HAL open science

Perverse sheaves and modular representation theory

Daniel Juteau, Carl Mautner, Geordie Williamson

To cite this version:

Daniel Juteau, Carl Mautner, Geordie Williamson. Perverse sheaves and modular representation theory. Séminaires et congrès, 2012, 24-II, pp.313-350. hal-00354963

HAL Id: hal-00354963

https://hal.science/hal-00354963v1

Submitted on 21 Jan 2009

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Perverse sheaves and modular representation theory

Daniel Juteau, Carl Mautner, and Geordie Williamson

Abstract. This paper is an introduction to the use of perverse sheaves with positive characteristic coefficients in modular representation theory. In the first part, we survey results relating singularities in finite and affine Schubert varieties and nilpotent cones to modular representations of reductive groups and their Weyl groups. The second part is a brief introduction to the theory of perverse sheaves with an emphasis on the case of positive characteristic and integral coefficients. In the final part, we provide some explicit examples of stalks of intersection cohomology complexes with integral or positive characteristic coefficients in nilpotent cones, mostly in type A. Some of these computations might be new.

Introduction

Representation theory has a very different flavour in positive characteristic. When one studies the category of representations of a finite group or a reductive group over a field of characteristic 0 (e.g. $\mathbb{C}$), one of the first observations to be made is that this category is semi-simple, meaning that every representation is isomorphic to a direct sum of irreducible representations. This fundamental fact helps answer many basic questions, e.g. the dimensions of simple modules, character formulae, and tensor product multiplicities. However, when one considers representations over fields of positive characteristic (often referred to as "modular" representations) the resulting categories are generally not semi-simple. This makes their study considerably more complicated and in many cases even basic questions remain unanswered.[1]

It turns out that some questions in representation theory have geometric counterparts. The connection is obtained via the category of

[1] For an introduction to the modular representation theory of finite groups we recommend the third part of [**Ser67**], and for that of reductive groups, [**Jan87**].

perverse sheaves, a certain category that may be associated to an algebraic variety and whose structure reflects the geometry of the underlying variety and its subvarieties. The category of perverse sheaves depends on a choice of coefficient field and, as in representation theory, different choices of coefficient field can yield very different categories.

Since the introduction of perverse sheaves it has been realised that many phenomena in Lie theory can be explained in terms of categories of perverse sheaves and their simple objects — intersection cohomology complexes. In studying representations of reductive groups and related objects, singular varieties arise naturally (Schubert varieties and their generalizations, nilpotent varieties, quiver varieties...). It turns out that the invariants of these singularities often carry representation theoretic information. For an impressive list of such applications, see [**Lus90**]. This includes constructing representations, computing their characters, and constructing nice bases for them.

However, most of these applications use a field k of characteristic zero for coefficients. In this paper, we want to give the reader a flavour for perverse sheaves and intersection cohomology with positive characteristic coefficients.

In the first section of this article we survey three connections between modular representation theory and perverse sheaves.

The geometric setting for the first result — known as the geometric Satake theorem — is a space (in fact an "ind-scheme") associated to a complex reductive group G. This space, $G(\mathbb{C}((t)))/G(\mathbb{C}[[t]])$, commonly referred to as the affine Grassmannian is a homogeneous space for the algebraic loop group $G(\mathbb{C}((t)))$. Under the action of $G(\mathbb{C}[[t]])$, it breaks up as a union of infinitely many finite-dimensional orbits. Theorems of Lusztig [**Lus83**], Ginzburg [**Gin95**], Beilinson-Drinfeld [**BD**], and Mirković-Vilonen [**MV07**] explain that encoded in the geometry of the affine Grassmannian and its orbit closures is the algebraic representation theory over any field (and even over the integers) of the split form of the reductive group $G^\vee$ with root data dual to that of G, also known as the Langlands dual group.

The second family of results that we discuss involves the geometry of the finite flag variety G/B where again G is a complex reductive group, and a generalization of it closely related to the affine Grassmannian known as the affine flag variety $G(\mathbb{C}((t)))/\mathcal{I}$. We describe theorems of Soergel [**Soe00**] and Fiebig [**Fie06, Fie07b, Fie07a, Fie08**] which show that the geometry of these spaces can be used to understand the modular representation theory of the Langlands dual group $G_k^\vee$ for k a field of characteristic larger than the Coxeter number of $G_k^\vee$. In

doing so, Fiebig is able to give a new proof of the celebrated Lusztig conjecture with an explicit bound on the characteristic.

The third theorem to be discussed is centered around the geometry of the variety $\mathcal{N}$ of nilpotent elements of a Lie algebra $\mathfrak{g}$, known as the nilpotent cone. The nilpotent cone has a natural resolution and, in 1976, Springer [**Spr76**] showed that the Weyl group acts on the ℓ-adic cohomology of the fibers of this resolution. He showed moreover that from this collection of representations one could recover all of the irreducible ℓ-adic representations and that they came with a natural labelling by a nilpotent adjoint orbit with an irreducible G-equivariant local system. This groundbreaking discovery was followed by a series of related constructions, one of which, based on the Fourier-Deligne transform, has recently been used by the first author [**Jut07**] to establish a modular version of the Springer correspondence.

The second goal of this article which occupies the second and third sections is to provide an introduction to "modular" perverse sheaves, in other words perverse sheaves with coefficients in a field of positive characteristic. We begin by recalling the theory of perverse sheaves, highlighting the differences between characteristic zero and characteristic p, and also the case of integer coefficients. We treat in detail the case of the nilpotent cone of $\mathfrak{sl}_2$.

In the last part, we treat more examples in nilpotent cones. We calculate all the IC stalks in all characteristics $\neq 3$ for the nilpotent cone of $\mathfrak{sl}_3$, and all the IC stalks in all characteristics $\neq 2$ for the subvariety of the nilpotent cone of $\mathfrak{sl}_4$ consisting of the matrices which square to zero. Before that, we recall how to deal with simple and minimal singularities in type A, for two reasons: we need them for the three-strata calculations, and they can be dealt with more easily than for arbitrary type (which was done in [**Jut08, Jut09**]). As a complement, we give a similar direct approach for a minimal singularity in the nilpotent cone of $\mathfrak{sp}_{2n}$.

The two first parts partly correspond to the talks given by the first and third author during the summer school, whose titles were "Intersection cohomology in positive characteristic I, II". The third part contains calculations that the three of us did together while in Grenoble. These calculations were the first non-trivial examples involving three strata that we were able to do.

It is a pleasure to thank Michel Brion for organizing this conference and allowing two of us to speak, and the three of us to meet. We would like to thank him, as well as Alberto Arabia, Peter Fiebig, Joel Kamnitzer and Wolfgang Soergel for very valuable discussions. The second author would also like to acknowledge the mathematics department at

the University of Texas at Austin and his advisor, David Ben-Zvi, for partially funding the travel allowing him to attend this conference and meet his fellow coauthors.

Contents

Introduction 1
1. Motivation 4
1.1. The geometric Satake theorem 5
1.2. Finite and affine flag varieties 8
1.3. The modular Springer correspondence 12
2. Perverse sheaves 18
2.1. Constructible sheaves 18
2.2. Perverse sheaves with coefficients in a field 20
2.3. Perverse sheaves over the integers 24
2.4. First example: the nilpotent cone of $\mathfrak{sl}_2$ 25
3. Some stalks in nilpotent cones 28
3.1. Semi-small resolutions of Richardson orbit closures 28
3.2. Minimal class in $\mathfrak{sl}_n$ 30
3.3. Minimal class in $\mathfrak{sp}_{2n}$ 32
3.4. Subregular class in $\mathfrak{sl}_n$ 34
3.5. IC stalks on the nilpotent cone of $\mathfrak{sl}_3$ for $p \neq 3$ 35
3.6. Some IC stalks on the nilpotent cone of $\mathfrak{sl}_4$ for $p \neq 2$ 37
References 39

1. Motivation

Perverse sheaves with coefficients in positive characteristic appear in a number of different contexts as geometrically encoding certain parts of modular representation theory. This section will provide a survey of three examples of this phenomenon: the geometric Satake theorem, the work of Soergel and Fiebig on Lusztig's conjecture, and the modular Springer correspondence. The corresponding geometry for the three pictures will be respectively affine Grassmannians, finite and affine flag varieties, and nilpotent cones.

Throughout, $G_{\mathbb{Z}}$ will denote a split connected reductive group scheme over $\mathbb{Z}$. Given a commutative ring k, we denote by G_k the split reductive group scheme over k obtained by extension of scalars

$$G_k = \operatorname{Spec} k \times_{\operatorname{Spec} \mathbb{Z}} G_{\mathbb{Z}}.$$

In Sections 1.1 and 1.2 we will consider $G_{\mathbb{C}}$, while in Section 1.3, we will consider $G_{\mathbb{F}_q}$.

We fix a split maximal torus $T_{\mathbb{Z}}$ in $G_{\mathbb{Z}}$. We denote by

$$(X^*(T_{\mathbb{Z}}), R^*, X_*(T_{\mathbb{Z}}), R_*)$$

the root datum of $(G_{\mathbb{Z}}, T_{\mathbb{Z}})$. We denote by $(G^\vee_{\mathbb{Z}}, T^\vee_{\mathbb{Z}})$ the pair associated to the dual root datum. Thus $G^\vee_{\mathbb{Z}}$ is the Langlands dual group. In Subsections 1.1 and 1.2, we will consider representations of $G^\vee_k = \operatorname{Spec} k \times_{\operatorname{Spec}\mathbb{Z}} G^\vee_{\mathbb{Z}}$, where k can be, for example, a field of characteristic p. We have $X^*(T^\vee_{\mathbb{Z}}) = X_*(T_{\mathbb{Z}})$ and $X_*(T^\vee_{\mathbb{Z}}) = X^*(T^\vee_{\mathbb{Z}})$.

We also fix a Borel subgroup $B_{\mathbb{Z}}$ of $G_{\mathbb{Z}}$ containing $T_{\mathbb{Z}}$. This determines a Borel subgroup $B^\vee_{\mathbb{Z}}$ of $G^\vee_{\mathbb{Z}}$ containing $T^\vee_{\mathbb{Z}}$. This also determines bases of simple roots $\Delta \subset R^*$ and $\Delta^\vee \subset R_*$. It will be convenient to choose $\Delta_* := -\Delta^\vee$ as a basis for R_* instead, so that the coroots corresponding to $B^\vee_{\mathbb{Z}}$ are the negative coroots $R^-_* = -R^+_*$.

1.1. The geometric Satake theorem. In this subsection and the next one, to simplify the notation, we will identify the group schemes $G_{\mathbb{C}} \supset B_{\mathbb{C}} \supset T_{\mathbb{C}}$ with their groups of $\mathbb{C}$-points $G \supset B \supset T$.

We denote by $\mathcal{K} = \mathbb{C}((t))$ the field of Laurent series and by $\mathcal{O} = \mathbb{C}[[t]]$ the ring of Taylor series. The affine (or loop) Grassmannian $\mathcal{G}r = \mathcal{G}r^G$ is the homogeneous space $G(\mathcal{K})/G(\mathcal{O})$. It has the structure of an ind-scheme. In what follows we will attempt to sketch a rough outline of this space and then briefly explain how perverse sheaves on it are related to the representation theory of $G^\vee_k$, where k is any commutative ring of finite global dimension. We refer the reader to [**BD, BL94a, LS97, MV07**] for more details and proofs.

We have a natural embedding of the coweight lattice $X_*(T) = \operatorname{Hom}(\mathbb{G}_m, T)$ into the affine Grassmannian: each $\lambda \in X_*(T)$ defines a point t^λ of $G(\mathcal{K})$ via

$$\operatorname{Spec}\mathcal{K} = \operatorname{Spec}\mathbb{C}((t)) \xrightarrow{c} \mathbb{G}_m = \operatorname{Spec}\mathbb{C}[t, t^{-1}] \xrightarrow{\lambda} T \xrightarrow{i} G$$

where c comes from the inclusion $\mathbb{C}[t, t^{-1}] \hookrightarrow \mathbb{C}((t))$ and $i : T \to G$ is the natural inclusion, and hence a point $[t^\lambda]$ in $\mathcal{G}r = G(\mathcal{K})/G(\mathcal{O})$.

For example, when $G = GL_n$ and T is the subgroup of diagonal matrices the elements of $X_*(T)$ consist of n-tuples of integers $\lambda = (\lambda_1, \ldots, \lambda_n)$ and they sit inside of $\mathcal{G}r$ as the points

$$\begin{pmatrix} t^{\lambda_1} & & & \\ & t^{\lambda_2} & & \\ & & \ddots & \\ & & & t^{\lambda_n} \end{pmatrix} \cdot G(\mathcal{O})$$

As in the finite case, one has a Cartan decomposition of $G(\mathcal{K})$

$$G(\mathcal{K}) = \bigsqcup_{\lambda \in X_*(T)^+} G(\mathcal{O}) t^\lambda G(\mathcal{O})$$

where $X_*(T)^+ = \{\lambda \in X_*(T) \mid \forall \alpha \in \Delta,\ \langle \alpha, \lambda \rangle \geq 0\}$ is the cone of dominant coweights. Thus the affine Grassmannian is the union of the $G(\mathcal{O})$-orbits of the points $[t^\lambda]$ for $\lambda \in X_*(T)^+$.

Another important feature of the affine Grassmannian is a special $\mathbb{C}^*$-action. As in the topological loop group, there is a notion of "loop rotation". In our case, this rotation comes in the form of $\mathbb{C}^*$ acting on $G(\mathcal{K})$ by composing a $\mathcal{K}$-point of G with the automorphism of $\operatorname{Spec} \mathcal{K}$ induced by the action of $\mathbb{C}^*$ on itself. More naively, this means replacing t by zt. This action clearly preserves the subgroup $G(\mathcal{O})$ and thus gives a well-defined action on the quotient $\mathcal{G}r$.

It is useful to get a sense of the geometry of the $G(\mathcal{O})$-orbits. As we saw above, these orbits are labelled by the dominant coweights. We begin by studying a subvariety of the $G(\mathcal{O})$-orbits. For $\lambda \in X_*(T)^+$, consider the G-orbit of $[t^\lambda]$. It turns out that for such a dominant coweight, the point $[t^\lambda]$ is fixed by a Borel subgroup. Thus, the G-orbit is a (partial) flag variety. In fact, the stabilizer of $[t^\lambda]$ in G is a parabolic subgroup P_λ with Levi factor corresponding to the roots $\alpha \in \Delta$ such that $\langle \alpha, \lambda \rangle = 0$. We conclude that $G \cdot [t^\lambda]$ is isomorphic to the (partial) flag variety G/P_λ. It is an easy exercise to check these claims for $G = GL_n$.

The points $[t^\lambda]$ of $\mathcal{G}r$ are in fact fixed by loop rotation. To see this, note that the reduced affine Grassmannian for T, $T(\mathcal{K})/T(\mathcal{O}) \cong X_*(T)$ is discrete, embeds in $\mathcal{G}r^G$ as the set of points $[t^\lambda]$ for $\lambda \in X_*(T)$, and is preserved by loop rotation. As $\mathbb{C}^*$ is connected and the subset $[t^\lambda]$ discrete, each such point is fixed under loop rotation. But of course the group $G \subset G(\mathcal{O})$ is certainly fixed by loop rotation, thus the G-orbit $G \cdot [t^\lambda]$ is fixed under loop rotation as well.

Not only are these G-orbits $G \cdot [t^\lambda]$ fixed, they form precisely the fixed point set of the action of loop rotation on the affine Grassmannian. Moreover, the $G(\mathcal{O})$-orbit $G(\mathcal{O}) \cdot [t^\lambda]$ is a vector bundle over $G \cdot [t^\lambda] \cong G/P_\lambda$. A proof of this statement involves considering the highest congruence subgroup of $G(\mathcal{O})$, defined as the preimage $\mathrm{ev}_0^{-1}(1)$ of 1 under the evaluation map $\mathrm{ev}_0 : G(\mathcal{O}) \to G, t \mapsto 0$. One can check that the orbit $\mathrm{ev}_0^{-1}(1) \cdot g \cdot [t^\lambda]$ is an affine space for any $g \in G$ and is isomorphic to a vector space on which loop rotation acts linearly by contracting characters. Combining this with the fact that $G(\mathcal{O}) = \mathrm{ev}_0^{-1}(1) \cdot G$, the claim follows. A corollary of this remark is that the $G(\mathcal{O})$-orbits are simply-connected.

For λ and μ dominant, the orbit $G(\mathcal{O}) \cdot [t^\lambda]$ is of dimension $2\rho(\lambda)$ (here ρ is half the sum of the positive roots) and is contained in $\overline{G(\mathcal{O}) \cdot [t^\mu]}$ if and only if $\lambda - \mu$ is a sum of positive coroots.

As a concrete example, it is instructive to consider the case $G = PSL_2$. Choose T to be the subgroup of diagonal matrices (up to scale) and B the upper triangular matrices (again, up to scale). The torus T is one dimensional, so the lattice $X_*(T)$ is isomorphic to a copy of the integers and $X_*(T)^+$ to the non-negative ones. Thus the $G(\mathcal{O})$-orbits are labelled by the non-negative integers. The parabolic subgroup corresponding to any positive number is the Borel subgroup B, and that correpsonding to the trivial weight is the whole group PSL_2. Thus the affine Grassmannian for PSL_2 is a union of a point and a collection of vector bundles over $\mathbb{P}^1$. Considering the remark of the previous paragraph, as the coroot lattice for PSL_2 is a subgroup of index two in the coweight lattice, the affine Grassmannian consists of two connected components.

Remark 1.1. One way to see some of the geometry is through the moment map with respect to the action of the torus T extended by loop rotation. This idea, from the differential point of view and in slightly different language, can be found in [**AP83**]. Yet another picture of the affine Grassmannian is provided by the spherical building, whose vertices are the $\mathbb{F}_q$-points of the affine Grassmannian where the Laurent and Taylor series are defined over $\overline{\mathbb{F}}_p$ instead of $\mathbb{C}$. This picture in the rank one case can be found in chapter 2 of [**Ser77**], although the affine Grassmannian is not mentioned explicitly.

Beginning with pioneering work of Lusztig, it was understood that the geometry of the affine Grassmannian is closely related to the representation theory of the Langlands dual group $G^\vee$, i.e. the reductive group with dual root data. In particular, Lusztig showed [**Lus83**] that the local intersection cohomology (with complex coefficients) of the $G(\mathcal{O})$-orbits was a refinement of the weight multiplicities of the corresponding representation of $G^\vee$.

This connection was further developed by Ginzburg [**Gin95**] (see also Beilinson-Drinfeld [**BD**]), who noted that the category of $G(\mathcal{O})$-equivariant perverse sheaves (with $\mathbb{C}$-coefficients) carried a convolution product and using Tannakian formalism on the total cohomology functor that the category was tensor equivalent to the category of representations of the Langlands dual group $G^\vee$. In other words:

Theorem 1.2. *There is an equivalence of tensor categories:*

$$(\mathbf{P}_{G(\mathcal{O})}(\mathcal{G}r^G; \mathbb{C}), *) \simeq (\mathrm{Rep}(G^\vee_{\mathbb{C}}), \otimes).$$

This result can be interpreted as a categorification of the much earlier work of Satake [**Sat63**] which identified the algebra of spherical functions (approximately bi-$G(\mathcal{O})$-invariant functions of $G(\mathcal{K})$) with W-invariant functions on the coweight lattice, $\mathbb{C}[\Lambda]^W$.

Remark 1.3. In the case of the affine Grassmannian, the category of $G(\mathcal{O})$-equivariant perverse sheaves is equivalent to the category of perverse sheaves constructible with respect to the $G(\mathcal{O})$-orbit stratification. For a proof, see the appendix to [**MV07**].

It was understood by Beilinson and Drinfeld [**BD**] that the affine Grassmannian described above should be thought of as associated to a point on an algebraic curve. Understood as such, there is a natural global analogue of the affine Grassmannian living over a configuration space of points on a curve. Using this Beilinson-Drinfeld Grassmannian, one can produce a natural commutativity constraint for the convolution product by identifying it as a "fusion product". >From this point of view, the geometric Satake theorem becomes identified with the local geometric Langlands conjecture.

For the remainder of this paragraph let k be a Noetherian commutative ring of finite global dimension. Mirković-Vilonen [**MV07**] generalized and rigidified the picture further by producing the analogue of the weight functors for perverse sheaves with coefficients in an arbitrary k. Consider the functor $F_\nu : \mathbf{P}_{G(\mathcal{O})}(\mathcal{G}r; k) \to k\text{-mod}$ for each $\nu \in X^*$ which takes compactly supported cohomology along the $N(\mathcal{K})$-orbit containing $[t^\nu]$. They prove that these cohomology groups vanish outside of degree $2\rho(\nu)$ and that the functors are exact. Summing over all ν, they prove that there is a natural equivalence of functors

$$H^* \cong \bigoplus_{\nu \in X_*(T)} F_\nu : \mathbf{P}_{G(\mathcal{O})}(\mathcal{G}r; k) \to k\text{-mod}.$$

This more refined fiber functor together with some delicate arithmetic work allowed them to prove that geometric Satake is true for any such k, meaning the category of $G(\mathcal{O})$-equivariant perverse sheaves with k-coefficients is tensor equivalent to the category of representations of the split form of the Langland dual group $G^\vee$ over k. In other words,

Theorem 1.4. *There is an equivalence of tensor categories:*

$$(\mathbf{P}_{G(\mathcal{O})}(\mathcal{G}r^G; k), *) \simeq (\mathrm{Rep}(G^\vee_k), \otimes).$$

1.2. Finite and affine flag varieties. In this subsection we give an overview of work of Soergel [**Soe00**] and Fiebig [**Fie06, Fie07b,**

Fie07a] relating the rational representation theory of reductive algebraic groups over a field k of positive characteristic to sheaves on complex Schubert varieties with coefficients in k.

Fix a field k of characteristic p. Recall that G denotes a reductive algebraic group over $\mathbb{C}$ and that $G_k^\vee$ is the split reductive algebraic group over a field k with root datum dual to that of G. In this section we assume that G is connected, simple and adjoint. It follows that $G_k^\vee$ is simply connected.

The previous section explained how one may give a geometric construction of the entire category of representations of $G_k^\vee$ in terms of G. The constructions which follow establish a relation between blocks (certain subcategories) of representations of $G_k^\vee$ and sheaves on (affine) Schubert varieties associated to G.

In order to explain this we need to recall some standard facts from representation theory which one may find in [**Jan87**]. Recall that we have also fixed a Borel subgroup and maximal torus $G_k^\vee \supset B_k^\vee \supset T_k^\vee$, that we write R_* and R_*^+ for the roots and positive roots of $(G_k^\vee, T_k^\vee)$ respectively, chosen so that $-R_*^+$ are the roots determined by $B_k^\vee$. By duality we may identify $X_*(T)$ and $X^*(T_k^\vee)$. We denote by $\operatorname{Rep} G_k^\vee$ the category of all finite dimensional rational representations of $G_k^\vee$.

To each weight $\lambda \in X^*(T_k^\vee)$ one may associate a $G_k^\vee$-equivariant line bundle $\mathcal{O}(\lambda)$ on $G_k^\vee/B_k^\vee$. Its global sections

$$H^0(\lambda) = H^0(G_k^\vee/B_k^\vee, \mathcal{O}(\lambda))$$

contain a unique simple subrepresentation $L(\lambda)$, and all simple $G_k^\vee$-modules arise in this way. The module $H^0(\lambda)$ is non-zero if and only if λ is dominant.

It is known that the characters of $H^0(\lambda)$ are given by the Weyl character formula and that $L(\mu)$ can only occur as a composition factor in $H^0(\lambda)$ if $\lambda - \mu \in \mathbb{N}R_*^+$. It follows that in order to determine the characters of the simple $G_k^\vee$-modules, it is enough to determine, for all dominant $\lambda, \mu \in X^*(T_k^\vee)$, the multiplicities:

$$[H^0(\lambda) : L(\mu)] \in \mathbb{N}. \tag{1}$$

In fact, many of these multiplicities are zero. Recall that the Weyl group W acts on $X^*(T_k^\vee)$ and we may consider the "dot action" given by

$$w \cdot \lambda = w(\lambda + \rho) - \rho$$

where ρ denotes the half-sum of the positive roots. Denote by $\widehat{W}$ the subgroup of all affine transformations of $X^*(T_k^\vee)$ generated by $(w\cdot)$ for $w \in W$ and $(+\mu)$ for all $\mu \in p\mathbb{Z}R_*$. As an abstract group this is isomorphic to the affine Weyl group associated to the root system of

$G_k^\vee$. The category of rational representations of $G_k^\vee$ decomposes into blocks[2]

$$\operatorname{Rep} G_k^\vee = \bigoplus_\Omega \operatorname{Rep}_\Omega G_k^\vee$$

where Ω runs over the orbits of $\widehat{W}$ on $X^*(T_k^\vee)$ and $\operatorname{Rep}_\Omega G_k^\vee$ denotes the full subcategory of $\operatorname{Rep} G_k^\vee$ whose objects are those representations all of whose simple factors are isomorphic to $L(\lambda)$ for $\lambda \in \Omega$.

Assume from now on that $p > h$, where h denotes the Coxeter number of the root system of $G_k^\vee$. Then the "translation principle" allows one to conclude that it is enough to know the characters of the simple modules in $\operatorname{Rep}_\Omega G_k^\vee$, where $\Omega = \widehat{W} \cdot 0$. This fact, combined with the Steinberg tensor product theorem, allows one to reduce the problem to calculating the multiplicities

$$[H^0(x \cdot 0) : L(y \cdot 0)] \in \mathbb{N} \tag{2}$$

where $x \cdot 0$ and $y \cdot 0$ lie in the "fundamental box":

$$I = \{\lambda \in X^*(T_k^\vee) \mid \langle \alpha^\vee, \lambda \rangle < p \text{ for all } \alpha^\vee \text{ simple}\}.$$

A celebrated conjecture of Lusztig expresses the multiplicity in (2) for $x \cdot 0$ and $y \cdot 0$ lying in I in terms of certain Kazhdan-Lusztig polynomials evaluated at 1. Kazhdan-Lusztig polynomials are the Poincaré polynomials of the local intersection cohomology of Schubert varieties (here in the affine case) [**KL80a**], and they can be defined through an inductive combinatorial procedure (which provided their original definition in [**KL79**]). This conjecture is known to hold for almost all p by work of Andersen, Jantzen and Soergel [**AJS94**].

We now return to geometry. Recall that G is a reductive algebraic group whose root system is dual to that of $G_k^\vee$. We identify the Weyl groups of G and $G_k^\vee$ in the obvious way. For each simple reflection $s \in S$ we may associate a minimal parabolic $B \subset P_s \subset G$ and we have a projection map

$$\pi_s : G/B \to G/P_s.$$

Let $D_c^b(G/B, k)$ denote the bounded derived category of constructible sheaves of k-vector spaces on G/B. In [**Soe00**], Soergel considers the category $\mathcal{K}$ defined to be the smallest additive subcategory of $D^b(G/B; k)$ such that:

(1) the skyscraper sheaf on $B/B \in G/B$ is in $\mathcal{K}$;

[2]A family of full subcategories $\mathcal{C}_\Omega$ of a category $\mathcal{C}$ yields a block decomposition (written $\mathcal{C} = \bigoplus \mathcal{C}_\Omega$) if every object M in $\mathcal{C}$ is isomorphic to a direct sum of objects $M_\Omega \in \mathcal{C}_\Omega$ and there are no morphisms between objects of $\mathcal{C}_\Omega$ and objects of $\mathcal{C}_{\Omega'}$ if $\Omega \neq \Omega'$.

(2) if $\mathcal{F} \in \mathcal{K}$ then so is $\pi_s^* \pi_{s*} \mathcal{F}$;
(3) if $\mathcal{F} \in \mathcal{K}$ then so is any object isomorphic to a shift of a direct summand of $\mathcal{F}$.

If k were of characteristic 0 then one could use the decomposition theorem to show that any indecomposable object in $\mathcal{K}$ isomorphic to a (shift of) an intersection cohomology complex of the closure of a B-orbit on G/B. However, as k is of positive characteristic this is not necessarily the case. Somewhat surprisingly, for each $x \in W$, it is still true that there exists up to isomorphism a unique indecomposable object $\mathcal{F}_x \in \mathcal{K}$ supported on $\overline{BxB/B}$ and such that $(\mathcal{F}_x)_{BxB/B} \simeq \underline{k}_{BxB/B}[\ell(w)]$. Each $\mathcal{F}_x$ is self-dual and any indecomposable object in $\mathcal{K}$ is isomorphic to $\mathcal{F}_x[m]$ for some $x \in W$ and $m \in \mathbb{Z}$.

Soergel goes on to establish a connection between $\mathcal{K}$ and the representation theory of $G_k^\vee$ as follows. Let

$$\rho = \frac{1}{2} \sum_{\alpha \in R_*^+} \alpha \in X^*(T_k^\vee)$$

and $st = (p-1)\rho$ be the Steinberg weight. He shows:

THEOREM 1.5 ([**Soe00**], Theorem 1.2). *With k as above, for $x, y \in W$, one has*

$$[H^0(st + x\rho) : L(st + y\rho)] = \dim(\mathcal{F}_y)_x$$

where $\dim(\mathcal{F}_y)_x$ denotes the total dimension of the cohomology of the stalk $(\mathcal{F}_y)_x$.

A disadvantage of the above approach is that it offers a geometric interpretation for only a small part of the representation theory of $G_k^\vee$. In recent work Fiebig has developed a more complete (and necessarily more complicated) picture.

Before we describe Fiebig's work we recall a construction of T-equivariant intersection cohomology due to Braden and MacPherson in [**BM01**]. Let $T \simeq (\mathbb{C}^*)^n$ be an algebraic torus and X a complex T-variety with finitely many zero- and one-dimensional orbits. To this situation one may associate a labelled graph called the "moment graph", which encodes the structure of the zero- and one-dimensional orbits (see the notes of Jantzen from this conference). Under some additional assumptions on X (the most important being a T-invariant stratification into affine spaces) Braden and MacPherson describe a method to calculate the T-equivariant intersection cohomology of X with coefficients in $\mathbb{Q}$. This involves the inductive construction of a "sheaf" $M(X, \mathbb{Q})$ on the moment graph of X; the equivariant intersection cohomology is then obtained by taking "global sections" of this sheaf.

As Fiebig points out, Braden and MacPherson's construction makes sense over any field k and produces a sheaf on the moment graph $M(X,k)$; however it is not clear if what one obtains in this way has anything to do with the intersection cohomology with coefficients in k.

Let G and B be as above and consider the affine flag variety $G((t))/\mathcal{I}$ where $\mathcal{I}$ denotes the Iwahori subgroup, defined as the preimage of B under the evaluation map $G(\mathcal{O}) \to G, t \mapsto 0$. As with the loop Grassmannian, $G((t))/\mathcal{I}$ may be given the structure of an ind-scheme. The $\mathcal{I}$-orbits on $G((t))/\mathcal{I}$ are affine spaces parametrized by $\widehat{W}$. Recall that earlier we defined the fundamental box $I \subset X^*(T_k^\vee)$. Define

$$\widehat{W}^{\mathrm{res}} = \{w \in \widehat{W} \mid -w \cdot 0 \in I\}.$$

Fiebig shows:

THEOREM 1.6. *If the stalks of the sheaf on the moment graph $M(\overline{\mathcal{I}x\mathcal{I}/\mathcal{I}}, k)$ are given by Kazhdan-Lusztig polynomials for all $x \in \widehat{W}^{\mathrm{res}}$ then Lusztig's conjecture holds for representations of $G_k^\vee$.*

For a more precise statement we refer the reader to [**Fie07a**]. This result enables Fiebig to give a new proof of the Lusztig's conjecture in almost all characteristics. Actually, Fiebig is able to give an explicit bound, although it is still very big [**Fie08**]. Moreover he is able to prove the multiplicity one case of the conjecture for all characteristics greater than the Coxeter number [**Fie06**].

It is expected that one may obtain the sheaf on the moment graph $M(\overline{\mathcal{I}x\mathcal{I}/\mathcal{I}}, k)$ by applying a functor similar to that considered by Braden and MacPherson to a sheaf $\mathcal{F}_x \in D^b_{G[[t]]}(G((t))/\mathcal{I}; k)$. (This sheaf should be analogous to the indecomposable sheaves considered by Soergel.) If this is the case then Fiebig's theorem asserts that Lusztig's conjecture would follow from a certain version of the decomposition theorem with coefficients in k.

1.3. The modular Springer correspondence. In this subsection, the base field is not $\mathbb{C}$ but $\mathbb{F}_q$, where q is a power of some prime p. Perverse sheaves still make sense in this context, using the étale topology [**BBD82**]. Now G will be $G_{\mathbb{F}_q}$, which we identify its set of $\overline{\mathbb{F}}_p$-points, endowed with a Frobenius endomorphism F. We denote by $\mathfrak{g}$ its Lie algebra, and by W its Weyl group. For simplicity, we assume that p is very good for G, so that the Killing form provides a non-degenerate G-invariant symmetric bilinear form on $\mathfrak{g}$. Thus we can identify $\mathfrak{g}$ with its dual $\mathfrak{g}'$.

In 1976, Springer established a link between the *ordinary* (that is, characteristic zero) representations of W, and the nilpotent cone $\mathcal{N}$ of $\mathfrak{g}$

[**Spr76**]. More precisely, he constructed the irreducible representations of W in the top cohomology (with ℓ-adic coefficients, ℓ being a prime different from p), of some varieties attached to the different nilpotent orbits, the Springer fibers. To each irreducible representation of W corresponds a nilpotent orbit and an irreducible G-equivariant local system on this orbit.

The modular Springer correspondence [**Jut07**] establishes such a link for *modular* representations of W, over a field of characteristic ℓ. The modular irreducible representations of W are still largely unknown, for example if W is a symmetric group of large rank. One would like to know their characters, and this is equivalent to determining the entries in the so-called decomposition matrix, relating ordinary and modular irreducible characters. Using the modular Springer correspondence, one can show that this decomposition matrix can be seen as a submatrix of a decomposition matrix for G-equivariant perverse sheaves on the nilpotent cone. As a result, just as the geometric Satake isomorphism implies that the modular representation theory of reductive groups is encoded in the singularities of the complex affine Grassmannian of the dual group, one can say that the modular representation theory of the Weyl group of a Lie algebra is encoded in the singularities of its nilpotent cone.

We fix a Borel subgroup B of G, with Lie algebra $\mathfrak{b}$. We denote by U the unipotent radical of B, and by $\mathfrak{u}$ the Lie algebra of U. Then $\mathfrak{u}$ is the orthogonal of $\mathfrak{b}$. The group G acts transitively on the set of Borel subalgebras of $\mathfrak{g}$, and the normalizer of the Borel subalgebra $\mathfrak{b}$ is B, so the flag variety $\mathcal{B} := G/B$ can be identified with the set of all Borel subalgebras of $\mathfrak{g}$. It is a smooth projective variety.

One then defines $\tilde{\mathcal{N}} := \{(x, gB) \in \mathcal{N} \times G/B \mid x \in \mathrm{Ad}(g)\mathfrak{b}\}$. One can check that the second projection makes it a G-equivariant vector bundle over $\mathcal{B} = G/B$, and that we have G-equivariant isomorphisms:

$$\tilde{\mathcal{N}} \simeq G \times^B \mathfrak{u} \simeq G \times^B \mathfrak{b}^\perp \simeq G \times^B (\mathfrak{g}/\mathfrak{b})^* \simeq T^*(G/B) = T^*\mathcal{B}$$

where the first isomorphism is given by $(y, gB) \mapsto g * \mathrm{Ad}(g^{-1})y$. The first projection gives a resolution $\pi_{\mathcal{N}} : \tilde{\mathcal{N}} \to \mathcal{N}$ of the nilpotent cone, called the Springer resolution.

Springer constructed an action of the Weyl group W on the ℓ-adic cohomology of the fibers $\mathcal{B}_x := \pi_{\mathcal{N}}^{-1}(x)$ of this resolution, which are called Springer fibers. These are connected projective varieties, which are usually singular. All their irreducible components have the same dimension $d_x := \frac{1}{2}\operatorname{codim}_{\mathcal{N}} G.x$. In particular, to each adjoint orbit of $\mathcal{N}$ one can associate the representation of W on the top cohomology of the corresponding Springer fiber. In type A, this is in fact a bijection

between the nilpotent orbits and the irreducible representations of $\mathfrak{S}_n$. Note that both are parametrized by the set of all partitions of the integer n. It turns out that the bijection is given by the conjugation of partitions.

More generally, for a point $x \in \mathfrak{g}$, let $C_G(x)$ denote the centralizer of x in G and $A_G(x) = C_G(x)/C_G(x)^o$ its component group; for $G = GL_n$ these groups are trivial. As $\pi_{\mathcal{N}}$ is G-equivariant, the centralizer $C_G(x)$ acts on the fiber $\mathcal{B}_x$, and the group $A_G(x)$ acts on the cohomology of $\mathcal{B}_x$. This action commutes with the action of W. Note that the action of $A_G(x)$ on the top cohomology of $\mathcal{B}_x$ is just the permutation representation of $A_G(x)$ on the set of irreducible components of $\mathcal{B}_x$. It turns out that $H^{2d_x}(\mathcal{B}_x)$ is an irreducible $W \times A_G(x)$-module. We can decompose it into $A_G(x)$-isotypic components:

$$H^{2d_x}(\mathcal{B}_x) = \bigoplus_{\rho} \rho \otimes V_{x,\rho}$$

where ρ runs over all irreducible representations of $A_G(x)$ such that the ρ-isotypic component of $H^{2d_x}(\mathcal{B}_x)$ is non-zero, and $V_{x,\rho}$ is a well-defined irreducible representation of W. Springer showed that the $V_{x,\rho}$, for x running over a set of representatives of the nilpotent orbits, form a complete collection of irreducible representations of the Weyl group W. So to each irreducible representation of W we can assign a pair (x, ρ). This is the Springer correspondence.

Later a number of related constructions were obtained by other mathematicians, as in [**Slo80a, KL80b**]. It turns out that the pairs (x, ρ) also parametrize the simple G-equivariant perverse sheaves on the nilpotent cone. Let us denote them by $\mathbf{IC}(x, \rho)$. Lusztig and Borho-McPherson gave a construction using perverse sheaves [**Lus81, BM83**]. Note, however, that all these approaches give a parametrization which differs from the original one obtained by Springer by tensoring with the sign character of W. On the other hand, other approaches, using some kind of Fourier transform, give the same parametrization as the one by Springer. One can use a Fourier transform for $\mathcal{D}$-modules [**HK84**] if the base field is $\mathbb{C}$, or a Fourier-Deligne transform in our context where the base field is $\mathbb{F}_q$ [**Bry86**], using perverse $\mathbb{K}$-sheaves, where $\mathbb{K}$ is a finite extension of $\mathbb{Q}_\ell$. One advantage of the latter approach is that it still makes sense over finite extensions $\mathbb{O}$ (resp. $\mathbb{F}$), of $\mathbb{Z}_\ell$ (resp. $\mathbb{F}_\ell$), so that it is possible to define a modular Springer correspondence [**Jut07**].

The Fourier-Deligne transform is an equivalence of derived categories of constructible sheaves between V and V', where V is a vector bundle $\xi : V \to S$ of constant rank r over a scheme S of finite type over

k, and $\xi' : V' \to S$ is its dual. In particular, if $S = \operatorname{Spec} \mathbb{F}_q$, then V is just an $\mathbb{F}_q$-vector space. For $\mathfrak{g} \to \operatorname{Spec} \mathbb{F}_q$, we get an auto-equivalence of $D^b_c(\mathfrak{g}, \mathbb{K})$, since we identify $\mathfrak{g}$ with its dual. Though we will use the Fourier-Deligne transform in the particular case $S = \operatorname{Spec} \mathbb{F}_q$, at some point we will also need to use the relative version, with $S = \mathcal{B}$.

This equivalence is a sheaf-theoretic version of the ordinary Fourier transform for functions on $\mathbb{R}^n$. Recall that the ordinary Fourier transform of a function $f : \mathbb{R}^n \to \mathbb{C}$ is the function $\hat{f} : (\mathbb{R}^n)^* \to \mathbb{C}$ given by the formula:

$$\hat{f}(\zeta) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \langle x, \zeta \rangle} dx.$$

In other words, the Fourier transform takes a function f on $\mathbb{R}^n$, and

- pulls it back by the first projection to the product $\mathbb{R}^n \times (\mathbb{R}^n)^*$;
- multiplies it by the pull-back of the exponential function $t \mapsto e^{-2\pi i t}$ via the evaluation map $\mathbb{R}^n \times (\mathbb{R}^n)^* \to \mathbb{R}$;
- pushes it forward to $(\mathbb{R}^n)^*$ by integrating along the fibres of the second projection.

In order to mimic this procedure sheaf-theoretically, we need to find a replacement for the exponential function. This role is played by an Artin-Schreier sheaf, and this is the reason why we use $\mathbb{F}_q$ as a base field, instead of a field of characteristic zero.

First, let us define a Fourier transform for $\mathbb{C}$-valued functions on $\mathbb{F}_q^n$. Instead of the exponential character $\mathbb{R} \to \mathbb{C}^*$, $t \mapsto e^{-2\pi i t}$, we have to choose a non-trivial character ψ of $\mathbb{F}_q$. For example, we can take the character $t \mapsto e^{-\frac{2\pi i}{p} t}$ of $\mathbb{F}_p$, and compose it with $\operatorname{Tr}_{\mathbb{F}_q/\mathbb{F}_p}$. For $f : \mathbb{F}_q^n \to \mathbb{C}$, we set

$$\hat{f}(\zeta) = \sum_{x \in \mathbb{F}_q^n} f(x) e^{-\frac{2\pi i}{p} \psi(\langle x, \zeta \rangle)}.$$

Let k denote either $\mathbb{K}$, $\mathbb{O}$ or $\mathbb{F}$ (see above). We can replace $\mathbb{C}$-valued functions by k-valued functions, as soon as $k^\times$ contains the pth roots of unity, which we assume from now on.

Let us now define the Fourier-Deligne transform. We consider the Artin-Schreier covering of the affine line: $\mathbb{A}^1 \to \mathbb{A}^1$, $t \mapsto t - t^q$. It is a Galois finite étale morphism, with Galois group $\mathbb{F}_q$. Thus to the character ψ corresponds a local system on $\mathbb{A}^1$, which we will denote by $\mathcal{L}_\psi$. We can pull it back to $V \times_S V'$ by the canonical pairing $\mu : V \times_S V' \to \mathbb{A}^1$.

As with functions, the Fourier-Deligne transform is a convolution against a kernel. It is defined as

$$\mathcal{F} = (\mathrm{pr}')_!(\mathrm{pr}^*(-) \otimes \mu^* \mathcal{L}_\psi) : D^b_c(V, k) \longrightarrow D^b_c(V', k),$$

where the notation is fixed by the following diagram:

$$\begin{tikzcd} & V \times_S V' \arrow[dl, "\mathrm{pr}"'] \arrow[dr, "\mathrm{pr}'"] \arrow[rr, "\mu"] & & \mathbb{A}^1 \\ V \arrow[dr] & & V' \arrow[dl] & \\ & S & & \end{tikzcd} \tag{3}$$

Most properties of the ordinary Fourier transform still hold for the Fourier-Deligne transform, if translated appropriately: it interchanges the extension by zero of the constant sheaf on the zero section with the constant sheaf in degree $-\dim V$ (and more generally interchanges the constant sheaf on a sub-bundle with the constant sheaf on its annihilator up to a shift), it "squares to the identity" up to a sign, which implies that it is an equivalence of triangulated categories, and it behaves well under base-change. Moreover, it takes perverse sheaves to perverse sheaves and, restricted to such, is in fact an equivalence of abelian categories.

We will now briefly describe the Springer correspondence using this Fourier-Deligne transform. Let us consider the adjoint quotient $\chi : \mathfrak{g} \to \mathfrak{g}//G$. The morphism χ is flat and surjective. By Chevalley's restriction theorem, we have $\mathfrak{g}//G \simeq \mathfrak{t}/W$. We denote by $\phi : \mathfrak{t} \to \mathfrak{t}/W$ the quotient morphism, which is finite and surjective. For $t \in \mathfrak{t}$, we will write $\phi(t) = \overline{t}$. One can see the nilpotent cone $\mathcal{N}$ as $\chi^{-1}(\overline{0})$. On the other hand, if t is a regular element in $\mathfrak{t}$, then $\chi^{-1}(\overline{t}) \simeq G/T$ is smooth. The fibers $\chi^{-1}(\overline{t})$, for $\overline{t}$ varying in $\mathfrak{t}/W$, interpolate these two extreme cases. Grothendieck found a way to obtain a resolution for all the fibers of χ simultaneously. We introduce $\tilde{\mathfrak{g}} := \{(x, gB) \in \mathfrak{g} \times G/B \mid x \in \mathrm{Ad}(g)\mathfrak{b}\}$. The second projection makes it a G-equivariant vector bundle over $\mathcal{B}$, isomorphic to $G \times^B \mathfrak{b}$. The first projection defines a proper surjective morphism $\pi : \tilde{\mathfrak{g}} \to \mathfrak{g}$. Then one can form a commutative diagram:

$$\begin{tikzcd} \tilde{\mathfrak{g}} \arrow[r, "\pi"] \arrow[d, "\theta"'] & \mathfrak{g} \arrow[d, "\chi"] \\ \mathfrak{t} \arrow[r, "\phi"'] & \mathfrak{t}/W \end{tikzcd}$$

where θ is the map $G \times^B \mathfrak{b} \to \mathfrak{b}/[\mathfrak{b}, \mathfrak{b}] \simeq \mathfrak{t}$. This is a smooth surjective morphism. Then for all $t \in \mathfrak{t}$, the morphism $\pi_t : \theta^{-1}(t) \to \chi^{-1}(\overline{t})$ is a resolution of singularities.

In the case of $\mathfrak{gl}_n$, one can see χ as the map taking a matrix to its eigenvalues (with multiplicities) up to ordering. On the other hand, the variety G/B can be identified with the variety of all complete flags $F_\bullet = (0 \subset F_1 \subset \cdots \subset F_n = \mathbb{C}^n)$. Then θ is identified with the map taking a pair $(x, F_\bullet)$ to $(a_1, \ldots, a_n)$ where a_i is the eigenvalue of x on F_i/F_{i-1}.

Inside of $\mathfrak{g}$, we have the dense open subvariety $\mathfrak{g}_{rs}$ of regular semi-simple elements. Consider for a moment the case of $\mathfrak{gl}_n$, then the semi-simple regular elements are those that are diagonalizable with pairwise distinct eigenvalues. For such a matrix consider the set of Borel subalgebras containing it. This is equivalent to the set of full flags preserved by the matrix. But the matrix decomposes $\mathbb{C}^n$ into a direct sum of n lines with distinct eigenvalues. Thus any subspace preserved by the matrix is a sum of such lines. It follows that the collection of such flags is a torsor for the symmetric group on the eigenspaces. More generally, the restriction $\pi_{\mathrm{rs}} : \tilde{\mathfrak{g}}_{\mathrm{rs}} \to \mathfrak{g}_{\mathrm{rs}}$ of π over $\mathfrak{g}_{\mathrm{rs}}$ is a principal Weyl group bundle: one can show that it is a Galois finite étale morphism with Galois group W.

A local system on $\mathfrak{g}_{\mathrm{rs}}$ can be identified with a representation of the fundamental group of $\mathfrak{g}_{\mathrm{rs}}$. By the above, W is a finite quotient of this fundamental group. Actually, the local systems which correspond to a representation factoring through W are those whose pull-back to $\tilde{\mathfrak{g}}_{\mathrm{rs}}$ is trivial. The local system $\pi_{\mathrm{rs}*}\underline{k}_{\tilde{\mathfrak{g}}_{\mathrm{rs}}}$ corresponds to the regular representation of the group algebra kW. Its endomorphism algebra is the group algebra kW.

It is known that $\pi_*\underline{k}_{\tilde{\mathfrak{g}}}[\dim \mathfrak{g}] = \mathbf{IC}(\mathfrak{g}, \pi_{\mathrm{rs}*}\underline{k}_{\tilde{\mathfrak{g}}_{\mathrm{rs}}})$ because π is a small proper morphism (this notion will be recalled in Subsection 2.2). But, in general, the IC complex is given by an intermediate extension functor $j_{\mathrm{rs}!*}$ which is fully faithful, where $j_{\mathrm{rs}} : \mathfrak{g}_{\mathrm{rs}} \to \mathfrak{g}$ is the open immersion. Thus the endomorphism algebra of $\pi_*\underline{k}_{\tilde{\mathfrak{g}}}[\dim \mathfrak{g}]$ is still kW.

We have

$$\mathcal{F}(\pi_{\mathcal{N}*}\underline{k}_{\tilde{\mathcal{N}}}[\dim \mathcal{N}]) \simeq \pi_*\underline{k}_{\tilde{\mathfrak{g}}}[\dim \mathfrak{g}]$$

(we ignore Tate twists). To see this one uses the descriptions of $\tilde{\mathcal{N}}$ and $\tilde{\mathfrak{g}}$ as $G \times^B \mathfrak{u}$ and $G \times^B \mathfrak{b}$ respectively. These are two orthogonal subbundles of the trivial bundle $\mathfrak{g} \times \mathcal{B}$ over $\mathcal{B}$, hence they are exchanged by the Fourier-Deligne transform with base $\mathcal{B}$. By base change, it follows that the perverse sheaves $\pi_{\mathcal{N}*}\underline{k}_{\tilde{\mathcal{N}}}[\dim \mathcal{N}]$ and $\pi_*\underline{k}_{\tilde{\mathfrak{g}}}[\dim \mathfrak{g}]$ are exchanged by the Fourier-Deligne transform on $\mathfrak{g}$ (here the former is considered as a perverse sheaf on $\mathfrak{g}$ by extension by zero). As the Fourier-Deligne transform is an equivalence of categories, we can conclude that the endomorphism algebra of $\pi_{\mathcal{N}*}\underline{k}_{\tilde{\mathcal{N}}}[\dim \mathcal{N}]$ is again kW.

Now, assume k is $\mathbb{K}$ or $\mathbb{F}$. Given a simple kW-module E, one can consider the corresponding local system $\mathcal{L}_E$ on $\mathfrak{g}_{\mathrm{rs}}$. Then one can show that $\mathcal{F}(\mathbf{IC}(\mathfrak{g}, \mathcal{L}_E))$ is a simple G-equivariant perverse sheaf supported on $\mathcal{N}$, thus we can associate to E a nilpotent orbit and a representation of the centralizer component group. In this way, one obtains a Springer correspondence for arbitrary characteristic: a map Ψ_k from $\operatorname{Irr} kW$ to the set $\mathfrak{P}_k$ of such pairs consisting of an orbit and representation of the component group (for characteristic zero coefficients, see [**Bry86**]; for characteristic ℓ, see [**Jut07**]). For example, in the case $G = GL_n$, the simple $k\mathfrak{S}_n$-modules are denoted D_μ, where μ runs over a subset of the partitions of n, whose elements are called ℓ-regular partitions, and it turns out that the modular Springer correspondence is given, as in characteristic zero, by the transposition of partitions [**Jut07**, §6.4].

The fact that there is an inverse Fourier transform implies that the map Ψ_k is an injection. The fact that the Fourier transform of the constant perverse sheaf is the sky-scraper sheaf concentrated in zero implies that the pair corresponding to the trivial representation consists of the trivial orbit and the trivial character.

Now, if E is a $\mathbb{K}W$-module, we can choose an $\mathbb{O}$-lattice $E_{\mathbb{O}}$ stable by W. Then $\mathbb{F} \otimes_{\mathbb{O}} E_{\mathbb{O}}$ is an $\mathbb{F}W$-module whose class in the Grothendieck group does not depend on the choice of the lattice [**Ser67**]. Thus we have well-defined multiplicities $d^W_{E,F} = [\mathbb{F} \otimes_{\mathbb{O}} E_{\mathbb{O}} : F]$ for F an $\mathbb{F}W$-module. One can similarly define decomposition numbers $d^{\mathcal{N}}_{(x,\rho),(y,\sigma)}$ for G-equivariant perverse sheaves on the nilpotent cone, where $(x, \rho) \in \mathfrak{P}_{\mathbb{K}}$ and $(y, \sigma) \in \mathfrak{P}_{\mathbb{F}}$. Then we have

$$d^W_{E,F} = d^{\mathcal{N}}_{\Psi_{\mathbb{K}}(E),\Psi_{\mathbb{F}}(F)}$$

which shows that the decomposition matrix of W is a submatrix of the decomposition matrix for G-equivariant perverse sheaves on the nilpotent cone [**Jut07**]. It follows that the stalks of IC sheaves in characteristic ℓ on the nilpotent singularities encode the modular representation theory of Weyl groups.

2. Perverse sheaves

2.1. Constructible sheaves. Throughout, k will denote a field or $\mathbb{Z}$. All varieties will be varieties over the complex numbers equipped with the classical topology and all morphisms will be morphisms of varieties. Dimension will always mean the complex dimension.

Let X be a variety. We will denote by $\mathcal{S}$ a decomposition

$$X = \bigsqcup_{S \in \mathcal{S}} S \tag{4}$$

of X into finitely many locally closed (in the Zariski topology) connected smooth subvarieties. A sheaf of k-vector spaces $\mathcal{F}$ on X will be called $\mathcal{S}$-constructible if the restriction of $\mathcal{F}$ to each $S \in \mathcal{S}$ is a local system (a sheaf of k-modules which is locally isomorphic to a constant sheaf with values in a finitely generated k-module). A sheaf $\mathcal{F}$ is constructible if there exists an $\mathcal{S}$ as above making it $\mathcal{S}$-constructible.

Let $D^b(X,k)$ denote the bounded derived category of sheaves of k-vector spaces. Given $\mathcal{K} \in D^b(X,k)$ we denote its cohomology sheaves by $\mathcal{H}^m(\mathcal{K})$. We denote by $D^b_c(X,k)$ (resp. $D^b_{\mathcal{S}}(X,k)$) the full subcategory of $D^b(X,k)$ with objects consisting of complexes $\mathcal{K} \in D^b(X,k)$ such that $\mathcal{H}^m(\mathcal{K})$ is constructible (resp. $\mathcal{S}$-constructible) for all m. We have truncation functors $\tau_{\leq i}$ and $\tau_{>i}$ on $D^b(X,k)$, $D^b_c(X,k)$ and $D^b_{\mathcal{S}}(X,k)$. For example, $\mathcal{H}^m(\tau_{\leq i}\mathcal{K})$ is isomorphic to $\mathcal{H}^m(\mathcal{K})$ if $m \leq i$ and is 0 otherwise.

We have internal bifunctors $\mathrm{R}\underline{\mathrm{Hom}}$ and $\otimes^L_k$ on $D^b_c(X,k)$, which are the derived functor of the usual bifunctors on categories of sheaves. For any morphism $f : X \to Y$ we have functors:

$$\begin{tikzcd} D^b_c(X,k) \arrow[r, bend left, "f_*, f_!"] & D^b_c(Y,k) \arrow[l, bend left, "f^*, f^!"] \end{tikzcd}$$

The functors f_* and $f_!$, sometimes denoted Rf_* and $Rf_!$, are the right derived functors of the direct image and direct image with compact support (both functors are left exact). The inverse image functor f^* for sheaves is exact, and passes trivially to the derived category. The pair (f^*, f_*) is adjoint. It turns out that the derived functor $f_!$ has a right adjoint, namely $f^!$.

In the case where Y is a point, for $\mathcal{F} \in D^b_c(X,k)$ we have

$$\begin{aligned} f_*\mathcal{F} &= \mathrm{R}\Gamma(X,\mathcal{F}), & f^*\underline{k}_{\mathrm{pt}} &= \underline{k}_X, \\ f_!\mathcal{F} &= \mathrm{R}\Gamma_{\mathrm{c}}(X,\mathcal{F}), & f^!\underline{k}_{\mathrm{pt}} &= \mathcal{D}_X, \end{aligned}$$

where $\mathcal{D}_X$ is the dualizing sheaf of X. It allows one to define the dualizing functor:

$$\mathbb{D} = \mathrm{R}\underline{\mathrm{Hom}}(-, \mathcal{D}_X) : D^b(X,k)^{\mathrm{op}} \longrightarrow D^b(X,k).$$

Its square is isomorphic to the identity functor. For example, if X is smooth of dimension d and $\mathcal{L}$ is a local system on X then $\mathbb{D}(\mathcal{L}[d]) \simeq \mathcal{L}^\vee[d]$, where $\mathcal{L}^\vee$ denotes the dual local system. In that case, we have $\mathcal{D}_X \simeq \underline{k}_X[2d]$, and $\underline{k}_X[d]$ is self-dual.

In general, we have isomorphisms $\mathbb{D}f_* \simeq f_!\mathbb{D}$ and $\mathbb{D}f^* \simeq f^!\mathbb{D}$. If Y is a point, X is smooth and $\mathcal{L}$ is a local system then the first isomorphism yields Poincaré duality between $H^*(X, \mathcal{L})$ and $H^*_c(X, \mathcal{L}^\vee)$.

In what follows, we will fix a decomposition $\mathcal{S}$ of X as in (4) and assume additionally that $\mathcal{S}$ is a Whitney stratification. For $\mathcal{S}$ to be a stratification, we require that the closure of a stratum is a union of strata. The Whitney conditions,[3] which we are not going to describe, will ensure that the functors induced by inclusions of unions of strata, and the duality, preserve the notion of $\mathcal{S}$-constructibility. Any stratification of X can be refined into a Whitney stratification.

As an important special case, if X is a G-variety with finitely many orbits, where G is a connected algebraic group, we can choose for $\mathcal{S}$ the set of G-orbits on X. In that case, we will be interested in G-equivariant sheaves. This notion will be explained below.

2.2. Perverse sheaves with coefficients in a field. Throughout this section we assume that k is a field. As in the previous section we fix a variety X with Whitney stratification $\mathcal{S}$. The category of perverse sheaves constructible with respect to $\mathcal{S}$, denoted $\mathbf{P}_{\mathcal{S}}(X, k)$, consists of the full subcategory of those objects $\mathcal{F} \in D^b_{\mathcal{S}}(X, k)$ such that:

(1) for all $S \in \mathcal{S}$, $i_S^*\mathcal{F}$ is concentrated in degrees $\leq -\dim S$,
(2) for all $S \in \mathcal{S}$, $i_S^!\mathcal{F}$ is concentrated in degrees $\geq -\dim S$.

Note that these two conditions are exchanged by $\mathbb{D}$. It follows that $\mathbb{D}$ preserves $\mathbf{P}_{\mathcal{S}}(X, k)$. We say that $\mathbf{P}_{\mathcal{S}}(X, k)$ is the heart of the t-structure $(D^{\leq 0}_{\mathcal{S}}(X, k), D^{\geq 0}_{\mathcal{S}}(X, k))$ where $D^{\leq 0}_{\mathcal{S}}(X, k)$, resp. $D^{\geq 0}_{\mathcal{S}}(X, k)$, is the full subcategory of $D^b_{\mathcal{S}}(X, k)$ with objects satisfying condition (1), resp. (2).

For $\mathcal{F} \in \mathbf{P}_{\mathcal{S}}(X, k)$, we are interested in the stalks of the cohomology sheaves of $\mathcal{F}$. An induction shows that any perverse sheaf $\mathcal{F}$ only has non-trivial stalks in degrees $\geq -d$, where $d = \dim X$. Hence, we see that $\mathcal{F}$ is perverse if and only if the cohomology sheaves of both $\mathcal{F}$ and $\mathbb{D}\mathcal{F}$ along strata are of the following form (see [**Ara01**]):

strata	$\dots$	$-d$	$-d+1$	$\dots$	-1	0
S_d	0	$*$	0	0	0	0
S_{d-1}	0	$*$	$*$	0	0	0
$\vdots$	0	$\vdots$	$\vdots$	$\ddots$	0	0
S_1	0	$*$	$*$	$\dots$	$*$	0
S_0	0	$*$	$*$	$\dots$	$*$	$*$

[3] See [**Whi65**] p. 540, [**B+08**] A'Campo §IV.1 p. 41, or [**Lip00**] §1–2.

Here S_m denotes the union of strata of dimension m and $*$ denotes the possibility of a non-trivial cohomology sheaf. For example, the first line tells us that the restriction of $\mathcal{F}$ to an open stratum is either zero or a local system concentrated degree $-d$. The last line tells us that the stalks and costalks of $\mathcal{F}$ at 0-dimensional strata can be non-trivial in degrees between $-d$ and 0.

The category $\mathbf{P}_{\mathcal{S}}(X,k)$ is an abelian category and every object has finite length. The exact sequences in $\mathbf{P}_{\mathcal{S}}(X,k)$ are those sequences

$$\mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3$$

which can be completed, via a map $\mathcal{F}_3 \to \mathcal{F}_1[1]$ (necessarily unique in this case), to distinguished triangles in $D^b_{\mathcal{S}}(X,k)$.

As in any abelian category of finite length, it is important to understand the simple objects. One has a bijection:

$$\left\{ \begin{array}{c} \text{simple objects} \\ \text{in } \mathbf{P}_{\mathcal{S}}(X,k) \end{array} \right\} \xrightarrow{\sim} \left\{ \begin{array}{c} \text{pairs } (S,\mathcal{L}) \text{ where } S \in \mathcal{S} \\ \text{and } \mathcal{L} \text{ is an irreducible} \\ \text{local system on } S \end{array} \right\}.$$

This bijection may be described as follows. Given a pair $(S,\mathcal{L})$ there exists a unique object $\mathbf{IC}(\overline{S},\mathcal{L}) \in \mathbf{P}_{\mathcal{S}}(X,k)$ such that

(1) $i_S^*\mathbf{IC}(\overline{S},\mathcal{L}) \simeq \mathcal{L}[d_S]$,
(2) $\mathbf{IC}(\overline{S},\mathcal{L})$ is supported on $\overline{S}$,

and, for all strata $T \subset \overline{S}$ with $T \neq S$,

(2) $i_T^*\mathbf{IC}(\overline{S},\mathcal{L})$ is concentrated in degrees $< -\dim T$,
(3) $i_T^!\mathbf{IC}(\overline{S},\mathcal{L})$ is concentrated in degrees $> -\dim T$.

The object $\mathbf{IC}(\overline{S},\mathcal{L})$ is called the intersection cohomology complex corresponding to $(S,\mathcal{L})$. A different convention is to shift this complex by $-\dim S$, so that it is concentrated in non-negative degrees. The normalization we use has the advantage that the the intersection cohomology complexes are perverse sheaves.

Note that we must have $\mathbb{D}\mathbf{IC}(\overline{S},\mathcal{L}) \simeq \mathbf{IC}(\overline{S},\mathcal{L}^\vee)$. This explains the existence of a Poincaré duality between $\mathrm{IH}^*(X,\mathcal{L})$ and $\mathrm{IH}^*_c(X,\mathcal{L}^\vee)$.

It is useful to note the special form of the restrictions to the strata of an intersection cohomology complex $\mathbf{IC}(\overline{S},\mathcal{L})$, depicted as before (note the zeroes on the diagonal). Let d_S denote the dimension of S, so that $S \subset S_{d_S}$.

strata	$-d$	$\dots$	$-d_S-1$	$-d_S$	$-d_S+1$	$\dots$	-1	0
S_d	$\mathbf{0}$	0	0	0	0	0	0	0
$\vdots$	0	$\mathbf{0}$	0	0	0	0	0	0
S_{d_S+1}	0	0	$\mathbf{0}$	0	0	0	0	0
S_{d_S}	0	0	0	$\mathbf{i_{S*}\mathcal{L}}$	0	0	0	0
S_{d_S-1}	0	0	0	$*$	$\mathbf{0}$	0	0	0
$\vdots$	0	0	0	$*$	$*$	$\mathbf{0}$	0	0
S_1	0	0	0	$*$	$*$	$*$	$\mathbf{0}$	0
S_0	0	0	0	$*$	$*$	$*$	$*$	$\mathbf{0}$

Let us make a small digression about equivariance. We follow [**Lus84**, §0]. Assume X is endowed with an action of a connected algebraic group G. Let $a : G \times X \to X$ and $\mathrm{pr} : G \times X \to X$ denote the action morphism and the second projection. A perverse sheaf $\mathcal{F}$ is called G-equivariant if there is an isomorphism $\alpha : a^*\mathcal{F} \xrightarrow{\sim} \mathrm{pr}^*\mathcal{F}$. This isomorphism is unique if we impose the condition that the induced isomorphism $i^*a^*\mathcal{F} \xrightarrow{\sim} i^*\pi^*\mathcal{F}$ is the identity of $\mathcal{F}$, where the morphism $i : X \to G \times X$, $x \mapsto (1, x)$ is a section of a and π, so that $i^*a^*\mathcal{F} = \mathcal{F}$ and $i^*\pi^*\mathcal{F} = \mathcal{F}$. This follows from [**BBD82**, Prop. 4.2.5] (Deligne). Then α satisfies the usual associativity condition. Note that this definition of G-equivariance for perverse sheaves does not work for arbitrary complexes.[4] It also works, though, for usual sheaves (complexes concentrated in one degree), and in particular for local systems. The G-equivariant local systems on an orbit S correspond bijectively to the finite dimensional representations of the finite group of components of the isotropy group of a point x in S. We are particularly interested in the case where G has finitely many orbits in X, and we take the stratification of X into its G-orbits. Then the simple G-equivariant perverse sheaves correspond to pairs consisting of an orbit and an irreducible G-equivariant local system on that orbit.

We now recall Deligne's construction of intersection cohomology complexes. As above let S_m denote the union of strata of dimension m and denote by X_m the union of all strata of dimension greater than or equal to m. We have a sequence of inclusions:

$$X_d \overset{j_{d-1}}{\hookrightarrow} X_{d-1} \overset{j_{d-2}}{\hookrightarrow} X_{d-2} \hookrightarrow \cdots \hookrightarrow X_1 \overset{j_0}{\hookrightarrow} X_0 = X.$$

[4] One should instead consider the equivariant derived category as defined in [**BL94b**].

Now let $\mathcal{L}$ be an irreducible local system on $S \in \mathcal{S}$. We still denote by $\mathcal{L}$ its extension by zero to X_{d_S}. One has an isomorphism

$$\mathbf{IC}(\overline{S}, \mathcal{L}) \simeq (\tau_{\leq -1} \circ j_{0*}) \circ (\tau_{\leq -2} \circ j_{1*}) \circ \cdots \circ (\tau_{\leq -d_S} \circ j_{d_S - 1*})(\mathcal{L}[d_S]).$$

This allows the calculation of $\mathbf{IC}(X, \mathcal{L})$ inductively on the strata. We will see examples of this construction below. However, the j_* functors are not easy to compute explicitly in general.

In characteristic zero, the decomposition theorem provides a much more powerful means of calculating the stalks of intersection cohomology complexes. Given a Laurent polynomial $P = \sum a_i v^i \in \mathbb{N}[v, v^{-1}]$ and $\mathcal{K} \in D^b(X, k)$, we define

$$P \cdot \mathcal{K} = \bigoplus \mathcal{K}[i]^{\oplus a_i}.$$

We call a complex $\mathcal{K} \in D^b_{\mathcal{S}}(X, k)$ semi-simple if one has an isomorphism

$$\mathcal{K} \simeq \bigoplus P_{S, \mathcal{L}_S} \cdot \mathbf{IC}(\overline{S}, \mathcal{L}_S)$$

for some $P_{S, \mathcal{L}_S} \in \mathbb{N}[v, v^{-1}]$, where the sum is over all pairs $(S, \mathcal{L})$ consisting of an irreducible local system on a stratum S. In other words, an object $\mathcal{K} \in D^b_{\mathcal{S}}(X, k)$ is semi-simple if it is isomorphic to a direct sum of shifts of irreducible intersection cohomology complexes. One version of the decomposition theorem is the following:

Theorem 2.1 ([**BBD82**], [**Sai89**], [**dCM05**]). *Let $\pi : Y \to X$ be a proper map from a smooth variety Y. If k is of characteristic zero, then $\pi_* \underline{k}_Y[\dim Y]$ is semi-simple.*

We will see examples below of the failure of the decomposition theorem in positive characteristic.

Some conditions on the dimensions of the fibers of a resolution, however, have consequences which hold for arbitrary coefficients. Let $\pi : Y \to X$ be a proper morphism between n-dimensional irreducible varieties, and assume X is endowed with a stratification $\mathcal{S}$ such that π is a weakly stratified mapping, that is, for each stratum S in $\mathcal{S}$, the restriction of π to $\pi^{-1}(S)$ is a topological fibration with base S and fibre F_S. Then π is said to be semi-small if, for all S, we have $\dim F_S \leq \frac{1}{2} \operatorname{codim}_X(S)$. A stratum S is relevant for π if equality holds. We say that π is small if the only relevant stratum is the dense one. The following proposition is well-known [**BM81, BM83, GM83**]:

Proposition 2.2. *Let $\pi : Y \to X$ be a proper morphism as above with Y smooth.*

(1) *If π is semi-small then $\pi_* \underline{k}_Y[\dim Y]$ is a perverse sheaf.*

(2) *If π is small then $\pi_* \underline{k}_Y[\dim Y]$ is an intersection cohomology complex.*

2.3. Perverse sheaves over the integers. In this section, we will give a flavour of the subtleties that occur when we take integer coefficients. See [**BBD82**, §3.3], and [**Jut09**] for a more detailed study.

Let us first consider the bounded derived category of constructible sheaves of $\mathbb{Z}$-modules on the point $\mathrm{pt} = \operatorname{Spec} \mathbb{C}$. These are just complexes of $\mathbb{Z}$-modules with finitely many non-zero cohomology groups, all of which are finitely generated over $\mathbb{Z}$. The duality functor is $\mathbb{D} = \mathrm{RHom}(-, \mathbb{Z})$. The perverse sheaves on pt are just finitely generated $\mathbb{Z}$-modules placed in degree zero. They form the heart of the usual t-structure, corresponding to the perversity p. We will see that this t-structure is not preserved by the duality.

Since $\mathbb{Z}$ is hereditary (of global dimension 1), any object of $D^b_c(\mathrm{pt}, \mathbb{Z})$ is isomorphic to the direct sum of its shifted cohomology objects. So the indecomposable objects in $D^b_c(\mathrm{pt}, \mathbb{Z})$ are concentrated in one degree, and they are isomorphic, up to shift, either to $\mathbb{Z}$ or to $\mathbb{Z}/\ell^a$ for some prime number ℓ, and some positive integer a.

First consider the indecomposable $\mathbb{Z}$. It is a free, hence projective $\mathbb{Z}$-module, and thus we have $\mathbb{D}(\mathbb{Z}) = \mathrm{RHom}(\mathbb{Z}, \mathbb{Z}) = \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) = \mathbb{Z}$. Here there is no problem. The dual of a torsion-free module remains in the heart of the natural t-structure.

Now consider the case of $\mathbb{Z}/\ell^a$. Here we cannot apply the functor $\mathrm{Hom}(-, \mathbb{Z})$ directly to the module $\mathbb{Z}/\ell^a$. First we have to replace $\mathbb{Z}/\ell^a$ by a projective resolution, as $\mathbb{Z} \xrightarrow{\ell^a} \mathbb{Z}$, with the first $\mathbb{Z}$ in degree -1. Now we can apply the functor $\mathrm{Hom}(-, \mathbb{Z})$, and we get the complex $\mathbb{Z} \xrightarrow{\ell^a} \mathbb{Z}$ with the last $\mathbb{Z}$ in degree 1. So the dual of $\mathbb{Z}/\ell^a$ is isomorphic to $\mathbb{Z}/\ell^a[-1]$. This is another way to say that $\mathrm{Hom}(\mathbb{Z}/\ell^a, \mathbb{Z}) = 0$, $\mathrm{Ext}^1(\mathbb{Z}/\ell^a, \mathbb{Z}) \simeq \mathbb{Z}/\ell^a$, and $\mathrm{Ext}^i(\mathbb{Z}/\ell^a, \mathbb{Z}) = 0$ for $i > 1$.

This shows that the usual t-structure (for the perversity p) is not stable by the duality. We see that the problem comes from the torsion. The duality exchanges a free module in degree n with a free module in degree $-n$, but it exchanges a torsion module in degree n with a torsion module in degree $1 - n$. The duality exchanges the usual t-structure $(D^{\leq 0}, D^{\geq 0})$ on $D^b_c(\mathrm{pt}, \mathbb{Z})$ with another t-structure $(D^{\leq 0^+}, D^{\geq 0^+})$, defined by:

$$K \in D^{\leq 0^+} \iff H^1(K) \text{ is torsion and } H^i(K) = 0 \text{ for i} > 1$$

$$K \in D^{\geq 0^+} \iff H^0(K) \text{ is torsion-free and } H^i(K) = 0 \text{ for i} < 0$$

Remember that we constructed the t-structure for the perversity p by taking on each stratum S the usual t-structure shifted by $\dim S$, and gluing them together. Over the integers, we can either do the same, or take on each stratum the dual of the usual t-structure, shifted by $\dim S$ (here we consider torsion versus torsion-free local systems on S), and then again glue them together. In the second case, we obtain the following t-structure, corresponding to the perversity p_+:

$$\mathcal{F} \in D_{\mathcal{S}}^{\leq 0^+}(X,\mathbb{Z}) \Longleftrightarrow \begin{cases} \mathcal{H}^m(i_S^*\mathcal{F}) \text{ is zero for } m > -\dim S + 1 \\ \text{and is torsion for } m = -\dim S + 1 \end{cases}$$

$$\mathcal{F} \in D_{\mathcal{S}}^{\geq 0^+}(X,\mathbb{Z}) \Longleftrightarrow \begin{cases} \mathcal{H}^m(i_S^!\mathcal{F}) = 0 \text{ is zero for } m < -\dim S \\ \text{and torsion-free for } m = -\dim S \end{cases}$$

We will denote the heart of the classical t-structure by $\mathbf{P}_{\mathcal{S}}(X,\mathbb{Z})$, and the heart of this new t-structure by $\mathbf{P}_{\mathcal{S}}^+(X,\mathbb{Z})$.

The abelian category $\mathbf{P}_{\mathcal{S}}(X,\mathbb{Z})$ is Noetherian but not Artinian (again, this is already the case for $X = pt$). However, given any local system $\mathcal{L}$ on a stratum S one still has a unique extension $\mathbf{IC}(\overline{S},\mathcal{L})$ satisfying the same conditions as for a field. Keeping the notation of the previous section, this may be defined by:

$$\mathbf{IC}(\overline{S},\mathcal{L}) := (\tau_{\leq -1} \circ j_{0*}) \circ (\tau_{\leq -2} \circ j_{1*}) \circ \cdots \circ (\tau_{\leq -d_S} \circ j_{d_S-1*})(\mathcal{L}[d_S]).$$

To obtain the dual of an intersection cohomology complex $\mathbf{IC}(\overline{S},\mathcal{L})$ in $\mathbf{P}_{\mathcal{S}}^+(X,k)$ one needs to consider a variant of the truncation functors on $D^b(X,\mathbb{Z})$, which we denote $\tau_{\leq i}^+$ and $\tau_{>i}^+$. If $\mathcal{K} \in D^b(X,\mathbb{Z})$ then $\mathcal{H}^m(\tau_{\leq i}^+\mathcal{K})$ is isomorphic to $\mathcal{H}^m(\mathcal{K})$ for $m \leq i$, to the torsion submodule of $\mathcal{H}^m(\mathcal{K})$ for $m = i+1$, and is zero otherwise. One defines

$$\mathbf{IC}^+(\overline{S},\mathcal{L}) := (\tau_{\leq -1}^+ \circ j_{0*}) \circ (\tau_{\leq -2}^+ \circ j_{1*}) \circ \cdots \circ (\tau_{\leq -d_S}^+ \circ j_{d_S-1*})(\mathcal{L}[d_S]).$$

As $\mathbf{IC}(\overline{S},\mathcal{L})$, the complex $\mathbf{IC}^+(\overline{S},\mathcal{L})$ may be characterized in terms of the stalks of i_S^* and $i_S^!$. As one might expect, the complexes $\mathbf{IC}(\overline{S},\mathcal{L})$ and $\mathbf{IC}^+(\overline{S},\mathcal{L}^\vee)$ are exchanged by the duality.

2.4. First example: the nilpotent cone of $\mathfrak{sl}_2$. Let $\mathfrak{sl}_2$ be the Lie algebra of 2×2 traceless matrices over $\mathbb{C}$ and let $\mathcal{N} \subset \mathfrak{sl}_2$ be its nilpotent cone. It is isomorphic to a quadratic cone inside affine 3-space:

$$\mathcal{N} = \left\{ \begin{pmatrix} x & y \\ z & -x \end{pmatrix} \middle| \, x^2 + yz = 0 \right\} \subset \mathfrak{sl}_2 \simeq \mathbb{A}^3$$

Note also that $\mathcal{N}$ is isomorphic to the quotient of a two dimensional vector space $V = \operatorname{Spec} \mathbb{C}[u,v]$ by the scalar action of $\{\pm 1\}$. If we choose

coordinates (u, v) on V then an isomorphism is given by

$$\begin{aligned} V/\{\pm 1\} &\longrightarrow \mathcal{N} \\ \pm(u,v) &\longmapsto \begin{pmatrix} uv & -v^2 \\ u^2 & -uv \end{pmatrix}. \end{aligned}$$

The conjugation action of $SL_2(\mathbb{C})$ on $\mathcal{N}$ has two orbits, $\mathcal{O}_{\mathrm{reg}}$ and $\{0\}$, and we let $\mathcal{S}$ denote the stratification of $\mathcal{N}$ into these two orbits. We will be interested in calculating $\mathbf{IC}(\mathcal{N}, k)$ for $k = \mathbb{Q}$, $\mathbb{Z}$ and $\mathbb{F}_p$ (of course $\mathbf{IC}(\{0\}, k)$ is always a skyscraper sheaf on $\{0\}$ in degree 0).

We will first examine Springer's resolution

$$T^*\mathbb{P}^1 \to \mathcal{N}.$$

Concretely, we may identify $T^*\mathbb{P}^1$ with pairs (ℓ, x) where $\ell \in \mathbb{P}^1$ is a line containing the image of $x \in \mathcal{N}$. The map π is then obtained by forgetting ℓ; it is clearly an isomorphism over $\mathcal{O}_{\mathrm{reg}}$ and has fibre $\mathbb{P}^1$ over $\{0\}$. Hence, for any k the stalks of $\pi_* \underline{k}_{T^*\mathbb{P}^1}[2]$ are given by

	-2	-1	0
$\mathcal{O}_{\mathrm{reg}}$	k	0	0
$\{0\}$	k	0	k

If k is of characteristic 0, then we know by the decomposition theorem that $\pi_* \underline{k}_{T^*\mathbb{P}^1}[2]$ is semi-simple and hence

$$\pi_* \underline{k}_{T^*\mathbb{P}^1}[2] \simeq \mathbf{IC}(\mathcal{N}, k) \oplus \mathbf{IC}(\{0\}, k).$$

It follows that $\mathbf{IC}(\mathcal{N}, k)$ is isomorphic to $\underline{k}_{\mathcal{N}}[2]$.

To handle the case $k = \mathbb{F}_p$ requires more care. Recall that the Deligne construction tells us that

$$\mathbf{IC}(\mathcal{N}, k) \simeq \tau_{\leq -1} \circ j_*(\underline{k}_{\mathcal{O}_{\mathrm{reg}}}[2]),$$

where $j : \mathcal{O}_{\mathrm{reg}} \hookrightarrow \mathcal{N}$ is the open immersion. Since this complex is an extension of $\underline{k}_{\mathcal{O}_{\mathrm{reg}}}[2]$, the only stalk that we have to compute is the stalk at zero. Let us compute $(j_* \underline{k}_{\mathcal{O}_{\mathrm{reg}}})_0$. Then it will be a trivial matter to shift and truncate.

We have

$$(j_* \underline{k}_{\mathcal{O}_{\mathrm{reg}}})_0 \simeq \lim_{V \ni 0} \mathrm{R}\Gamma(V \setminus \{0\}, k)$$

where V runs over the open neighbourhoods of 0 in $\mathcal{N}$.

We can replace this limit by a limit over a basis of neighbourhoods of zero, as for example V_n, $n \geq 1$, the intersection of $\mathcal{N}$ with the open ball of radius $1/n$ centered at 0 in $\mathbb{C}^3$. But $\mathcal{N}$ is a cone: it is stable by multiplication by a scalar in $\mathbb{C}^*$, and in particular in $\mathbb{R}_{>0}$. We can use

this to see that all the $V_n \setminus \{0\}$ are homeomorphic: they are actually homeomorphic to $\mathcal{N} \setminus \{0\} = \mathcal{O}_{\mathrm{reg}}$ itself. Thus we have

$$(j_* \underline{k}_{\mathcal{O}_{\mathrm{reg}}})_0 \simeq \lim_n \mathrm{R}\Gamma(V_n \setminus \{0\}, k) \simeq \mathrm{R}\Gamma(\mathcal{O}_{\mathrm{reg}}, k).$$

(this argument applies for any cone). However, as we observed above, $\mathcal{N} \simeq V/\{\pm 1\}$ and hence $\mathcal{O}_{\mathrm{reg}} \simeq (V \setminus \{0\})/\{\pm 1\} = (\mathbb{C}^2 \setminus \{0\})/\{\pm 1\}$, which is homotopic to $S^3/\{\pm 1\} = \mathbb{RP}^3$. Thus

$$(j_* \underline{k}_{\mathcal{O}_{\mathrm{reg}}})_0 \simeq \mathrm{R}\Gamma(\mathbb{RP}^3, k) \simeq (k \xrightarrow{0} k \xrightarrow{2} k \xrightarrow{0} k),$$

the latter complex being concentrated in degrees between 0 and 3.

Hence, if k is a field of characteristic p, the stalks of $j_* \underline{k}_{\mathcal{O}_{\mathrm{reg}}}[2]$ are given by:

	-2	-1	0	1
$\mathcal{O}_{\mathrm{reg}}$	k	0	0	0
$\{0\}$	k	$(k)_2$	$(k)_2$	k

where $(k)_2$ means k if $p = 2$, and 0 otherwise. We obtain the stalks of $\mathbf{IC}(\mathcal{N}, k)$ by truncating:

	-2	-1	0
$\mathcal{O}_{\mathrm{reg}}$	k	0	0
$\{0\}$	k	$(k)_2$	0

In fact, the decomposition theorem holds here if and only if $p \neq 2$.

One may calculate the stalks of $\mathbf{IC}(\mathcal{N}, \mathbb{Z})$ and $\mathbf{IC}^+(\mathcal{N}, \mathbb{Z})$ and one obtains:

	-2	-1	0
$\mathcal{O}_{\mathrm{reg}}$	$\mathbb{Z}$	0	0
$\{0\}$	$\mathbb{Z}$	0	0

	-2	-1	0
$\mathcal{O}_{\mathrm{reg}}$	$\mathbb{Z}$	0	0
$\{0\}$	$\mathbb{Z}$	0	$\mathbb{Z}/2\mathbb{Z}$

>From this information, one calculate the decomposition numbers for GL_2-equivariant perverse sheaves on its nilpotent cone [**Jut09**], and we get:

$$\begin{array}{cc} & \begin{array}{cc} (1^2) & (2) \end{array} \\ \begin{array}{c} (1^2) \\ (2) \end{array} & \begin{pmatrix} \mathbf{1} & 0 \\ \mathbf{1} & 1 \end{pmatrix} \end{array}$$

from which we can extract the decomposition matrix for $\mathfrak{S}_2$ in characteristic 2:

$$\begin{array}{cc} & D_{(2)} \\ \begin{array}{c} S_{(2)} \\ S_{(1^2)} \end{array} & \begin{pmatrix} \mathbf{1} \\ \mathbf{1} \end{pmatrix} \end{array}$$

3. Some stalks in nilpotent cones

In this section we give more examples of calculations of stalks of intersection cohomology sheaves on nilpotent cones, motivated by the modular Springer correspondence. Actually, in type A the nilpotent singularities also occur in the affine Grassmannian [**Lus81**], so the geometric Satake theorem is another motivation.

First we recall a parabolic generalization of Springer's resolution, providing a resolution of Richardson orbit closures. Then we deal with the minimal nilpotent orbit closure in $\mathfrak{sl}_n$ and $\mathfrak{sp}_{2n}$, and with the singularity of the nilpotent cone of $\mathfrak{sl}_n$ at the subregular orbit, by a direct approach. Note that minimal and simple singularities have been dealt with in all types in [**Jut08, Jut09**], but in the special cases we treat here, the calculation can be done quickly. Finally, we give more computations of stalks of intersection cohomology complexes in characteristic p, which might be new: we give all the stalks in the nilpotent cone of $\mathfrak{sl}_3$ for $p \neq 3$, and all the stalks in the subvariety of the nilpotent cone of $\mathfrak{sl}_4$ consisting of the matrices which square to zero.

All our calculations will be completed for varieties over $\mathbb{C}$ in the metric topology, however all the results and most proofs can be translated into the étale situation: we mostly use basic facts about the cohomology of projective spaces and flag varieties, Gysin sequences etc. which have direct translations in étale cohomology. We will always outline how a calculation can be performed in the étale topology when such a direct translation is not possible.

Throughout we will use the following notation (already used in the last section). If k is a field of characteristic p and n is an integer then

$$(k)_n = \begin{cases} k & \text{if } p \text{ divides } n, \\ 0 & \text{otherwise.} \end{cases}$$

3.1. Semi-small resolutions of Richardson orbit closures. Remember the Springer resolution:

$$\begin{array}{rcccl} \pi_{\mathcal{N}} : T^*(G/B) & = & G \times^B \mathfrak{u} & \longrightarrow & \mathcal{N} = \overline{\mathcal{O}}_{\mathrm{reg}} \\ & & g *_B x & \longmapsto & (\operatorname{Ad} g)(x) \end{array}$$

We will see a generalization of this resolution, where we replace the Borel subgroup B by a parabolic subgroup P, with unipotent radical U_P. We denote by $\mathfrak{u}_P$ the Lie algebra of U_P. We can naturally define the proper morphism:

$$\begin{array}{rcccl} \pi_P : T^*(G/P) & = & G \times^P \mathfrak{u}_P & \longrightarrow & \mathcal{N} \\ & & g *_P x & \longmapsto & (\operatorname{Ad} g)(x) \end{array}$$

but what is the image of π_P ?

Since $\mathfrak{u}_P$ is irreducible and there are only finitely many nilpotent orbits, there is a unique nilpotent orbit $\mathcal{O}$ such that $\mathcal{O} \cap \mathfrak{u}_P$ is dense in $\mathfrak{u}_P$. This orbit is called the Richardson orbit associated to P. One can see that the image of π_P is the closure of $\mathcal{O}$. The morphism π_P induces a semi-small resolution of $\overline{\mathcal{O}}$, which is used in [**BM81**]. In type A, all nilpotent orbits are Richardson, but in general this is not so. The regular nilpotent orbit, though, is always Richardson: it is the one associated to B.

Let us describe the situation in the case $G = SL_n$. If $\lambda = (\lambda_1, \dots, \lambda_s)$ is a partition of n, let $\mathcal{O}_\lambda$ denote the nilpotent orbit consisting of the nilpotent matrices whose Jordan normal form x_λ has Jordan blocks of sizes given by the parts of λ. We denote by P_λ the parabolic subgroup of SL_n stabilizing the standard partial flag of shape λ:

$$F^\lambda_\bullet := (0 \subset F^\lambda_1 \subset \cdots \subset F^\lambda_s = \mathbb{C}^n)$$

where F^λ_i is spanned by the $\lambda_1 + \cdots + \lambda_i$ first elements of the canonical basis of $\mathbb{C}^n$. The partial flag variety G/P_λ can be interpreted as the variety $\mathcal{F}_\lambda$ of all partial flags of shape λ, that is, sequences of subspaces $F_\bullet = (0 \subset F_1 \subset \cdots \subset F_s = \mathbb{C}^n)$ with $\dim F_i = \lambda_1 + \cdots + \lambda_i$. We have

$$\begin{array}{rcl} G \times^P \mathfrak{u}_{P_\lambda} & \xrightarrow{\sim} & \{(x, F_\bullet) \in \mathcal{N} \times \mathcal{F}_\lambda \mid x(F_i) \subset F_{i-1}\} \\ g *_{P_\lambda} x & \longmapsto & (\operatorname{Ad} g(x), g(F^\lambda_\bullet)) \end{array}$$

For $x \in \mathcal{N}$, we have

$$x \in \mathcal{O}_\lambda \Longleftrightarrow \forall i,\ \dim \operatorname{Ker} x^i = \lambda'_1 + \cdots + \lambda'_i$$

and

$$x \in \overline{\mathcal{O}}_\lambda \Longleftrightarrow \forall i,\ \dim \operatorname{Ker} x^i \geq \lambda'_1 + \cdots + \lambda'_i$$

where λ' is the partition conjugate to λ. Let us note that we have

$$\mathcal{O}_\mu \subset \overline{\mathcal{O}}_\lambda \Longleftrightarrow \forall i,\ \mu'_1 + \cdots + \mu'_i \geq \lambda'_1 + \cdots + \lambda'_i \Longleftrightarrow \mu' \geq \lambda' \Longleftrightarrow \mu \leq \lambda$$

where $\leq$ is the usual dominance order on partitions.

Now assume x is in the image of $\pi_{P_{\lambda'}}$. Then there is a flag $F_\bullet$ of type λ' such that $x(F_i) \subset F_{i-1}$ for all i. In particular, we have $F_i \subset \operatorname{Ker} x^i$ for all i, and hence $\dim \operatorname{Ker} x^i \geq \lambda'_1 + \dots \lambda'_i$ for all i. Thus $x \in \overline{\mathcal{O}}_\lambda$. Consequently, the image of $\pi_{P_{\lambda'}}$ is included in $\overline{\mathcal{O}}_\lambda$.

If $x \in \mathcal{O}_\lambda$, then there is a unique flag $F_\bullet$ of type λ' such that $x(F_i) \subset F_{i-1}$, namely $F_i = \operatorname{Ker} x^i$. Thus the image of $\pi_{P_{\lambda'}}$ contains $\mathcal{O}_\lambda$, and hence $\overline{\mathcal{O}}_\lambda$, as it is a proper morphism.

Thus the image of $\pi_{P_{\lambda'}}$ is equal to $\overline{\mathcal{O}}_\lambda$, and $\pi_{P_{\lambda'}}$ is an isomorphism over $\mathcal{O}_\lambda$. Since it is proper, it is a resolution of singularities.

Note that this gives, in principle, a method to compute all the IC stalks with $\mathbb{Q}$ coefficients of closures of nilpotent orbits in $\mathfrak{sl}_n$, by induction, using the decomposition theorem: the direct image $\pi_{P_{\lambda'}*}\underline{\mathbb{Q}}[\dim \mathcal{O}_\lambda]$ decomposes as a direct sum of $\mathbf{IC}(\overline{\mathcal{O}}_\lambda, k)$ and some copies of IC sheaves for lower strata, which we know by induction. The stalks of the direct image are given by the cohomology of the fibers. One finds the stalks of the IC sheaf of $\overline{\mathcal{O}}_\lambda$ by removing the stalks of the other summands. It is a nice exercise to do that for small ranks. We will see some examples below. Note, however, that all stalks of G-equivariant IC complexes on nilpotent cones are known in characteristic zero, as there is an algorithm to compute them [**Lus86**, V], as soon as one has determined the generalized Springer correspondence defined in [**Lus84**], which has also been done, by work of several authors. In the case of GL_n, the answer (which is in terms of Kostka polynomials) has been known since [**Lus81**].

With $\mathbb{Z}$ or $\mathbb{F}_p$ coefficients, however, one cannot use the decomposition theorem, and the calculations are much more difficult. To find the answer in general is a very deep and important problem; in particular, such information would be sufficient to determine the decomposition matrices of the symmetric groups, which is a central problem in the modular representation theory of finite groups. We will see some examples below where the calculations can be done.

3.2. Minimal class in $\mathfrak{sl}_n$. The goal of this paragraph will be to generalize the calculation 2.4 of the stalk at the origin of the IC sheaf on the nilpotent cone for $\mathfrak{sl}_2$ to that of the closure of the minimal non-trivial nilpotent orbit in $\mathfrak{sl}_n$. The minimal orbit $\mathcal{O}_{\min} = \mathcal{O}_{(2,1^{n-2})}$ is the set of nilpotent matrices with 1-dimensional image and $(n-1)$-dimensional kernel. Let us apply the considerations of the last subsection. Here $\lambda = (2, 1^{n-2})$, and $\lambda' = (n-1, 1)$. The parabolic subgroup $P = P_{(n-1,1)}$ is the stabilizer of a hyperplane. The partial flag variety G/P is identified with the projective space $\mathbb{P}^{n-1}$ of hyperplanes in $\mathbb{C}^n$. We get a resolution of $\overline{\mathcal{O}}_{\min}$ by taking pairs (x, H) consisting of a nilpotent element x (necessarily in $\overline{\mathcal{O}}_{\min}$) and a hyperplane H contained in the kernel of x. (Dually, we could have considered pairs (x, ℓ) where x is nilpotent and ℓ is a line such that $\mathrm{Im}(x) \subset \ell$.)

Thus we have a proper morphism:

$$T^*\mathbb{P}^{n-1} \to \overline{\mathcal{O}}_{\min} = \mathcal{O}_{\min} \cup \{0\}$$

which is an isomorphism over $\mathcal{O}_{\min}$ and has fibre $\mathbb{P}^{n-1}$ over $\{0\}$. Hence, for any k the stalks of $\pi_*\underline{k}_{T^*\mathbb{P}^{n-1}}[2n-2]$ are given by:

	$-2n+2$	$-2n+1$	$-2n+2$	$\dots$	-1	0
$\mathcal{O}_{\min}$	k	0	0	$\dots$	0	0
$\{0\}$	k	0	k	$\dots$	0	k

If k is of characteristic 0, then we know by the decomposition theorem that $\pi_*\underline{k}_{T^*\mathbb{P}^{n-1}}[2n-2]$ is semi-simple and hence

$$\pi_*\underline{k}_{T^*\mathbb{P}^{n-1}}[2n-2] \simeq \mathbf{IC}(\overline{\mathcal{O}}_{\min}, k) \oplus \mathbf{IC}(\{0\}, k).$$

It follows that the cohomology of $\mathbf{IC}(\overline{\mathcal{O}}_{\min}, k)_0$ is isomorphic to k in even degrees between $-2n+2$ and -2, and zero otherwise.

As in the $\mathfrak{sl}_2$ case, to handle the cases $k = \mathbb{F}_p$ or $\mathbb{Z}$, we use Deligne's construction:

$$\mathbf{IC}(\overline{\mathcal{O}}_{\min}, k) \simeq \tau_{\leq -1} \circ j_*(\underline{k}_{\mathcal{O}_{\min}}[2n-2])$$

where j is the open immersion $\mathcal{O}_{\min} \hookrightarrow \overline{\mathcal{O}}_{\min}$. Again, $\overline{\mathcal{O}}_{\min}$ is a cone and thus we have:

$$(j_*\underline{k}_{\mathcal{O}_{\min}})_0 \simeq \mathrm{R}\Gamma(\mathcal{O}_{\min}, k).$$

Now we have observed that $\mathcal{O}_{\min}$ is isomorphic to the complement of the zero section to the cotangent bundle of $\mathbb{P}^{n-1}$. The cohomology can thus be read off the Gysin sequence:

$$\cdots \longrightarrow H^{i-2n+2} \xrightarrow{e} H^i \longrightarrow H^i(\mathcal{O}_{\min}) \longrightarrow H^{i-2n+3} \xrightarrow{e} H^{i+1} \longrightarrow \cdots$$

where $H^i := H^i(\mathbb{P}^{n-1}, \mathbb{Z})$ and e is the Euler class of the cotangent bundle. But this is simply the Euler characteristic $n = \chi(\mathbb{P}^{n-1})$ times a generator of $H^{2n-2}(\mathbb{P}^{n-1}, \mathbb{Z}) \simeq \mathbb{Z}$. We deduce that $H^i(\mathcal{O}_{\min}, \mathbb{Z})$ is isomorphic to $\mathbb{Z}$ for $i = 0,\ 2,\ \dots,\ 2n-4$, to $\mathbb{Z}/n$ for $i = 2n-2$, and then again to $\mathbb{Z}$ for $i = 2n-1,\ 2n+1,\ \dots,\ 4n-5$, and zero otherwise:

0	1	2	$\cdots$	$2n-3$	$2n-2$	$\cdots$	$4n-3$	$4n-4$	$4n-5$
$\mathbb{Z}$	0	$\mathbb{Z}$	$\cdots$	0	$\mathbb{Z}/n$	$\cdots$	$\mathbb{Z}$	0	$\mathbb{Z}$

We have $\mathrm{R}\Gamma(\mathcal{O}_{\min}, k) = k \otimes^L_{\mathbb{Z}} \mathrm{R}\Gamma(\mathcal{O}_{\min}, \mathbb{Z})$. Thus each copy of $\mathbb{Z}$ is replaced by k. For $k = \mathbb{Q}$ or $\mathbb{F}_p$ with $p \nmid n$, the torsion group $\mathbb{Z}/n$ in degree $2n-2$ is killed. For $k = \mathbb{F}_p$ with $p \mid n$, $\mathbb{Z}/n$ is replaced by two copies of $\mathbb{F}_p$, one in degree $2n-3$, one in degree $2n-2$.

We obtain the stalks of $\mathbf{IC}(\overline{\mathcal{O}}_{\min}, k)$ by shifting and truncating. For $k = \mathbb{F}_p$, we get:

	$-2n+2$	$-2n+3$	$-2n+4$	$\cdots$	-2	-1	0
$\mathcal{O}_{\min}$	k	0	0	$\cdots$	0	0	0
$\{0\}$	k	0	k	$\cdots$	k	$(k)_n$	0

For $k = \mathbb{Z}$, we get:

	$-2n+2$	$-2n+3$	$-2n+4$	$\cdots$	-2	-1	0
$\mathcal{O}_{\min}$	$\mathbb{Z}$	0	0	$\cdots$	0	0	0
$\{0\}$	$\mathbb{Z}$	0	$\mathbb{Z}$	$\cdots$	$\mathbb{Z}$	0	0

For $\mathbf{IC}^+(\mathcal{O}_{\min}, k)$, one adds a copy of $\mathbb{Z}/n$ in degree 0 at $\{0\}$.

The above calculations give a geometric proof that the natural representation of $\mathfrak{S}_n$ (which corresponds to the minimal orbit) remains irreducible modulo ℓ if and only if $\ell \nmid n$, while if $\ell \mid n$ its modular reduction involves the trivial representation once.

3.3. Minimal class in $\mathfrak{sp}_{2n}$. In this section, we will treat the case of the minimal class $\mathcal{O}_{\min}$ in $\mathfrak{g} = \mathfrak{sp}_{2n}$. We view $G = Sp_{2n}$ as the subgroup of GL_{2n} stabilizing some symplectic form on $\mathbb{C}^{2n}$ given by a matrix Q with respect to the canonical basis.

A matrix $M \in GL_{2n}$ will be in Sp_{2n} if and only if ${}^tMQM = Q$. Then $\mathfrak{g}$ can be identified with the Lie algebra of the matrices $H \in \mathfrak{gl}_{2n}$ such that the following identity holds in $GL_{2n}(\mathbb{C}[\varepsilon])$:

$$(1 + \varepsilon {}^tH)Q(1 + \varepsilon H) = Q$$

where $\mathbb{C}[\varepsilon] = \mathbb{C}[X]/(X^2)$ and ε is the image of X in this quotient. This is equivalent to

$$ {}^tHQ + QH = 0 \tag{5}$$

in $\mathfrak{gl}_{2n}$. Now, the minimal class $\mathcal{O}_{\min}$ consists of those matrices in $\mathfrak{sp}_{2n}$ which are nilpotent with Jordan type $(2, 1^{2n-2})$. They are also characterized in $\mathfrak{sp}_{2n}$ by the fact that they are of rank one (all matrices in $\mathfrak{sp}_{2n}$ have zero trace).

A matrix $H \in \mathfrak{gl}_{2n}$ of rank one is of the form $H = u{}^tv$, where u and v are non-zero vectors in $\mathbb{C}^{2n}$. Moreover, u and v are uniquely determined up to multiplying u by some non-zero scalar λ and dividing v by the same scalar λ. Now suppose H is in $\mathcal{O}_{\min}$. Then (5) writes

$$v{}^tuQ + Qu{}^tv = 0$$

that is,

$$(Qu){}^tv = v{}^t(Qu)$$

using the fact that Q is anti-symmetric. Since Q is non-degenerate, we have $Qu \neq 0$, which implies that v is proportional to Qu.

Let E denote the bundle $\{(H, \ell) \in \mathfrak{sp}_{2n} \times \mathbb{P}^{2n-1} \mid \operatorname{Im} H \subset \ell\}$ over $\mathbb{P}^{2n-1}$. The first projection gives a morphism

$$\pi : E \to \overline{\mathcal{O}}_{\min} = \mathcal{O}_{\min} \sqcup \{0\}$$

which is a resolution of singularities. It is an isomorphism over $\mathcal{O}_{\min}$, and the exceptional fiber is the null section.

The above discussion shows that we have

$$E \simeq \mathcal{O}(-1) \otimes_{\mathcal{O}} \mathcal{O}(-1) \simeq \mathcal{O}(-2).$$

Let H^i denote $H^i(\mathbb{P}^{2n-1}, \mathbb{Z})$, and let $t \in H^2$ be the first Chern class of $\mathcal{O}(-1)$. Then we have $H^*(\mathbb{P}^{2n-1}, \mathbb{Z}) \simeq \mathbb{Z}[t]/t^{2n}$. The Euler class e of E is $2t$. As $\mathcal{O}_{\min}$ is isomorphic to E minus the null section, we have a Gysin sequence:

$$\cdots \longrightarrow H^{i-2} \xrightarrow{e} H^i \longrightarrow H^i(\mathcal{O}_{\min}, \mathbb{Z}) \longrightarrow H^{i-1} \xrightarrow{e} H^{i+1} \longrightarrow \cdots$$

As the cohomology of $\mathbb{P}^{2n-1}$ is concentrated in even degrees, for i even, we get $H^i(\mathcal{O}_{\min}, \mathbb{Z}) \simeq \mathrm{Coker}(e : H^{i-2} \to H^i)$ which is isomorphic to $\mathbb{Z}$ for $i = 0$, to $\mathbb{Z}/2$ if i is an even integer between 2 and $4n-2$, and 0 otherwise. For i odd, we have $H^i(\mathcal{O}_{\min}, \mathbb{Z}) \simeq \mathrm{Ker}(e : H^{i-1} \to H^{i+1})$ which is isomorphic to $\mathbb{Z}$ for $i = 4n-3$, and zero otherwise.

With $\mathbb{F}_p$ coefficients, p odd, $\mathcal{O}_{\min}$ has the cohomology of a sphere. With $\mathbb{F}_2$ coefficients, it has cohomology $\mathbb{F}_2[u]/u^{4n}$, with u in degree one.

Again, $\overline{\mathcal{O}}_{\min}$ is a cone, so we have

$$\mathbf{IC}(\overline{\mathcal{O}}_{\min}, k)_0 = \tau_{\leq -1} \circ j_*(\underline{k}_{\mathcal{O}_{\min}}[2n])_0 \simeq \tau_{\leq -1}(\mathrm{R}\Gamma(\mathcal{O}_{\min}, k)[2n])$$

where $j : \mathcal{O}_{\min} \hookrightarrow \overline{\mathcal{O}}_{\min}$ is the open immersion. Thus for $k = \mathbb{F}_p$, the stalks of $\mathbf{IC}(\overline{\mathcal{O}}_{\min}, k)$ are as follows:

	$-2n$	$-2n+1$	$-2n+2$	$\cdots$	-2	-1	0
$\mathcal{O}_{\min}$	k	0	0	$\cdots$	0	0	0
$\{0\}$	k	$(k)_2$	$(k)_2$	$\cdots$	$(k)_2$	$(k)_2$	0

For $k = \mathbb{Z}$, we get:

	$-2n$	$-2n+1$	$-2n+2$	$\cdots$	-2	-1	0
$\mathcal{O}_{\min}$	$\mathbb{Z}$	0	0	$\cdots$	0	0	0
$\{0\}$	$\mathbb{Z}$	0	$\mathbb{Z}/2$	$\cdots$	$\mathbb{Z}/2$	0	0

and, for the p_+ version, one has to add a copy of $\mathbb{Z}/2$ in degree 0 for the trivial orbit.

Let us give an alternative point of view, in more concrete terms. If we take $Q = \left(\begin{smallmatrix} 0 & -J \\ J & 0 \end{smallmatrix}\right)$, where

$$J = \begin{pmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{pmatrix}$$

then we have a morphism $q : \mathbb{C}^{2n} \to \overline{\mathcal{O}}_{\min}$ which sends the vector $(x_1, \dots, x_n, y_1, \dots, y_n)$ to the matrix

$$\begin{pmatrix} -x_1y_n & \cdots & -x_1y_1 & x_1x_n & \cdots & x_1^2 \\ \vdots & \iddots & \vdots & \vdots & \iddots & \vdots \\ -x_ny_n & \cdots & -x_ny_1 & x_n^2 & \cdots & x_nx_1 \\ -y_1y_n & \cdots & -y_1^2 & y_1x_n & \cdots & y_1x_1 \\ \vdots & \iddots & \vdots & \vdots & \iddots & \vdots \\ y_n^2 & \cdots & -y_ny_1 & y_nx_n & \cdots & y_nx_1 \end{pmatrix}$$

This can be identified with the quotient by $\{\pm 1\}$. In particular, this explains why $\overline{\mathcal{O}}_{\min}$ is p-smooth for $p \neq 2$.

The map q induces $q^0 : \mathbb{C}^{2n} \setminus \{0\} \to \mathcal{O}_{\min}$, which is the quotient by $\{\pm 1\}$, which implies that $\mathcal{O}_{\min} \simeq (\mathbb{C}^{2n} \setminus \{0\})/\{\pm 1\}$ is homotopic to $\mathbb{RP}^{4n-1}$. We have

$$\mathrm{R}\Gamma(\mathcal{O}_{\min}, k) = \mathrm{R}\Gamma(\mathbb{RP}^{4n-1}, k) = (k \xrightarrow{0} k \xrightarrow{2} k \xrightarrow{0} \dots \xrightarrow{2} k \xrightarrow{0} k)$$

where the last complex has a copy of k in each degree between 0 and $4n-1$, and the differential is alternatively 0 and 2. We recover the preceding calculation.

Let us note that, for $n = 1$, we recover the $\mathfrak{sl}_2$ calculation.

3.4. Subregular class in $\mathfrak{sl}_n$. Let us consider the nilpotent cone $\mathcal{N}$ of $\mathfrak{sl}_n$. The regular nilpotent orbit $\mathcal{O}_{\mathrm{reg}} = \mathcal{O}_{(n)}$ is open dense in $\mathcal{N}$. There is a unique open dense orbit $\mathcal{O}_{\mathrm{subreg}} = \mathcal{O}_{(n-1,1)}$ in its complement. It is of codimension 2. Let $U := \mathcal{O}_{\mathrm{reg}} \cup \mathcal{O}_{\mathrm{subreg}}$. In this section we will compute the stalks of $\mathbf{IC}(\mathcal{N}, k)$ restricted to U and find a condition on the characteristic of k for this restriction to be a constant sheaf.

By [**Bri71, Slo80a, Slo80b**], the singularity of $\mathcal{N}$ along $\mathcal{O}_{\mathrm{subreg}}$ is a simple surface singularity of type A_{n-1}. This is the singularity at 0 of the variety $S = \mathbb{C}^2/\mu_n$, where μ_n is the group of nth roots of unity in $\mathbb{C}^*$, and $\zeta \in \mu_n$ acts on $\mathbb{C}^2$ by $\left(\begin{smallmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{smallmatrix}\right)$. This implies that $\mathbf{IC}(U, k)_x[-\dim \mathcal{N}] \simeq \mathbf{IC}(S, k)_0[-2]$, where $x \in \mathcal{O}_{\mathrm{subreg}}$. We have

$$S = \operatorname{Spec} \mathbb{C}[u, v]^{\mu_n} = \operatorname{Spec} \mathbb{C}[u^n, v^n, uv] = \operatorname{Spec} \mathbb{C}[x, y, z]/(xy - z^n)$$

and there is a $\mathbb{C}^*$-action on $\mathbb{C}^3 = \operatorname{Spec} \mathbb{C}[x, y, z]$, contracting to the origin, stabilizing S, given by $t.(x, y, z) = (t^n x, t^n y, t^2 z)$. The same argument as with a cone shows that

$$(j_* \underline{k}_{S \setminus \{0\}})_0 = \mathrm{R}\Gamma(S \setminus \{0\}, k) = \mathrm{R}\Gamma(S^3/\mu_n, k) = (k \xrightarrow{0} k \xrightarrow{n} k \xrightarrow{0} k)$$

(a complex in degrees between 0 and 3), where $j : S \setminus \{0\} \hookrightarrow S$ is the open immersion.

Thus if k is a field of characteristic p, then the stalks of $\mathbf{IC}(U,k)$ are given by:

	$-\dim\mathcal{N}$	$-\dim\mathcal{N}+1$	$-\dim\mathcal{N}+2$
$\mathcal{O}_{\mathrm{reg}}$	k	0	0
$\mathcal{O}_{\mathrm{subreg}}$	k	$(k)_n$	0

For $k=\mathbb{Z}$ and for the perversity p, we get the constant sheaf on X in degree $-\dim\mathcal{N}$. For the perversity p_+, we get an extra stalk $\mathbb{Z}/n$ in degree $-\dim\mathcal{N}+2$ for $\mathcal{O}_{\mathrm{subreg}}$.

Remark 3.1. Here and in the previous section, we have relied on knowledge of the cohomology ring of real projective spaces and lens spaces in order to calculate the cohomology of spaces like $(\mathbb{C}^m\setminus\{0\})/\mu_n$. Thus it is not immediately clear how to proceed in the étale situation. However, instead one can use the fact that $\mathrm{R}\Gamma(\mathbb{C}^m\setminus\{0\},k)$ is a perfect complex of $k\mu_n$-modules, because μ_n acts freely on $\mathbb{C}^m\setminus\{0\}$. As there are only two non-trivial cohomology groups, there is only one possibility up to quasi-isomorphism. Then one can take derived invariants to recover $\mathrm{R}\Gamma((\mathbb{C}^m\setminus\{0\})/\mu_n,k)$. This proof makes sense if we replace $\mathbb{C}$ by $\mathbb{F}_q$. Alternatively, one could also use comparison theorems.

For completeness, and for future use, let us describe the Springer fiber $\mathcal{B}_x$. It is the union of $n-1$ projective lines $L_1,\ldots,L_{n-1}$, where L_i is identified with the variety of flags $F_\bullet$ such that $F_j=\operatorname{Im}x^{n-1-j}$ for $1\le j\le i-1$ and $F_j=\operatorname{Ker}x^{j-1}$ for $i+1\le j\le n$. These projective lines intersect as the Dynkin diagram of type A_{n-1}. Let $\pi_U:\tilde{U}\to U$ be the restriction of the Springer resolution to U. Then the stalks of $\pi_{U*}\underline{k}_{\tilde{U}}[\dim\mathcal{N}]$ are given by:

	$-\dim\mathcal{N}$	$-\dim\mathcal{N}+1$	$-\dim\mathcal{N}+2$
$\mathcal{O}_{\mathrm{reg}}$	k	0	0
$\mathcal{O}_{\mathrm{subreg}}$	k	0	k^{n-1}

When k is of characteristic zero, we can use the decomposition theorem for another proof of the above calculation in that case.

Let us note that, for $n=2$, we recover the $\mathfrak{sl}_2$ calculation.

3.5. IC stalks on the nilpotent cone of $\mathfrak{sl}_3$ for $p\neq 3$. Let us consider the nilpotent cone in $\mathfrak{sl}_3$ and a field of coefficients k of characteristic p different from 3. We have three nilpotent orbits, indexed by the partitions (3), (21) and (1^3): the regular orbit $\mathcal{O}_{\mathrm{reg}}$ of dimension 6, the minimal orbit $\mathcal{O}_{\mathrm{min}}$ of dimension 4 and the trivial orbit $\{0\}$.

The complex $\mathbf{IC}(\{0\},k)$ is trivial. By Subsection 3.2, we know the stalks of $\mathbf{IC}(\overline{\mathcal{O}}_{\mathrm{min}},k)$. We wish to compute the stalks of $\mathbf{IC}(\mathcal{N},k)$.

Let $U := \mathcal{O}_{\mathrm{reg}} \cup \mathcal{O}_{\mathrm{subreg}}$. By Subsection 3.4, we have:

$$\mathbf{IC}(U, k) = \underline{k}_U[6]$$

since k is assumed to be of characteristic $p \neq 3$. In other words, U is p-smooth if $p \neq 3$. This fact will allow us to compute the stalk $\mathbf{IC}(\mathcal{N}, k)_0$ as in the preceding cases, where the complement of the origin was smooth. Let $j : U \hookrightarrow \mathcal{N}$ denote the open immersion. Since $\mathcal{N}$ is a cone, we have:

$$(j_* \underline{k}_U)_0 = \mathrm{R}\Gamma(U, k)$$

As in Subsection 3.4, we will consider the restriction of the Springer resolution $\pi_U : \tilde{U} \to U$. We will consider the truncation distinguished triangle:

$$\tau_{\leq -6} \pi_{U*} \underline{k}_{\tilde{U}}[6] \longrightarrow \pi_{U*} \underline{k}_{\tilde{U}}[6] \longrightarrow \tau_{>-6} \pi_{U*} \underline{k}_{\tilde{U}}[6] \rightsquigarrow$$

By what we said in Subsection 3.4, if $i : \mathcal{O}_{\min} \hookrightarrow U$ denotes the (closed) inclusion this is isomorphic to the triangle:

$$\underline{k}_U[6] \longrightarrow \pi_{U*} \underline{k}_{\tilde{U}}[6] \longrightarrow i_* \underline{k}^2_{\mathcal{O}_{\min}}[4] \rightsquigarrow \tag{6}$$

Since all the objects involved are actually perverse sheaves, we have a short exact sequence of perverse sheaves:

$$0 \longrightarrow \underline{k}_U[6] \longrightarrow \pi_{U*} \underline{k}_{\tilde{U}}[6] \longrightarrow i_* \underline{k}^2_{\mathcal{O}_{\min}}[4] \longrightarrow 0$$

with left and right terms intersection cohomology complexes. We will see that the sequence splits. For this, we have to show that the degree one morphism of the triangle above is zero. But we have

$$\begin{array}{rcll} \mathrm{Hom}(i_* \underline{k}_{\mathcal{O}_{\min}}[4], \underline{k}_U[7]) & = & \mathrm{Hom}(\underline{k}_U[5], i_* \underline{k}_{\mathcal{O}_{\min}}[4]) & \text{by duality} \\ & = & \mathrm{Hom}(i^* \underline{k}_U[5], \underline{k}_{\mathcal{O}_{\min}}[4]) & \text{by adjunction} \\ & = & \mathrm{Hom}(\underline{k}_{\mathcal{O}_{\min}}, \underline{k}_{\mathcal{O}_{\min}}[-1]) & \\ & = & 0. & \end{array}$$

This is another way of saying that $\mathrm{Ext}^1(i_* \underline{k}_{\mathcal{O}_{\min}}[4], \underline{k}_U[6]) = 0$. Thus we have

$$\pi_{U*} \underline{k}_{\tilde{U}}[6] = \underline{k}_U[6] \oplus i_* \underline{k}^2_{\mathcal{O}_{\min}}[4].$$

Taking global sections yields

$$\mathrm{R}\Gamma(\tilde{U}, k)[6] = \mathrm{R}\Gamma(U, k)[6] \oplus \mathrm{R}\Gamma(\mathcal{O}_{\min}, k)^2[4].$$

Finally if we can compute the cohomology of $\tilde{U}$ and $\mathcal{O}_{\min}$ with coefficients in k, then we will have computed the cohomology of U and equivalently, as U is p-smooth, its intersection cohomology.

Recall that $\tilde{U}$ is the cotangent bundle to the flag variety G/B with the zero section removed. We can then apply the Gysin sequence

$$\cdots \longrightarrow H^{i-6} \xrightarrow{e} H^i \longrightarrow H^i(\tilde{U}, k) \longrightarrow H^{i-5} \xrightarrow{e} H^{i+1} \longrightarrow \cdots$$

where $H^i := H^i(G/B, k)$ and e denotes multiplication by the Euler class of $T^*(G/B)$, which in this case is given by $\chi(G/B) = 6$ times a generator of H^6. Recalling that the cohomology of G/B is as follows:

0	1	2	3	4	5	6
k	0	k^2	0	k^2	0	k

we find the cohomology of of the complex $\mathrm{R}\Gamma(\tilde{U}, k)[6]$ is given by:

-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6
k	0	k^2	0	k^2	$(k)_6$	$(k)_6$	k^2	0	k^2	0	k	0

We have computed the cohomology of $\mathcal{O}_{\min}$ in Subsection 3.2. Since $p \neq 3$, the cohomology of the complex $\mathrm{R}\Gamma(\mathcal{O}_{\min}, k)[4]$ is given by:

-4	-3	-2	-1	0	1	2	3
k	0	k	0	0	k	0	k

Subtracting two copies of the cohomology of $\mathrm{R}\Gamma(\mathcal{O}_{\min}, k)[4]$ from the cohomology of $\mathrm{R}\Gamma(\tilde{U}, k)[6]$, we obtain the cohomology of $\mathrm{R}\Gamma(U, k)[6]$ for $p \neq 3$:

-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6
k	0	0	0	0	$(k)_2$	$(k)_2$	0	0	0	0	k	0

By truncation, we obtain the stalk $\mathbf{IC}(\mathcal{N}, k)_0$ for $p \neq 3$:

THEOREM 3.2. *Let k be a field of characteristic $p \neq 3$. Then*

$$\mathbf{IC}(\mathcal{N}, k)_0 = \begin{cases} k[6] \oplus k[1] & \text{if } p = 2, \\ k[6] & \text{if } p > 3. \end{cases}$$

This completes the geometric determination of the decomposition matrix of GL_3-equivariant perverse sheaves on its nilpotent cone, and hence of the symmetric group $\mathfrak{S}_3$, in characteristic 2:

$$\begin{array}{c} \\ 1^3 \\ 21 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1^3 & 21 & 3 \end{array} \\ \begin{pmatrix} \mathbf{1} & \mathbf{0} & 0 \\ \mathbf{0} & \mathbf{1} & 0 \\ \mathbf{1} & \mathbf{0} & 1 \end{pmatrix} \end{array} \qquad \begin{array}{c} \\ S_{(3)} \\ S_{(21)} \\ S_{(1^3)} \end{array} \begin{array}{c} \begin{array}{cc} D_{(3)} & D_{(21)} \end{array} \\ \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix} \end{array}$$

3.6. Some IC stalks on the nilpotent cone of $\mathfrak{sl}_4$ for $p \neq 2$. In this subsection, we calculate the stalks of the intersection cohomology complex on $\overline{\mathcal{O}}_{(2^2)}$ in characteristic $p \neq 2$. The closure $\overline{\mathcal{O}}_{(2^2)}$ consists of three orbits: $\mathcal{O}_{(2^2)}$, $\mathcal{O}_{\min} = \mathcal{O}_{(2,1^2)}$ and $\{0\}$ of dimensions 8, 6 and 0. Let $U' = \mathcal{O}_{(2^2)} \cup \mathcal{O}_{\min}$.

Kraft and Procesi show [**KP81**] that the singularity of U' along $\mathcal{O}_{\min}$ is equivalent to a simple A_1 singularity, that of the nilpotent cone of $\mathfrak{sl}_2$ that we studied in Section 2.4. It follows that the stalks of $\mathbf{IC}(\overline{\mathcal{O}}_{(2^2)}, k)$ restricted to U' are as follows:

	-8	-7	-6
$\mathcal{O}_{(2^2)}$	k	0	0
$\mathcal{O}_{\min}$	k	$(k)_2$	0

>From now on, let k be a field of characteristic $p \neq 2$ (so that U' is p-smooth). Using the Deligne construction and the fact that $\overline{\mathcal{O}}_{(2^2)}$ is a cone we have

$$\mathbf{IC}(\overline{\mathcal{O}}_{(2^2)}, k)_0 = (\tau_{\leq -1} j_* \underline{k}_{U'}[8])_0 = \tau_{\leq -1}(\mathrm{R}\Gamma(U', k)[8])$$

where $j : U' \hookrightarrow \overline{\mathcal{O}}_{(2^2)}$ denotes the inclusion. We will proceed as in the previous section, calculating $\mathrm{R}\Gamma(U', k)$ with the help of the resolution

$$\pi : T^*(G/P) \to \overline{\mathcal{O}}_{(2^2)}$$

defined in Section 3.1. Here $G = SL_4$ and P is the parabolic stabilizing $\mathbb{C}^2 \subset \mathbb{C}^4$. The fiber over a nilpotent N with Jordan blocks $(2, 1^2)$ is the collection of 2-planes $V \in \mathrm{Gr}(2, 4)$ which are in the kernel of N and contain its image. The image of N is 1-dimensional and contains the kernel which is 3-dimensional. Thus the collection of such 2-planes forms a projective line $\mathbb{P}^1$. Similarly to above, let $\tilde{U}'$ denote the preimage of U' under π and denote by π_U the restriction of π to U'.

We consider the push-forward $\pi_{U*}\underline{k}_{U'}[8]$. The last paragraph implies that the stalks of this push-forward are as follows:

	-8	-7	-6
$\mathcal{O}_{\mathrm{reg}}$	k	0	0
$\mathcal{O}_{\mathrm{subreg}}$	k	0	k

Let $i : \mathcal{O}_{\min} \hookrightarrow U$ denote the closed inclusion. We have a truncation distinguished triangle analogous to Section 3.5:

$$\underline{k}_{U'}[8] \longrightarrow \pi_{U*}\underline{k}_{\tilde{U}'}[8] \longrightarrow i_*\underline{k}_{\mathcal{O}_{\min}}[6] \rightsquigarrow$$

or equivalently a short exact sequence of perverse sheaves:

$$0 \longrightarrow \underline{k}_{U'}[8] \longrightarrow \pi_{U*}\underline{k}_{\tilde{U}'}[8] \longrightarrow i_*\underline{k}_{\mathcal{O}_{\min}}[6] \longrightarrow 0$$

This sequence splits by the same calculation as in Section 3.5. Hence

$$\pi_{U*}\underline{k}_{\tilde{U}'}[8] = \underline{k}_{U'}[8] \oplus i_*\underline{k}_{\mathcal{O}_{\min}}[6]$$

and

$$\mathrm{R}\Gamma(\tilde{U}', k)[8] = \mathrm{R}\Gamma(U', k)[8] \oplus \mathrm{R}\Gamma(\mathcal{O}_{\min}, k)[6]. \tag{7}$$

We already know the cohomology of $\mathcal{O}_{\min}$ by Section 3.2. Hence all that remains is to compute $\mathrm{R}\Gamma(\tilde{U}', k)[8]$. Recall that the cohomology

of $\mathrm{Gr}(2,4)$ has a integral basis given by Schubert cells. In particular its cohomology over $\mathbb{Z}$ is:

0	1	2	3	4	5	6	7	8
$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}^2$	0	$\mathbb{Z}$	0	$\mathbb{Z}$

The space $\tilde{U}'$ is the cotangent bundle to $\mathrm{Gr}(2,4)$ with the zero section removed. Using the Gysin sequence we can compute the cohomology of $\tilde{U}'$ from that of $\mathrm{Gr}(2,4)$. Thus the cohomology of $\underline{k}_{\tilde{U}'}[8]$ is given by:

-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7
k	0	k	0	k^2	0	k	$(k)_6$	$(k)_6$	k	0	k^2	0	k	0	k

We computed the cohomology of $\mathcal{O}_{\min}$ in Section 3.2 (shifted here by 6) for $p \neq 2$ to be:

-6	-5	-4	-3	-2	-1	0	1	2	3	4	5
k	0	k	0	k	0	0	k	0	k	0	k

By (7) we obtain the desired cohomology of U' (for $p \neq 2$) shifted by 8:

-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7
k	0	0	0	k	0	0	$(k)_3$	$(k)_3$	0	0	k	0	0	0	k

Finally, all that remains is to truncate:

THEOREM 3.3. *Let k be a field of characteristic $p \neq 2$. Then*

$$\mathbf{IC}(\overline{\mathcal{O}}_{(2^2)}, k)_0 = \begin{cases} k[8] \oplus k[4] \oplus k[1] & \text{if } p = 3, \\ k[8] \oplus k[4] & \text{if } p > 3. \end{cases}$$

We get the following parts of the decomposition matrices for $p = 3$:

$$\begin{array}{c} \\ 1^4 \\ 21^2 \\ 2^2 \end{array} \begin{array}{c} \begin{array}{ccc} 1^4 & 21^2 & 2^2 \end{array} \\ \begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{1} \end{pmatrix} \end{array} \qquad \begin{array}{c} \\ S_{(4)} \\ S_{(31)} \\ S_{(2^2)} \end{array} \begin{array}{c} \begin{array}{ccc} D_{(4)} & D_{(31)} & D_{(2^2)} \end{array} \\ \begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{1} \end{pmatrix} \end{array}$$

References

[AJS94] H. H. Andersen, J. C. Jantzen, and W. Soergel. Representations of quantum groups at a pth root of unity and of semisimple groups in characteristic p: independence of p. *Astérisque*, (220):321, 1994.

[AP83] M. F. Atiyah and A. N. Pressley. Convexity and loop groups. In *Arithmetic and geometry, Vol. II*, volume 36 of *Progr. Math.*, pages 33–63. Birkhäuser Boston, Boston, MA, 1983.

[Ara01] A. Arabia. Faisceaux pervers sur les variétés algébriques complexes. Correspondance de Springer (d'après Borho-MacPherson). Lecture notes, 67 pages, `http://people.math.jussieu.fr/~arabia/math/Pervers.pdf`, 2001.

[B+08] A. Borel et al. *Intersection cohomology*. Modern Birkhäuser Classics. Birkhäuser Boston Inc., Boston, MA, 2008. Notes on the seminar held at the University of Bern, Bern, 1983, Reprint of the 1984 edition.

[BBD82] A. Beĭlinson, J. Bernstein, and P. Deligne. Faisceaux pervers. In *Analyse et topologie sur les espaces singuliers, I (Luminy, 1981)*, volume 100 of *Astérisque*, pages 5–171. Soc. Math. France, Paris, 1982.

[BD] A. Beĭlinson and V. Drinfel'd. Quantization of Hitchin integrable system and Hecke eigensheaves. 384 pages, preprint `http://www.math.uchicago.edu/~mitya/langlands/hitchin/BD-hitchin.pdf`.

[BL94a] A. Beauville and Y. Laszlo. Conformal blocks and generalized theta functions. *Comm. Math. Phys.*, 164(2):385–419, 1994.

[BL94b] J. Bernstein and V. Lunts. *Equivariant sheaves and functors*, volume 1578 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1994.

[BM81] W. Borho and R. MacPherson. Représentations des groupes de Weyl et homologie d'intersection pour les variétés nilpotentes. *C. R. Acad. Sci. Paris Sér. I Math.*, 292(15):707–710, 1981.

[BM83] W. Borho and R. MacPherson. Partial resolutions of nilpotent varieties. In *Analyse et topologie sur les espaces singuliers, II, III (Luminy, 1981)*, volume 101 of *Astérisque*, pages 23–74. Soc. Math. France, Paris, 1983.

[BM01] T. Braden and R. MacPherson. From moment graphs to intersection cohomology. *Math. Ann.*, 321(3):533–551, 2001.

[Bri71] E. Brieskorn. Singular elements of semi-simple algebraic groups. In *Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 2*, pages 279–284. Gauthier-Villars, Paris, 1971.

[Bry86] J.-L. Brylinski. Transformations canoniques, dualité projective, théorie de Lefschetz, transformations de Fourier et sommes trigonométriques. *Astérisque*, 140-141:3–134, 1986. Géométrie et analyse microlocales.

[dCM05] M. A. A. de Cataldo and L. Migliorini. The Hodge theory of algebraic maps. *Ann. Sci. École Norm. Sup. (4)*, 38(5):693–750, 2005.

[Fie06] P. Fiebig. Multiplicity one results in Kazhdan-Lusztig theory and equivariant intersection cohomology. Preprint arXiv:math/0607501, 2006.

[Fie07a] P. Fiebig. Lusztig's conjecture as a moment graph problem. Preprint arXiv:0712.3909, 2007.

[Fie07b] P. Fiebig. Sheaves on affine Schubert varieties, modular representations and Lusztig's conjecture. Preprint arXiv:0711.0871, 2007.

[Fie08] P. Fiebig. An upper bound on the exceptional characteristics for Lusztig's character formula. Preprint arXiv:0811.1674, 2008.

[Gin95] V. Ginzburg. Perverse sheaves on a loop group and langlands duality. Preprint arXiv:alg-geom/9511007, 1995.

[GM83] M. Goresky and R. MacPherson. Intersection homology. II. *Invent. Math.*, 72(1):77–129, 1983.

[HK84] R. Hotta and M. Kashiwara. The invariant holonomic system on a semisimple Lie algebra. *Invent. Math.*, 75(2):327–358, 1984.

[Jan87] J. C. Jantzen. *Representations of algebraic groups*, volume 131 of *Pure and Applied Mathematics*. Academic Press Inc., Boston, MA, 1987.

[Jut07] D. Juteau. *Modular Springer correspondence and decomposition matrices*. PhD thesis, Université Paris 7 Denis Diderot, 2007. Available at `www.math.jussieu.fr/~juteau/manuscrit/these-juteau.pdf`.

[Jut08] D. Juteau. Cohomology of the minimal nilpotent orbit. *Transform. Groups*, 13(2):355–387, 2008.

[Jut09] D. Juteau. Decomposition numbers for perverse sheaves. *Ann. Inst. Fourier*, 59, 2009. To appear.

[KL79] D. Kazhdan and G. Lusztig. Representations of Coxeter groups and Hecke Algebras. *Invent. Math.*, 53:165–184, 1979.

[KL80a] D. Kazhdan and G. Lusztig. Schubert varieties and Poincaré duality. *Proc. Symposia in Pure Math.*, 36:185–203, 1980.

[KL80b] D. Kazhdan and G. Lusztig. A topological approach to Springer's representations. *Adv. in Math.*, 38(2):222–228, 1980.

[KP81] H. Kraft and C. Procesi. Minimal singularities in $\mathrm{GL}\mathfrak{s}n$. *Invent. Math.*, 62(3):503–515, 1981.

[Lip00] J. Lipman. Equisingularity and simultaneous resolution of singularities. In *Resolution of singularities (Obergurgl, 1997)*, volume 181 of *Progr. Math.*, pages 485–505. Birkhäuser, Basel, 2000.

[LS97] Y. Laszlo and C. Sorger. The line bundles on the moduli of parabolic G-bundles over curves and their sections. *Ann. Sci. École Norm. Sup. (4)*, 30(4):499–525, 1997.

[Lus81] G. Lusztig. Green polynomials and singularities of unipotent classes. *Adv. Math.*, 42:169–178, 1981.

[Lus83] G. Lusztig. Singularities, character formulas, and a q-analog of weight multiplicities. In *Analyse et topologie sur les espaces singuliers, II, III (Luminy, 1981)*, volume 101-102 of *Astérisque*, pages 208–229. Soc. Math. France, Paris, 1983.

[Lus84] G. Lusztig. Intersection cohomology complexes on a reductive group. *Invent. Math.*, 75:205–272, 1984.

[Lus86] G. Lusztig. Character sheaves. *Adv. Math.*, 1985-1986. I 56:193–237, 1985 ; II 57:226–265, 1985 ; III 57:266–315, 1985 ; IV 59:1–63, 1986 ; V 61:103–155, 1986.

[Lus90] G. Lusztig. *Intersection cohomology methods in representation theory.* ICM-90. Mathematical Society of Japan, Tokyo, 1990. A plenary address presented at the International Congress of Mathematicians held in Kyoto, August 1990.

[MV07] I. Mirković and K. Vilonen. Geometric Langlands duality and representations of algebraic groups over commutative rings. *Ann. of Math. (2)*, 166(1):95–143, 2007.

[Sai89] M. Saito. Introduction to mixed Hodge modules. *Astérisque*, (179-180):10, 145–162, 1989. Actes du Colloque de Théorie de Hodge (Luminy, 1987).

[Sat63] I. Satake. Theory of spherical functions on reductive algebraic groups over $\mathfrak{p}$-adic. *Publ. Math. I.H.E.S.*, 18:5–69, 1963.

[Ser67] J.-P. Serre. *Représentations linéaires des groupes finis.* Hermann, Paris, 1967.

[Ser77] J.-P. Serre. *Arbres, amalgames,* SL_2. Société Mathématique de France, Paris, 1977. Avec un sommaire anglais, Rédigé avec la collaboration de Hyman Bass, Astérisque, No. 46.

[Slo80a] P. Slodowy. *Four lectures on simple groups and singularities*, volume 11 of *Communications of the Mathematical Institute, Rijksuniversiteit Utrecht.* Rijksuniversiteit Utrecht Mathematical Institute, Utrecht, 1980.

[Slo80b] P. Slodowy. *Simple singularities and simple algebraic groups*, volume 815 of *Lecture Notes in Mathematics*. Springer, Berlin, 1980.

[Soe00] W. Soergel. On the relation between intersection cohomology and representation theory in positive characteristic. *J. Pure Appl. Algebra*, 152(1-3):311–335, 2000.

[Spr76] T. A. Springer. Trigonometric sums, Green functions of finite groups and representations of Weyl groups. *Invent. Math.*, 36:173–207, 1976.

[Whi65] H. Whitney. Tangents to an analytic variety. *Ann. of Math. (2)*, 81:496–549, 1965.

LMNO, Université de Caen Basse-Normandie, CNRS, BP 5186, 14032 Caen, France

E-mail address: juteau@math.unicaen.fr

Mathematics department, University of Texas at Austin, 1 University Station C1200, Austin TX, 78712 USA

E-mail address: cmautner@math.utexas.edu

Mathematical Institute, University of Oxford, 24-29 St Giles', Oxford, OX1 3LB, UK

E-mail address: williamsong@maths.ox.ac.uk