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The supercover of an m-flat is a discrete analytical object

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Abstract

The aim of this paper is to show that the supercover of an m-flat (i.e. a Euclidean affine subspace of dimension m) in Euclidean n-space is a discrete analytical object. The supercover of a Euclidean object F is a discrete object consisting of all the voxels that intersect F. A discrete analytical object is a set of discrete points that is defined by a finite set of inequalities. A method to determine the inequalities defining the supercover of an m-flat is provided.

Keywords: discrete geometry, computer graphics, supercover, m-flat, discrete analytical object, arbitrary dimension.

1 Introduction

Classically, in computer graphics, a discrete primitive (such as a 2D line segment [10], a 3D line segment [21, 1, 14], a 3D polygon [22], etc.) is defined as the result of a local approximation algorithm applied to the corresponding Euclidean primitive. Such a generation algorithm typically defines the discrete primitive without any global analytical characterisation in the discrete domain. This approach has some drawbacks [11]. While the algorithms are designed to be efficient, it is difficult to study the properties of the generated discrete object. There is also no simple way to extend such primitives to higher dimensions and different discrete primitives are rarely coherent with each other. For instance, Kaufman's discrete 3D polygon [22] is not a piece of a discrete 3D plane as defined by other authors [24, 18, 25, 4] and its edges do not correspond to most notions of discrete 3D line segments [23, 21, 1, 17, 14, 5].
A more recent approach is the so-called *discrete analytical geometry approach* introduced by J-P. Reveillès [25]. A discrete primitive is defined, in this approach, as a *discrete analytical object*, which is a discrete object defined by a finite set of *discrete inequalities*. A *discrete inequality* is an inequality, with coefficients in $\mathbb{R}$, of which we retain only the integer coordinate solutions (in $\mathbb{Z}^n$). The properties of such a discrete primitive are relatively easy to study because of its global analytical description. Extension to higher dimensions is often straightforward. The discrete analytical geometry approach has been used to define and study different classes of discrete primitives such as discrete 2D lines [25], naive [15, 4] and standard 3D planes [2, 19, 20, 4], discrete 3D lines [17, 5], discrete hyperplanes [26, 28, 4], discrete circles and hyperspheres [7], discrete standard hyperplanes, m-flats, m-simplices [8, 9] etc.

In this paper we examine, within the framework of the discrete analytical geometry approach, an old discretisation scheme [12], called the *supercover*, which was studied by Cohen and Kaufman for volume graphics purposes [13]. The *supercover* of a Euclidean object F is the set of discrete points consisting of all the *voxels* that intersect F. The *voxel* associated with a discrete point, in $\mathbb{Z}^n$, is a unit hypercube, in $\mathbb{R}^n$, centered on the point with corresponding integer coordinates. In fact, supercovers have interesting properties under set operations [13]. It has been shown that the supercovers of a 2D line and a 3D plane [3], a 3D line, a 3D triangle and a 3D tetrahedron [5, 6] are discrete analytical objects.
The aim of this paper is to show that the supercover of an m-flat in dimension n is a discrete analytical object. An *m-flat* in a space of dimension n is a Euclidean affine subspace of dimension m. We first propose a simple discrete analytical characterisation, through discrete inequalities, of the supercover of a hyperplane (i.e. supercover of an $(n-1)$-flat) and the supercover of a point supercover (i.e. supercover of a 0-flat). We show that the supercover and the *orthogonal projection* commute. A similar result holds for the *orthogonal extrusion*. Using this result, we prove that the supercover of an m-flat F, for $0 \leq m \leq n-2$, is equal to the intersection of the supercovers of orthogonal extrusions of F. From this we deduce a method of determining a set of discrete inequalities that characterise the supercover of an m-flat. This proves that the supercover of an m-flat is a discrete analytical object.
The paper is organised as follows. In section 2 we present the notations and the main definitions used in this paper. We also present some straightforward results that will be useful in subsequent sections. In section 3, we introduce the notion of supercover and some of its general properties. The main results of this paper can be found in the fourth section. Section 4 is divided into 4 subsections. In subsection 1 we provide a discrete analytical characterisation of the supercover of a hyperplane (i.e. $(n-1)$-flat). In subsection 2 we do the same for the supercover of a point (i.e. 0-flat). In the third subsection, we prove a result that characterises the supercover of an m-flat. In a fourth part of section 4 we show that the supercover of an m-flat is a discrete analytical object. We provide a method that gives a discrete analytical characterisation of the supercover of an m-flat. We conclude in section 5 with some discussion and a conjecture on the discrete analytical characterisation of a simplex in dimension n.

2 Preliminaries

2.1 Basic notations

Let $\mathbb{Z}^n$ be the subset of the nD Euclidean space $\mathbb{R}^n$ that consists of all the integer coordinate points. A *discrete* (resp. *Euclidean*) *point* is an element of $\mathbb{Z}^n$ (resp. $\mathbb{R}^n$). A *discrete* (resp. *Euclidean*) *object* is a set of discrete (resp. Euclidean) points. A *discrete inequality* is an inequality with coefficients in $\mathbb{R}$ from which we retain only the integer coordinate solutions. A *discrete analytical object* is a discrete object defined by a finite set of discrete inequalities [7]. An *m-flat* is a Euclidean affine subspace of dimension m. A Euclidean hyperplane is an $(n-1)$-flat. A Euclidean point is a 0-flat.
We denote by p_i the i-th coordinate of a point or vector p. For p a discrete point or vector in dimension n, we denote by $\gcd(p)$ the greatest common divisor of $p_1, \ldots, p_n$. For $0 \leq k \leq n$, Two discrete points p and q in dimension n are *k-neighbours*, if $|p_i - q_i| \leq 1$ and $k \leq n - \sum_{i=1}^{n} |p_i - q_i|$. A k-path is a sequence of discrete points such that two consecutive points of the sequence are k-neighbours. We denote by σ^n the set of all the permutations of the set $\{1, 2, \ldots, n\}$.
We denote by $\mathbb{J}_m^n$ the set of all the strictly growing sequences of m integers all between 1 and n: $\mathbb{J}_m^n = \{j \in \mathbb{Z}^m \,|\, 1 \leq j_1 < j_2 < \ldots < j_m \leq n\}$. This defines a set of *multi-indices*.
Let us consider an object F in the n-dimensional Euclidean space $\mathbb{R}^n$, with $n > 1$.

The orthogonal projection is defined by:

$$\pi_i(q) = (q_1, \ldots, q_{i-1}, q_{i+1}, \ldots, q_n), \text{ for } 1 \leq i \leq n \text{ and } q \in \mathbb{R}^n;$$
$$\pi_j(q) = (\pi_{j_1} \circ \pi_{j_2} \circ \cdots \circ \pi_{j_m})(F) \text{ and } \pi_j(F) = \{\pi_j(q) \,|q \in F\} \text{ for } j \in \mathbb{J}_m^n.$$

The orthogonal extrusion is defined by:

$$\varepsilon_j(F) = \pi_j^{-1}(\pi_j(F)), \text{ for } j \in \mathbb{J}_m^n.$$

It is easy to see that, for two Euclidean objects $X, Y \subseteq \mathbb{R}^n$, the following three statements are equivalent (i) $\pi_j(X) \cap \pi_j(Y) \neq \varnothing$ (ii) $\varepsilon_j(X) \cap Y \neq \varnothing$ (iii) $X \cap \varepsilon_j(Y) \neq \varnothing$. For any discrete or Euclidean object F, we have $\pi_i \circ \pi_i^{-1}(F) = F$ while in general $\pi_i^{-1} \circ \pi_i(F) = \varepsilon_i(F) \neq F$.
Let σ^n denote the set of all permutations of $\{1, 2, \ldots, n\}$. For each $\sigma \in \sigma^n$, we define the corresponding axis permutation r_σ by:

$$r_\sigma : \mathbb{R}^n \to \mathbb{R}^n$$
$$x \mapsto \left(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}\right)$$

and the corresponding multi-index axis permutation r_j by $r_j = r_{\sigma_j}$, for $j \in \mathbb{J}_m^n$, where the permutation $\sigma_j \in \sigma^n$ is defined by:

$$\sigma_j(j_i) = i, \text{ for } 1 \leq i \leq m,$$
$$\sigma_j(k_i) = i + m, \text{ for all } 1 \leq i \leq n - m.$$

where $k_1 < k_2 < \cdots < k_{n-m}$ are the elements of $\{1, 2, \ldots, n\} \setminus \{j_i \,|1 \leq i \leq m\}$ in ascending order. Informally, we can say that $r_j(x)$ is obtained from x by placing the ith coordinate of x in the j_i^{th} position, the $m + 2^{nd}$ coordinate of x in the next remaining position, and so on.

Lemma 1 *: Let us consider a Euclidean object F in $\mathbb{R}^n$ and a multi-index axis permutation r_j, with $j \in \mathbb{J}_m^n$. Then:*

$$\pi_j(F) = \pi_{(1,2,\ldots,m)}\left(r_j^{-1}(F)\right)$$
$$\varepsilon_j(F) = r_j\left(\varepsilon_{(1,2,\ldots,m)}\left(r_j^{-1}(F)\right)\right)$$

The proof of this lemma is straightforward. In fact we have designed r_j to satisfy these two properties. This lemma will be helpful in section 4.
The *voxel* $\mathbb{V}(p) \subset \mathbb{R}^n$ of a discrete nD point p is defined by $\mathbb{V}(p) = \left[p_1 - \frac{1}{2}, p_1 + \frac{1}{2}\right] \times \cdots \times \left[p_n - \frac{1}{2}, p_n + \frac{1}{2}\right]$. In 1D and 2D a voxel is also called a *pixel*. For a discrete object F, $\mathbb{V}(F) = \bigcup_{p \in F} \mathbb{V}(p)$.
It is easy to see that, for two discrete points p and q and a multi-index $j \in \mathbb{J}_m^n$, we have $\mathbb{V}(p) \times \mathbb{V}(q) = \mathbb{V}(p \times q)$, $\mathbb{V}(\pi_j(p)) = \pi_j(\mathbb{V}(p))$ and $\mathbb{V}(\varepsilon_j(p)) = \varepsilon_j(\mathbb{V}(p))$.
It is also easy to see that, for a discrete point p and an integer m that satisfies $1 \leq m < n$, we have $\mathbb{V}(p) = \bigcap_{j \in \mathbb{J}_m^n} \varepsilon_j(\mathbb{V}(p))$.

3 General properties of the Supercover

In this section we will introduce the notion of supercover, recall some of its known properties [13] and present several original properties that seem not to have been mentioned in the literature.

Definition 2 *: The supercover $\mathbb{S}(F) \subset \mathbb{Z}^n$ of a Euclidean object F is defined by:*

$$\mathbb{S}(F) = \{p \in \mathbb{Z}^n \,|\mathbb{V}(p) \cap F \neq \varnothing\}$$

There exists an alternative way, proposed by Tajine et al. [27], of defining the supercover with the distance d_∞: $\mathbb{S}(F) = \left\{p \in \mathbb{Z}^n \,\middle|\, d_\infty(p, F) \leq \frac{1}{2}\right\}$ for a Euclidean object F.
The supercover has many basic properties [13]. Let us consider two Euclidean objects F and G. Then: $\mathbb{S}(F) = \bigcup_{\alpha \in F} \mathbb{S}(\alpha)$, $\mathbb{S}(F \cup G) = \mathbb{S}(F) \cup \mathbb{S}(G)$, $\mathbb{S}(F \cap G) \subseteq \mathbb{S}(F) \cap \mathbb{S}(G)$ and if $F \subseteq G$ then $\mathbb{S}(F) \subseteq \mathbb{S}(G)$. One can say that the operators $\mathbb{S}$ and $\cup$ commute. There are several other known properties of the supercover such as the $(n-1)$-connectivity of a supercover, the separation property, etc. See [13] for more details.
Let us examine now some original properties of the supercover:

Proposition 3 *: For any two Euclidean objects F, G, we have: $\mathbb{S}(F \times G) = \mathbb{S}(F) \times \mathbb{S}(G)$.*

One can say that the operators $\mathbb{S}$ and $\times$ commute.

Proof. Let us consider two Euclidean objects $F \subset \mathbb{R}^m$ and $G \subset \mathbb{R}^n$.

a) Is $\mathbb{S}(F) \times \mathbb{S}(G) \subset \mathbb{S}(F \times G)$ true ?
Let us consider two discrete points $p \in \mathbb{S}(F)$ and $q \in \mathbb{S}(G)$. The discrete point p belongs to $\mathbb{S}(F)$ if and only if there exists a Euclidean point α belonging to $\mathbb{V}(p) \cap F$. The discrete point q belongs to $\mathbb{S}(G)$ if and only if there exists a Euclidean point β belonging to $\mathbb{V}(q) \cap G$. This means that $(\alpha, \beta) \in \mathbb{V}(p) \times \mathbb{V}(q)$ and thus, since $\mathbb{V}(p \times q) = \mathbb{V}(p) \times \mathbb{V}(q)$, we have $(\alpha, \beta) \in \mathbb{V}(p \times q)$. With $(\alpha, \beta) \in F \times G$, we have $\mathbb{V}(p \times q) \cap (F \times G) \neq \varnothing$ and therefore $(p, q) \in \mathbb{S}(F \times G)$. This proves the first part of our proposition.

b) Is $\mathbb{S}(F \times G) \subset \mathbb{S}(F) \times \mathbb{S}(G)$ true ?
Let us consider a discrete point $r \in \mathbb{S}(F \times G)$. We can write r in the form $(p, q) \in \mathbb{Z}^{m+n}$, with $p \in \mathbb{Z}^m$ and $q \in \mathbb{Z}^n$. The discrete point r belongs to $\mathbb{S}(F \times G)$ if and only if there exists a Euclidean point $\gamma = (\alpha, \beta)$, with $\alpha \in \mathbb{R}^m$ and $\beta \in \mathbb{R}^n$, belonging to $\mathbb{V}(p \times q) \cap (F \times G)$. We have $\gamma = (\alpha, \beta) \in \mathbb{V}(p \times q) = \mathbb{V}(p) \times \mathbb{V}(q)$ and $(\alpha, \beta) \in F \times G$. Therefore we have $\alpha \in \mathbb{V}(p) \cap F$ and $\beta \in \mathbb{V}(q) \cap G$. This proves that $r = (p, q) \in \mathbb{S}(F) \times \mathbb{S}(G)$. ■

The following proposition shows that the supercover operator commutes with the operators π^{-1}, π and ε.

Proposition 4 *: For any multi-index $j \in \mathbb{J}_m^n$. Then:*

(i) For $F \subseteq \mathbb{R}^{n-m}$, we have $\pi_j^{-1}(\mathbb{S}(F)) = \mathbb{S}\left(\pi_j^{-1}(F)\right)$.

(ii) For $F \subseteq \mathbb{R}^n$, we have $\pi_j(\mathbb{S}(F)) = \mathbb{S}(\pi_j(F))$ and $\varepsilon_j(\mathbb{S}(F)) = \mathbb{S}(\varepsilon_j(F))$.

Proof. We can suppose without loss of generality that j belongs to $\mathbb{J}_1^n$, i.e. that j is an integer that satisfies $1 \leq j \leq n$. If the assertions (i) and (ii) of proposition 4 hold for an arbitrary integer j then they hold for every multi-index j.

a) $\pi_j^{-1}(\mathbb{S}(F)) = \mathbb{S}\left(\pi_j^{-1}(F)\right)$ is an easy consequence of proposition 3.

b) Let us show that $\mathbb{S}(\pi_j(F)) = \pi_j(\mathbb{S}(F))$. We can suppose without loss of generality that $j = n$.
$\mathbb{S}(\pi_n(F)) = \mathbb{S}\left(\left\{\alpha' \in \mathbb{R}^{n-1} \,|\alpha = (\alpha', \alpha_n) \in F\right\}\right)$ and thus
$\mathbb{S}(\pi_n(F)) = \left\{p' \in \mathbb{Z}^{n-1} \,|\forall \alpha = (\alpha', \alpha_n) \in F \text{ and } \alpha' = \pi_n(\alpha) \in \mathbb{V}(p')\right\}$.
On the other hand $\pi_n(\mathbb{S}(F)) = \pi_n\{p = (p', p_n) \in \mathbb{Z}^n \,|\mathbb{V}(p) \cap F \neq \varnothing\}$
and thus $\pi_n(\mathbb{S}(F)) = \left\{p' \in \mathbb{Z}^{n-1} \,|\forall \alpha = (\alpha', \alpha_n) \in F \text{ and } \alpha' = \pi_n(\alpha) \in \mathbb{V}(p')\right\}$.
This proves that $\pi_n(\mathbb{S}(F)) = \mathbb{S}(\pi_n(F))$.

c) $\varepsilon_j(\mathbb{S}(F)) = \mathbb{S}(\varepsilon_j(F))$ is an immediate consequence of a) and b). ■

The following proposition is important. It shows that the supercover is independent of any axis permutation. There is no privileged direction. This is not true for most other digitisation schemes [10, 21, 1, 22, 26, 28, 15, 17].

Proposition 5 *: Let $F \subseteq \mathbb{R}^n$ and $\sigma \in \sigma^n$. Then:*

$$r_\sigma(\mathbb{S}(F)) = \mathbb{S}(r_\sigma(F))$$

The proof of this proposition is obvious. One can say that the operators $\mathbb{S}$ and r_σ commute.
The following proposition tells us that the problem of determining the supercover of an extrusion is essentially a problem of determining a supercover in the corresponding projection space.

Proposition 6 *: Let $F \subseteq \mathbb{R}^n$ and $j \in \mathbb{J}_m^n$. Then:*

$$\mathbb{S}(\varepsilon_j(F)) = r_j(\mathbb{Z}^m \times \mathbb{S}(\pi_j(F)))$$

Proof. We know from lemma 1 that $\varepsilon_j(F) = r_j\left(\varepsilon_{(1,2,\ldots,m)}\left(r_j^{-1}(F)\right)\right)$ for r_j a multi-index axis permutation with $j \in \mathbb{J}_m^n$. It is easy to see that $\varepsilon_{(1,\ldots,m)}(F) = \pi_{(1,\ldots,m)}^{-1}\left(\pi_{(1,\ldots,m)}(F)\right) = \mathbb{R}^m \times \pi_{(1,\ldots,m)}(F)$. Therefore, $\varepsilon_{(1,\ldots,m)}\left(r_j^{-1}(F)\right) = \mathbb{R}^m \times \pi_{(1,\ldots,m)}\left(r_j^{-1}(F)\right)$. We know from lemma 1 that $\pi_j(F) = \pi_{(1,\ldots,m)}\left(r_j^{-1}(F)\right)$. This leads to $\varepsilon_j(F) = r_j\left(\varepsilon_{(1,2,\ldots,m)}\left(r_j^{-1}(F)\right)\right) = r_j(\mathbb{R}^m \times \pi_j(F))$ and therefore $\mathbb{S}(\varepsilon_j(F)) = \mathbb{S}(r_j(\mathbb{R}^m \times \pi_j(F)))$. Since r_j is also an axis permutation, we can apply proposition 5 and thus, $\mathbb{S}(\varepsilon_j(F)) = r_j(\mathbb{S}(\mathbb{R}^m \times \pi_j(F)))$. Now, because of proposition 3, we have $\mathbb{S}(\varepsilon_j(F))$ equal to $r_j(\mathbb{S}(\mathbb{R}^m) \times \mathbb{S}(\pi_j(F)))$. It is easy to see that $\mathbb{S}(\mathbb{R}^m) = \mathbb{Z}^m$, which proves the proposition. ■

4 The supercover of an m-flat

In this section we will study the supercover of an m-flat in dimension n. We will show that the supercover of an m-flat is a discrete analytical object and give a method that provides the corresponding discrete inequalities.
This section is organized in four subsections. In the first subsection we show that the supercover of a hyperplane (i.e. $(n-1)$-flat) is described by 2 discrete inequalities. In the second subsection we show that the supercover of a point (i.e. 0-flat) is described by $2n$ discrete inequalities. In subsection 3, we show that the supercover of an m-flat is the intersection of supercovers of its orthogonal extrusions. Note that of these orthogonal extrusions is an l-flat for some $l \geq m$. In subsection 4, we show that if $m < n-1$, the supercover of an m-flat is a discrete analytical object by providing a method that determines the discrete inequalities describing the supercover of an m-flat.

4.1 Supercover of a hyperplane

The following theorem characterises the supercover of a Euclidean hyperplane (i.e. a $(n-1)$-flat) as the set of discrete points that satisfy a double inequality.

Theorem 7 *: Let H be a Euclidean hyperplane in $\mathbb{R}^n$ defined by: $H : e + \sum_{i=1}^n A_i x_i = 0$, with $A \in \mathbb{R}^n, e \in \mathbb{R}$. Then the supercover of H is:*

$$\mathbb{S}(H) = \left\{ X \in \mathbb{Z}^n \,\middle|\, -\frac{\sum_{i=1}^n |A_i|}{2} \leq e + \sum_{i=1}^n A_i X_i \leq \frac{\sum_{i=1}^n |A_i|}{2} \right\}$$

Proof. Let us suppose, without loss of generality, that $A_{i+1} \geq A_i \geq 0$ for $1 \leq i \leq n-1$. Here "if" follows from the fact that $e + \sum_{i=1}^{n} A_i \left(p_i - \frac{1}{2}\right) \leq 0$ and $e + \sum_{i=1}^{n} A_i \left(p_i + \frac{1}{2}\right) \geq 0$ would imply that the segment whose endpoints are $p - \left(\frac{1}{2}, \cdots, \frac{1}{2}\right)$ and $p + \left(\frac{1}{2}, \cdots, \frac{1}{2}\right)$ must intersect H, and hence that $\mathbb{V}(p)$ (which is convex and therefore contains that segment) intersects H. A discrete point p belongs to $\mathbb{S}(H)$ if and only if $\mathbb{V}(p) \cap H \neq \varnothing$. We have $\mathbb{V}(p) \cap H \neq \varnothing$ for some discrete point p if and only if $e + \sum_{i=1}^{n} A_i \left(p_i - \frac{1}{2}\right) \leq 0$ and $e + \sum_{i=1}^{n} A_i \left(p_i + \frac{1}{2}\right) \geq 0$. Here "if" follows from the fact that $e + \sum_{i=1}^{n} A_i \left(p_i - \frac{1}{2}\right) \leq 0$ and $e + \sum_{i=1}^{n} A_i \left(p_i + \frac{1}{2}\right) \geq 0$ would imply that the segment whose endpoints are $p - \left(\frac{1}{2}, \cdots, \frac{1}{2}\right)$ and $p + \left(\frac{1}{2}, \cdots, \frac{1}{2}\right)$ must intersect H, and hence that $\mathbb{V}(p)$ (which is convex and therefore contains that segment) intersects H. The "only if" part follows from the fact that, for $1 \leq i \leq n$, $p_i - \frac{1}{2} \leq x_i \leq p_i + \frac{1}{2}$ for all $x \in \mathbb{V}(p)$: if $e + \sum_{i=1}^{n} A_i \left(p_i - \frac{1}{2}\right) > 0$ then we would have that $e + \sum_{i=1}^{n} A_i x_i > 0$ for all $x \in \mathbb{V}(p)$, so that $\mathbb{V}(p) \cap H = \varnothing$; symmetrically we would have that $\mathbb{V}(p) \cap H = \varnothing$ if $e + \sum_{i=1}^{n} A_i \left(p_i - \frac{1}{2}\right) < 0$. ■

Corollary 8 *: The supercover of a hyperplane is a discrete analytical object.*

When we compare the Supercover of a hyperplane to other discrete hyperplanes, we notice that it is not, in general, a Reveillès discrete analytical hyperplane [25, 2, 4]. There is a small difference in the definitions. A Reveillès discrete analytical hyperplane H is a discrete object satisfying a double inequality [25, 2, 4]:

$$H = \left\{ p \in \mathbb{Z}^n \middle| \alpha \leq e + \sum_{i=1}^{n} A_i p_i < \beta \right\}; \text{ where } \alpha, \beta, e \in \mathbb{R} \text{ and } A \in \mathbb{R}^n$$

One of the characteristics of a A Reveillès discrete analytical hyperplane is its arithmetical thickness defined by $\omega = \beta - \alpha$. There is however an interesting subset of hyperplanes, the rational hyperplanes, for which the supercover are Reveillès discrete analytical hyperplane [4]. A hyperplane or discrete analytical hyperplane is said to be rational if it has a normal vector whose components are all integers. See [25] for a study of 2D rational discrete analytical straight lines, [2, 19, 20] for a discussion on a specific class of 3D discrete analytical planes and [4] for a study of the rational discrete analytical hyperplanes in arbitrary dimensions. Theorem 7 implies that the supercover of any rational hyperplane is a discrete analytical hyperplane.

Corollary 9 *: Let $H : e + \sum_{i=1}^{n} A_i x_i = 0$ be a rational hyperplane in $\mathbb{R}^n$ with $A \in \mathbb{Z}^n, e \in \mathbb{R}$. Then the supercover of H is equal to:*

$$\mathbb{S}(H) = \left\{ X \in \mathbb{Z}^n \middle| \left\lceil -\frac{\sum_{i=1}^{n} |A_i|}{2} - e \right\rceil \leq \sum_{i=1}^{n} A_i X_i \leq \left\lfloor \frac{\sum_{i=1}^{n} |A_i|}{2} - e \right\rfloor \right\}$$

The proof of corollary 9 is immediate. Thus rational hyperplanes have a supercover that is characterised by Diophantine inequalities. One of the reasons why this is of interest is of course that, in a finite part of space, a discrete hyperplane is always rational.

Corollary 10 *: The arithmetical thickness of the supercover of the rational hyperplane H : $e + \sum_{i=1}^{n} A_i x_i = 0$ is $\omega = \sum_{i=1}^{n} |A_i|$ or $\omega = \sum_{i=1}^{n} |A_i| + 1$.*

The proof of this result is simple. The arithmetical thickness is equal to $\sum_{i=1}^{n} |A_i| + 1$, when $\sum_{i=1}^{n} |A_i|$ is even and e is an integer, or, when $\sum_{i=1}^{n} |A_i|$ is odd and $e - \frac{1}{2}$ is an integer. In all other cases the arithmetical thickness is equal to $\sum_{i=1}^{n} |A_i|$. Discrete analytical hyperplanes that have an arithmetical thickness equal to $\sum_{i=1}^{n} |A_i|$ are called *standard hyperplanes* [19]. This is an important class of discrete analytical hyperplanes. See [4, 19, 20, 8, 9] for details on standard hyperplanes.

4.2 Supercover of a point

The supercover of a Euclidean point in dimension n is given by:

Proposition 11 *: The supercover of a Euclidean point $y \in \mathbb{Z}^n$ is equal to:*

$$\mathbb{S}(y) = \mathbb{Z}^n \cap \left[\left\lceil y_1 - \frac{1}{2}\right\rceil, \left\lfloor y_1 + \frac{1}{2}\right\rfloor\right] \times \cdots \times \left[\left\lceil y_n - \frac{1}{2}\right\rceil, \left\lfloor y_n + \frac{1}{2}\right\rfloor\right]$$

Proof. We first consider the case in which $y \in \mathbb{Z}^1$. The point y is a 0-flat $\in \mathbb{Z}^1$. A normal vector of the 0-flat is $A = 1$ and therefore y is a rational 0-flat. We can apply corollary 9 (with $n = 1$, $A_1 = 1$ and $e = -y$) to deduce that the supercover of y is equal to $\mathbb{S}(y) = \left\{x \in \mathbb{Z} \middle| \left\lceil y - \frac{1}{2}\right\rceil \leq x \leq \left\lfloor y + \frac{1}{2}\right\rfloor\right\}$. This and Proposition 3 imply Proposition 11. ■

The supercover of a Euclidean point can be equal to more than one single discrete point. This is what we call a "bubble" [6, 8, 9]. This is one of the reasons why, despite its properties, the supercover is not often used as a discretization scheme. Classically, in computer graphics [10], one of the discrete points is, arbitrarily, chosen as the discretisation of the Euclidean point. This leads to asymmetries and the loss of many of the properties of the supercover [13]. The way to obtain a model without bubbles is given by the *standard analytical discretization model* [8, 9].

Another way of considering proposition 11 is to see it as a special case of Theorem 13 (that is presented in the next section) since proposition 4 tell us that $p \in \mathbb{S}(F)$ iff for all $j \in \mathbb{J}^n_{n-(m+1)}$, $\pi_j(p) \in \mathbb{S}(\pi_j(F))$.

Corollary 12 *: The supercover of a point is a discrete analytical object.*

This is a straightforward corollary of proposition 11.

4.3 Supercover of an m-flat for $m \leq n-2$

The m-flat supercover, for $0 \leq m \leq n-2$, is given by:

Theorem 13 *: Let F be an m-flat in $\mathbb{R}^n$, where $0 \leq m \leq n-2$. Then:*

$$\mathbb{S}(F) = \bigcap_{j \in \mathbb{J}^n_{n-(m+1)}} \mathbb{S}(\varepsilon_j(F))$$

We have that $\mathbb{S}(F \cup G)$ is equal to $\mathbb{S}(F) \cup \mathbb{S}(G)$ but in general $\mathbb{S}(F \cap G)$ is not equal to $\mathbb{S}(F) \cap \mathbb{S}(G)$. Theorem 13 provides an example of a collection of sets for which the supercover of the intersection is the intersection of the sets' supercovers. Indeed, since $F \subseteq \varepsilon_j(F)$ it is evident that $\mathbb{S}(F) \subseteq \mathbb{S}\left(\bigcap_{j \in \mathbb{J}^n_{n-(m+1)}} \varepsilon_j(F)\right) \subseteq \bigcap_{j \in \mathbb{J}^n_{n-(m+1)}} \mathbb{S}(\varepsilon_j(F))$ and, in view of Theorem 13, this implies that $\mathbb{S}(F) = \mathbb{S}\left(\bigcap_{j \in \mathbb{J}^n_{n-(m+1)}} \varepsilon_j(F)\right) = \bigcap_{j \in \mathbb{J}^n_{n-(m+1)}} \mathbb{S}(\varepsilon_j(F))$ holds.

Proof. Since $F \subseteq \varepsilon_j(F)$ for any j, we see that $\mathbb{S}(F) \subset \mathbb{S}(\varepsilon_j(F))$ and therefore $\mathbb{S}(F) \subset \bigcap_{j \in \mathbb{J}^n_{n-(m+1)}} \mathbb{S}(\varepsilon_j(F))$.

To prove the reverse inclusion, let p be any point in $\bigcap_{j\in\mathbb{J}^n_{n-(m+1)}} \mathbb{S}(\varepsilon_j(F))$. We need to show that $p \in \mathbb{S}(F)$. Since $p \in \bigcap_{j\in\mathbb{J}^n_{n-(m+1)}} \mathbb{S}(\varepsilon_j(F))$, we have that $\mathbb{V}(p)\cap\varepsilon_j(F) \neq \varnothing$ for every $j \in \mathbb{J}^n_{n-(m+1)}$. Equivalently, we have that

$$\text{for all } j \in \mathbb{J}^n_{n-(m+1)}, \varepsilon_j(\mathbb{V}(p)) \cap F \neq \varnothing \tag{1}$$

Now, if j and k are any subsequences of $\{1, 2, \ldots, n\}$, then $\varepsilon_j(\mathbb{V}(p)) \cap \varepsilon_k(\mathbb{V}(p)) = \varepsilon_{j\cap k}(\mathbb{V}(p))$. Moreover, the intersection of any $m+1$ distinct members of $\mathbb{J}^n_{n-1}$ is a member of $\mathbb{J}^n_{n-(m+1)}$. It follows from these two observations that the intersection of any $m+1$ distinct members of the collections of sets $\left\{\varepsilon_j(\mathbb{V}(p)) \cap F \,\middle|\, j \in \mathbb{J}^n_{n-1}\right\}$ is a member of the collection of sets $\left\{\varepsilon_j(\mathbb{V}(p)) \cap F \,\middle|\, j \in \mathbb{J}^n_{n-(m+1)}\right\}$ and therefore is non-empty, by (1).
Since $\left\{\varepsilon_j(\mathbb{V}(p)) \cap F \,\middle|\, j \in \mathbb{J}^n_{n-1}\right\}$ is a collection of convex sets in the m-dimensional affine set F, and we have just seen that every subcollection of $m+1$ of these sets has a nonempty intersection, it follows from Helly's theorem [16] that the entire collection of sets has a nonempty intersection. As the intersection is $F\cap \bigcap_{j\in\mathbb{J}^n_{n-1}} \varepsilon_j(\mathbb{V}(p)) = F \cap \mathbb{V}(p)$, we have that $F \cap \mathbb{V}(p) \neq \varnothing$. This proves that $p \in \mathbb{S}(F)$. ■

Corollary 14 *: Let F be an m-flat in $\mathbb{R}^n$, where $m \leq n-2$, and let l be an integer such that $m < l < n$. Then:*

$$\mathbb{S}(F) = \bigcap_{j\in\mathbb{J}^n_{l-m}} \mathbb{S}(\varepsilon_j(F))$$

Proof. The non-trivial "$\supseteq$" part of corollary 14 follows immediately from theorem 13; indeed, if p lies in the intersection of corollary 14 then p lies in the intersection of theorem 13 since, if j is any subsequence of $\{1, 2, \ldots, n\}$ of length $n-1-m$ then there is a subsequence j' of j of length $l-m$ and we have that $p \in \mathbb{S}(\varepsilon_{j'}(F)) \subseteq \mathbb{S}(\varepsilon_j(F))$. ■

4.4 Discrete analytical characterization of the supercover of an m-flat

Let us show that the supercover of an m-flat, for $0 \leq m \leq n-1$, is a discrete analytical object. For that we will present a method that determines explicitely all the discrete inequalities characterising the supercover of an m-flat.

Recursive method : Let F be an m-flat in $\mathbb{R}^n$ specified in a parametric form.

a) If F is a 0-flat in $\mathbb{R}^n$, we directly apply Proposition 11 which gives us $2n$ discrete inequalities that characterise $\mathbb{S}(F)$.

b) If F is an $(n-1)$-flat in $\mathbb{R}^n$, we transform the parametric form of F into an analytical form. Then apply Theorem 7, which gives us 2 discrete inequalities that characterise $\mathbb{S}(F)$.

c) If F is an m-flat in $\mathbb{R}^n$, with $0 < m < n-1$, use Theorem 13 as follows. For each $j \in \mathbb{J}^n_{n-1-m}$, we recursively apply the method with $\mathbb{R}^{m+1}$ in place of $\mathbb{R}^n$ and with a parametric specification of $\pi_j(F)$ in place of the parametric specification of F, to produce a discrete analytical characterisation of $\mathbb{S}(\pi_j(F))$. From the discrete analytical characterisation of $\mathbb{S}(\pi_j(F))$ in $\mathbb{R}^{m+1}$, we use Proposition 6 to derive a discrete analytical characterisation of $\mathbb{S}(\varepsilon_j(F))$ for all $j \in \mathbb{J}^n_{n-1-m}$; it is a trivial matter to compute r_j. From the discrete

analytical characterisation of $\mathbb{S}(\varepsilon_j(F))$ for all $j \in \mathbb{J}^n_{n-1-m}$, we use Theorem 13 to derive a discrete analytical characterisation of $\mathbb{S}(F)$.

Theorem 15 *: The supercover of an m-flat is a discrete analytical object.*

Proof. Let us consider an m-flat F and let us apply the recursive method described above. Let us prove that the method terminates always with a discrete analytical description of $\mathbb{S}(F)$. This will prove that the supercover of an m-flat is a discrete analytical object.
In case a) and case b) we obtain an explicit discrete analytical description of $\mathbb{S}(F)$ in $\mathbb{R}^n$. This is also true in case c), assuming that the recursive calls which are made in that case all terminate. Note that the dimension n of the ambient space must be greater than 2 in case c), the only case which makes recursive calls. Moreover, when case c) makes a recursive call it sets the dimension of the mabient space in the recursive call to $m = 1$, which is strictly less than the dimension n of the ambient space in the caller. It follows that the depth of recursion is bounded by $N - 2$, where N is the dimension of the ambient space in the outermost call. So the recursive call must indeed terminate. ■

5 Conclusion

In this paper we propose to take a new look at an old discretisation method called the supercover. The supercover of a Euclidean object F is the set of discrete points consisting of all the voxels that intersect F. The supercover has the advantage, over many other discretisation schemes used in computer graphics, of being defined in an arbitrary dimension and for arbitrary objects. The principal aim of this paper is to prove that the supercover of an m-flat is a discrete analytical object. A method is proposed to determine a set of discrete inequalities that define the supercover of an m-flat.
The next step will consist in looking at the supercovers of finite objects, such as simplices of dimension k in a space of dimension n. Finite discrete objects are, in practice, more useful in computer graphics than infinite discrete objects. They can be generated and an important part of the planned work on the supercovers of simplices will consist in designing efficient generation algorithms. In order to devise such generation algorithms, it would be helpful to have a discrete analytical characterisation of the supercover of a simplex.
We will propose now in conclusion, as a conjecture, such a discrete analytical characterisation.
Let us consider a simplex A of dimension k in a space of dimension n defined by $k+1$ linearly independant Euclidean points: $A = simp\left(P^0, \ldots, P^k\right)$ with $P^i \in \mathbb{R}^n$ for $0 \leq i \leq k$. Let us denote by $\overline{A}$ the k-flat defined by $P^0, \ldots, P^k$. If $n = k$ then, by definition, $\overline{A} = \mathbb{R}^k$. Let us denote by $E\left(A, P^i\right)$ the half-space whose border is the $(n-1)$-flat defined by $\left\{P^0, \ldots, P^{i-1}, P^{i+1}, \ldots, P^n\right\}$, and which contains the point P^i.

Conjecture 16 *: For any symplex $A = simp\left(P^0, \ldots, P^k\right) \subset \mathbb{R}^n$,*

- *If $k = n$ then* $\mathbb{S}(A) = \left(\bigcap_{i=0}^n \mathbb{S}\left(E\left(A, P^i\right)\right)\right) \cap \left(\bigcap_{j=1}^n \mathbb{S}\left(\varepsilon_j(A)\right)\right)$;
- *If $k = n-1$ then* $\mathbb{S}(A) = \mathbb{S}\left(\overline{A}\right) \cap \left(\bigcap_{j=1}^n \mathbb{S}\left(\varepsilon_j(A)\right)\right)$;
- *If $k \leq n-2$ then* $\mathbb{S}(A) = \bigcap_{j \in \mathbb{J}^n_{n-k-1}} \mathbb{S}\left(\varepsilon_j(A)\right)$.

There is some information about this conjecture in dimensions 2 and 3 in [3, 5, 6].

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