1}, \ldots, Z_T^{N,q})|L_T \geq l_0 + 1) \\ &\qquad \times \mathbb{P}(L_T \geq l_0 + 1).\end{aligned}$$

It is sufficient for the proof of (3.2) to show that $\mathbb{P}(L_T \geq l)$ is of order $\leq 1/N^l$, $\forall l \in \mathbb{N}^*$.

We define piecewise constant processes $(\widehat{K}'_s)_{s\geq 0}$, $(\widehat{L}'_s)_{s\geq 0}$ (in $\mathbb{N}$) such that $\widehat{K}'_0 = q$, $\widehat{L}'_0 = 0$. Their jump times are $(\widehat{T}'_k)_{k\geq 0}$ defined recursively by $\widehat{T}'_0 = 0$ and

$$\widehat{T}'_k = \inf\left\{ s : \int_{\widehat{T}'_{k-1}}^{s} \Lambda \widehat{K}'_u du \geq \widehat{U}_k \right\} .$$

In $\widehat{T}'_k$:

- If $\widehat{A}_k \leq \frac{(N-\widehat{K}'_{\widehat{T}'_k-})_+}{N-1}$, then we perform the following jump: $\widehat{K}'_{\widehat{T}'_k} = \widehat{K}'_{\widehat{T}'_k-}+1$, $\widehat{L}'_{\widehat{T}'_k} = \widehat{L}'_{\widehat{T}'_k-}$.
- If $\widehat{A}_k > \frac{(N-\widehat{K}'_{\widehat{T}'_k-})_+}{N-1}$, then we perform the following jump: $\widehat{K}'_{\widehat{T}'_k} = \widehat{K}'_{\widehat{T}'_k-}+1$, $\widehat{L}'_{\widehat{T}'_k} = \widehat{L}'_{\widehat{T}'_k-} + 1$.

Notice that we use there the same variables $\widehat{U}_k$'s and $\widehat{A}_k$'s coming from the definition of $(\widehat{Z}^N)$. We have:

$$\begin{aligned}
&\text{as } \forall t,\ \omega,\ \frac{2(N-\widehat{K}_t(\omega))_+ + \widehat{K}_t(\omega) - 1}{2(N-1)} \leq 1 \\
&\qquad \text{then } \forall t,\ \omega,\ \widehat{K}_t(\omega) \leq \widehat{K}'_t(\omega) , \\
&\text{as } \forall t,\ \omega,\ \frac{2(N-\widehat{K}_t(\omega))_+}{2(N-\widehat{K}_t(\omega))_+ + \widehat{K}_t(\omega) - 1} \geq \frac{(N-\widehat{K}'_t(\omega))_+}{N-1} \\
&\qquad \text{then } \ \forall t,\ \omega,\ \widehat{L}_t(\omega) \leq \widehat{L}'_t(\omega) , \\
&\forall t,\ \omega,\ \widehat{L}'_t(\omega) \leq \widehat{K}'_t(\omega) .
\end{aligned} \tag{3.3}$$

The process $(\widehat{K}'_s)_{s\geq 0}$ is equal in law to the sum of q in dependant Yule processes $Y_s^{(1)}, \ldots, Y_s^{(q)}$ and its law is thus independent of N (see [AN72], p. 102-109, p. 109 for the law of the Yule process). We have $\mathbb{P}(Y_s^{(1)} = k) = e^{-s\Lambda}(1-e^{-s\Lambda})^{k-1}$ and so:

$$\begin{aligned}
\mathbb{P}(\widehat{K}'_t = k) &= \mathbb{P}(Y_t^{(1)} + \cdots + Y_t^{(q)} = k) \\
&\leq \sum_{i=1}^{q} \mathbb{P}(Y_t^{(i)} \geq \lceil k/q \rceil) \\
&\leq q(1-e^{-t\Lambda})^{k/q-1} .
\end{aligned} \tag{3.4}$$

Notice that $\forall t \in [0,T]$

$$\mathbb{P}(\Delta\widehat{L}'_t = 1 | \Delta\widehat{K}'_t = 1, (\widehat{K}'_t)_{t\geq 0}) = 1 - \frac{(N-\widehat{K}'_{t-})_+}{N-1} \leq \frac{\widehat{K}'_T}{N-1} . \tag{3.5}$$

We decompose

$$\begin{aligned}
\mathbb{P}(L_T = l) &\leq \mathbb{P}(L_T \geq l) \\
\text{(by Lem. 2.8)} \quad &= \mathbb{P}(\widehat{L}_T \geq l) \\
&\leq \mathbb{P}(\widehat{L}'_T \geq l) \\
&\leq \mathbb{P}(\widehat{K}'_T \geq \lfloor\sqrt{N-1}\rfloor) + \mathbb{P}(\ \widehat{L}'_T \geq l, \widehat{K}'_T \leq \lfloor\sqrt{N-1}\rfloor)\ ,
\end{aligned}$$

and we compute

$$\begin{aligned}
&\mathbb{P}(\widehat{L}'_T \geq l, \widehat{K}'_T \leq \lfloor\sqrt{N-1}\rfloor) \\
&\quad = \sum_{l\leq r\leq \lfloor\sqrt{N-1}\rfloor} \mathbb{P}(\widehat{L}'_T = r, \widehat{K}'_T \leq \lfloor\sqrt{N-1}\rfloor) \\
&\quad = \sum_{l\leq r\leq \lfloor\sqrt{N-1}\rfloor} \sum_{r\leq k\leq \lfloor\sqrt{N-1}\rfloor} \mathbb{P}(\widehat{L}'_T = r|\widehat{K}'_T = k)\mathbb{P}(\widehat{K}'_T = k) \\
\text{by (3.4), (3.5)} \quad &\leq \sum_{l\leq r\leq \lfloor\sqrt{N-1}\rfloor} \sum_{r\leq k\leq \lfloor\sqrt{N-1}\rfloor} C_k^r \left(\frac{k}{N-1}\right)^r q(1-\alpha)^{k/q-1} \\
&\quad \leq \sum_{l\leq k\leq \lfloor\sqrt{N-1}\rfloor} \sum_{l\leq r\leq k} \frac{k^{2r}}{r!(N-1)^r} q(1-\alpha)^{k/q-1}\ .
\end{aligned}$$

As, for $k \leq \lfloor\sqrt{N-1}\rfloor$, $\frac{k^2}{N-1} \leq 1$, we get

$$\begin{aligned}
\mathbb{P}(\widehat{L}'_T \geq l, \widehat{K}'_T \leq \lfloor\sqrt{N-1}\rfloor) &\leq \sum_{l\leq k\leq \lfloor\sqrt{N-1}\rfloor} k\frac{k^{2l}}{l!(N-1)^l} q(1-\alpha)^{k/q-1} \\
&\leq \frac{q}{l!(N-1)^l} \sum_{l\leq k} k(k+1)(k+2)\dots(k+2l) \\
&\qquad \times((1-\alpha)^{1/q})^{k-1}(1-\alpha)^{1/q-1} \\
&\leq \frac{q(1-\alpha)^{1/q-1}}{l!(N-1)^l} \frac{(2l+1)!}{(1-(1-\alpha)^{1/q})^{2l+2}}\ .
\end{aligned}$$

We also have

$$\begin{aligned}
\mathbb{P}(\widehat{K}'_T \geq \lfloor\sqrt{N-1}\rfloor) &\leq \sum_{k\geq \lfloor\sqrt{N-1}\rfloor} q(1-\alpha)^{k/q-1} \\
&= \frac{q(1-\alpha)^{\lfloor\sqrt{N-1}\rfloor/q-1}}{(1-(1-\alpha)^{1/q})} \\
&\leq \frac{q}{(1-(1-\alpha)^{1/q})} \\
&\qquad \times \sup_{N\geq 1} \left((1-\alpha)^{\lfloor\sqrt{N-1}\rfloor/q-1}(N-1)^l\right) \frac{1}{(N-1)^l}\ .
\end{aligned}$$

□

4 Wick formula

We now define an auxiliary system $(\widetilde{Z}_t^i)_{0\le t\le T, i\ge 1}$ with an infinite number of particles. We start in $s=0$ with $\widetilde{C}_0^i=\{i\}$, $\forall i\in[q]$. We set $\forall i$, $\widetilde{K}_0^i=\#\widetilde{C}_0^i$. For $1\le i<j\le N$, we define processes $(\widetilde{N}_s^{i,j}=\widetilde{N}_s^{j,i})_{s\ge 0, i,j\in\mathbb{N}^*}$, $(\widetilde{C}_s^i)_{s\ge 0, 1\le i\le q}$, $(\widetilde{K}_s^i)_{s\ge 0, 1\le i\le q}$ (respectively in $\mathbb{N},\mathcal{P}(\mathbb{N}),\mathbb{N}$) by the following. The processes $(\widetilde{N}^{i,j}),(\widetilde{C}^i),(\widetilde{K}^i)$ are piecewise constant. We set

$$\widetilde{K}_s=\#(\widetilde{C}_s^1+\cdots+\widetilde{C}_s^q)\ .$$

We define the jump times recursively by (taking $(\widetilde{U}_k)_{1\le i\le q,1\le k}$ i.i.d. $\sim\mathcal{E}(1)$), $\widetilde{T}_0=0$ and

$$\begin{aligned}\widetilde{T}_k' &= \inf\left\{\widetilde{T}_{k-1}\le s\le T:\int_{\widetilde{T}_{k-1}}^s \Lambda\widetilde{K}_u-\frac{\Lambda K_u(N-K_u)_+}{N-1}du\ge\widetilde{U}_k\right\}\\ \widetilde{T}_k &= \inf(\widetilde{T}_k',\inf\{T_l:T_l>\widetilde{T}_{k-1}\})\end{aligned}$$

(recall that the process (K_t) and the T_k's are defined in Subsection 2.2). Notice that $\{T_k,k\ge 0\}\subset\{\widetilde{T}_k,k\ge 0\}$. In $\widetilde{T}_k$:

- If $\widetilde{T}_k\notin\{T_l'',l\ge 1\}$, we take $\widetilde{r}(k)$ uniformly in $\widetilde{C}_{T_k-}^1\cup\cdots\cup\widetilde{C}_{T_k-}^q$ and $\widetilde{j}(k)$ uniformly in $\mathbb{N}^*\backslash(\widetilde{C}_{T_k-}^1\cup\cdots\cup\widetilde{C}_{T_k-}^q)$. For any i such that $\widetilde{r}(k)\in\widetilde{C}_{T_k-}^i$, we perform the jumps: $\widetilde{C}_{T_k}^i=\widetilde{C}_{T_k-}^i\cup\{\widetilde{j}(k)\}$, $\widetilde{K}_{T_k}^i=\widetilde{K}_{T_k-}^i+1$ and $\widetilde{N}_{T_k}^{\widetilde{r}(k),\widetilde{j}(k)}=\widetilde{N}_{T_k-}^{\widetilde{r}(k),\widetilde{j}(k)}+1$.
- If $\widetilde{T}_k\in\{T_l'',l\ge 1\}$, we take $\widetilde{r}(k)$ uniformly in $\widetilde{C}_{T_k-}^1\cup\cdots\cup\widetilde{C}_{T_k-}^q$ and $\widetilde{j}(k)$ uniformly in $\widetilde{C}_{T_k-}^1\cup\cdots\cup\widetilde{C}_{T_k-}^q\backslash\{\widetilde{r}(k)\}$. For any i,i' such that $\widetilde{r}(k)\in\widetilde{C}_{T_k-}^i$, $\widetilde{j}(k)\in\widetilde{C}_{T_k-}^{i'}$, we perform the jumps: $\widetilde{C}_{T_k}^i=\widetilde{C}_{T_k-}^i\cup\{\widetilde{j}(k)\}$, $\widetilde{K}_{T_k}^i=\widetilde{K}_{T_k-}^i+1$, $\widetilde{C}_{T_k}^{i'}=\widetilde{C}_{T_k-}^{i'}\cup\{\widetilde{r}(k)\}$, $\widetilde{K}_{T_k}^{i'}=\widetilde{K}_{T_k-}^{i'}+1$ and $\widetilde{N}_{T_k}^{\widetilde{r}(k),\widetilde{j}(k)}=\widetilde{N}_{T_k-}^{\widetilde{r}(k),\widetilde{j}(k)}+1$

Notice that $\forall i\in[q]$, the processes $\widetilde{K}^i$ and $\#\widetilde{C}^i$ are not equal. The following lemma is a consequence of Lemma 2.5.

Lemma 4.1. *1. The process $(\widetilde{K}_s)_{s\ge 0}$ is piecewise constant, has jumps of size 1 and satisfies $\forall 0\le s\le t$*

$$\mathbb{P}(\widetilde{K}_t=\widetilde{K}_s|\widetilde{K}_s)=\exp(-\Lambda\widetilde{K}_s)\ .$$

And so it has the same law as $(\widehat{K}_s')_{s\ge 0}$.

2. For all t, $\widetilde{K}_t\ge K_t$, a.s.

3. If $T_1=\widetilde{T}_1,\ \ldots,\ T_k=\widetilde{T}_k$ then $\widetilde{K}_{T_k}=K_{T_k}$.

Definition 4.2. *The interaction times of the $\widetilde{Z}^i$ are $\{T - \widetilde{T}_k, k \geq 1\}$ (we say they are defined backward in time).*

- *The $(\widetilde{Z}_0^i)$ are i.i.d. $\sim \widetilde{P}_0$.*
- *Between the jump times, the $\widetilde{Z}^i$ evolve independently from each other according to the Markov generator L.*
- *At a jump time $T - \widetilde{T}_k$, $(\widetilde{Z})$ undergo a jump like in Definition 2.1, (3), with i, j replaced by $\widetilde{r}(k), \widetilde{j}(k)$.*

We define $(\forall t \geq 0)$ the event G and the trajectories $\mathcal{K}_t$, $\widetilde{\mathcal{K}}_t$ by

$$\begin{aligned} G &= \{\forall k \geq 1 \text{ such that } \widetilde{T}_k \leq T,\ \widetilde{T}_k = T_k\} \\ \mathcal{K}_t &= \{(K_s^i)_{1 \leq s \leq t}, i \in [q]\} \\ \widetilde{\mathcal{K}}_t &= \{(\widetilde{K}_s^i)_{1 \leq s \leq t}, i \in [q]\} \end{aligned}$$

and, for q even, we set $\{T_{k_1} \leq T_{k_2} \leq \dots\} = \{T_k : k \geq 1, T_k = T_k''\}$, $\{\widetilde{T}_{\widetilde{k}_1} \leq \widetilde{T}_{\widetilde{k}_2} \leq \dots\} = \{\widetilde{T}_k : k \geq 1, \exists l, \widetilde{T}_k = T_l''\}$,

$$\begin{aligned} A &= \{\#\{k \geq 1 : T_k = T_k'', T_k < T\} = q/2\} \\ &\quad \cap \{r(k_1) \in C^1_{T_{k_1}-}, j(k_1) \in C^2_{T_{k_1}-}, \dots \\ &\qquad \dots, r(k_{q/2}) \in C^{q-1}_{T_{k_{q/2}}-}, j(k_{q/2}) \in C^q_{T_{k_{q/2}}-}\} \\ \widetilde{A} &= \{\#\{k \geq 1 : \exists l, \widetilde{T}_k = T_l'', \widetilde{T}_k < T\} = q/2\} \\ &\quad \cap \{\widetilde{r}(\widetilde{k}_1) \in \widetilde{C}^1_{\widetilde{T}_{\widetilde{k}_1}-}, \widetilde{j}(\widetilde{k}_1) \in \widetilde{C}^2_{\widetilde{T}_{\widetilde{k}_1}-}, \dots \\ &\qquad \dots, \widetilde{r}(\widetilde{k}_{q/2}) \in \widetilde{C}^{q-1}_{\widetilde{T}_{\widetilde{k}_{q/2}}-}, \widetilde{j}(\widetilde{k}_{q/2}) \in \widetilde{C}^q_{\widetilde{T}_{\widetilde{k}_{q/2}}-}\} \end{aligned}$$

(recall the $i(k)$'s, $r(k)$'s are defined in Subsection 2.2). For $q \in \mathbb{N}^*$, we say that $F : (\mathbb{R}^d)^q \to \mathbb{R}$ is symmetric if for all σ in the set of permutation of $[q]$ (denoted by $\mathcal{S}_q$), $\forall x_1, \dots, x_q \in \mathbb{R}^d$, $F(x_{\sigma(1)}, \dots, x_{\sigma(q)}) = F(x_1, \dots, x_q)$. We define

$$\mathcal{B}_0^{sym}(q) = \Big\{ F : (\mathbb{R}^d)^q \to \mathbb{R}, F \text{ measurable, symmetric, bounded,} \int_{x_1, \dots, x_q \in \mathbb{R}^d} F(x_1, \dots, x_q) \widetilde{P}_T(dx_q) = 0 \Big\} .$$

Proposition 4.3. *For $F \in \mathcal{B}_0^{sym}(q)$, we have:*

- *for $k < \frac{q}{2}$, $\Delta_{q,T}^{N,k}(F) = 0$,*
- *for q even,*

$$N^{q/2} \mathbb{E}((\eta_T^N)^{\odot q}(F)) \underset{N \to +\infty}{\longrightarrow} \frac{q!}{2^{q/2}(q/2)!} \mathbb{E}\Big(\mathbb{E}(F(\widetilde{Z}_T^1, \dots, \widetilde{Z}_T^q)) | \widetilde{\mathcal{K}}_t, \widetilde{A}) \times \prod_{1 \leq i \leq q/2} \int_0^T \Lambda \widetilde{K}_s^{2i-1} \widetilde{K}_s^{2i} ds \Big) . \quad (4.1)$$

Proof. In the following computations, we set $R = \sup\{k : \widetilde{T}_k \leq T\}$. We have

$$\begin{aligned}\mathbb{E}(G|\mathcal{K}_T) &= \mathbb{E}(\mathbf{1}_{\widetilde{T}_1=T_1} \dots \mathbf{1}_{\widetilde{T}_R=T_R}|\mathcal{K}_T) \\ &= \mathbb{E}(\mathbf{1}_{\widetilde{T}_1=T_1} \dots \mathbf{1}_{\widetilde{T}_{R-1}=T_{R-1}} \\ &\qquad \times \mathbb{E}(\mathbf{1}_{\widetilde{T}_R=T_R}|\mathcal{K}_T, \widetilde{T}_1 = T_1, \dots, \widetilde{T}_{R-1} = T_{R-1})) \ .\end{aligned}$$

On the event $\{\widetilde{T}_1 = T_1, \dots, \widetilde{T}_{R-1} = T_{R-1}\}$, we have $\widetilde{K}_t = K_t$, $\forall t : T_{R-1} \leq t < T_R$. And so we have:

$$\begin{aligned}&\mathbb{E}(\mathbf{1}_{\widetilde{T}_R=T_R}|\mathcal{K}_T, T_{k-1}, \widetilde{T}_1 = T_1, \dots, \widetilde{T}_{R-1} = T_{R-1}) \\ &= \mathbb{P}\left(\int_{T_{R-1}}^{T_R} \Lambda K_u - \frac{\Lambda K_u (N-K_u)_+}{N-1} du \leq \widetilde{U}_k \right. \\ &\qquad \left. |\, \mathcal{K}_T, \widetilde{T}_1 = T_1, \dots, \widetilde{T}_{R-1} = T_{R-1}\right) \\ &\geq \mathbb{P}\left(\int_{T_{R-1}}^{T_R} \frac{\Lambda (K_T)^2}{N-1} du \leq \widetilde{U}_k \middle| \mathcal{K}_T, \widetilde{T}_1 = T_1, \dots, \widetilde{T}_{R-1} = T_{R-1}\right) \\ &= \exp\left(-\frac{\Lambda (K_T)^2}{N-1}(T_R - T_{R-1})\right) \ .\end{aligned}$$

So, by recurrence,

$$\mathbb{E}(G|\mathcal{K}_T) \geq \exp\left(-\frac{\Lambda K_T^2}{N-1} T\right) \ . \tag{4.2}$$

When $\exists k, r, s : N^{k,r}$ jumps at $s, k \in C_s^i, r \in C_s^j$ and $s \in [0,T]$, we say there is a loop at s between C^i and C^j. If $i = j$, we say that C^i has a loop with itself at s. We can define in the same way loops between $\widetilde{C}^i$, $\widetilde{C}^j$. Notice that $[\exists i,\ C^i$ has two loops on $[0,T]$ and $L_T \leq q/2] \Rightarrow [\ \exists j,\ C^j$ has no loop on $[0,T]]$. Conditionally on $\{C^1$ has no loop on $[0,T]\}$ (i.e. C^1 has no loop with any other C^j and has no loop with itself), $Z_T^{N,1}$ is independent of $Z_T^{N,2}, \dots, Z_T^{N,q}$ and has the law $\widetilde{P}_T$ (by Theorem 3.1 applied for $q = 1$ and Lemma 2.8). Suppose $k < q/2$, as $F \in \mathcal{B}_0^{sym}(q)$, we then get

$$\begin{aligned}&E(F(Z_T^{N,1}, \dots, Z_T^{N,q})|L_T = k) \\ &\quad = \mathbb{E}(\mathbb{E}(F(Z_T^{N,1}, \dots, Z_T^{N,q})|C^1 \text{ has no loop on } [0,T], L_T = k)|L_T = k) \\ &\quad = 0 \end{aligned} \tag{4.3}$$

We suppose by now that q is even. Because of Theorem 3.1, we then get:

$$\begin{aligned}N^{q/2}\mathbb{E}((\eta_T^N)^{\odot q}(F)) &= N^{q/2}\mathbb{E}(F(\overline{Z}_T^{N,1}, \dots, \overline{Z}_T^{N,q})) \\ &= N^{q/2}\mathbb{E}(F(Z_T^{N,1}, \dots, Z_T^{N,q}))\end{aligned}$$

which has same limit as $N^{q/2}\mathbb{E}(F(Z_T^{N,1}, \dots, Z_T^{N,q})\mathbf{1}_{L_T=q/2})$ when $N \to +\infty$. As $\frac{q!}{2^{q/2}(q/2)!}$ is the number of ways of partitioning $[q]$ into $q/2$ couples, we get

$$N^{q/2}\mathbb{E}(F(Z_T^{N,1},\dots,Z_T^{N,q})\mathbf{1}_{L_T=q/2}) \\ = \quad N^{q/2}\tfrac{q!}{2^{q/2}(q/2)!}\mathbb{E}(F(Z_T^{N,1},\dots,Z_T^{N,q})\mathbf{1}_A)\ .$$

Noticing that $\mathcal{L}((Z_T^{N,1},\dots,Z_T^{N,q})|G) = \mathcal{L}((\widetilde{Z}_T^{N,1},\dots,\widetilde{Z}_T^{N,q})|G)$, we get:

$$\begin{aligned} &N^{q/2}\mathbb{E}(F(Z_T^{N,1},\dots,Z_T^{N,q})\mathbf{1}_{L_T=q/2}) \\ &= N^{q/2}\frac{q!}{2^{q/2}(q/2)!}\Big[\mathbb{E}(F(\widetilde{Z}_T^{N,1},\dots,\widetilde{Z}_T^{N,q})\mathbf{1}_{\widetilde{A}}) \\ &\qquad +\mathbb{E}((F(Z_T^{N,1},\dots,Z_T^{N,q})\mathbf{1}_A\mathbf{1}_{G^c}) \\ &\qquad -\mathbb{E}(F(\widetilde{Z}_T^{N,1},\dots,\widetilde{Z}_T^{N,q})\mathbf{1}_{\widetilde{A}}\mathbf{1}_{G^c})\Big]\ . \end{aligned}$$

Notice that, knowing $\mathcal{K}_T$, for $i\neq j$, the number of loops between C^i and C^j on $[0,t]$ is a non-homogeneous Poisson process of intensity $\left(\frac{\Lambda K_t^i K_t^j}{N-1}\right)_{0\le t\le T}$. Notice also that, knowing $\widetilde{\mathcal{K}}_T$, for $i\neq j$, the number of loops between $\widetilde{C}^i$ and $\widetilde{C}^j$ on $[0,t]$ is a non-homogeneous Poisson process of intensity $\left(\frac{\Lambda \widetilde{K}_t^i \widetilde{K}_t^j}{N-1}\right)_{0\le t\le T}$. So we have:

$$\mathbb{P}(\text{ no loop between } C^i \text{ and } C^j|\mathcal{K}_T) = \exp\left(-\int_0^T \frac{\Lambda K_s^i K_s^j}{N-1}ds\right) =: \alpha(i,j)\ ,$$

$$\mathbb{P}(\text{ no loop between } \widetilde{C}^i \text{ and } \widetilde{C}^j|\widetilde{\mathcal{K}}_T) = \exp\left(-\int_0^T \frac{\Lambda \widetilde{K}_s^i \widetilde{K}_s^j}{N-1}ds\right) =: \widetilde{\alpha}(i,j)\ .$$

We set

$$\begin{aligned} B \quad = \quad &\{\text{at least one loop between } C^1 \text{ and } C^2\}\cap\dots \\ &\cdots\cap\{\text{at least one loop between } C^{q-1} \text{ and } C^q\}\ . \end{aligned}$$

We have

$$\mathbb{P}(A|\mathcal{K}_T) = \mathbb{P}(B|\mathcal{K}_T) - \mathbb{P}(B\backslash A|\mathcal{K}_T)\ . \tag{4.4}$$

and

$$\begin{aligned} \mathbb{P}(B|\mathcal{K}_T) \quad &= \quad \prod_{1\le j\le q/2}(1-\alpha(2j-1,2j)) \\ &\le \quad \frac{1}{(N-1)^{q/2}}\prod_{1\le j\le q/2}(T\Lambda K_T^{2j-1}K_T^{2j}) \\ \text{(by Lem. 4.1)} \quad &\le \quad \frac{1}{(N-1)^{q/2}}\prod_{1\le j\le q/2}(T\Lambda \widetilde{K}_t^{2j-1}\widetilde{K}_T^{2j})\ . \end{aligned} \tag{4.5}$$

So, we have:

$$
\begin{aligned}
&N^{q/2}\mathbb{E}(|(F(Z_T^{N,1},\dots,Z_T^{N,q})\mathbf{1}_A\mathbf{1}_{G^c}|)\\
&\leq N^{q/2}\|F\|_\infty\mathbb{E}(\mathbb{E}(\mathbf{1}_A\mathbf{1}_{G^c}|\mathcal{K}_T))\\
&= N^{q/2}\|F\|_\infty\mathbb{E}(\mathbb{E}(\mathbf{1}_A|\mathcal{K}_T)\mathbb{E}(\mathbf{1}_{G^c}|\mathcal{K}_T))\\
&\text{(using (4.2), (4.4), (4.5))}\\
&\leq N^{q/2}\|F\|_\infty\mathbb{E}\left(\frac{1}{(N-1)^{q/2}}\prod_{1\leq j\leq q/2}(T\Lambda\widetilde{K}_T^{2j-1}\widetilde{K}_T^{2j})\right.\\
&\qquad\left.\times\left(1-\exp\left(-\frac{\Lambda(K_T)^2T}{N-1}\right)\right)\right)\\
&\text{(using Lemma 4.1)}\\
&\leq N^{q/2}\|F\|_\infty\mathbb{E}\left(\frac{1}{(N-1)^{q/2}}\prod_{1\leq j\leq q/2}(T\Lambda\widetilde{K}_T^{2j-1}\widetilde{K}_T^{2j})\right.\\
&\qquad\left.\times\left(1-\exp\left(-\frac{\Lambda(\widetilde{K}_T)^2T}{N-1}\right)\right)\right)\ ,
\end{aligned}
$$

For a fixed ω,

$$
\begin{aligned}
&N^{q/2}\frac{1}{(N-1)^{q/2}}\prod_{1\leq j\leq q/2}(t\Lambda\widetilde{K}_T^{2j-1}(\omega)\widetilde{K}_T^{2j}(\omega))\\
&\qquad\times\left(1-\exp\left(-\frac{\Lambda(\widetilde{K}_T(\omega))^2T}{N-1}\right)\right)\\
&\qquad\underset{N\to+\infty}{\longrightarrow}0
\end{aligned}
\tag{4.6}
$$

and

$$
\begin{aligned}
0 &\leq N^{q/2}\frac{1}{(N-1)^{q/2}}\prod_{1\leq j\leq q/2}(t\Lambda\widetilde{K}_T^{2j-1}\widetilde{K}_T^{2j})\times\left(1-\exp\left(-\frac{\Lambda(\widetilde{K}_T)^2T}{N-1}\right)\right)\\
&\leq 2^{q/2}\prod_{1\leq j\leq q/2}(t\Lambda\widetilde{K}_T^{2j-1}\widetilde{K}_T^{2j})
\end{aligned}
$$

which is of finite expectation by (3.4) and Lemma 4.1. So, by dominated convergence:

$$N^{q/2}\mathbb{E}(|(F(Z_T^{N,1},\dots,Z_T^{N,q})\mathbf{1}_A\mathbf{1}_{G^c}|)\underset{N\to+\infty}{\longrightarrow}0\ .$$

We can show in the same way :

$$N^{q/2}\mathbb{E}(|(F(\widetilde{Z}_T^1,\dots,\widetilde{Z}_T^q)\mathbf{1}_{\widetilde{A}}\mathbf{1}_{G^c}|)\underset{N\to+\infty}{\longrightarrow}0\ .$$

We have:

$$N^{q/2}\mathbb{E}(F(\widetilde{Z}_T^1,\dots,\widetilde{Z}_T^q)\mathbf{1}_{\widetilde{A}}) \;=\; \mathbb{E}(\mathbb{E}(F(\widetilde{Z}_T^1,\dots,\widetilde{Z}_T^q)|\widetilde{\mathcal{K}}_T,\widetilde{A}))N^{q/2}\mathbb{P}(\widetilde{A}|\widetilde{\mathcal{K}}_T)) \;,$$

We set

$$\begin{aligned}\widetilde{B} \;=\; &\{\text{at least one loop between } \widetilde{C}^1 \text{ and } \widetilde{C}^2\}\cap\dots\\ &\cdots\cap\{\text{at least one loop between } \widetilde{C}^{q-1} \text{ and } \widetilde{C}^q\}\;.\end{aligned}$$

We decompose

$$\mathbb{P}(\widetilde{A}|\widetilde{C}_T)=\mathbb{P}(\widetilde{B}|\widetilde{\mathcal{K}}_T)-\mathbb{P}(\widetilde{B}\backslash\widetilde{A}|\widetilde{\mathcal{K}}_T)\;.$$

We have

$$\begin{aligned}N^{q/2}\mathbb{P}(\widetilde{B}|\widetilde{\mathcal{K}}_T) \;&=\; N^{q/2}\prod_{1\le j\le q/2}(1-\widetilde{\alpha}(2j-1,2j))\\ &\xrightarrow[N\to+\infty]{\text{a.s.}}\; \prod_{1\le i\le q/2}\int_0^T\Lambda\widetilde{K}_s^{2i-1}\widetilde{K}_s^{2i}ds\end{aligned}$$

and

$$\mathbb{P}(\text{ at least two loops between } \widetilde{C}^1 \text{ and } \widetilde{C}^2|\widetilde{\mathcal{K}}_T)\le(1-\widetilde{\alpha}(1,2))^2\;.$$

So we get, computing very roughly:

$$\begin{aligned}\mathbb{P}(\widetilde{B}\backslash\widetilde{A}|\widetilde{\mathcal{K}}_T) \;&\le\; \sum_{1\le r\le q/2}\;\sum_{1\le i_1,\dots i_r\le q/2}\Bigg[\prod_{j\in\{i_1,\dots,i_r\}}(1-\widetilde{\alpha}(2j-1,2j))^2\\ &\qquad\qquad\times\prod_{j\notin\{i_1,\dots,i_r\}}(1-\widetilde{\alpha}(2j-1,2j))\Bigg]\\ &\le\; \sum_{1\le r\le q/2}C_{q/2}^r\prod_{1\le j\le q/2}\left[\left(T\Lambda\widetilde{K}_T^{2j-1}\widetilde{K}_T^{2j}\right)^2\vee(T\Lambda\widetilde{K}_T^{2j-1}\widetilde{K}_T^{2j})\right]\\ &\qquad\qquad\times\frac{1}{(N-1)^{q/2+1}}\\ &\le\; \frac{2^{q/2}}{(N-1)^{q/2+1}}\\ &\qquad\times\prod_{1\le j\le q/2}\left[\left(T\Lambda\widetilde{K}_T^{2j-1}\widetilde{K}_T^{2j}\right)^2\vee(T\Lambda\widetilde{K}_T^{2j-1}\widetilde{K}_T^{2j})\right]\;,\end{aligned}$$

so $N^{q/2}\mathbb{P}(\widetilde{B}\backslash\widetilde{A}|\widetilde{\mathcal{K}}_T)\xrightarrow[N\to+\infty]{\text{a.s.}}0$. And so:

$$N^{q/2}\mathbb{P}(\widetilde{A}|\widetilde{\mathcal{K}}_T)\;\xrightarrow[N\to+\infty]{\text{a.s.}}\;\prod_{1\le i\le q/2}\int_0^T\Lambda\widetilde{K}_s^{2i-1}\widetilde{K}_s^{2i}ds\;,$$

and again by dominated convergence, we get the result.

□

We denote by $\mathcal{I}_q$ the set of partitions of $[q]$ into pairs.

Corollary 4.4. *[Wick formula] For $F \in \mathcal{B}_0^{sym}(q)$ of the form $(f_1 \otimes \cdots \otimes f_q)_{sym}$ and q even,*

$$N^{q/2}\mathbb{E}((\eta_t^n)^{\odot q}(F)) \underset{N\to+\infty}{\longrightarrow} \sum_{I_q \in \mathcal{I}_q} \prod_{\{i,j\}\in I_q} \mathbb{E}(V_T^B(f_i,f_j)) \ ,$$

with

$$V_T^B(f_i,f_j) = \mathbb{E}(f_i(\widetilde{Z}_t^1)f_j(\widetilde{Z}_T^2)|\widetilde{A},\widetilde{\mathcal{K}}_t) \times \int_0^T \Lambda \widetilde{K}_s^1 \widetilde{K}_s^2 ds \ .$$

5 Convergence theorems

All the theorems of this section are valid under assumptions (1), (2).

5.1 Almost sure convergence

Theorem 5.1. *For any measurable bounded f, $T \geq 0$,*

$$\eta_T^N(f) \underset{N\to+\infty}{\overset{p.s.}{\longrightarrow}} \widetilde{P}_T(f) \ .$$

Proof. We recall the notations of [DPR09a]. For any empirical measure $m(x) = \frac{1}{N}\sum_{i=1}^N \delta_{x^i}$ (based on N points $x^1, x^2, \ldots, x^N$), any q,

$$m(x)^{\otimes q} := \frac{1}{N^q} \sum_{a\in[N]^{[q]}} \delta_{(x^{a(1)},\cdots,x^{a(q)})} \ ,$$

where $[N]^{[q]} = a : [q] \to [N]$. Notice that for any F

$$m(x)^{\otimes q}(F) = m(x)^{\otimes q}(F_{\text{sym}}) \ .$$

We define, $\forall 1 \leq p \leq q$, $[q]_p^{[q]} := \{a \in [q]^{[q]}, \#\text{Im}(a) = p\}$ and $(\forall k \leq q)$

$$\partial^k L_q = \sum_{q-k\leq p\leq q} s(p,q-k)\frac{1}{(q)_p} \sum_{a\in[q]_p^{[q]}} a$$

(the $s(.,.)$ are the Stirling numbers of the first kind) and $\forall F$(of q variables), $\forall b \in [q]^{[q]}$,

$$D_b(F)(x^1,\ldots,x^q) = F(x^{b(1)},\ldots,x^{b(q)}) \ ,$$

$$D_{\partial^k L_q}(F) = \sum_{q-k\leq p\leq q} s(p,q-k)\frac{1}{(q)_p} \sum_{a\in[q]_p^{[q]}} F(x^{a(1)},\cdots,x^{a(q)}) \ .$$

We then have, by Corollary 2.3 p. 789 of [DPR09a], for any empirical measure $m(x)$ (based on N points), any F of q variables,

$$m(x)^{\otimes q}(F) = m(x)^{\odot q}\left(\sum_{0\leq k<q} \frac{1}{N^k} D_{\partial^k L_q}(F)\right) .$$

Suppose $F \in \mathcal{B}_0^{sym}(q)$,

$$\mathbb{E}\left((\eta_T^N)^{\otimes q}(F)\right) = \sum_{0\leq k<q} \frac{1}{N^k} \sum_{q-k\leq p\leq q} s(p,q-k) \sum_{a\in[q]_p^{[q]}} \mathbb{E}\left((\eta_T^N)^{\odot q}(D_a(F))\right) .$$

Notice that for $0 \leq k < q$, $q-k \leq p \leq q$, $a \in [q]_p^{[q]}$, $\#\{i \in [q], \#a(\{i\}) = 1\} \geq q-2k$. Suppose by now that q is even. If $k \leq q/2$, using Theorem 3.2 and reasoning in the same way as we did to prove (4.3), we get (for some constant C which will change from line to line):

$$\mathbb{E}\left((\eta_T^N)^{\odot q}(D_a(F))\right) \leq \frac{C}{N^{\frac{q}{2}-k}} .$$

And so

$$\mathbb{E}\left((\eta_T^N)^{\otimes q}(F)\right) \leq \frac{C}{N^{\frac{q}{2}}} .$$

Suppose we take a bounded function $f : E \to \mathbb{R}$. We set $\bar{f} = f - \widetilde{P}_T(f)$. We then have (with the notation $\bar{f}^{\otimes q}(x^1,\ldots,x^q) := \bar{f}(x^1)\times\cdots\times\bar{f}(x^q)$)

$$\begin{aligned}\mathbb{E}(((\eta_T^N(f) - \widetilde{P}_T(f))^q) &= \mathbb{E}((\eta_T^N(\bar{f}))^q) \\ &= \mathbb{E}((\eta_T^N)^{\otimes q}(\bar{f}^{\otimes q})) \\ &\leq \frac{C}{N^{\frac{q}{2}}} .\end{aligned}$$

Provided we take q big enough, we can apply Borel-Cantelli Lemma to finish the proof.

□

5.2 Central-limit theorem

Theorem 5.2. *For all $f_1,\ldots,f_q \in \mathcal{B}_0^{sym}(1)$, $\forall T \geq 0$,*

$$N^{q/2}(\eta_T^N(f_1),\ldots,\eta_T^N(f_q)) \xrightarrow[N\to+\infty]{law} \mathcal{N}(0,K) ,$$

with $K(i,j) = \widetilde{P}_T(f_i f_j) + V_T^B(f_i,f_j)$.

Proof. For any $u_1, \dots, u_q$, we have:

$$\begin{aligned}
&\mathbb{E}\left(\exp\left(N\eta_T^N\left(\log\left(1+\frac{iu_1f_1+\cdots+iu_qf_q}{\sqrt{N}}\right)\right)\right)\right)\\
&=\mathbb{E}\left(\exp\left(\sum_{k\geq 1}\frac{(-1)^{k+1}}{k}N^{1-k/2}\eta_T^N((iu_1f_1+\cdots+iu_qf_q)^k)\right)\right)\\
&\underset{N\to+\infty}{\sim}\mathbb{E}(\exp(\sqrt{N}(iu_1\eta_T^N(f_1)+\cdots+iu_q\eta_t^N(f_q))))\\
&\qquad\times\exp\left(-\frac{1}{2}\widetilde{P}_T((u_1f_1+\cdots+u_qf_q)^2)\right)\ .
\end{aligned}$$

We also have

$$\begin{aligned}
&\mathbb{E}\left(\exp\left(N\eta_T^N\left(\log\left(1+\frac{iu_1f_1+\dots iu_qf_q}{\sqrt{N}}\right)\right)\right)\right)\\
&=\mathbb{E}\left(\prod_{j=1}^N\left(1+\frac{iu_1f_1(\xi_t^j)+\cdots+iu_qf_q(\xi_t^j)}{\sqrt{N}}\right)\right)\\
&=\mathbb{E}\left(\sum_{0\leq k\leq N}\frac{1}{N^{k/2}}\sum_{1\leq j_1,\dots,j_k\leq q}i^ku_{j_1}\dots u_{j_k}\right.\\
&\qquad\left.\times\sum_{1\leq i_1<\cdots<i_k\leq N}f_{j_1}(\xi_t^{i_1})\dots f_{j_k}(\xi_t^{i_k})\right)\\
&=\mathbb{E}\left(\sum_{0\leq k\leq N}\frac{(N)_k}{N^{k/2}}\sum_{1\leq j_1,\dots,j_k\leq q}i^ku_{j_1}\dots u_{j_k}\frac{1}{k!}(\eta_T^N)^{\odot k}(f_{j_1}\otimes\cdots\otimes f_{j_k})\right)\\
&\underset{N\to+\infty}{\longrightarrow}\sum_{k\geq 0,k\text{ even}}(-1)^{k/2}\sum_{1\leq j_1,\dots,j_k\leq q}\frac{u_{j_1}\dots u_{j_k}}{k!}\sum_{I_k\in\mathcal{I}_k}\prod_{\{a,b\}\in I_k}V_T^B(f_a,f_b)\\
&=\sum_{k\geq 0,k\text{ even}}\frac{(-1)^{k/2}}{2^{k/2}(k/2)!}\\
&\qquad\times\sum_{1\leq j_1,\dots,j_k\leq q}u_{j_1}\dots u_{j_k}\mathbb{E}(V_T^B(f_{j_1},f_{j_2}))\dots\mathbb{E}(V_t^B(f_{j_{k-1}},f_{j_k}))\\
&=\sum_{k\geq 0,k\text{ even}}\frac{(-1)^{k/2}}{2^{k/2}(k/2)!}\left(\sum_{1\leq j_1,j_2\leq q}u_{j_1}u_{j_2}\mathbb{E}(V_T^B(f_{j_1},f_{j_2}))\right)^{k/2}\\
&=\exp\left(-\frac{1}{2}\sum_{1\leq j_1,j_2\leq q}u_{j_1}u_{j_2}\mathbb{E}(V_T^B(f_{j_1},f_{j_2}))\right)\ .
\end{aligned}$$

□

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