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# SOME REMARKS ABOUT SEMICLASSICAL TRACE INVARIANTS AND QUANTUM NORMAL FORMS

VICTOR GUILLEMIN AND THIERRY PAUL

ABSTRACT. In this paper we explore the connection between semi-classical and quantum Birkhoff canonical forms (BCF) for Schrödinger operators. In particular we give a "non-symbolic" operator theoretic derivation of the quantum Birkhoff canonical form and provide an explicit recipe for expressing the quantum BCF in terms of the semi-classical BCF.

## 1. INTRODUCTION

Let  $X$  be a compact manifold and  $H : L^2(X) \rightarrow L^2(X)$  a self-adjoint first order elliptic pseudodifferential operator with leading symbol  $H(x, \xi)$ . From the wave trace

$$(1.1) \quad \sum_{E_k \in \text{Spec}(H)} e^{itE_k},$$

one can read off many properties of the "classical dynamical system" associated with  $H$ , i.e. the flow generated by the vector field

$$(1.2) \quad \xi_H = \sum \frac{\partial H}{\partial \xi_i} \frac{\partial}{\partial x_i} - \frac{\partial H}{\partial x_i} \frac{\partial}{\partial \xi_i}.$$

For instance it was observed in the '70's' by Colin de Verdière, Chazarain and Duistermaat-Guillemin that (1.1) determines the period spectrum of (1.2) and the linear Poincaré map about a non-degenerate periodic trajectory,  $\gamma$ , of (1.2) ([3], [2], [4]).

More recently it was shown by one of us [5] that (1.1) determines the entire Poincaré map about  $\gamma$ , i.e. determines, up to isomorphism, the classical dynamical system associated with  $H$  in a formal neighborhood of  $\gamma$ . The proof of this result involved a microlocal Birkhoff canonical form for  $H$  in a formal neighborhood of  $\gamma$  and an algorithm for computing the wave trace invariants associated with  $\gamma$  from the microlocal Birkhoff canonical form. Subsequently a more compact and elegant algorithm for computing these invariants from the Birkhoff canonical form was discovered by Zelditch [11] [12] making the computation of these local trace invariants extremely simple and explicit.

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In this paper we will discuss some semiclassical analogues of these results. In our set-up  $H$  can either be the Schrödinger operator on  $\mathbb{R}^n$

$$-\hbar^2\Delta + V$$

with  $V \rightarrow \infty$  as  $x$  tends to infinity, or more generally a self-adjoint semiclassical elliptic pseudodifferential operator

$$H(x, \hbar D_x)$$

whose symbol,  $H(x, \xi)$ , is proper (as a map from  $T^*X$  into  $\mathbb{R}$ ). Let  $E$  be a regular value of  $H$  and  $\gamma$  a non-degenerate periodic trajectory of period  $T_\gamma$  lying on the energy surface  $H = E$ <sup>1</sup>.

Consider the Gutzwiller trace (see [6])

$$(1.3) \quad \sum \psi \left( \frac{E - E_i}{\hbar} \right)$$

where  $\psi$  is a  $C^\infty$  function whose Fourier transform is compactly supported with support in a small neighborhood of  $T_\gamma$  and is identically one in a still smaller neighborhood. As shown in [8], [9] (1.3) has an asymptotic expansion

$$(1.4) \quad e^{i\frac{S_\gamma}{\hbar} + \sigma_\gamma} \sum_{k=0}^{\infty} a_k \hbar^k$$

and we will show below how to compute the terms of this expansion to all orders in terms of a microlocal Birkhoff canonical form for  $H$  in a formal neighborhood of  $\gamma$  by means of a Zelditch-type algorithm<sup>2</sup>.

If  $\gamma$  is non-degenerate so are all its iterates. Then, for each of these iterates one gets an expansion of (1.3) similar to (1.4)

$$(1.5) \quad e^{i\frac{S_\gamma}{\hbar} + \sigma_\gamma} \sum_{k=0}^{\infty} a_{k,r} \hbar^k$$

and for these expansions as well the coefficients  $a_{k,r}$  can be computed from the microlocal Birkhoff canonical form theorem for  $H$  in a formal neighborhood of  $\gamma$ . Conversely one can show

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<sup>1</sup> For simplicity we will consider periodic trajectories of elliptic type in this paper however our results are true for non-degenerate periodic of all types, hyperbolic, mixed elliptic hyperbolic, focus-focus, etc. Unfortunately however the Zelditch algorithm depends upon the type of the trajectory and in dimension  $n$  there are roughly as many types of trajectories as there are Cartan subalgebras of  $Sp(2n)$  (See for instance [1]) i.e. the number of types can be quite large

<sup>2</sup>For elliptic trajectories non-degeneracy means that the numbers

$$\theta_1, \dots, \theta_n, 2\pi$$

are linearly independent over the rationals,  $e^{i\theta_\kappa}$ ,  $\kappa = 1, \dots, n$  being the eigenvalues of the Poincaré map about  $\gamma$ . the results above are true to order  $O(\hbar^r)$  providing

$$\kappa_1\theta_1 + \dots + \kappa_n\theta_n + l2\pi \neq 0$$

for all  $|\kappa_1| + \dots + |\kappa_n| \leq r$ , i.e. providing there are no resonances of order  $\leq r$ .

**Theorem 1.1.** *the constants  $a_{k,r}, \kappa, r = 0, 1, \dots$  determine the microlocal Birkhoff canonical form for  $H$  in a formal neighborhood of  $\gamma$  (and hence, a fortiori, determine the the classical Birkhoff canonical form).*

This result is due to A. Iantchenko, J. Sjostrand and M. Zworski (see [7]). In this paper we will give a new proof of this result which involves an expansion of the quantum canonical form in a basis of Hermite functions and relations between Weyl and Wick symbols of semiclassical operators. The paper is organized as follows. In section 2 we review the standard proof of the Birkhoff canonical form theorem for classical Hamiltonians and in section 3 the adaption of this construction to the quantum case in the references [5], [11], [12] and [7] that we cited above. The key results of this paper are in sections 4 and 5 where we give a direct quantum construction of the normal form and a formula linking the two constructions. Finally in section 6 we give the proof of theorem 1.1.

## 2. THE CLASSICAL BIRKHOFF CANONICAL FORM THEOREM

Let  $M$  be a  $2n + 2$  dimensional symplectic manifold,  $H$  a  $C^\infty$  function and

$$(2.1) \quad \xi_H = \sum \frac{\partial H}{\partial \xi_i} \frac{\partial}{\partial x_i} - \frac{\partial H}{\partial x_i} \frac{\partial}{\partial \xi_i}$$

the Hamiltonian vector field associated with  $H$ . Let  $E$  be a regular value of  $H$  and  $\gamma$  a non-degenerate elliptic periodic trajectory of  $\xi_H$  lying on the energy surface,  $H = E$ . Without loss of generality one can assume that the period of  $\gamma$  is  $2\pi$ . In this section we will review the statement (and give a brief sketch of the proof) of the classical Birkhoff canonical form theorem for the pair  $(H, \gamma)$ .

Let  $(x, \xi, t, \tau)$  be the standard cotangent coordinates on  $T^*(\mathbb{R}^n \times S^1)$  and let

$$(2.2) \quad \begin{aligned} p_i &= x_i^2 + \xi_i^2 \\ &\text{and} \\ q_i &= \arg(x_i + \sqrt{-1} y_i) \end{aligned}$$

**Theorem 2.1.** *There exists a symplectomorphism,  $\varphi$ , of a neighborhood of  $\gamma$  in  $M$  onto a neighborhood of  $p = \tau = 0$  such that  $\varphi \circ \gamma(t) = (0, 0, t, 0)$  and*

$$(2.3) \quad \varphi^* H = H_1(p, \tau) + H_2(x, \xi, t, \tau)$$

$H_2$  vanishing to infinite order at  $p = \tau = 0$ .

We break the proof of this up into the following five steps.

Step 1 For  $\epsilon$  small there exists a periodic trajectory,  $\gamma_\epsilon$ , on the energy surface,  $H = \overline{E} + \epsilon$ , which depends smoothly on  $\epsilon$  and is equal to  $\gamma$  for  $\epsilon = 0$ . The union of these trajectories is a 2 dimensional symplectic submanifold,  $\Sigma$ , of  $M$  which is invariant under the flow of  $\xi_H$ . Using the Weinstein tubular neighborhood theorem one can map a neighborhood of  $\gamma$  symplectically onto a neighborhood of  $p = \tau = 0$  in  $T^*(\mathbb{R}^n \times S^1)$  such that  $\Sigma$  gets mapped onto  $p = 0$  and  $\varphi \circ \gamma(t) = (0, 0, t, 0)$ . Thus we can henceforth assume that  $M = T^*(\mathbb{R}^n \times S^1)$  and  $\Sigma$  is the set,  $p = 0$ .

Step 2 We can assume without loss of generality that the restriction of  $H$  to  $\Sigma$  is a function of  $\tau$  alone, i.e.  $H = h(\tau)$  on  $\Sigma$ . With this normalization,

$$(2.4) \quad H = E + h(\tau) + \sum \theta_i(\tau)p_i + O(p^2)$$

where  $h(\tau) = \tau + O(\tau^2)$  and

$$(2.5) \quad \theta_i = \theta_i(0), \quad i = 1, \dots, n$$

are the rotation angles associated with  $\gamma$ . Since  $\gamma$  is non- degenerate,  $\theta_1, \dots, \theta_n, 2\pi$  are linearly independent over the rationals.

Step 3 Theorem 2.1 can be deduced from the following result (which will also be the main ingredient in our proof of the "microlocal" Birkhoff canonical theorem in the next section).

**Theorem 2.2.** *Given a neighborhood,  $\mathcal{U}$ , of  $p = \tau = 0$  and  $G = G(x, \xi, t, \tau) \in C^\infty(\mathcal{U})$ , there exist functions  $F, G_1, R \in C^\infty(\mathcal{U})$  with the properties*

- i.  $G_1 = G_1(p, \tau)$*
- ii.  $\{H, F\} = G + G_1 + R$*
- iii.  $R$  vanishes to infinite order on  $p = \tau = 0$ .*

*Moreover, if  $G$  vanishes to order  $\kappa$  on  $p = \tau = 0$ , one can choose  $F$  to have this property as well.*

Proof of the assertion: Theorem 2.2  $\Rightarrow$  Theorem 2.1:

By induction one can assume that  $H$  is of the form,  $H = H_0(p, \tau) + G(x, \xi, t, \tau)$ , where  $G$  vanishes to order  $\kappa$  on  $p = \tau = 0$ . We will show that  $H$  can be conjugated to a Hamiltonian of the same form with  $G$  vanishing to order  $\kappa + 1$  on  $p = \tau = 0$ . By Theorem 2.2 there exists an  $F, G$  and  $R$  such that  $F$  vanishes to order  $\kappa$  and  $R$  to order  $\infty$  on  $p = \tau = 0$ ,  $G_1 = G_1(p, \tau)$  and

$$\{H, F\} = G + G_1 + R.$$

Thus

$$\begin{aligned} (\exp \xi_F)^* H &= H + \{F, H\} + \frac{1}{2!} \{F, \{F, H\}\} + \dots \\ &= H_0(p, \tau) - G_1(p, \tau) + \dots \end{aligned}$$

the "dots" indicating terms which vanish to order  $\kappa + 1$  on  $p = \tau = 0$ .

Step 4 Theorem 2.2 follows (by induction on  $\kappa$ ) from the following slightly weaker result:

**Lemma 2.3.** *Given a neighborhood,  $\mathcal{U}$ , of  $p = \tau = 0$  and a function,  $G \in C^\infty(\mathcal{U})$ , which vanishes to order  $\kappa$  on  $p = \tau = 0$ , there exist functions  $F, G_1, R \in C^\infty(\mathcal{U})$  such that*

- i.  $G_1 = G_1(p, \tau)$*
- ii.  $\{H, F\} = G + G_1 + R$*
- iii.  $F$  vanishes to order  $\kappa$  and  $R$  to order  $\kappa + 1$  on  $p = \tau = 0$ .*

Step 5 Proof of Lemma 2.3: In proving Lemma 2.3 we can replace  $H$  by the Hamiltonian

$$H_0 = E + \tau + \sum \theta_i p_i$$

since  $H(p, q, t, \tau) - H_0(p, q, t, \tau)$  vanishes to second order in  $\tau, p$ . Consider now the identity

$$\{H_0, F\} = G + G_1(p, \tau) + O(p^\infty).$$

Introducing the complex coordinates,  $z = x + \sqrt{-1}\xi$ , and  $\bar{z} = x - \sqrt{-1}\xi$ , this can be written

$$\sqrt{-1} \sum_{i=1}^n \theta_i \left( z_i \frac{\partial}{\partial z_i} - \bar{z}_i \frac{\partial}{\partial \bar{z}_i} \right) F + \frac{\partial}{\partial t} = G + G_1 + O(p^\infty).$$

Expanding  $F$ ,  $G$  and  $G_1$  in Fourier-Taylor series about  $z = \bar{z} = 0$ :

$$\begin{aligned} F &= \sum_{\mu \neq \nu} a_{\mu, \nu, m}(\tau) z^\mu \bar{z}^\nu e^{2\pi i m t} \\ G &= \sum b_{\mu, \nu}(\tau) z^\mu \bar{z}^\nu e^{2\pi i m t} \\ G_1 &= \sum_{\mu} c_{\mu}(\tau) z^\mu \bar{z}^\mu \end{aligned}$$

one can rewrite this as the system of equations

$$(2.6) \quad \sqrt{-1} \left( \sum_{i=1}^n \theta_i (\mu_i - \nu_i) + 2\pi m \right) a_{\mu, \nu, m}(\tau) = b_{\mu, \nu, m}(\tau)$$

for  $\mu \neq \nu$  or  $\mu = \nu$  and  $m \neq 0$ , and

$$(2.7) \quad -c_{\mu}(\tau) = b_{\mu, \mu, 0}(\tau)$$

for  $\mu = \nu$  and  $m = 0$ . By assumption the numbers,  $\theta_1, \dots, \theta_n, 2\pi$ , are linearly independent over the rationals, so this system has a unique solution. Moreover, for  $\mu$  and  $\nu$  fixed

$$\sum b_{\mu, \nu, m}(\tau) e^{2\pi i m t}$$

is the  $(\mu, \nu)$  Taylor coefficient of  $G(z, \bar{z}, t, \tau)$  about  $z = \bar{z} = 0$ ; so, with  $\mu$  and  $\nu$  fixed and  $\kappa \gg 0$

$$|b_{\mu, \nu, m}(\tau)| \leq C_{\mu, \nu, \kappa} m^{-\kappa}$$

for all  $m$ . Hence, by (2.6)

$$|a_{\mu, \nu, m}| \leq C'_{\mu, \nu, \kappa} m^{-\kappa-1}$$

for all  $m$ . Thus

$$a_{\mu, \nu}(t, \tau) = \sum a_{\mu, \nu, m}(\tau) e^{2\pi i m t}$$

is a  $C^\infty$  function of  $t$  and  $\tau$ . Now let  $F(z, \bar{z}, t, \tau)$  and  $G_1(p, \tau)$  be  $C^\infty$  functions with Taylor expansion:

$$\sum_{\mu \neq \nu} a_{\mu, \nu}(t, \tau) z^\mu \bar{z}^\nu$$

and

$$\sum_{\mu} c_{\mu}(\tau) z^{\mu} \bar{z}^{\mu}$$

about  $z = \bar{z} = 0$ . Note, by the way that, if  $G$  vanishes to order  $\kappa$  on  $p = \tau = 0$ , so does  $F$  and  $G$ ; so we have proved Theorem 2.2 (and, a fortiori Lemma 2.6) with  $H$  replaced by  $H_0$ .

### 3. THE SEMICLASSICAL VERSION OF THE BIRKHOFF CANONICAL FORM THEOREM

Let  $X$  be an  $(n + 1)$ -dimensional manifold and  $H : C_0^{\infty}(X) \rightarrow C^{\infty}(X)$  a semiclassical elliptic pseudo-differential operator with leading symbol,  $H(x, \xi)$ , and let  $\gamma$  be a periodic trajectory of the bicharacteristic vector field (2.1). As in Section 1 we will assume that  $\gamma$  is elliptic and non-degenerate, with rotation numbers (2.4). Let  $P_i$  and  $D_t$  be the differential operators on  $\mathbb{R}^n \times S^1$  associated with the symbols (2.2) and  $\tau$  i.e.

$$\begin{aligned} P_i &= \hbar^2 D_{x_i}^2 + x_i^2 \\ \text{and} \\ D_t &= -i\hbar \partial_t \end{aligned}$$

We will prove below the following semiclassical version of Theorem 2.1

**Theorem 3.1.** *There exists a semiclassical Fourier integral operator  $A_{\varphi} : C_0^{\infty}(X) \rightarrow C^{\infty}(\mathbb{R}^n \times S^1)$  implementing the symplectomorphism (2.3) such that microlocally on a neighborhood,  $\mathcal{U}$ , of  $p = \tau = 0$*

$$(3.1) \quad A_{\varphi}^* = A_{\varphi}^{-1}$$

and

$$(3.2) \quad A_{\varphi} H A_{\varphi}^{-1} = H'(P_1, \dots, P_n, D_t, \hbar) + H''$$

the symbol of  $H''$  vanishing to infinite order on  $p = \tau = 0$ .

*Proof.* Let  $B_{\varphi}$  be any Fourier integral operator implementing  $\varphi$  and having the property (3.1). Then, by Theorem 2.1, the leading symbol of  $B_{\varphi} H B_{\varphi}^{-1}$  is of the form

$$(3.3) \quad H'_0(p, \tau) + H''_0(p, q, t, \tau)$$

$H''_0(p, q, t, \tau)$  being a function which vanishes to infinite order on  $p = \tau = 0$ . Thus the symbol,  $H_0$ , of  $B_{\varphi} H B_{\varphi}^{-1}$  is of the form

$$(3.4) \quad H'_0(p, \tau) + H''_0(p, q, t, \tau) + \hbar H_1(p, q, t, \tau) + O(\hbar^2).$$

By Theorem 2.2 there exists a function,  $F(p, q, t, \tau)$ , with the property

$$(3.5) \quad \{H_0, F\} = H_1(p, q, t, \tau) - H'_1(p, \tau) - H''_1(p, q, t, \tau)$$

where  $H''_1$  vanishes to infinite order on  $p = \tau = 0$ .

Let  $Q$  be a self-adjoint pseudo-differential operator with leading symbol  $F$  and consider the unitary pseudo-differential operator

$$\mathcal{U}_s = e^{isQ}.$$

Let

$$\begin{aligned} H_s &= (\mathcal{U}_s B_\varphi) H (\mathcal{U}_s B_\varphi)^{-1} \\ &= \mathcal{U}_s (B_\varphi H B_\varphi^{-1}) \mathcal{U}_{-s} \end{aligned}$$

Then

$$(3.6) \quad \frac{\partial}{\partial s} H_s = i[Q, H_s]$$

so  $\frac{\partial}{\partial s} H_s$  is of order  $-1$ , and hence the leading symbol of  $H_s$  is independent of  $s$ . In particular the leading symbol of  $\frac{\partial}{\partial s} H_s$  is equal, by (3.6) to the leading symbol of  $i[Q, H_s]$  which, by (3.5), is:

$$-\hbar (H_1(p, q, t, \tau) + H_1'(p, \tau) + H_1''(p, q, t, \tau)).$$

Thus by (3.4) and (3.5) the symbol of

$$(\mathcal{U}_1 B_\varphi) H (\mathcal{U}_1 B_\varphi)^{-1} = B_\varphi H B_\varphi^{-1} + \int_0^1 \frac{\partial}{\partial s} H_s ds$$

is of the form

$$(3.7) \quad H_0'(p, \tau) + \hbar H_1'(p, \tau) + (H_0'' + \hbar H_1'') + O(\hbar^2)$$

the term in parenthesis being a term which vanishes to infinite order on  $p = \tau = 0$ .

By repeating the argument one can successively replace the terms of order  $\hbar^2, \dots, \hbar^r$  etc in (3.7) by expressions of the form

$$\hbar^r (H_r'(p, \tau) + H_r''(p, q, t, \tau))$$

with  $H_r''$  vanishing to infinite order on  $p = \tau = 0$ . □

#### 4. A DIRECT CONSTRUCTION OF THE QUANTUM BIRKHOFF FORM

In this section we present a “quantum” construction of the quantum Birkhoff normal form which is in a sense algebraically equivalent to the classical one of Section 2. To do this we will need to define for operators the equivalent of “a Taylor expansion which vanishes at a given order”.

We will first start in the  $L^2(\mathbb{R}^n \times S^1)$  setting, and show at the end of the section the link with Theorem 3.1.

**Definition 4.1.** Let us consider on  $L^2(\mathbb{R}^n \times S^1, dxdt)$  the following operators:

- $a_i = \frac{1}{\sqrt{2}}(x_i + \hbar \partial_{x_i})$
- $a_i^+ = \frac{1}{\sqrt{2}}(x_i - \hbar \partial_{x_i})$
- $D_t = -i\hbar \frac{\partial}{\partial t}$



We will say that an operator  $A$  on  $L^2(\mathbb{R}^n \times S^1)$  is a “word of length greater than  $p \in \mathbb{N}$ ” (WLG( $p$ )) if there exists  $P \in \mathbb{N}$  such that:

$$(4.1) \quad H = \sum_{i=p}^P \sum_{j=0}^{[\frac{i}{2}]} \alpha_{ij}(t, \hbar) D_t^j \prod_{l=1}^{i-2j} b_l$$

with,  $\forall l, b_l \in \{a_1, a_1^+, \dots, a_n, a_n^+\}$  and  $\alpha_{ij} \in C^\infty(S_1 \times [0, 1])$ .

In (4.1)  $\prod_{l=1}^{i-2j} b_l$  is meant to be the ordered product  $b_1 \dots b_{i-2j}$ .

The meaning of this definition is clarified by the following:

**Lemma 4.2.** *Let  $\mu \in \mathbb{N}^{n+1}$ . Let  $H_\mu$  denote the basis of  $L^2(\mathbb{R}^n \times S^1)$  defined by  $H_\mu(x, t) = \hbar^{-n/4} h_{\mu_1}(x_1/\sqrt{\hbar}) \dots h_{\mu_n}(x_n/\sqrt{\hbar}) e^{i\mu_{n+1}t}$  where the  $h_j$  are the (normalized) Hermite functions.*

*Then, if  $A$  is a WLG( $p$ ), we have:*

$$\|AH_{\mu,m}\|_{L^2} \leq C_p |\mu \hbar|^{\frac{p}{2}}.$$

where  $|\mu \hbar| = \sqrt{\mu^2 \hbar^2}$ .

*Proof.* The proof follows immediately from the two well known facts (expressed here in one dimension):

$$a^\pm H_\mu = \sqrt{(\mu \pm 1)\hbar} H_{\mu \pm 1}$$

and

$$D_t e^{imt} = m \hbar e^{imt}.$$

□

For the rest of this section we will need the following collection of results.

**Proposition 4.3.** *Let  $A$  be a (Weyl)pseudodifferential operator on  $L^2(\mathbb{R}^n \times S^1)$  with symbol of type  $S_{1,0}$ . Then,  $\forall L \in \mathbb{N}$ , there exists a WLG(1)  $A_L$  such that:*

$$\|(A - A_L)H_{\mu,m}\|_{L^2} = O(|\mu \hbar|^{\frac{L+1}{2}}).$$

*Moreover, if the principal symbol of  $A$  is of the form:*

$$a_0(x_1, \xi_1, \dots, x_n, \xi_n, t, \tau) = \sum \theta_i(x_i^2 + \xi_i^2) + \tau + h.o.t.,$$

*(or is any function whose symbol vanishes to first order at  $x = \xi = \tau = 0$ ) then  $A_L$  is a WLG(2).*

*Proof.* Let us take the  $L$ th order Taylor expansion of the (total) symbol of  $A$  in the variables  $x, \xi, \tau, \hbar$  near the origin. Noticing that a pseudodifferential operator with polynomial symbol in  $x, \xi, \tau, \hbar$  is a word, we just have to estimate the action, on  $H_{\mu,m}$ , of a pseudo-differential operator whose symbol vanishes at the origin to order  $L$  in the variable  $x, \xi, \tau$ . The result is easily obtained for the  $\tau$  part, as the “ $t$ ” part of  $H_{\mu,m}$  is an exponential. For the  $\mu$  part we will prove this result in one dimension, the extension to  $n$  dimensions being straightforward.

Let us define a coherent state at  $(q, p)$  to be a function of the form  $\psi_{qp}^a(x) := \hbar^{-1/4} a\left(\frac{x-q}{\sqrt{\hbar}}\right) e^{i\frac{px}{\hbar}}$ , for  $a$  in the Schwartz class and  $\|a\|_{L^2} = 1$ . Let us also set  $\varphi_{qp} = \psi_{qp}^a$  for  $a(\eta) = \pi^{-1/2} e^{-\eta^2/2}$ . It is well known, and easy to check using the generating function of the Hermite polynomials, that:

$$H_\mu = \hbar^{-\frac{1}{4}} \int_{S^1} e^{-i\frac{t}{2}} \varphi_{q(t)p(t)} dt,$$

where  $q(t) + ip(t) = e^{it}(q + ip)$ ,  $q^2 + p^2 = (\mu + \frac{1}{2})\hbar$ . Therefore, for any operator  $A$ ,

$$(4.2) \quad \|AH_\mu\| = O\left(\sup_{p^2+q^2=(\mu+\frac{1}{2})\hbar} \hbar^{-\frac{1}{4}} \|A\varphi_{qp}\|\right).$$

**Lemma 4.4.** *let  $H$  a pseudodifferential operator whose (total) Weyl symbol vanishes at the origin to order  $M$ . Then, if  $\frac{\hbar}{q^2+p^2} = O(1)$ :*

$$\|H\psi_{qp}^a\| = O\left((p^2 + q^2)^{\frac{M}{2}}\right).$$

Before proving the Lemma we observe that the proof of the Proposition follows easily from the Lemma using (4.2).

*Proof.* An easy computation shows that, if  $h$  is the (pseudodifferential) symbol of  $H$ , then  $H\psi_{qp}^a = \psi_{qp}^b$  with

$$(4.3) \quad b(\eta) = \int_{\mathbb{R}} h(q + \sqrt{\hbar}\eta, p + \sqrt{\hbar}\nu) e^{i\eta\nu} \hat{a}(\nu) d\nu,$$

where  $\hat{a}$  is the ( $\hbar$  independent) Fourier transform of  $a$ .

Developing (4.3) we get that  $H\psi_{qp}^a = \sum_{k=0}^{k=K} \hbar^{\frac{k}{2}} D_k h(q, p) \psi_{qp}^{b_k} + O(\hbar^{\frac{k+1}{2}})$ , where  $b_k \in \mathcal{S}$  and  $D_k$  is an homogeneous differential operator of order  $k$ . It is easy to conclude, thanks to the hypothesis  $\frac{\hbar}{q^2+p^2} = O(1)$ , that

$$\hbar^{-\frac{k}{2}} (q^2 + p^2)^{\frac{M-k}{2}} = O((q^2 + p^2)^{\frac{M}{2}}), \quad \hbar^{\frac{M+1}{2}} = O((q^2 + p^2)^{\frac{M}{2}}).$$

□

□

This Proposition is crucial for the rest of this Section, as it allows us to reduce all computations to the polynomial setting. For example  $A$  may have a symbol bounded at infinity (class  $S(1)$ ), an assumption which we will need for the application below of Egorov's Theorem), but, with respect to the algebraic equations we will have to solve, one can consider it as a "word". In order to simplify our proofs, we will omit the distinction between pseudodifferential operators and their "word" approximations.

**Lemma 4.5.** *Let  $A$  be a  $WLG(1)$  on  $L^2(\times \mathbb{R}^n \times S^1)$ . Let us suppose that  $A$  is a symmetric operator. For  $P \in \mathbb{N}$  (large), let*

$$(4.4) \quad A_P := A + (|D_\theta|^2 + |x|^2 + |D_x|^2)^P$$

*Then  $A_P$  is an elliptic selfadjoint pseudo-differential operator. Therefore  $e^{is\frac{A_P}{\hbar}}$  is a family of unitary Fourier integral operators.*

*Proof.* it is enough to observe that  $A_P$  is, defined on the domain of  $|D_\theta|^2 + |x|^2 + |D_x|^2$ , a selfadjoint pseudodifferential operator with symbol of type  $S_{1,0}$ .  $\square$

**Lemma 4.6.** *Let  $H_0$  the operator*

$$H_0 = \sum_1^n \theta_i a_i a_i^+ + D_t$$

*then, if  $W$  is a  $WLG(r)$ , so is  $\frac{[H_0, W]}{i\hbar}$ .*

*Proof.*  $\frac{[H_0, W]}{i\hbar} = \frac{d}{ds} e^{isH_0/\hbar} W e^{isH_0/\hbar} |_{s=0}$  which, since  $H_0$  is quadratic, is the same polynomial as  $W$  modulo the substitution  $a_i \rightarrow e^{is} a_i$ ,  $a_i^+ \rightarrow e^{-is} a_i^+$  and shifting of the coefficients in  $t$  by  $s$ . Therefore the result is immediate.  $\square$

More generally:

**Lemma 4.7.** *For any  $H$  and  $W$  of type  $WLG(m)$  and  $WLG(r)$  respectively,  $\frac{[H, W]}{i\hbar}$  is a  $WLG(m+r-2)$ .*

The proof is immediate noting that  $[a_i, a_j^+] = \hbar \delta_{ij}$  and that, for any  $C^\infty$  function  $a(t)$ ,  $[D_t, a] = i\hbar a'$ .

We can now state the main result of this section:

**Theorem 4.8.** *Let  $H$  be a (Weyl) pseudo-differential operator on  $L^2(\mathbb{R}^n \times S^1)$  whose principal symbol is of the form:*

$$H_0(x, \xi; t, \tau) = \sum_1^n \theta_i (x_i^2 + \xi_i^2) + \tau + H_2,$$

*where  $H_2$  vanishes to third order at  $x = \xi = \tau = 0$  and  $\theta_1, \dots, \theta_n, 2\pi$  are linearly independent over the rationals. Let us define, as before,  $P_i = -\hbar^2 \frac{\partial^2}{\partial x_i^2} + x_i^2$  and*

$$D_t = -i\hbar \frac{\partial}{\partial t}.$$

*Then there exists a family of unitary operators  $(U_L)_{L=3\dots}$  and constants  $(C_L)_{L=3\dots}$ , and a  $C^\infty$  function  $h(p_1, \dots, p_n, \tau, \hbar)$  such that:*

$$\| (U_L H U_L^{-1} - h(P_1, \dots, P_n, D_t, \hbar)) H_\mu \|_{L^2(\mathbb{R}^n \times S^1)} \leq C_L |\mu \hbar|^{\frac{L+1}{2}}.$$

*Proof.* The proof of Theorem 4.8 will be a consequence of the following:

**Theorem 4.9.** *Let  $H$  be as before, and let  $G$  be a  $WLG(3)$ . Then there exists a function  $G_1(p_1, \dots, p_n, \tau, \hbar)$ , a word  $F$  and an operator  $R$  such that:*

- i.  $\frac{[H, F]}{i\hbar} = G + G_1 + R$*
- ii.  $R$  satisfies:  $\|RH_\mu\| = O(|\mu\hbar|^{\frac{L+1}{2}})$ ,  $\forall \mu \in \mathbb{N}^{n+1}$*
- iii if  $G$  is a  $WLG(\kappa)$  so is  $F$*
- iv. if  $G$  is a symmetric operator, so is  $F$  and  $G_1$  is real.*

Let us first prove that Theorem 4.9 implies Theorem 4.8:

by induction, as in the ‘‘classical’’ case and thanks to Proposition 4.3, one can assume that  $H$  is of the form  $H = H_0 + G$ , where  $G$  is a  $WLG(\kappa)$ . Let us consider the operators  $e^{i\frac{F_P}{\hbar}} H e^{-i\frac{F_P}{\hbar}}$  and  $H(s) := e^{is\frac{F_P}{\hbar}} H e^{-is\frac{F_P}{\hbar}}$ , where  $F$  satisfies Theorem 4.9 and  $F_P$  is defined by (4.4) for  $P$  large enough. By Egorov’s Theorem  $H(s)$  is a family of pseudodifferential operators. Since we are in an iterative perturbative setting, it is easy to check by taking  $P$  large enough that we can omit the subscript  $P$  in  $H(s)$  and let  $e^{\pm i\frac{F_P}{\hbar}}$  stand for  $e^{\pm i\frac{F}{\hbar}}$  in the rest of the computation. We have:

$$\begin{aligned}
 e^{i\frac{F}{\hbar}} H e^{-i\frac{F}{\hbar}} &= H + \frac{[F, H]}{i\hbar} + \frac{[F, \frac{[F, H]}{i\hbar}]}{i\hbar} + \frac{[F, [F, [F, \int_0^1 \int_0^t \int_0^s H(u) du ds dt]/i\hbar]/i\hbar]}{i\hbar} \\
 (4.5) \qquad &= H_0 + G + \frac{[F, H]}{i\hbar} + \frac{[F, \frac{[F, H]}{i\hbar}]}{i\hbar} + \tilde{R} \\
 &= H_0 - G_1 + R + \frac{[F, \frac{[F, H]}{i\hbar}]}{i\hbar} + \tilde{R}.
 \end{aligned}$$

Since  $H(s)$  is a pseudodifferential operator, so is  $\int_0^1 \int_0^t \int_0^s H(u) du ds dt$ . By Proposition 4.3, Lemma 4.7 and Lemma 4.2 we have, since  $G$  is a  $WLG(\kappa)$ ,

$$\|\tilde{R}H_\mu\| = O(|\mu\hbar|^{\kappa+1}).$$

By the same argument,  $\frac{[F, \frac{[F, H]}{i\hbar}]}{i\hbar}$  satisfies the same estimate. Developing  $\tilde{R}$  by the Lagrange formula (4.5) to arbitrary order, we get, thanks to Lemma 4.7,  $\tilde{R} = \tilde{G} + R$  where  $\tilde{G}$  is a  $WLG(\kappa + 1)$  and

$$\|RH_\mu\| = O(|\mu\hbar|^{\frac{L+1}{2}}).$$

Therefore, letting  $G' = \frac{[F, \frac{[F, H]}{i\hbar}]}{i\hbar} + \tilde{G}$ , we have:

$$e^{i\frac{F_P}{\hbar}} H e^{-i\frac{F_P}{\hbar}} = H_0 + G_1 + G' + R,$$

with  $G'$  a  $WLG(\kappa + 1)$ . By induction Theorem 4.8 follows.

Proof of Theorem 4.9:

let us first prove the following

**Lemma 4.10.** *Let  $H_0$  be as before and let  $G$  be a  $WLG(r)$ . Then there exists a  $WLG(r)$   $F$  and  $G_1 = G_1(p_1, \dots, p_n, D_t, \hbar)$ , such that:*

$$(4.6) \quad \frac{[H_0, F]}{i\hbar} = G + G_1.$$

*Proof.* By Lemma 4.6, if  $F$  is a word, it must be a  $WLG(r)$ , since the left hand side of (4.6) is  $WLG(r)$ . Let us take the matrix elements of (4.6) relating the  $H_\mu$ s. We get:

$$-i\Theta.(\mu - \nu) \langle \mu | F | \nu \rangle = \langle \mu | G + G_1 | \nu \rangle + \langle \mu | R | \nu \rangle,$$

where  $\Theta.(\mu - \nu) := \sum_1^n \theta_i \mu_i + \mu_{n+1}$  and  $\langle \mu | \cdot | \nu \rangle = (H_\mu, \cdot H_\nu)$ . We get immediately that  $G_1(\mu\hbar, \hbar) = -\langle \mu | G | \mu \rangle$ . Moreover, let us define  $F$  by:

$$\langle \mu | F | \nu \rangle := \frac{\langle \mu | G + G_1 | \nu \rangle}{-i\Theta.(\mu - \nu)},$$

which exists by the non-resonance condition. To show that  $F$  is a word one just has to decompose  $G = \sum G_l$  in monomial words  $G^l = \alpha(t) D_t^j b_1 \dots b_m$ ,  $b_i \in \{a_1, a_1^+, \dots, a_n, a_n^+\}$ . Then, for each  $\nu$  there is only one  $\mu$  for which  $\langle \mu | G + G_1 | \nu \rangle \neq 0$  and the difference  $\mu - \nu$  depends obviously only on  $G^l$ , not on  $\nu$ . Let us call this difference  $\rho_{G_l}$ . Then  $F$  is given by the sum:

$$F = \sum \frac{1}{-i\Theta.\rho_{G_l}} G_l.$$

□

It is easy to check that one can pass from Lemma 4.10 to Theorem 4.9 by induction, writing  $[H, F + F'] = [H, F] + [H_0, F'] + [H - H_0, F] + [H - H_0, F']$ . □

We will show finally that Theorems 4.8 and 3.1 are equivalent. Once again we can start by considering an Hamiltonian on  $L^2(\mathbb{R}^n \times S^1)$  since any Fourier integral operator  $B_\varphi$ , as defined in the beginning of the proof of Theorem 3.1, intertwines the original Hamiltonian  $H : C_0^\infty(X) \rightarrow C^\infty(X)$  of Section 3 with a pseudodifferential operator on  $L^2(\mathbb{R}^n \times S^1)$  satisfying the hypothesis of Theorem 4.8.

Let us remark first of all that if  $U_L = e^{i\frac{W_3}{\hbar}} e^{i\frac{W_4}{\hbar}} \dots e^{i\frac{W_L}{\hbar}}$ , all  $e^{i\frac{W_l}{\hbar}}$  being Fourier integral operators, then so is  $U_L$ . Secondly we have

**Proposition 4.11.** *Let  $A$  be a pseudodifferential operator of total Weyl symbol  $a(x, \xi, t, \tau, \hbar)$ . Then*

*a vanishes to infinite order at  $p = \tau = 0$  if and only if  $\|AH_\mu\|_{L^2(\mathbb{R}^n \times S^1)} = O(|\mu\hbar|^\infty)$ .*

*Proof.* the “if” part is exactly Proposition 4.3. For the “only if” part let us observe that, if the total symbol didn’t vanish to infinite order, then it would contain terms of the form  $\alpha_{kmnr}(t)\hbar^k(x + i\xi)^m(x - i\xi)^n\tau^r$ . Let us prove this can’t happen in dimension 1, the extension to dimension  $n$  being straightforward.

Each term of the form  $(x + iD_x)^m(x - iD_x)^n = a^m(a^+)^n$  gives rise to an operator  $A_{m,n}$  such that:

$$A_{m,n}H_\mu = \hbar^{\frac{|m+n|}{2}} \sqrt{(\mu+1)\dots(\mu+n)(\mu+n-1)\dots(\mu+n-m)} H_{\mu+m-n} \sim |\mu\hbar|^{\frac{m+n}{2}} H_{\mu+m-n}.$$

Therefore  $\sum c_{mn}A_{m,n}H_\mu = \sum c_{m,m-l}H_{\mu+l} \sim \sum c_{m,m-l}|\mu\hbar|^{\frac{2m-l}{2}}H_{\mu+l}$ . In particular:

$$\|\sum c_{mn}A_{m,n}H_\mu\|^2 \sim \sum |\mu\hbar|^{2m-l}$$

so  $\|\sum c_{mn}A_{m,n}H_\mu\| = O(|\mu\hbar|^\infty)$  implies  $C_{mn} = 0$ . It is easy to check that the same argument is also valid for any ordered product of  $a$ 's and  $a^+$ 's.  $\square$

In the next section we will show how the functions  $H'$  of Theorem 4.8 and  $h$  of Theorem 3.1 are related.

## 5. LINK BETWEEN THE TWO QUANTUM CONSTRUCTIONS

Consider a symbol (on  $\mathbb{R}^{2n}$ ) of the form

$$h(p_1, \dots, p_n)$$

with  $p_i = \frac{\xi_i^2 + x_i^2}{2}$ . There are several ways of quantizing  $h$ : one of them consists in associating to  $h$ , by the spectral theorem, the operator

$$h(P_1, \dots, P_n) = h(P)$$

where  $P_i = \frac{-\hbar^2 \partial_{x_i}^2 + x_i^2}{2}$ . Another one is the Weyl quantization procedure.

In this section we want to compute the Weyl symbol  $h^{we}$  of  $h(P_1, \dots, P_n)$  and apply the result to the situation of the preceding sections. By the metaplectic invariance of the Weyl quantization and the fact that  $h(P_1, \dots, P_n)$  commutes with all the  $P_i$ 's we know that  $h^{we}$  has the form

$$h^{we}(p_1, \dots, p_n) = h^{we}(p),$$

that is, is function of the classical harmonic oscillators  $p_i := \xi_i^2 + x_i^2$ .

To see how this  $h^{we}$  is related to the  $h$  above we note that  $H$  is diagonal on the Hermite basis  $h_j$ . Therefore

$$h((j + \frac{1}{2})\hbar) = \langle h_j, H h_j \rangle = \int h^{we} \left( \left( \frac{x+y}{2} \right)^2 + \xi^2 \right) e^{i \frac{x\xi}{\hbar}} h_j(x) h_j(\xi) \frac{dx d\xi}{\hbar^{n/2}}.$$

We now claim

**Proposition 5.1.** *let  $h$  be either in the Schwartz class, or a polynomial function. Let  $\hat{h}(s) = \frac{1}{(2\pi)^n} \int h(p) e^{-is \cdot p} dp$  be the Fourier transform of  $h$ . Then*

$$(5.1) \quad h^{we}(p) = \int \hat{h}(s) e^{\frac{2i \tan(s\hbar/2) \cdot p}{\hbar}} \Phi(s) ds$$

where  $\tan(s\hbar/2) \cdot p$  stands for  $\sum_i \tan(s_i \hbar/2) p_i$  and  $\Phi(s) = \prod_{i=1}^n (1 - 2i \tan(s_i \hbar/2))$ , and where (5.1) has to be interpreted in the sense of distribution, that is, for each  $\varphi$  in the Schwartz's class of  $\mathbb{R}$ ,

$$\int h^{we}(p) \varphi(p) dp = \int \int \hat{h}(s) e^{\frac{2i \tan(s\hbar/2) \cdot p}{\hbar}} \Phi(s) ds \varphi(p) dp = \int \hat{h}(s) \Phi(s) \hat{\varphi} \left( \frac{2i \tan(s\hbar/2)}{\hbar} \right) ds.$$

Finally, as  $\hbar \rightarrow 0$ ,

$$(5.2) \quad h^{we} \sim h + \sum_{l=1}^{\infty} c_l \hbar^{2l}$$

*Proof.* Let  $h(P) = \int \hat{h}(s) e^{is \cdot P} ds$ , where  $e^{is \cdot P}$  is a zeroth order semiclassical pseudo-differential operator whose Weyl symbol will be computed from its Wick symbol (see 5.5 below for the definition). Let us first remark that since  $e^{is \cdot P} = \prod_{i=1}^n e^{is_i \cdot P_i}$  it is enough to prove the Theorem in the one-dimensional case.

Let  $\varphi_{x\xi}$  be a coherent state at  $(x, \xi)$ , that is

$$\varphi_{x\xi}(y) = (\pi\hbar)^{-\frac{1}{4}} e^{i\frac{\xi y}{\hbar}} e^{-\frac{(y-x)^2}{2\hbar}}$$

Let  $z = \frac{\xi+ix}{\sqrt{2}}$ ,  $z' = \frac{\xi'+ix'}{\sqrt{2}}$  and  $z(t) = \frac{\xi(t)+ix(t)}{\sqrt{2}}$ . A straightforward computation gives

$$(5.3) \quad (\varphi_{x\xi}, \varphi_{x'\xi'}) = e^{\frac{2zz' - |z|^2 - |z'|^2}{2\hbar}}.$$

Moreover decomposing  $\varphi_{x\xi}$  on the Hermite basis leads to

$$(5.4) \quad e^{isP} \varphi_{x\xi} = e^{i\frac{s}{2}\hbar} \varphi_{x(\hbar s)\xi(\hbar s)}$$

where  $P = \frac{-\hbar^2 \partial_x^2 + x^2}{2}$  and  $z(t) = e^{it} z$ .

The Wick symbol of  $e^{isP}$  is defined as

$$(5.5) \quad \sigma^{wi}(e^{isP})(x, \xi) := (\varphi_{x\xi}, e^{isP} \varphi_{x\xi})$$

which, by (5.3) and (5.4), is equal to

$$e^{-\frac{1-e^{-ihs}}{\hbar} \left( \frac{x^2 + \xi^2}{2} \right) + i\frac{s}{2}\hbar}$$

Moreover, using the Weyl quantization formula, it is immediate to see that the Weyl and Wick symbols are related by

$$\sigma^{wi} = e^{-\frac{\hbar\Delta}{4}} \sigma^{we}$$

where  $\Delta = -\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial \xi^2} \right)$ .

It is a standard fact the the Wick symbol determines the operator: indeed the function  $e^{\frac{-2zz' + |z|^2 + |z'|^2}{2\hbar}} (\varphi_{x\xi}, e^{isP} \varphi_{x'\xi'})$  obviously determines  $e^{isP}$ . Moreover it is easily seen to be analytic in  $z$  and  $z'$ . Therefore it is determined by its values on the diagonal  $z = z'$  i.e., precisely, the Wick symbol of  $e^{isP}$ . A straightforward calculation shows that, for  $\frac{s}{2\hbar} \neq \frac{(2k+1)\pi}{2}$ ,  $k \in \mathbb{Z}$ ,

$$(5.6) \quad (1 - 2i \tan(s\hbar/2)) e^{-\frac{\hbar\Delta}{4}} e^{\frac{2i \tan(s\hbar/2) x^2 + \xi^2}{\hbar}} = e^{-\frac{1-e^{-ihs}}{\hbar} \left( \frac{x^2 + \xi^2}{2} \right) + i\frac{s}{2}\hbar}$$

This shows that, for  $\frac{s}{2\hbar} \neq \frac{(2k+1)\pi}{2}$ ,  $k \in \mathbb{Z}$ , we have

$$\sigma^{we}(e^{isP})(p) = (1 - 2i \tan(s\hbar/2)) e^{\frac{2i \tan(s\hbar/2) x^2 + \xi^2}{\hbar}}.$$

Let us now take  $\varphi$  in the Schwartz's class of  $\mathbb{R}$ , and  $B_\varphi$  be the operator of (total) Weyl symbol  $\varphi(\frac{x^2+\xi^2}{2})$ . Let:

$$f(s) := 2\pi \int \sigma^{we}(e^{isP})(p)\varphi(p)pdp = \text{Trace}[e^{isP}B_\varphi].$$

**Lemma 5.2.**

$$f \in C^\infty(\mathbb{R}).$$

*Proof.* by metaplectic invariance we know that  $B_\varphi$  is diagonal on the Hermite basis. Therefore,  $\forall k \in \mathbb{N}$ ,

$$(-i)^k \frac{d^k}{ds^k} f(s) := \text{Trace}[e^{isP}P^k B_\varphi] = \sum \langle h_j, B_\varphi h_j \rangle \left( (j + \frac{1}{2})\hbar \right)^k e^{is(j+\frac{1}{2})\hbar}.$$

Since  $h_j$  is microlocalized on the circle of radius  $(j + \frac{1}{2})\hbar$  and  $\varphi$  is in the Schwartz class, the sum is absolutely convergent for each  $k$ .  $\square$

Therefore  $f(s) = 2\pi \int (1 - 2i \tan(s\hbar/2)) e^{\frac{2i \tan(s\hbar/2) \frac{x^2+\xi^2}{2}}{\hbar}} \varphi(p)pdp$  and (5.6) is valid in the sense of distribution (in the variable  $p$ ) for all  $s \in \mathbb{R}$ . This expression gives (5.1) immediately for  $h$  in the Schwartz class. When  $h$  is a polynomial function it is straightforward to check that, since  $\hat{h}$  is a sum of derivatives of the Dirac mass and  $e^{isP}$  is a Weyl operator whose symbol is  $C^\infty$  with respect of  $s$ , the formula also holds in this case. The asymptotic expansion (5.2) is obtained by expanding  $e^{\frac{2itg(s\hbar/2) \frac{x^2+\xi^2}{2}}{\hbar}}$  near  $e^{is \frac{x^2+\xi^2}{2}}$ .  $\square$

Formula (5.1) shows clearly that  $h^{we}$  depends only on the  $\frac{2\pi}{\hbar}$  periodization of  $\hat{h}(s)e^{i\frac{s\hbar}{2}}$ , therefore

**Corollary 5.3.**  $h^{we}$  depends only of the values  $h((k + \frac{1}{2})\hbar)$ ,  $k \in \mathbb{N}$ .

We mention one application of formula (5.1). Let us suppose first that we have computed the quantum normal form at order  $K$ , that is

$$h_K(p) = \sum_{|k|=k_1+\dots+k_n \leq K} c_k p^k := \sum_{|k|=k_1+\dots+k_n \leq K} c_k p_1^{k_1} \dots p_n^{k_n}$$

and let us define  $h_K^{we}$  as the Weyl symbol of  $h_K(P)$ .

**Corollary 5.4.**

$$\begin{aligned} h_K^{we}(p) &= \sum_{|k|=k_1+\dots+k_n \leq K} c_k \frac{\partial^K}{\partial s^k} \left( \Phi(s) e^{\frac{2i \tan(s\hbar/2)p}{\hbar}} \right) \Big|_{s=0} \\ &:= \sum_{|k|=k_1+\dots+k_n \leq K} c_k \frac{\partial^K}{\partial s_1^{k_1} \dots \partial s_n^{k_n}} \left( \Phi(s) e^{2i \frac{\tan(s_1 \hbar/2) p_1 + \dots + \tan(s_n \hbar/2) p_n}{\hbar}} \right) \Big|_{s=0}. \end{aligned}$$



Let us come back now to the comparison between the two constructions of Sections 2 and 3.

Clearly the " $\theta$ " part doesn't play any role, as the Weyl quantization of any function  $f(\tau)$  is exactly  $f(D_\theta)$ . therefore we have the following

**Theorem 5.5.** *The functions  $H'$  of Theorem 3.1 and  $h$  of Theorem 4.8 are related by the formula*

$$H'(P_1, \dots, P_n, D_t, \hbar) = \int \hat{H}^*(s, D_t, \hbar) e^{\frac{2i \tan(s\hbar/2) \cdot p}{\hbar}} \Phi(s) ds$$

where  $\hat{H}^*$  is the Fourier transform of  $H^*$  with respect to the variables  $p_i$ . In particular

$$H' - H^* = O(\hbar^2).$$

*Proof.* the proof follows immediately from Proposition 5.1, and the unicity of the (quantum) Birkhoff normal form.  $\square$

## 6. THE COMPUTATION OF THE SEMICLASSICAL BIRKHOFF CANONICAL FORM FROM THE ASYMPTOTICS OF THE TRACE FORMULA

Let  $X$  and  $H$  be as in the introduction. Let  $\gamma$  be a periodic trajectory of the vector field (2.1) of period  $2\pi$ .

For  $l \in \mathbb{Z}$  let  $\varphi_l$  be a Schwartz function on the real line whose Fourier transform  $\hat{\varphi}_l$  is supported in a neighborhood of  $2\pi l$  containing no other period of (2.1). The semiclassical trace formula gives an asymptotic expansion for  $\text{Trace } \varphi_l \left( \frac{H-E}{\hbar} \right)$  of the form:

$$(6.1) \quad \text{Trace } \varphi_l \left( \frac{H-E}{\hbar} \right) \sim \sum_{m=0}^{\infty} d_l^m \hbar^m$$

where the  $d_l$ 's are distributions acting on  $\hat{\varphi}_l$  with support concentrated at  $\{2\pi l\}$ .

We will show that the knowledge of the  $d_l$ s determine the quantum semiclassical Birkhoff form of Section 2, and therefore the classical one.

Let us first rewrite the l.h.s of (6.1) as

$$(6.2) \quad \text{Trace} \left( \int \hat{\varphi}(t) e^{it \frac{H-E}{\hbar}} dt \right)$$

Since  $\hat{\varphi}$  is supported near a single period of (2.1) we know from the general theory of Fourier integral operators that one can microlocalize (6.1) near  $\gamma$  modulo error term of order  $O(\hbar^\infty)$ .

Therefore we can conjugate (6.2) by the semiclassical Fourier integral operator  $A_\varphi$  of Theorem 3.1. This leads to the computation of

$$(6.3) \quad \begin{aligned} & \text{Trace} \left( A_\varphi \int \hat{\varphi}(t) e^{it \frac{H-E}{\hbar}} dt A_\varphi^{-1} \right) \\ &= \text{Tr} \left( \int \hat{\varphi}(t) \rho(P_1 + \dots + P_n) e^{it \frac{H'(P_1 + \dots + P_n, D_t, \hbar) + H'' - E}{\hbar}} dt \right) \end{aligned}$$

where  $\rho \in C_0^\infty(\mathbb{R}^{n+1})$  with  $\rho = 1$  in a neighborhood of 0 and  $\text{Tr}$  stands for the Trace in  $L^2(\mathbb{R}^n \times S^1)$ .

Let us write  $H'(P_1 + \dots + P_n, D_t, \hbar)$  as

$$(6.4) \quad E + D_t + \sum \theta_i P_i + \sum_{r \in \mathbb{N}^n, s \in \mathbb{Z}} c_{r,s}(\hbar) P^r D_t^s.$$

We will first prove

**Proposition 6.1.** *let  $g_{r,s}^l(t, \theta)$  be the function defined by*

$$(6.5) \quad g_{r,s}^l(t, \theta) = \left(-i \frac{\partial}{t \partial \theta}\right)^r \left(-i \frac{\partial}{\partial t}\right)^s \left[ \frac{e^{it \frac{\theta_1 + \dots + \theta_n}{2}}}{\prod_i (1 - e^{it \theta_i})} t \hat{\varphi}(t) \right]$$

Let us fix  $l \in \mathbb{Z}$ . Then the knowledge of all the  $d_l^m$ s for  $m < M$  in (6.1) determines the following quantities

$$(6.6) \quad \sum_{|r|+s=m} c_{r,s}(\hbar) g_{r,s}^l(2\pi l, \theta)$$

for all  $m < M$ .

*Proof.* the r.h.s. of (6.3) can be computed thanks to (6.4) using

$$\begin{aligned} \text{spectrum } P_i &= \left\{ \left(\mu_i + \frac{1}{2}\right) \hbar, \mu_i \in \mathbb{N} \right\} \\ \text{spectrum } D_t &= \{ \nu \hbar, n \in \mathbb{Z} \} \end{aligned}$$

Thus the r.h.s of (6.3) can be written as

$$(6.7) \quad \int \hat{\varphi}_l(t) \sum_{\mu, \nu} \rho \left( \left( |\mu| + \frac{n}{2} \right) \hbar \right) e^{it[\nu + \theta \cdot (\mu + \frac{1}{2})]} \sum_{k=0}^{\infty} \frac{(it)^k}{k!} \left( \sum_{r,s} c_{r,s}(\hbar) \left( \mu + \frac{1}{2} \right)^r \nu^s \hbar^{|\mu|+s-1} \right)^k dt$$

since the support of  $\hat{\varphi}_l$  contains only one period, and therefore the trace can be microlocalized infinitely close to the periodic trajectory, making the role of  $H''$  inessential.

Using the following remark of S. Zelditch:

$$\left(\mu + \frac{1}{2}\right)^r \nu^s = \left(-i \frac{\partial}{t \partial \theta}\right)^r \left(-i \frac{\partial}{\partial t}\right)^s e^{it[\nu + \theta \cdot (\mu + \frac{1}{2})]}$$

we get, mod( $\hbar^\infty$ ),

$$(6.8) \quad \int \hat{\varphi}_l(t) \sum_{k=0}^{\infty} \frac{(it)^k}{k!} \left( \sum_{r,s} \hbar^{|\mu|+s-1} c_{r,s}(\hbar) \left(-i \frac{\partial}{t \partial \theta}\right)^r \left(-i \frac{\partial}{\partial t}\right)^s \right)^k \sum_{\mu, \nu} e^{it[\nu + \theta \cdot (\mu + \frac{1}{2})]} dt.$$

Since  $\sum_{\nu \in \mathbb{Z}} e^{it\nu} = 2\pi \sum_l \delta(t - 2\pi l)$ , and  $\sum_{\mu \in \mathbb{N}^n} e^{it\theta \cdot (\mu + \frac{1}{2})} = \frac{e^{it\frac{\theta_1 + \dots + \theta_n}{2}}}{\prod_i (1 - e^{it\theta_i})}$ , together with the fact that  $\hat{\varphi}$  is supported near  $2\pi l$ , we get that (6.8) is equal to

$$2\pi \left[ \sum_{k=0}^{\infty} \frac{(i)^k}{k!} \left( \sum_{r,s} \hbar^{|\mu|+s-1} c_{r,s}(\hbar) \left( -i \frac{\partial}{t\partial\theta} \right)^r \left( -i \frac{\partial}{\partial t} \right)^s \right)^k \left( t^k \hat{\varphi}_l(t) \frac{e^{it\frac{\theta_1 + \dots + \theta_n}{2}}}{\prod_i (1 - e^{it\theta_i})} \right) \right]_{t=2\pi l}.$$

Rearranging terms in increasing powers of  $\hbar$  shows that the quantities (6.6) can be computed recursively.  $\square$

The fact that one can compute the  $c_{r,s}(\hbar)$  from the quantities (6.6) is an easy consequence of the rational independence of the  $\theta_i$ s and the Kronecker theorem, and is exactly the same as in [5].

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