Oblique poles of $\int_X |f|^{2\lambda}|g|^{2\mu}$

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ABSTRACT. Existence of oblique polar lines for the meromorphic extension of the current valued function $\int |f|^{2\lambda}|g|^{2\mu}\square$ is given under the following hypotheses: f and g are holomorphic function germs in $\mathbb{C}^{n+1}$ such that g is non-singular, the germ $S := \{\mathrm{d}f \wedge \mathrm{d}g = 0\}$ is one dimensional, and $g|_S$ is proper and finite. The main tools we use are interaction of strata for f (see [4]), monodromy of the local system $H^{n-1}(u)$ on S for a given eigenvalue $\exp(-2i\pi u)$ of the monodromy of f, and the monodromy of the cover $g|_S$. Two non-trivial examples are completely worked out.

INTRODUCTION

In the study of a holomorphic function f defined in an open neighbourhood of $0 \in \mathbb{C}^{n+1}$ with one dimensional critical locus S started in [4] and completed in [5], the main tool was to restrict f to hyperplane sections transverse to $S^* := S \setminus \{0\}$ and examine, for a given eigenvalue $\exp(-2i\pi u)$ of the monodromy of f, the local system $H^{n-1}(u)$ on S^* formed by the corresponding spectral subspaces. Higher order poles of the current valued meromorphic function $\int |f|^{2\lambda}\square$ at $-u-m$, some $m \in \mathbb{N}$, are detected by non-extendable sections of $H^{n-1}(u)$ to S. An important part of this local system remained unexplored in [4] and [5] because only the eigenvalue 1 of the monodromy Θ of the local system $H^{n-1}(u)$ has been considered, via the spaces $H^0(S^*, H^{n-1}(u))$ and $H^1(S^*, H^{n-1}(u))$. In this paper, we will focus on the other eigenvalues of Θ. Let us introduce an auxiliary function t with the following properties:

(1) the function t is non-singular near 0;
(2) the set $\Sigma := \{\mathrm{d}f \wedge \mathrm{d}t = 0\}$ is a curve ;
(3) the restriction $t|_S : S \to \mathbb{D}$ is proper and finite ;
(4) $t|_S^{-1}(0) = \{0\}$ and $t|_{S^*}$ is a finite cover of $\mathbb{D}^* := \mathbb{D} \setminus \{0\}$.

Remark that condition (4) may always be acheived by localization near 0 when conditions (1), (2) and (3) are satisfied. These conditions are satisfied in a neighbourhood of the origin if (f, t) forms an isolated complete intersection singularity (icis) with one dimensional critical locus. But we allow also the case where Σ has branches in $\{f = 0\}$ not contained in S.

The direct image of the constructible sheaf $H^{n-1}(u)$ supported in S by t will be denoted by $\mathcal{H}$; it is a local system on $\mathbb{D}^*$. Let $\mathcal{H}_0$ be the fibre of $\mathcal{H}$ at $t_0 \in \mathbb{D}^*$ and Θ_0 its monodromy which is an automorphism of $\mathcal{H}_0$. In case where S is smooth, it is possible to choose the function t in order that $t|_S$ is an isomorphism and Θ_0 may be identified with the monodromy Θ of $H^{n-1}(u)$ on S^*. In general, Θ_0 combines Θ and

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the monodromy of the cover $t|_{S^*}$.
If $S =: \cup_{i\in I} S_i$ is the decomposition of S into irreducible branches, we have an analoguous decomposition on S^* of the local system $H^{n-1}(u) = \oplus_i H^{n-1}(u)^i$ with $\Theta = \oplus_i \Theta^i$.

Take an eigenvalue $\exp(-2i\pi l/k) \neq 1$ of Θ, with $l \in [1, k-1]$ and $(l,k) = 1$. We define an anologue of the interaction of strata in this new context. The auxiliary non singular function t is used to realize analytically the rank one local system on S^* with monodromy $\exp(-2i\pi l/k)$. To perform this we need the degree of t at the origin on the irreducible branch S_i we are interested in to be relatively prime to k. Of course this is the case when S is smooth and t transversal to S at the origin. Using then a $k-$th root of t we can lift our situation to the case where we consider the invariant section of the complex of vanishing cycles of the lifted function $\tilde{f}$ (see Theorem 3.2) and then use already known results from [4].

With the help of elementary properties of meromorphic functions of two variables detailed in paragraph 2, we deduce from interaction of strata above the existence of oblique polar lines for the meromorphic extension of $\int_X |f|^{2\lambda}|t|^{2\mu}\Box$. This result is new and consists in a first step toward the comprehension of the polar structure of such an extension.

1. Polar structure of $\int_X |f|^{2\lambda}\,\Box$

Theorem 1.1. Bernstein & Gelfand. *For m and $p \in \mathbb{N}^*$, let Y be an open subset in $\mathbb{C}^m$, $f : Y \to \mathbb{C}^p$ a holomorphic map and X a relatively compact open set in Y. Then there exists a finite set $P(f) \subset \mathbb{N}^p$ such that, for any form $\phi \in \Lambda^{m,m}C_c^\infty(X)$ with compact support, the holomorphic map in $\{\Re\lambda_1 > 0\} \times \cdots \times \{\Re\lambda_p > 0\}$ given by*

$$(\lambda_1, \ldots, \lambda_p) \mapsto \int_X |f_1|^{2\lambda_1} \ldots |f_p|^{2\lambda_p}\phi \tag{1.1}$$

has a meromorphic extension to $\mathbb{C}^p$ with poles contained in the set

$$\bigcup_{a\in P(f), l\in\mathbb{N}^*} \{\langle a \mid \lambda\rangle + l = 0\}.$$

Proof. For sake of completeness we recall the arguments of [8].
Using desingularization of the product $f_1 \ldots f_p$, we know [10] that there exists a holomorphic manifold $\tilde{Y}$ of dimension m and a holomorphic proper map $\pi : \tilde{Y} \to Y$ such that the composite functions $\tilde{f}_j := f_j \circ \pi$ are locally expressible as

$$\tilde{f}_k(y) = y_1^{a_1^k} \ldots y_m^{a_m^k} u_k(y), 1 \leqslant k \leqslant p, \tag{1.2}$$

where $a_j^k \in \mathbb{N}$ and u_k is a holomorphic nowhere vanishing function. Because $\pi^{-1}(X)$ is relatively compact, it may be covered by a finite number of open set where (1.2) is valid. For $\varphi \in \Lambda^{m,m}C_c^\infty(X)$ and $\Re\lambda_1, \ldots, \Re\lambda_p$ positive, we have

$$\int_X |f_1|^{2\lambda_1} \ldots |f_p|^{2\lambda_p}\phi = \int_{\pi^{-1}(X)} |\tilde{f}_1|^{2\lambda_1} \ldots |\tilde{f}_p|^{2\lambda_p}\pi^*\phi.$$

Using partition of unity and setting $\mu_k := a_k^1\lambda_1 + \cdots + a_k^p\lambda_p$, $1 \leqslant k \leqslant m$, we are reduced to give a meromorphic extension to

$$(\mu_1, \ldots, \mu_m) \to \int_{\mathbb{C}^m} |y_1|^{2\mu_1} \ldots |y_m|^{2\mu_m}\omega(\mu, y), \tag{1.3}$$

where ω is a C^∞ form of type (m, m) with compact support in $\mathbb{C}^m$ valued in the space of entire functions on $\mathbb{C}^m$. Of course, (1.3) is holomorphic in $\{\Re\mu_1 > -1, \ldots, \Re\mu_m > -1\}$. The relation

$$(\mu_1 + 1).|y_1|^{2\mu_1} = \partial_1(|y_1|^{2\mu_1}.y_1)$$

implies by partial integration in y_1

$$\int_{\mathbb{C}^m} |y_1|^{2\mu_1} \ldots |y_m|^{2\mu_m}\omega(\mu, y) = \frac{-1}{\mu_1 + 1}\int_{\mathbb{C}^m} |y_1|^{2\mu_1}.y_1.|y_2|^{2\mu_2} \ldots |y_m|^{2\mu_m}\partial_1\omega(\mu, y).$$

Because $\partial_1\omega$ is again a C^∞ form of type (m, m) with compact support in $\mathbb{C}^m$ valued in the space of entire functions on $\mathbb{C}^m$, we may repeat this argument for each coordinate $y_2, \ldots, y_m$ and obtain

$$\int_{\mathbb{C}^m} |y_1|^{2\mu_1} \ldots |y_m|^{2\mu_m}\omega(\mu, y) =$$
$$\frac{(-1)^m}{(\mu_1 + 1)\ldots(\mu_m + 1)}\int_{\mathbb{C}^m} |y_1|^{2\mu_1}.y_1.|y_2|^{2\mu_2}.y_2 \ldots |y_m|^{2\mu_m}.y_m.\partial_1 \ldots \partial_m\omega(\mu, y).$$

The integral on the right hand side is holomorphic for $\Re\mu_1 > -3/2, \ldots, \Re\mu_m > -3/2$. Therefore the function (1.3) is meromorphic in this domain with only possible poles in the union of the hyperplanes $\{\mu_1 + 1 = 0\}, \ldots, \{\mu_m + 1 = 0\}$.
Iteration of these arguments concludes the proof. □

Remark 1.2. An alternate proof of Theorem 1.1 has been given for $p = 1$ by Bernstein [7], Björk [9], Barlet-Maire [6], and by Loeser [11] and Sabbah [12] in general.

In case where $f_1, \ldots, f_p$ define an isolated complete intersection singularity (icis), Loeser and Sabbah gave morover the following information on the set $P(f)$ of the "slopes" of the polar hyperplanes in the meromorphic extension of the function (1.1): it is contained in the set of slopes of the discriminant Δ of f, which in this case is an hypersurface in $\mathbb{C}^p$. More precisely, take the $(p-1)-$skeleton of the fan associated to the Newton polyhedron of Δ at 0 and denote by $P(\Delta)$ the set of directions associated to this $(p-1)-$skeleton union $\{(a_1, \ldots, a_p) \in \mathbb{N}^p \mid a_1 \ldots a_p = 0\}$. Then

$$P(f) \subseteq P(\Delta).$$

In particular, if the discriminant is contained in the hyperplanes of coordinates, then there are no polar hyperplanes with direction in $(\mathbb{N}^*)^p$.
The results of Loeser and Sabbah above have the following consequence for an icis which is proved below directly by elementary arguments, after we have introduced some terminology.

Definition 1.3. Let $f_1, \ldots, f_p$ be holomorphic functions on an open neighbourhood X of the origin in $\mathbb{C}^m$. We shall say that a polar hyperplane $H \subset \mathbb{C}^p$ for the meromorphic extension of $\int_X |f_1|^{2\lambda_1} \ldots |f_p|^{2\lambda_p}\square$ *is supported by the closed set* $F \subset X$, when H is not

a polar hyperplane for the meromorphic extension of $\int_{X\setminus F}|f_1|^{2\lambda_1}\dots|f_p|^{2\lambda_p}\square$. We shall say that a polar direction is *supported in* F if any polar hyperplane with this direction is supported by F.

Proposition 1.4. *Assume $f_1,\dots,f_p$ are quasi-homogeneous for the weights $w_1,\dots,w_p$, of degree $a_1,\dots,a_p$. Then if there exists a polar hyperplane direction supported by the origin for (1.1) in $(\mathbb{N}^*)^p$ it is $(a_1,\dots,a_p)$ and the corresponding poles are at most simple. In particular, for $p=2$, and if (f_1,f_2) is an icis, all oblique poles have direction (a_1,a_2).*

Proof. Quasi-homogeneity gives $f_k(t^{w_1}x_1,\dots,t^{w_m}x_m)=t^{a_k}f_k(x),\ k=1,\dots,p$.
Let $\Omega:=\sum_1^m(-1)^{j-1}w_jx_j\,\mathrm{d}x_0\wedge\dots\wedge\widehat{\mathrm{d}x_j}\wedge\dots\wedge\mathrm{d}x_m$ so that $\mathrm{d}\Omega=(\sum w_j)\,\mathrm{d}x$.
¿From Euler's relation,because $f_k^{\lambda_k}$ is quasi-homogeneous of degree $a_k\lambda_k$:

$$\mathrm{d}f_k^{\lambda_k}\wedge\Omega=a_k\lambda_kf_k^{\lambda_k}\,\mathrm{d}x,$$
$$\mathrm{d}x^\delta\wedge\Omega=\langle w\mid\delta\rangle x^\delta\,\mathrm{d}x,\ \forall\delta\in\mathbb{N}^m.$$

Take $\rho\in\mathcal{C}_c^\infty(\mathbb{C}^m)$; then, with $\mathbf{1}=(1,\dots,1)\in\mathbb{N}^p$ and $\varepsilon\in\mathbb{N}^m$:

$$\mathrm{d}(|f|^{2\lambda}x^\delta\bar{x}^\varepsilon\rho\Omega\wedge\mathrm{d}\bar{x})=(\langle a\mid\lambda\rangle+\langle w\mid\delta+\mathbf{1}\rangle)|f|^{2\lambda}x^\delta\bar{x}^\varepsilon\rho\,\mathrm{d}x\wedge\mathrm{d}\bar{x}+|f|^{2\lambda}x^\delta\bar{x}^\varepsilon\,\mathrm{d}\rho\wedge\Omega\wedge\mathrm{d}\bar{x}.$$

Using Stokes' formula we get

$$(\langle a\mid\lambda\rangle+\langle w\mid\delta+\mathbf{1}\rangle)\int|f|^{2\lambda}x^\delta\bar{x}^\varepsilon\rho\,\mathrm{d}x\wedge\mathrm{d}\bar{x}=-\int|f|^{2\lambda}x^\delta\bar{x}^\varepsilon\,\mathrm{d}\rho\wedge\Omega\wedge\mathrm{d}\bar{x}.$$

For $\rho=1$ near 0, $\mathrm{d}\rho=0$, near 0. Therefore the right hand side has no poles supported by the origin. Now the conclusion comes from the Taylor expansion at 0 of the test function. □

The question of whether a polar hyperplane is effectively present in the meromorphic extension of the function (1.1) for a least one ϕ has been addressed in case $p=1$ under the name "contribution effective" in a sequence of papers by D. Barlet [2], [1], [3] etc ... For $p>1$ no general geometric conditions are known to produce poles with direction in $(\mathbb{N}^*)^p$. In the following paragraphs, we examine the case $p=2$.

2. Existence of polar oblique lines

In this paragraph, we consider two holomorphic functions $f,g:Y\to\mathbb{C}$, where Y is an open subset in $\mathbb{C}^m$ and fix a relatively compact open subset X of Y. Without loss of generality, we assume $0\in X$. We study the possible oblique poles of the meromorphic extension of the current valued function

$$(\lambda,\mu)\mapsto\int_X|f|^{2\lambda}|g|^{2\mu}\square. \tag{2.1}$$

The following elementary lemma is basic.

Lemma 2.1. *Let M be a meromorphic function in $\mathbb{C}^2$ with poles along a family of lines with directions in $\mathbb{N}^2$. For $(\lambda_0,\mu_0)\in\mathbb{C}^2$, assume*

(i) *$\{\lambda=\lambda_0\}$ is a polar line of order $\leqslant k_0$ of M,*
(ii) *$\{\mu=\mu_0\}$ is not a polar line of M,*
(iii) *$\lambda\mapsto M(\lambda,\mu_0)$ has a pole of order at least k_0+1 at λ_0.*

Then there exists $(a,b)\in(\mathbb{N}^*)^2$ *such that the function* M *has a pole along the (oblique) line* $\{a\lambda+b\mu=a\lambda_0+b\mu_0\}$.

Proof. If M does not have an oblique pole through (λ_0,μ_0), then the function $(\lambda,\mu)\mapsto(\lambda-\lambda_0)^{k_0}M(\lambda,\mu)$ is holomorphic near (λ_0,μ_0). Therefore, $\lambda\mapsto M(\lambda,\mu_0)$ has at most a pole of order k_0 at λ_0. Contradiction. □

It turns out that to check the first condition in the above lemma for the function (2.1), it is sufficient to examine the poles of the meromorphic extension of the current valued function

$$\lambda\mapsto\int_X|f|^{2\lambda}\square. \tag{2.2}$$

Such a simplification does not hold for general meromorphic functions. For example,

$$(\lambda,\mu)\mapsto\frac{\lambda+\mu}{\lambda^2}$$

has a double pole along $\{\lambda=0\}$ but its restricition to $\{\mu=0\}$ has only a simple pole at 0.

Proposition 2.2. *If the meromorphic extension of the current valued function* (2.2) *has a pole of order* k *at* $\lambda_0\in\mathbb{R}_-$, i.e., *it has a principal part of the form*

$$\frac{T_k}{(\lambda-\lambda_0)^k}+\cdots+\frac{T_1}{\lambda-\lambda_0},$$

at λ_0, *then the meromorphic extension of the function* (2.1) *has a pole of order*

$$k_0:=\max\{0\leqslant l\leqslant k\mid \operatorname{supp}T_l\not\subseteq\{g=0\}\} \tag{2.3}$$

along the line $\{\lambda=\lambda_0\}$.

Proof. As a consequence of the Bernstein identity (see [9]), there exists $N\in\mathbb{N}$ such that the extension of $\int_X|f|^{2\lambda}\phi$ in $\{\Re\lambda>\lambda_0-1\}$ can be achieved for $\phi\in\Lambda^{m,m}\mathcal{C}^N_c(X)$. Our hypothesis implies that this function has a pole of order $\leqslant k$ at λ_0. Because $|g|^{2\mu}\phi$ is of class $\mathcal{C}^N$ for $\Re\mu$ large enough, the function

$$\lambda\mapsto\int_X|f|^{2\lambda}|g|^{2\mu}\phi$$

has a meromorphic extension in $\{\Re\lambda>\lambda_0-1\}$ with a pole of order $\leqslant k$ at λ_0. We have proved that (2.1) has a pole of order $\leqslant k$ along the line $\{\lambda=\lambda_0\}$.
Near λ_0, the extension of $\int_X|f|^{2\lambda}\phi$ writes

$$\frac{\langle T_k,\phi\rangle}{(\lambda-\lambda_0)^k}+\cdots+\frac{\langle T_1,\phi\rangle}{\lambda-\lambda_0}+\cdots.$$

Hence that of $\int_X|f|^{2\lambda}|g|^{2\mu}\phi$ looks

$$\frac{\langle T_k|g|^{2\mu},\phi\rangle}{(\lambda-\lambda_0)^k}+\cdots+\frac{\langle T_1|g|^{2\mu},\phi\rangle}{\lambda-\lambda_0}+\cdots.$$

If $\operatorname{supp}T_k\subseteq\{g=0\}$, then the first term vanishes for $\Re\mu$ large enough, because T_k is of finite order (see the beginning of the proof). So the order of the pole along the line $\{\lambda=\lambda_0\}$ is $\leqslant k_0$.

Take $x_0 \in \operatorname{supp} T_{k_0}$ such that $g(x_0) \neq 0$ and V a neighborhood of x_0 in which g does not vanish. From the definition of the support, there exists $\psi \in \Lambda^{m,m} C_c^\infty(V)$ such that $\langle T_{k_0}, \psi \rangle \neq 0$. With $\phi := \psi |g|^{-2\mu} \in \Lambda^{m,m} C_c^\infty(V)$, we get

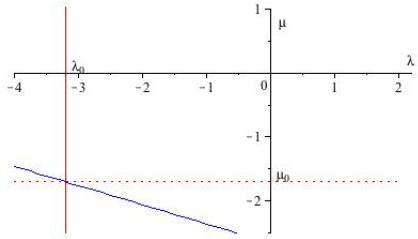
$$\langle T_{k_0} |g|^{2\mu}, \phi \rangle = \langle T_{k_0}, \psi \rangle \neq 0.$$

Therefore, the extension of (2.1) has a pole of order k_0 along the line $\{\lambda = \lambda_0\}$. □

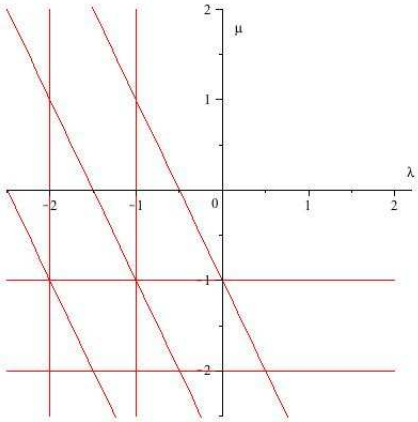
Corollary 2.3. *For $(\lambda_0, \mu_0) \in (\mathbb{R}_-)^2$, assume*

(i) *the extension of the current valued function (2.2) has a pole of order k at λ_0,*
(ii) *μ_0 is not an integer translate of a root of the Bernstein polynomial of g,*
(iii) *$\lambda \mapsto \mathrm{Pf}(\mu = \mu_0, \int_X |f|^{2\lambda} |g|^{2\mu} \square)$ has a pole of order $l_0 > k_0$ where k_0 is defined in (2.3) at λ_0.*

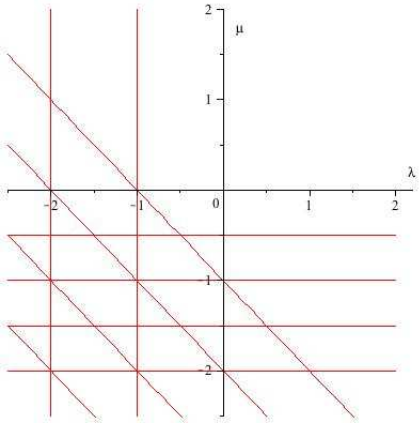
Then the meromorphic extension of the current valued function (2.1) has at least $l_0 - k_0$ oblique lines, counted with multiplicities, through (λ_0, μ_0).



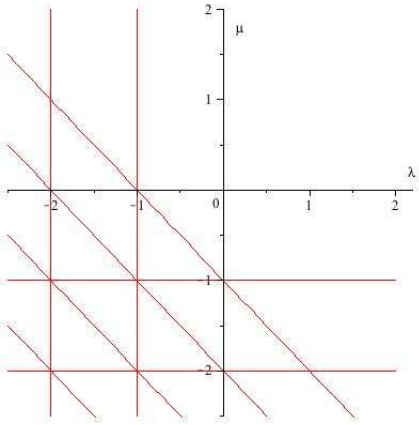
Example 2.4. $m = 3$, $f(x, y, z) = x^2 + y^2 + z^2$, $g(x, y, z) = z$.



Example 2.5. $m = 4$, $f(x, y, z) = x^2 + y^2 + z^2 + t^2$, $g(x, y, z, t) = t^2$.



Example 2.6. $m = 3,\ f(x,y,z) = x^2 + y^2,\ g(x,y,z) = y^2 + z^2$.



In this last example, Corollary 2.3 does not apply because for $\lambda_0 = -1$ we have $k_0 = l_0$. Existence of an oblique polar line through $(-1, 0)$ is obtained by computation of the extension of $\lambda \mapsto \mathrm{Pf}(\mu = 1/2, \int_X |f|^{2\lambda}|g|^{2\mu}\square)$.

3. Pullback and interaction

In this paragraph, we give by pullback a method to verify condition (iii) of Corollary 2.3 when g is a coordinate. As a matter of fact the function $\lambda \mapsto \int_X |f|^{2\lambda}|g|^{2\mu_0}\square$ is only known by meromorphic extension (via Bernstein identity) when μ_0 is negative; it is in general difficult to exhibit some of its poles.
We consider therefore only one holomorphic function $f : Y \to \mathbb{C}$, where Y is an open subset in $\mathbb{C}^{n+1}$ and fix a relatively compact open subset X of Y. The coordinates in $\mathbb{C}^{n+1}$ will be denoted by $x_1, \ldots, x_n, t$. Let us introduce also the finite map

$$p : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1} \text{ such that } p(x_1, \ldots, x_n, \tau) = (x_1, \ldots, x_n, \tau^k) \tag{3.1}$$

for some fixed integer k. Finally, put $\tilde{f} := f \circ p : \tilde{X} \to \mathbb{C}$ where $\tilde{X} := p^{-1}(X)$.

Proposition 3.1. *With the above notations and $\lambda_0 \in \mathbb{R}_-$ suppose*

(a) *the extension of the current valued function (2.2) has a pole of order $\leqslant 1$ at λ_0,*
(b) *$\lambda \mapsto \int_{\tilde{X}} |\tilde{f}|^{2\lambda}\square$ has a double pole at λ_0.*

Then there exists $l \in [1, k-1]$ *such that the extension of the current valued function* $\lambda \mapsto \int_X |f|^{2\lambda}|t|^{-2l/k}\square$ *has a double pole at* λ_0.

Proof. Remark that the support of the polar part of order 2 of $\int_{\tilde{X}} |\tilde{f}|^{2\lambda}\square$ at λ_0 is contained in $\{\tau = 0\}$ because we assume (a) and p is a local isomorphism outside $\{\tau = 0\}$. After hypothesis (b) there exists $\varphi \in \Lambda^{n+1,n+1}\mathcal{C}_c^\infty(\tilde{X})$ such that

$$A := P_2\big(\lambda = \lambda_0, \int_{\tilde{X}} |\tilde{f}|^{2\lambda}\varphi\big) \neq 0.$$

Consider the Taylor expansion of φ along $\{\tau = 0\}$

$$\varphi(x,\tau) = \sum_{j+j' \leqslant N} \tau^j \bar{\tau}^{j'} \varphi_{j,j'}(x) \wedge d\tau \wedge d\bar{\tau} + o(|\tau|^N)$$

where N is larger than the order of the current defined by P_2 on a compact set K containing the support of φ. Therefore

$$A = \sum_{j+j' \leqslant N} P_2\big(\lambda = \lambda_0, \int_{\tilde{X}} |\tilde{f}|^{2\lambda}\tau^j\bar{\tau}^{j'}\varphi_{j,j'}(x)\chi(|\tau|^{2k}) \wedge d\tau \wedge d\bar{\tau}\big)$$

where χ has support in K and is equal to 1 near 0. Because A does not vanish there exists $(j,j') \in \mathbb{N}^2$ with $j + j' \leqslant N$ such that

$$A_{j,j'} := P_2\big(\lambda = \lambda_0, \int_{\tilde{X}} |\tilde{f}|^{2\lambda}\tau^j\bar{\tau}^{j'}\varphi_{j,j'}(x)\chi(|\tau|^{2k}) \wedge d\tau \wedge d\bar{\tau}\big) \neq 0.$$

The change of variable $\tau \to \exp(2i\pi/k)\tau$ that leaves $\tilde{f}$ invariant, shows that $A_{j,j'} = \exp(2i\pi(j-j')/k)A_{j,j'}$. Hence $A_{j,j'} = 0$ for $j - j' \notin k\mathbb{Z}$. We then get the existence of $(j,j') \in \mathbb{N}^2$ verifying $j' = j + k\nu$ with $\nu \in \mathbb{Z}$ and $A_{j,j'} \neq 0$.
The change of variable $t = \tau^k$ in the computation of $A_{j,j'}$ gives

$$P_2\big(\lambda = -\lambda_0, \int_X |f|^{2\lambda}|t|^{2(j-k+1)/k}\bar{t}^\nu\varphi_{j,j+k\nu}(x)\chi(|t|^2) \wedge dt \wedge d\bar{t}\big) \neq 0.$$

This ends the proof with $-l = j - k + 1$ if $\nu \geqslant 0$ and with $-l = j' - k + 1$ if $\nu < 0$. Notice that $l < k$ in all cases. Necessarily $l \neq 0$ because from hypothesis (a), we know that the extension of the function (2.2) does not have a double pole at λ_0. □

Theorem 3.2. *For* Y *open in* $\mathbb{C}^{n+1}$ *and* X *relatively compact open subset of* Y, *let* $f : Y \to \mathbb{C}$ *be holomorphic and* $g(x,t) = t$. *Assume* (f,g) *satisfy properties* (1) *to* (4) *of the Introduction. Moreover suppose*

(a) $\int |f|^{2\lambda}\square$ *has a at most a simple pole at* $\lambda_0 - \nu$, $\forall \nu \in \mathbb{N}$;
(b) $e^{2\pi i\lambda_0}$ *is a eigenvalue of the monodromy of* f *acting on the* H^{n-1} *of the Milnor fiber of* f *at the generic point of a connected component* S_i^* *of* S^*, *and there exist a non zero eigenvector such its monodromy around* 0 *in* S_i *is a primitive* $k-$*root of unity, with* $k \geqslant 2$;
(c) the degree d_i *of the covering* $t : S_i^* \to \mathbb{D}^*$ *is prime to* k;
(d) $e^{2\pi i\lambda_0}$ *is not an eigenvalue of the monodromy of* $\tilde{f}$ *acting on the* H^{n-1} *of the Milnor fiber of* $\tilde{f}$ *at* 0 , *where* $\tilde{f}(x,\tau) = f(x_1,\ldots,x_n,\tau^k)$.

Then there exists an oblique pole of $\int |f|^{2\lambda}|g|^{2\mu}\square$ *through* $(\lambda_0 - j, -l/k)$, *some* $j \in \mathbb{N}$, *and some* $l \in [1, k-1]$.

Remark that condition (b) implies that $\lambda_0 \notin \mathbb{Z}$ because of the result of [1].

Proof. Notice first that $(\lambda, \mu) \mapsto \int |f|^{2\lambda}|g|^{2\mu}\square$ has a simple pole along $\{\lambda = \lambda_0\}$. Indeed the support of the residue current of $\lambda \mapsto \int |f|^{2\lambda}\square$ at λ_0 contains S_i^* where t does not vanish and Proposition 2.2 applies.
Denote by z a local coordinate on the normalization of S_i. The function t has a zero of order d_i on this normalization and hence d_i is the degree of the cover $S_i^* \to \mathbb{D}^*$ induced by t. Without loss of generality, we may suppose $t = z^{d_i}$ on the normalization of S_i.

Lemma 3.3. *Let* $k, d \in \mathbb{N}^*$ *and put*

$$Y_d := \{(z,\tau) \in \mathbb{D}^2 / \tau^k = z^d\}, Y_d^* := Y_d \setminus \{0\}.$$

If k *and* d *are relatively prime, then the first projection* $\mathrm{pr}_{1,d} : Y_d^* \to \mathbb{D}^*$ *is a cyclic cover of degree* k.

Proof. Let us prove that the cover defined by $\mathrm{pr}_{1,d}$ is isomorphic to the cover defined by $\mathrm{pr}_{1,1}$, that may be taken as definition of a cyclic cover of degree k.
After Bézout's identity, there exist $a, b \in \mathbb{Z}$ such that

$$ak + bd = 1. \tag{3.2}$$

Define $\varphi : Y_d \to \mathbb{C}^2$ by $\varphi(z,\tau) = (z, z^a\tau^b)$. From (3.2), we have $\varphi(Y_d) \subseteq Y_1$ and clearly $\mathrm{pr}_{1,1} \circ \varphi = \mathrm{pr}_{1,d}$.
The map φ is injective because

$$\varphi(z,\tau) = \varphi(z,\tau') \Longrightarrow \tau^k = \tau'^k \text{ and } \tau^b = \tau'^b,$$

hence $\tau = \tau'$, after (3.2). It is also surjective: take $(z,\sigma) \in Y_1^*$; the system

$$\tau^b = \sigma z^{-a},\ \tau^k = z^d, \text{ when } \sigma^k = z,$$

has a unique solution because the compatibility condition $\sigma^k z^{-ak} = z^{bd}$ is satisfied. $\square$

End of proof of Theorem 3.2. Take the eigenvector with monodromy $\exp(-2i\pi l/k)$ with $(l,k) = 1$ given by condition (b). Its pullback by p becomes invariant under the monodromy of τ because of the condition (c) and the lemma given above. After (d), this section does not extend through 0. So we have interaction of strata (see [4]) and a double pole for $\lambda \mapsto \int |\tilde{f}|^{2\lambda}\square$ at $\lambda_0 - j$ with some $j \in \mathbb{N}$. It remains to use Proposition 3.1 and Corollary 2.3. $\square$

4. Interaction of strata revised

In this paragraph notations and hypotheses are those of the Introduction. Here, the function t is the last coordinate. We suppose that the eigenvalue $\exp(-2i\pi u)$ of the monodromy of f is simple at each point of S^*. Therefore, this eigenvalue is also simple for the monodromy acting on the group H^{n-1} of the Milnor fibre of f at 0. In order to compute the constructible sheaf $H^{n-1}(u)$ on S we may use the complex $(\Omega_X[f^{-1}], \delta_u)$, that is the complex of meromorphic forms with poles in $f^{-1}(0)$ equipped with the differential $\delta_u := d - u\frac{\mathrm{d}f}{f}\wedge$ along S. This corresponds to the case $k_0 = 1$ in [4].

We use the isomorphisms

$$r^{n-1} : h^{n-1} \to H^{n-1}(u) \text{ over } S \quad \text{and} \quad \tau_1 : h^{n-1} \to h^n \text{ over } S^*, \tag{4.1}$$

where h^{n-1} [resp. h^n] denotes the $(n-1)-$th [resp. $n-$th] cohomology sheaf of the complex $(\Omega_X[f^{-1}], \delta_u)$.

In order to look at the eigenspace for the eigenvalue $\exp(-2i\pi l/k)$ of the monodromy Θ of the local system $H^{n-1}(u)$ on S^*, it will be convenient to consider the complex of sheaves

$$\Gamma_l := \big(\Omega^\bullet[f^{-1}, t^{-1}], \delta_u - \frac{l}{k}\frac{dt}{t}\wedge\big)$$

which is locally isomorphic along S^* to $(\Omega_X[f^{-1}], \delta_u)$ via the choice of a local branch of $t^{l/k}$ and the morphism of complexes $(\Omega_X[f^{-1}], \delta_u) \to \Gamma_l$ given by $\omega \mapsto t^{l/k}.\omega$ which satisfies

$$\delta_u(t^{l/k}\omega) - \frac{l}{k}\frac{dt}{t} \wedge t^{l/k}\omega = t^{l/k}\delta_u(\omega).$$

But notice that this complex Γ_l is also defined near the origin. Of course, a global section $\sigma \in H^0(S^*, h^{n-1}(\Gamma_l))$ gives, via the above local isomorphism, a multivalued global section on S^* of the local system $H^{n-1}(u) \simeq h^{n-1}$ with monodromy $\exp(-2i\pi l/k)$ (as multivalued section).

So a global meromorphic differential $(n-1)-$form ω with poles in $\{f.t = 0\}$ such that $d\omega = u\frac{\mathrm{d}f}{f}\wedge\omega + \frac{l}{k}\frac{dt}{t}\wedge\omega$ defines such a σ, and an element in $\mathcal{H}_0$ with monodromy $\exp(-2i\pi l/k)$.

We shall use also the morphism of complexes of degree $+1$

$$\check{\tau}_1 : \Gamma_l \to \Gamma_l$$

given by $\check{\tau}_1(\sigma) = \frac{df}{f}\wedge\sigma$. It is an easy consequence of [4] that in our situation $\check{\tau}_1$ induces an isomorphisme $\check{\tau}_1 : h^{n-1}(\Gamma_l) \to h^n(\Gamma_l)$ on S^*, because we have assumed that the eigenvalue $\exp(-2i\pi u)$ for the monodromy of f is simple along S^*.

Our first objective is to build for each $j \in \mathbb{N}$ a morphism of sheaves on S^*

$$r_j : h^{n-1}(\Gamma_l) \to \underline{H}^n_{[S]}(\mathcal{O}_X), \tag{4.2}$$

via the meromorphic extension of $\int_X |f|^{2\lambda}|t|^{2\mu}\Box$. Here $\underline{H}^n_{[S]}(\mathcal{O}_X)$ denotes the subsheaf of the moderate cohomology with support S of the sheaf $\underline{H}^n_S(\mathcal{O}_X)$. It is given by the $n-$th cohomology sheaf of the Dolbeault-Grothendieck complex with support S :

$$\underline{H}^n_{[S]}(\mathcal{O}_X) \simeq \mathcal{H}^n(\underline{H}^0_S(\mathcal{D}b^{0,\bullet}_X), \mathrm{d}'').$$

Let w be a $(n-1)-$meromorphic form with poles in $\{f = 0\}$, satisfying $\delta_u(w) = \frac{l}{k}\frac{\mathrm{d}t}{t}\wedge w$ on an open neighbourghood $U \subset X \setminus \{t = 0\}$ of a point in S^*. Put for $j \in \mathbb{N}$:

$$\overline{r_j(w)} := \mathrm{Res}\,\big(\lambda = -u, \mathrm{Pf}(\mu = -l/k, \int_U |f|^{2\lambda}|t|^{2\mu}\bar{f}^{-j}\frac{\mathrm{d}f}{f}\wedge w \wedge \Box)\big).$$

These formula define $\mathrm{d}'-$closed currents of type $(n,0)$ with support in $S^* \cap U$.

Indeed it is easy to check that the following formula holds in the sense of currents on U:

$$\mathrm{d}'\big[\,\mathrm{Pf}\,\big(\lambda=-u,\mathrm{Pf}(\mu=-l/k,\int_U |f|^{2\lambda}|t|^{2\mu}\bar f^{-j}w\wedge\square)\big)\big]=$$
$$\mathrm{Res}\,\big(\lambda=-u,\mathrm{Pf}(\mu=-l/k,\int_U |f|^{2\lambda}|t|^{2\mu}\bar f^{-j}\frac{\mathrm{d}f}{f}\wedge w\wedge\square)\big).$$

On the other hand, if $w=\delta_u(v)-\frac{l}{k}\frac{\mathrm{d}t}{t}\wedge v$ for $v\in\Gamma(U,\Omega^{n-2}[f^{-1}])$, then

$$\mathrm{d}'\big[\,\mathrm{Res}\,\big(\lambda=-u,\mathrm{Pf}(\mu=-l/k,\int_U |f|^{2\lambda}|t^{2\mu}\bar f^{-j}\frac{\mathrm{d}f}{f}\wedge v\wedge\square\big)\big]=$$
$$\mathrm{Res}\,\big(\lambda=-u,\mathrm{Pf}(\mu=-l/k,\int_U |f|^{2\lambda}|t|^{2\mu}\bar f^{-j}\frac{\mathrm{d}f}{f}\wedge w\wedge\square)\big)$$

because the meromorphic extension of $\int_X |f|^{2\lambda}\square$ has no double poles at $\lambda\in -u-\mathbb{N}$ along S^*, since $\exp(-2i\pi u)$ is a simple eigenvalue of the monodromy of f along S^*. It follows that the morphism of sheaves (4.2) is well defined on S^*.
By direct computation we show the following equality between sections on S^* of the sheaf $\underline{H}^n_{[S]}(\Omega^1_X)$:

$$\mathrm{d}'r_j(w)=-(u+j)\,\mathrm{d}f\wedge r_{j+1}(w)$$

where $\mathrm{d}':\underline{H}^n_{[S]}(\mathcal{O}_X)\to\underline{H}^n_{[S]}(\Omega^1_X)$ is the morphism induced by the de Rham differential $\mathrm{d}:\mathcal{O}_X\to\Omega^1_X$.
Because $\underline{H}^n_{[S]}(\mathcal{O}_X)$ is a sheaf of $\mathcal{O}_X-$modules, it is possible to define the product $g.r_j$ for g holomorphic near a point of S^* and the usual rule holds

$$\mathrm{d}'(g.r_j)=\mathrm{d}g\wedge r_j+g.\,\mathrm{d}'\rho_j.$$

Now we shall define, for each irreducible component S_i of S such that the local system $H^{n-1}(u)^i$ has $\exp(-2i\pi l/k)$ as eigenvalue for its monodromy Θ^i, linear maps

$$\rho^i_j:\mathrm{Ker}\,\big(\Theta^i-\exp(-2i\pi l/k)\big)\to H^0(S^*_i,\underline{H}^n_{[S]}(\mathcal{O}_X))$$

as follows :
Let $s_i\in S^*_i$ be a base point and let $\gamma\in H^{n-1}(u)_{s_i}$ be such that $\Theta^i(\gamma)=\exp(2i\pi l/k).\gamma$. Denote by $\sigma(\gamma)$ the multivalued section of the local system $H^{n-1}(u)^i$ on S^*_i defined by γ. Near each point of $s\in S^*_i$ we can induce σ by a meromorphic $(n-1)-$form w_0 which is δ_u-closed. Choose a local branch of $t^{1/k}$ near the point s and put $w:=t^{l/k}w_0$. Then it is easy to check that we define in this way a global section $\Sigma(\gamma)$ on S^*_i of the sheaf $h^{n-1}(\Gamma_l)$ which is independent of our choices. Now set

$$\rho^i_j(\gamma):=r_j(\Sigma(\gamma)).$$

Like in paragraph 3, define $\tilde f:\tilde X\to\mathbb{C}$ by $\tilde f:=f\circ p$ with p of (3.1). The singular locus $\tilde S$ of $\tilde f$ is again a curve, but it may have components contained in $\{\tau=0\}$ (see for instance Example 5.1). Let $\tilde S^*:=\tau^{-1}(\mathbb{D}^*)$ (so in $\tilde S$ we forget about the components that are in $\tau^{-1}(0)$) and define the local system $\tilde{\mathcal{H}}$ on $\mathbb{D}^*$ as $\tau_*(\tilde H^{n-1}(u)|_{\tilde S^*})$. Denote its fiber $\tilde{\mathcal{H}}_0$ at some τ_0 with $\tau_0^k=t_0$ and the monodromy $\tilde\Theta_0$ of $\tilde{\mathcal{H}}_0$.
We have

$$\tilde\Theta_0=(\pi_*)^{-1}\circ\Theta_0\circ\pi_*,\ \text{where}\ \pi(\tau):=\tau^k,$$

and $\pi_* : \tilde{\mathcal{H}}_0 \to \mathcal{H}_0$ is the isomorphism induced by π.

Choose now the base points s_i of the connected components S_i^* of S^* in $\{t = t_0\}$ where t_0 is the base point of $\mathbb{D}^*$. Moreover choose the base point $\tau_0 \in \mathbb{C}^*$ such that $\tau_0^k = t_0$.
In order to use the results of [4], we need to guarantee that for the component S_i^* of S^*, the map $p^{-1}(S_i^*) \to S_i^*$ is the cyclic cover of degree k.
Fix a base point $\tilde{s}_i \in p^{-1}(S_i^*)$ such that $p(\tilde{s}_i) = s_i$. The local system $\tilde{H}^{n-1}(u)$ on the component $\tilde{S}_i^*$ of $\tilde{S}^*$ containing $\tilde{s}_i$ is given by $\tilde{H}^{n-1}(u)_{\tilde{s}_i}$ which is isomorphic to $H^{n-1}(u)_{s_i}$, and the monodromy automorphism $\tilde{\Theta}^i$. In case $p : \tilde{S}_i^* \to S_i^*$ is the cyclic cover of degree k, we have $\tilde{\Theta}^i = (\Theta^i)^k$.
After Lemma 3.3, this equality is true if k is prime to the degree d_i of the covering $t : S_i^* \to \mathbb{D}^*$.
Let $\tilde{\gamma}$ be the element in $(\tilde{H}^{n-1}(u)^i)_{\tilde{s}_i}$ whose image by p is γ. Let $\sigma(\tilde{\gamma})$ the multivalued section of the local system $\tilde{H}^{n-1}(u)^i$ on $\tilde{S}_i^*$ given by $\tilde{\gamma}$ on $\tilde{S}_i^*$. By construction, if $(k, d_i) = 1$, we get $\tilde{\Theta}^i\tilde{\gamma} = \tilde{\gamma}$. Therefore $\sigma(\tilde{\gamma})$ is in fact a global (singlevalued) section of the local system $\tilde{H}^{n-1}(u)^i$ over $\tilde{S}_i^*$.

Theorem 4.1. *Notations and hypotheses are those introduced above. Take $\gamma \in H^{n-1}(u)_{s_i}$ such that $\Theta^i(\gamma) = \exp(-2i\pi l/k)\gamma$ where Θ^i is the monodromy of $H^{n-1}(u)_{s_i}$ and l is an integer prime to k, between 1 and $k-1$.*
Assume that k is relatively prime to the degree of the cover $t|_{S_i^*}$ of $\mathbb{D}^*$.
If the section $\sigma(\tilde{\gamma})$ of $\tilde{H}^{n-1}(u)^i$ on $\tilde{S}_i^*$ defined by γ is the restriction to $\tilde{S}_i^*$ of a global section W on $\tilde{S}$ of the constructible sheaf $\tilde{H}^{n-1}(u)$ then there exists $\omega \in \Gamma(X, \Omega_X^{n-1})$ such that the following properties are satisfied:

(1) $\mathrm{d}\omega = (m+u)\dfrac{\mathrm{d}f}{f} \wedge \omega + \dfrac{l}{k}\dfrac{\mathrm{d}t}{t} \wedge \omega$, *for some $m \in \mathbb{N}$;*

(2) The $(n-1)-$meromorphic δ_u-closed form $t^{-l/k}\omega/f^m$ induces a section on S of the sheaf $h^{n-1}(\Gamma_l)$ whose restriction to S_i^* is given by $\Sigma(\gamma)$;

(3) the current

$$T_j := \mathrm{Res}\left(\lambda = -m-u, Pf(\mu = -l/k, \int_X |f|^{2\lambda}\bar{f}^{m-j}|t|^{2\mu}\frac{\mathrm{d}f}{f} \wedge \omega \wedge \square)\right)$$

satisfies $\mathrm{d}'T_j = \mathrm{d}'K_j$ for some current K_j supported in the origin and $T_j - K_j$ is a $(n,0)-$current supported in S whose conjugate induces a global section on S of the sheaf $\underline{H}^n_{[S]}(\mathcal{O}_X)$ which is equal to $r_j(\gamma)$ on S_i^*.

Proof. After [2] and [4], there exist an integer $m \geqslant 0$ and a $(n-1)-$holomorphic form $\tilde{\omega}$ on X verifying the following properties:

(i) $\mathrm{d}\tilde{\omega} = (m+u)\dfrac{\mathrm{d}\tilde{f}}{\tilde{f}} \wedge \tilde{\omega}$;

(ii) along $\tilde{S}$ the meromorphic δ_u-closed form $\dfrac{\tilde{\omega}}{\tilde{f}^m}$ induces the section W;

(iii) the current

$$\tilde{T}_j := \mathrm{Res}\left(\lambda = -m-u, \int_{\tilde{X}} |\tilde{f}|^{2\lambda}\bar{\tilde{f}}^{m-j}\frac{\mathrm{d}\tilde{f}}{\tilde{f}} \wedge \tilde{\omega} \wedge \square\right)$$

satisfies $\mathrm{d}'\tilde{T}_j = \mathrm{d}'\tilde{K}_j$ for some current $\tilde{K}_j$ supported in the origin and $\tilde{T}_j - \tilde{K}_j$ is a $(n,0)-$current supported in $\tilde{S}$ whose conjugate induces the element $r_j(W)$ of $H^n_{\tilde{S}}(\tilde{X}, \mathcal{O}_{\tilde{X}})$.

On $\tilde{X}$ we have an action of the group $\mathfrak{G}_k$ of $k-$th roots of unity that is given by $Z.(x,\tau) := (x, \zeta\tau)$ where $\zeta := \exp(2i\pi/k)$. Then X identifies to the complex smooth quotient of $\tilde{X}$ by this action. In particular every $\mathfrak{G}_k-$invariant holomorphic form on $\tilde{X}$ is the pullback of a holomorphic form on X. For the holomorphic form $\tilde{\omega} \in \Gamma(\tilde{X}, \Omega^{n-1}_{\tilde{X}})$ above, we may write

$$\tilde{\omega} = \sum_{l=0}^{k-1} \tilde{\omega}_\ell, \qquad \text{with} \quad Z^*\tilde{\omega}_\ell = \zeta^\ell \tilde{\omega}_\ell.$$

Indeed, $\tilde{\omega}_\ell = \frac{1}{k}\sum_{j,\ell=0}^{k-1} \zeta^{-j\ell}(Z^j)^*\tilde{\omega}$ does the job. Because $\tau^{k-\ell}\tilde{\omega}_\ell$ is $\mathfrak{G}_k-$invariant, there exist holomorphic forms $\omega_0, \ldots, \omega_{k-1}$ on X such that

$$\tilde{\omega} = \sum_{\ell=0}^{k-1} \tau^{\ell-k} p^*\omega_\ell. \tag{4.3}$$

Put $\omega := \omega_{k-l}$. Because property (i) above is $Z-$invariant, each ω_ℓ verifies it and hence ω, whose pullback by p is $\tau^{k-l}\tilde{\omega}_{k-l}$, will satisfy the first condition of the Theorem, after the injectivity of p^*.
The action of Z on $\tilde{\gamma}$ is $Z\tilde{\gamma} = \zeta^{-l}\tilde{\gamma}$; therefore $\tilde{\omega}_{k-l}$ verifies (ii) above and hence for $\ell \neq k-l$, the form $\tilde{\omega}_\ell$ induces 0 in $\tilde{H}^{n-1}(u)$ along $\tilde{S}$.
Let us prove property (3) of the Theorem. When $\tilde{\omega}$ is replaced by $\tilde{\omega}_{k-l}$ in the definition of $\tilde{T}_j$, the section it defines on $\tilde{S}$ does not change. On the other hand, the action of Z on this section is given by multiplication by ζ^{-l}. Because this section extends through 0, the same is true for $\tau^{k-l}\tilde{T}_j$ whose conjugate will define a $\mathfrak{G}_k-$invariant section of $\underline{H}^n_{[S]}(\mathcal{O}_{\tilde{X}})$ extendable through 0. Condition (3) follows from the isomorphism of the subsheaf of $\mathfrak{G}_k-$invariant sections of $\underline{H}^n_{[\tilde{S}]}(\mathcal{O}_{\tilde{X}})$ and $\underline{H}^n_{[S]}(\mathcal{O}_X)$. □

Our next result treats the case where there is a section $\tilde{W}$ of $\tilde{H}^{n-1}(u)$ on $\tilde{S}^*$ which is *not extendable at the origin* and induces $\tilde{\gamma}$ on $\tilde{S}^*_i$. Remark that there always exists a global section on $\tilde{S}^*$ inducing $\tilde{\gamma}$ on $\tilde{S}^*_i$: just put 0 on the branches $\tilde{S}^*_{i'}$ for each $i' \neq i$. The next theorem shows that is this case we obtain an oblique pole of $\int_X |f|^{2\lambda}|g|^{2\mu}\square$.

Remark that in any case we may apply the previous theorem or the next one. When S^* is not connected, it is possible that both apply, because it may exist at the same time a global section on $\tilde{S}^*$ of the sheaf $\tilde{H}^{n-1}(u)$ inducing $\tilde{\gamma}$ on $\tilde{S}^*_i$ which is extendable at the origin, and another one which is not extendable at the origin.

Theorem 4.2. *Under the hypotheses of Theorem* 4.1*, assume that we have a global section $\tilde{W}$ on $\tilde{S}^*$ of the local system $\tilde{H}^{n-1}(u)$ inducing $\tilde{\gamma}$ on $\tilde{S}^*_i$ which is not extendable at the origin. Then there exists $\Omega \in \Gamma(X, \Omega^n_X)$ and $l' \in [1,k]$ with the following properties:*

(1) $\mathrm{d}\Omega = (m+u)\dfrac{\mathrm{d}f}{f} \wedge \Omega + \dfrac{l'}{k}\dfrac{\mathrm{d}t}{t} \wedge \Omega$*;*

(2) along S^* the $n-$meromorphic $(\delta_u - \frac{l'}{k}\frac{dt}{t}\wedge)-$closed form Ω/f^m induces $\check{\tau}_1(\sigma) = \frac{df}{f}\wedge\sigma$ in the sheaf $h^n(\Gamma_{l'})$, for some global section σ on S^* of the sheaf $h^{n-1}(\Gamma_{l'})$;
(3) the current on X of type $(n+1,0)$ with support $\{0\}$:

$$P_2\big(\lambda = -m-u, Pf(\mu = -(k-l')/k, \int_X |f|^{2\lambda}\bar{f}^{m-j}|t|^{2\mu}\frac{df}{f}\wedge\Omega\wedge\Box)\big)$$

defines a non zero class in $H^{n+1}_{[0]}(X,\mathcal{O}_X)$ for j large enough in $\mathbb{N}$.

Remark 4.3. As a consequence, with the aid of Corollary 2.3, we get an oblique pole of $\int_X |f|^{2\lambda}|g|^{2\mu}\Box$ through $(-m-u-j, -l'/k)$ for $j \gg 1$, provided $\int_X |f|^{2\lambda}\Box$ does not have double pole at $-u-m$, for all $m \in \mathbb{N}$.

Proof. As our assumption implies that $H^1_0(\tilde{S}, \tilde{H}^{n-1}(u)) \neq 0$, Proposition 10 and Theorem 13 of [4] imply the existence of $\tilde{\Omega} \in \Gamma(\tilde{X}, \Omega^n_{\tilde{X}})$ verifying

(i) $d\tilde{\Omega} = (m+u)\frac{d\tilde{f}}{f}\wedge\tilde{\Omega}$, for some $m \in \mathbb{N}$;
(ii) $\tilde{ob}_1([\tilde{\Omega}]) \neq 0$, that is $\tilde{\Omega}/\tilde{f}^m$ induces, via the isomorphisms (4.1), an element in $H^0(\tilde{S}^*, \tilde{H}^{n-1}(u))$ which is not extendable at the origin;
(iii) $Z\tilde{\Omega} = \zeta^{-l'}\Omega$, for some $l' \in [1,k]$ and $\zeta := \exp(-2i\pi/k)$.

Define then $\gamma' \in \mathcal{H}_0$ by the following condition: $(\pi_*)^{-1}\gamma'$ is the value at τ_0 of $\widetilde{\mathrm{ob}}_1([\tilde{\Omega}])$. After condition (iii) we have $\Theta_0(\gamma') = \zeta^{-l'}\gamma'$.
As we did in (4.3), we may write

$$\tilde{\Omega} = \sum_{\ell=0}^{k-1}\tau^{\ell-k}p^*\Omega_\ell.$$

Put $\Omega := \Omega_{k-l'}$. Because $p^*\Omega = \tau^{k-l'}\tilde{\Omega}_{k-l'}$ and $\tilde{\Omega}_\ell$ satisties $d\tilde{\Omega}_\ell = (m+u)\frac{d\tilde{f}}{f}\wedge\tilde{\Omega}_\ell$ for any ℓ, property (1) of the Theorem is satisfied thanks to injectivity of p^*.
Relation (iii) implies that $\tilde{\Omega}_{k-l'}$ induces $\tilde{\gamma}' := (\pi_*)^{-1}\gamma'$ and $\tilde{\Omega}_\ell$ induces 0 for $\ell \neq k-l'$. Hence condition (2) of the Theorem is satisfied.
In order to check condition (3), observe that the image of $r_j(\tilde{\gamma}')$ in $H^{n+1}_{[0]}(\tilde{X}, \Omega^{n+1}_{\tilde{X}})$ is equal to the conjugate of

$$d'\,\mathrm{Res}(\lambda = -m-u, \int_{\tilde{X}} |\tilde{f}|^{2\lambda}\bar{\tilde{f}}^{-j}\tilde{\Omega}_{k-l'}\wedge\Box) = P_2(\lambda = -m-u, \int_{\tilde{X}} |\tilde{f}|^{2\lambda}\bar{\tilde{f}}^{-j}\frac{d\tilde{f}}{\tilde{f}}\wedge\tilde{\Omega}_{k-l'}\wedge\Box).$$

After [4], this current is an analytic nonzero functional supported in the origin in $\tilde{X}$. There exists therefore $\tilde{w} \in \Gamma(\tilde{X}, \tilde{\Omega}^{n+1}_{\tilde{X}})$ such that

$$P_2(\lambda = -m-u, \int_{\tilde{X}} |\tilde{f}|^{2\lambda}\bar{\tilde{f}}^{-j}\frac{d\tilde{f}}{\tilde{f}}\wedge\tilde{\Omega}_{k-l'}\wedge\chi\bar{\tilde{w}}) \neq 0,$$

for any cutoff χ equal to 1 near 0. The change of variable $\tau \mapsto \zeta\tau$ shows that $\tilde{w}$ may be replaced by its component $\tilde{w}_{k-l'}$ in the above relation. With $w \in \Gamma(X, \Omega_X^{n+1})$ such that $p^*w = \tau^{k-l'}\tilde{w}_{k-l'}$ we get

$$P_2(\lambda = -m-u, \int_X |f|^{2\lambda}\bar{f}^{-j}|t|^{-2(k-l')/k}\frac{\mathrm{d}f}{f} \wedge \Omega_{k-l'} \wedge \chi\bar{w}) \neq 0.$$

$\square$

Remark 4.4. The case $l' = k$ is excluded if $\int_X |f|^{2\lambda}\square$ has only simple poles at $-m-u$ for all $m \in \mathbb{N}$. Indeed, if $l' = k$, the class $[\Omega]$ in $H^n(u)$ satisfies $\mathrm{ob}_1([\Omega]) \neq 0$; from Theorem 13 of [4] interaction of strata is present and gives rise to poles of order $\geqslant 2$.

5. Examples

Example 5.1. $n = 2$, $f(x,y,t) = tx^2 - y^3$. The extension of $\int_X |f|^{2\lambda}|t|^{2\mu}\square$ presents an oblique polar line of direction $(3,1)$ through $(-5/6-j, -1/2)$, for $j \gg 1$. In fact it follows from general facts that $j = 2$ is large enough because here X is a neighborhood of 0 in $\mathbb{C}^3$.

Proof. We verify directly that the standard generator of $H^1(5/6)$ (which is a local system of rank 1) on $S^* := \{x = y = 0\} \cap \{t \neq 0\}$ has monodromy $-1 = \exp(2i\pi 1/2)$. We take therefore $k = 2$ and we have $\tilde{f}(x,y,\tau) := \tau^2x^2 - y^3$.
Put

$$\tilde{S}^* = \tilde{S}_1^* \cup \tilde{S}_2^* \text{ with } \tilde{S}_1^* := \{x = y = 0\} \cap \{\tau \neq 0\},\ \tilde{S}_2^* := \{\tau = y = 0\} \cap \{x \neq 0\}.$$

The form $\tilde{\omega} := 3x\tau\, dy - 2y\,\mathrm{d}(x\tau)$ verifies

$$\mathrm{d}\tilde{\omega} = \frac{5}{6}\frac{\mathrm{d}\tilde{f}}{\tilde{f}} \wedge \tilde{\omega} \tag{5.1}$$

and $\tilde{\omega}$ induces a nonzero element in the H^1 of the Milnor fibre of $\tilde{f}$ at 0 because it induces on $\tilde{S}_1^*$ the pullback of the multivalued section of $H^1(5/6)$ we started with. It follows that the form ω of Theorem 4.1 is

$$\omega = 3xt\, dy - 2yt\, dx - xy\, dt.$$

It verifies $p^*\omega = \tau\tilde{\omega}$ and hence

$$\mathrm{d}\omega = \frac{5}{6}\frac{\mathrm{d}f}{f} \wedge \omega + \frac{1}{2}\frac{dt}{t} \wedge \omega.$$

One way to see interaction of strata for $\tilde{f}$ and $\exp(2i\pi 5/6)$ consists in looking at the form $\tilde{\Omega} := \frac{d\tau}{\tau} \wedge \tilde{\omega} = d\tau \wedge (3x\, dy - 2y\, dx)$ that verifies $\mathrm{d}\tilde{\Omega} = \frac{5}{6}\frac{\mathrm{d}\tilde{f}}{\tilde{f}} \wedge \tilde{\Omega}$. Along $\tilde{S}_1^*$ we have

$$\tilde{\Omega} = \mathrm{d}(\tilde{\omega}\log\tau) - \frac{5}{6}\frac{\mathrm{d}\tilde{f}}{\tilde{f}} \wedge \tilde{\omega}\log\tau,$$

after (5.1). Hence $\mathrm{ob}_1(\tilde{\Omega}) \neq 0$ in $H^1(S_1^*, \tilde{H}^1(5/6))$. Interaction of strata is proved.

It turns out that Theorem 4.2 may also be used to see existence of an oblique pole as follows. Construct a section on $\tilde{S}^*$ of $\tilde{H}^1(5/6)$ that does not extend through 0 by setting 0 on $\tilde{S}_2^*$ and the restricition of $\tilde{\omega}$ to $\tilde{S}_1^*$. This section does not extend because otherwise its value at the origin should be not 0 in H^1 of $\tilde{f}$ (because not 0 along $\tilde{S}_1^*$) on one hand and should be 0, because of its value on $\tilde{S}_2^*$, on the other hand.
Notice also that the meromorphic extension of $\int_X |f|^{2\lambda}\square$ does not have a double pole at $-5/6-j$, for all $j \in \mathbb{N}$, because interaction of strata is not present for $\exp(-2i\pi 5/6)$: the monodromies for H^1 and H^2 of the Milnor fibre of f do not have the eigenvalue $\exp(-2i\pi 5/6)$ because they are of order 3 thanks to homogeneity. □

Example 5.2. $n = 2$, $f(x,y,t) = x^4 + y^4 + tx^2y$. The extension of $\int_X |f|^{2\lambda}|t|^{2\mu}\square$ presents an oblique polar line of direction $(4,1)$ through $(-5/8, -1/2)$.

Proof. The Jacobian ideal of f relative to t, denoted by $J_/(f)$, is generated by

$$\frac{\partial f}{\partial x} = 4x^3 + 2txy \quad \text{and} \quad \frac{\partial f}{\partial y} = 4y^3 + tx^2.$$

We have

$$t\frac{\partial f}{\partial x} - 4x\frac{\partial f}{\partial y} = 2(t^2 - 8y^2)xy. \tag{5.2}$$

Put $\delta := t^2 - 8y^2$ and notice that for $t \neq 0$ the function δ is invertible at $(t,0,0)$. We use notations and results of [5]. Recall that $\mathbb{E} := \Omega^2_/ / \mathrm{d}_/ f \wedge \mathrm{d}_/ \mathcal{O}$ is equipped with two operations a and b defined by $a\xi = \xi f$, $b(\mathrm{d}_/\xi) := \mathrm{d}_/ f \wedge \xi$ and a $t-$connection $b^{-1}.\nabla : \mathbb{P} \to \mathbb{E}$ that commutes to a and b where

(1) $\nabla : \mathbb{E} \to \mathbb{E}$ is given by $\nabla(\mathrm{d}_/\xi) := \mathrm{d}_/ f \wedge \frac{\partial \xi}{\partial t} - \frac{\partial f}{\partial t} d\xi$,
(2) $\mathbb{P} := \{\alpha \in \mathbb{E} \mid \nabla(\alpha) \in b\mathbb{E}\}$.

Relation (5.2) gives

$$2xy\delta = \mathrm{d}_/ f \wedge (t\,dy + 4x\,dx) = \mathrm{d}_/ f \wedge \mathrm{d}_/(ty + 2x^2); \tag{5.3}$$

hence $xy\delta = 0 \in \mathbb{E}$, and $xy \in J_/(f)$ for $t \neq 0$. As a consequence, for $t \neq 0$ fixed, x^3 and y^4 belong to $J(f_t)$. Therefore the $(a,b)-$module $\mathbb{E}_{t_0} := \mathbb{E}/(t - t_0)\mathbb{E}$ has rank 5 over $\mathbb{C}[[b]]$. The elements $1, x, y, x^2, y^2$ form a basis of this module.
We compute now the structure of $(a,b)-$module of $\mathbb{E}$ over the open set $\{t \neq 0\}$, *i.e.*, compute the action of a on the basis. Let us start with

$$a(y^2) = x^4y^2 + y^6 + x^2y^3.$$

Relation (5.2) yields

$$2x^2y\delta = \mathrm{d}_/ f \wedge (tx\,dy + 4x^2\,dx) = t.b(1) \tag{5.4}$$

and also

$$x^2y = b(\mathrm{d}(\frac{tx\,dy + 4x^2\,dx}{2\delta})) = \frac{1}{2t}b(1) + \frac{4}{t^3}b(y^2) + b^2.\mathbb{E}. \tag{5.5}$$

From

$$b(1) = \mathrm{d}_/ f \wedge (x\,dy) = 4x^4 + 2tx^2y = \mathrm{d}_/ f \wedge (-y\,dx) = 4y^4 + tx^2y$$

we get

$$4x^4 = -\frac{8}{t^2}b(y^2) + b^2\mathbb{E}. \tag{5.6}$$

Therefore

$$4y^4 = b(1) - tx^2y = \frac{1}{2}b(1) - \frac{4}{t^2}b(y^2) + b^2\mathbb{E}. \tag{5.7}$$

The relation

$$x^4y^2 = \mathrm{d}_/f \wedge \frac{x^3y\,dy + 4x^4y\,dx}{2\delta}$$

deduced from (5.3) shows $x^4y^2 \in b^2\mathbb{E}$.
Relation (5.4) rewritten as $2t^2x^2y = tb(1) + 16x^2y^3$ yields

$$x^2y^3 = \frac{t^2}{8}x^2y - \frac{t}{16}b(1).$$

Moreover

$$y^3(4y^3 + tx^2) = y^3\frac{\partial f}{\partial y} = \mathrm{d}_/f \wedge (-y^3\,dx) = 3b(y^2)$$

and hence

$$4y^6 = -tx^2y^3 + 3b(y^2).$$

On the other hand

$$b(y^2) = \mathrm{d}_/f \wedge (xy^2\,dy) = 4x^4y^2 + 2tx^2y^3 = 2tx^2y^3 + b^2\mathbb{E}.$$

Finally

$$a(y^2) = x^4y^2 + y^6 + tx^2y^3 = -\frac{t}{4}x^2y^3 + \frac{3}{4}b(y^2) + tx^2y^3 + b^2\mathbb{E} = \frac{9}{8}b(y^2) + b^2\mathbb{E}.$$

Now, after (5.6), (5.7) and (5.5) we obtain successively

$$\begin{aligned} a(1) = x^4 + y^4 + tx^2y &= -\frac{2}{t^2}b(y^2) + \frac{1}{8}b(1) - \frac{1}{t^2}b(y^2) + \frac{1}{2}b(1) + \frac{4}{t^2}b(y^2) + b^2\mathbb{E} \\ &= \frac{5}{8}b(1) + \frac{1}{t^2}b(y^2) + b^2\mathbb{E}, \end{aligned}$$

$$a(1 - \frac{2y^2}{t^2}) = \frac{5}{8}b(1) + \frac{1}{t^2}b(y^2) - \frac{2}{t^2}\frac{9}{8}b(y^2) + b^2\mathbb{E} = \frac{5}{8}b(1 - \frac{2y^2}{t^2}) + b^2\mathbb{E}.$$

Some more computations of the same type left to the reader give

$$\begin{aligned} a(x) &= b(x) + b^2\mathbb{E}, \\ a(y) &= \frac{7}{8}b(y) + b^2\mathbb{E}, \\ a(x^2) &= \frac{11}{8}b(x^2) + b^2\mathbb{E}. \end{aligned}$$

Let us compute the monodromy M of t on the eigenvector $v_0 := 1 - \frac{2y^2}{t^2} + b\mathbb{E}$. Because $b^{-1}\nabla = \frac{\partial}{\partial t}$ it is given by

$$M = \exp(2i\pi t b^{-1}\nabla).$$

We have

$$\nabla(1) = -x^2 y = -b(\frac{1}{2t}1 + \frac{4y^2}{t^3}) + b^2\mathbb{E}$$

and hence

$$t\frac{\partial}{\partial t}(1) = \frac{-1}{2}1 - \frac{4y^2}{t^2} + b\mathbb{E}.$$

Also

$$\nabla(y^2) = -x^2y^3 = -\frac{t^2}{8}x^2y + \frac{t}{16}b(1) = -\frac{t^2}{8}\big(\frac{1}{2t}b(1) + \frac{4}{t^3}b(y^2)\big) + \frac{t}{16}b(1) + b^2\mathbb{E}$$

gives

$$t\frac{\partial}{\partial t}(y^2) = -\frac{1}{2}y^2 + b\mathbb{E}.$$

Hence

$$t\frac{\partial}{\partial t}(1 - \frac{2y^2}{t^2}) = -\frac{1}{2}(1 - \frac{2y^2}{t^2}) + b\mathbb{E}$$

from what we deduce $Mv = -v$.

An analogous computation with $\tau^2 = t$, shows that the eigenvector $\tilde{v}$ is invariant under $\tilde{M}$. On the other hand, the relation

$$\tilde{v} = 1 - \frac{2y^2}{\tau^4} + b\tilde{\mathbb{E}}$$

where $\tilde{\mathbb{E}}$ is associated to the pair $(\tilde{f}, \tau)$, with $\tilde{f}(x, y, \tau) := x^4 + y^4 + \tau^2x^2y$, shows that $\tilde{v}$ does not extend through 0 as a section of $\tilde{\mathbb{E}}$.

This last assertion may be proved directly. It suffices to show that there does not exist a holomorphic non-trival[1] $1-$form $\tilde{\omega}$ near 0 such that

$$\mathrm{d}\tilde{\omega} = \frac{5}{8}\frac{\mathrm{d}\tilde{f}}{\tilde{f}} \wedge \tilde{\omega}. \tag{5.8}$$

Because $\tilde{f}$ is quasi-homogeneous of degree 8 with the weights $(2, 2, 1)$ and because $\tilde{\omega}/\tilde{f}^{5/8}$ is homogeneous of degree 0, the form $\tilde{\omega}$ must be homogeneous of degree 5. So we may write

$$\tilde{\omega} = (\alpha_0 + \alpha_1\tau^2 + \alpha_2\tau^4)d\tau + \tau\beta_0 + \tau^3\beta_1$$

where α_i and β_i are respectively $0-$ and $1-$homogeneous forms of degree $2 - i$ with respect to x, y. Setting $\beta := \beta_0 + \tau^2\beta_1$, we get

$$\mathrm{d}\tilde{f} \wedge \tilde{\omega} = \mathrm{d}_/\tilde{f} \wedge \tau\beta \quad \text{and} \quad \mathrm{d}\tilde{\omega} = \tau d_/\beta \quad \text{modulo} \quad d\tau \wedge \square.$$

With (5.8) we deduce

$$8\tilde{f}\,\mathrm{d}_/\beta = 5\,\mathrm{d}_/\tilde{f} \wedge \beta$$

and an easy computation shows that this can hold only if $\beta = 0$. In that case $\alpha = 0$ also and the assertion follows. $\square$

[1]that is, not inducing 0 in the Milnor fibre of $\tilde{f}$ at 0

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